§3 IRROTATIONAL FLOWS, aka POTENTIAL FLOWS

Irrotational flows are also known as 'potential flows' because the velocity field can be taken to be the gradient of a

§3.1 Velocity potential.

That is, an irrotational flow has a velocity field $\mathbf{u}(\mathbf{x},t)$ that can be represented in the form

$$\boxed{\mathbf{u} \;=\; \nabla\phi\ ,}$$

for some scalar field $\phi(\mathbf{x},t)$. The field $\phi(\mathbf{x},t)$ is called the *potential*, or *velocity potential*, for $\mathbf{u}$.

> **Note** the sign convention, opposite to the usual sign convention for *force* $\mathbf{F}$ and *force potential* Φ.
>
> **Note also**: it will prove useful to include cases in which $\phi(\mathbf{x},t)$ is a multi-valued function of its arguments.

A velocity field of the form $\mathbf{u} = \nabla\phi$ is, indeed, irrotational — it has vorticity $\boldsymbol{\omega} = 0$ — because curl grad of any scalar field is zero: $\boldsymbol{\omega} = \nabla\times\mathbf{u} = \nabla\times(\nabla\phi) = 0$.

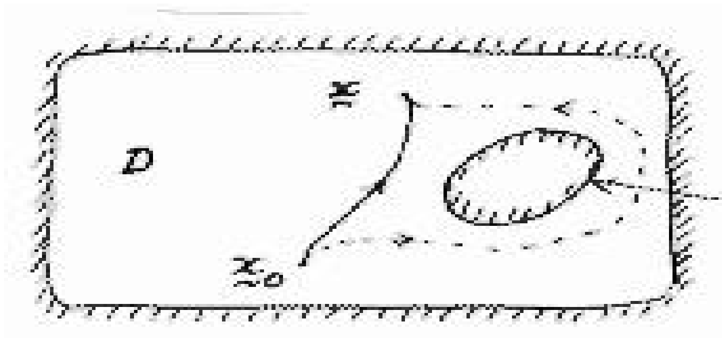
The converse — that any irrotational velocity field $\mathbf{u}(\mathbf{x},t)$ can be written as $\nabla\phi$ for some $\phi(\mathbf{x},t)$ — is also true, though less obvious.

Given $\mathbf{u}(\mathbf{x},t)$ in a *simply-connected* domain $\mathcal{D}$, we can construct $\phi(\mathbf{x},t)$ as a *single-valued* function. This will now be shown.

We are given an irrotational velocity field $\mathbf{u}(\mathbf{x},t)$. Choose some fixed point $\mathbf{x}_0 \in \mathcal{D}$ — $\mathbf{x}_0$ could be the origin of coordinates, or any other convenient choice — then define

$$\phi(\mathbf{x},t) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{u}(\mathbf{x}',t).\,d\mathbf{l}'\ ,$$

where the integral is taken along any path within $\mathcal{D}$ that joins $\mathbf{x}_0$ to $\mathbf{x}$, with $d\mathbf{l}'$ denoting the line element of the path at position $\mathbf{x}'$. This function $\phi(\mathbf{x},t)$ is an ordinary, single-valued function, being independent of the path of integration as long as $\mathcal{D}$ is simply-connected. The path-independence then follows from the fact that any two paths make up a closed curve around which the circulation $\mathcal{C} = \oint\mathbf{u}.\,d\mathbf{l}$ must be zero, by Stokes' theorem. (Since we are given that $\boldsymbol{\omega} = \nabla\times\mathbf{u} = 0$ everywhere, we have, in particular, that $\nabla\times\mathbf{u} = 0$ on a surface spanning the two paths.)



A simply-connected domain $\mathcal{D}$ is one in which any closed curve can be shrunk continuously to a point while staying in $\mathcal{D}$. The simplest example of a domain that is not simply connected is a 2-D domain with an island, as in the picture on the left. A closed curve encircling such an island cannot be shrunk to a point in the way envisaged, and there is no surface spanning such a curve and lying within the domain. A domain that is not simply-connected, as in the picture, is called multiply-connected.

In a multiply-connected domain, the path for the integral defining ϕ can go to one side or the other of an island; indeed it can wind round an island any number of times. Then ϕ can be multi-valued, and will be multi-valued whenever the circulation $\mathcal{C}$ for a closed curve encircling an island is nonzero: $\mathcal{C} = \oint_{\text{island}} \mathbf{u} \, . \, d\mathbf{l} \neq 0$. An example in which such nonzero circulation $\mathcal{C}$ of crucial importance, making multi-valued potentials ϕ useful, is the flow round a lifting aerofoil (bottom of p. 17, details in §3.8 below).

Kelvin's circulation theorem suggests — and experiments confirm — that there is a significant set of circumstances in which irrotational, or potential, flows are good models of real flows. Very often this is because, to good approximation, the flow has started from rest and because, furthermore, viscosity is negligible on all the relevant closed curves within the fluid domain $\mathcal{D}$.

Examples:

- Airflow just ahead of an aircraft, and above and below the wings
- Flow of water toward a small drainage hole in the bottom of a large tank containing water previously at rest
- Flow towards the end of a straw (thin tube) through which air is being sucked
- Water wave motion, for waves propagating into water previously at rest

All these examples depend — of course — on viscosity being small enough. Potential flow would be a bad, a grossly inaccurate, model for everyday (domestic-scale) flows of fluids like golden syrup or treacle.
*There is one exception to the last statement: viscous flow in a narrow gap between two planes.

For irrotational flow to be a good model we need $\nabla \times \mathbf{F}$ negligible in the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\mathbf{u} + \nabla \times \mathbf{F} \ .$$

where $\mathbf{F}$ includes, besides gravity, the force per unit mass from internal friction (viscosity).

In the simplest case of a uniformly viscous flow under gravity or other conservative force field, the viscous contribution to $\mathbf{F}$ can be shown to be equal to $\nabla^2\mathbf{u}$ times a constant coefficient called the 'kinematic viscosity' having dimensions of length^2/time. In that case, $\nabla \times \mathbf{F} \ \propto \ \nabla^2\boldsymbol{\omega}$ (which happens to be zero for potential flows — a fact that historically seems to have caused some confusion, for perhaps a century or so, until early in the twentieth century when pioneers in aerodynamics (especially Ludwig Prandtl) saw the importance of what happens to material curves very close to boundaries, where flow is hardly ever irrotational, and began to develop the 'boundary layer theory' needed to understand this. For the time being, in these lectures, we can adopt the usual attitude to scientific model-making: simply adopt a set of assumptions, including, in this case, the assumption that $\boldsymbol{\omega} = 0$ everywhere, then ask how far the resulting model fits reality in various cases. (For further discussion of, and insight into, scientific model-making as such — and such insight has great importance in today's and tomorrow's world — see the animated image on my home page, `http://www.atmos-dynamics.damtp.cam.ac.uk/people/mem/`, and associated links).

Let us now look at some basic properties of irrotational flows.

Mass conservation $\nabla.\mathbf{u} = 0$ gives $\nabla.(\nabla\phi) = 0$, i.e.,

$$\boxed{\nabla^2\phi \ = \ 0 \ ,}$$

i.e., ϕ is a harmonic function, in the sense of satisfying Laplace's equation.

The boundary condition for impermeable boundaries, also called the 'kinematic boundary condition', is that

$$\mathbf{U}.\mathbf{n} = \mathbf{u}.\mathbf{n} = \mathbf{n}.(\nabla\phi) \equiv \frac{\partial\phi}{\partial n} \ ,$$

where $\mathbf{U}$ is the velocity of points on boundary as before. *This exemplifies what is called a Neumann boundary condition for Laplace's equation.* If the flow is irrotational, this is the only boundary condition we need.

We can now apply techniques from Part IB Methods. For complex geometries, we can solve numerically — see Part II(a) or (b) Numerical Analysis.

§3.2 Some examples

Because ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, we can apply all the mathematical machinery of potential theory, developed in 1B Methods:

A: Spherical geometry, axisymmetric flow

Recall the general axisymmetric solution of Laplace's equation obtained by separation-of-variable methods in the usual spherical polar coordinates, by trying $\phi = R(r)\,\Theta(\theta)$, etc.,

$$\phi = \sum_{n=0}^{\infty} \left\{ A_n r^n + B_n r^{-(n+1)} \right\} P_n(\cos\theta),$$

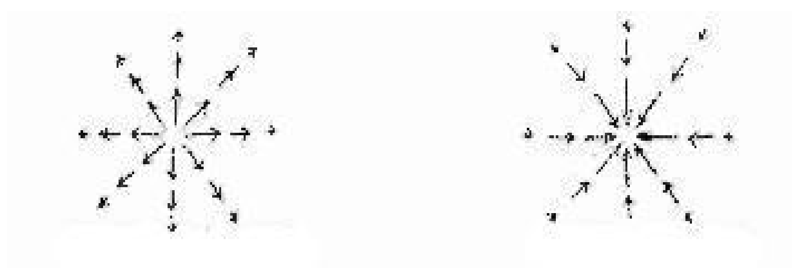
where A_n, B_n are arbitrary constants, P_n is the Legendre polynomial of degree n, r is the spherical radius ($r^2 = x^2 + y^2 + z^2$), so that $z = r\cos\theta$, and θ is the co-latitude. (Recall that $P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3}{2}\mu^2 - \frac{1}{2}$, etc.)

Let us look more closely at the kinds of flows represented by the first three nontrivial terms, those with coefficients B_0, A_1, and B_1.

(i) all A's, B's zero except B_0 [Notation: $\mathbf{e}_r$ = unit radial vector = $\mathbf{x}/|\mathbf{x}| = \mathbf{x}/r$]:

$$\phi = \frac{B_0}{r} = \frac{B_0}{|\mathbf{x}|}\,, \qquad \Rightarrow \qquad \mathbf{u} = \nabla\phi = -\frac{B_0\mathbf{x}}{|\mathbf{x}|^3} = -\frac{B_0\mathbf{e}_r}{r^2}\ .(*)$$

This represents source or sink flow, i.e., depending on the sign of the coefficient B_0, it represents radial outflow from a point source ($B_0 < 0$), or inflow to a point sink ($B_0 > 0$), at the origin. Mass appears or disappears, at the singularity at the origin, according as $B_0 \lessgtr 0$. The velocity field is radially symmetric in both cases:



source flow ($B_0 < 0$) sink flow ($B_0 > 0$)

Let us check that incompressibility and mass conservation are satisfied: the outward mass flux across any surface containing the origin should be independent of the choice of surface. For simplicity, take a sphere of radius R. The outward mass flux is $\rho\nabla\phi = -\rho B_0/R^2$ per unit area, hence $-(\rho B_0/R^2)(4\pi R^2)$ in total, $= -4\pi\rho B_0$, independent of R as expected.

(ii) all A's, B's zero except A_1:

$$\phi = A_1 r\cos\theta = A_1 z\,, \qquad \Rightarrow \qquad \mathbf{u} = \nabla\phi = A_1\mathbf{e}_z$$

where z is the Cartesian co-ordinate parallel to the symmetry axis, and $\mathbf{e}_z = (0,0,1)$ a unit axial vector. So this simply represents uniform flow in axial or z direction, a $\mathbf{u}$ field like this:

$$\begin{array}{cccccccccccccccc} \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{array} \qquad \longrightarrow z$$

(iii) all A's, B's zero except B_1:

$$\phi = \frac{B_1 \cos\theta}{r^2} = \frac{B_1 z}{r^3}\ , \qquad \Rightarrow \qquad \mathbf{u} = B_1 \left(\frac{\mathbf{e}_z}{r^3} - \frac{3z\mathbf{e}_r}{r^4} \right) (**)$$

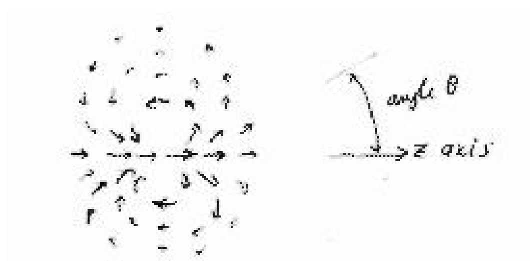
Note that apart from the different coefficient B_1 this is $-\partial/\partial z$ of case (i), eqs. $(*)$ above. (To check this, hold x and y constant in $r^2 = x^2 + y^2 + z^2$, then $2rdr = 2zdz \;\Rightarrow\; \partial r/\partial z = z/r \;\Rightarrow\; \partial(r^{-1})/\partial z = -z/r^3$, etc.)

So the solution $(**)$ can be regarded as the limiting case of a superposition — note that we *can* use superposition, because Laplace's equation is linear — a superposition of two copies of the solution $(*)$ above with their origins separated by a infinitesimal distance δz and with equal and opposite coefficients $B_0 = \pm B_1/\delta z$.

Specifically, $\phi = B_1 z/r^3$ can be regarded as the result of adding together the two potentials

$$\phi = \frac{-B_1/\delta z}{\{x^2 + y^2 + (z + \frac{1}{2}\delta z)^2\}^{1/2}} \qquad \text{and} \qquad \phi = \frac{+B_1/\delta z}{\{x^2 + y^2 + (z - \frac{1}{2}\delta z)^2\}^{1/2}}\ .$$

and then taking the limit $\delta z \downarrow 0$. The resulting solution $\phi = B_1 z/r^3$ is often, therefore, called a 'dipole', or occasionally a 'doublet' (and case (i), the single mass source or sink, is correspondingly called a 'monopole') (rarely a 'singlet'). The velocity field of the dipole looks like this, when $B_1 < 0$:



(iv) Uniform flow past a sphere $r = a$; this turns out to be a combination of (ii) and (iii) above:

Irrotational and incompressible, so $\nabla^2\phi = 0$ in $r > a$.

Uniform flow at ∞, so $\phi \to Ur\cos\theta$ as $r \to \infty$, i.e., tends toward case (ii) as $r \to \infty$.

No normal flow across surface of sphere, so $\partial\phi/\partial r = 0$ on $r = a$.

Now $\cos\theta$ $(= P_1(\cos\theta))$ is the only Legendre polynomial $\propto \cos\theta$. Therefore it is worth trying a solution that superposes (ii) and (iii) above, i.e. of form

$$\phi = (A_1 r + \frac{B_1}{r^2})\cos\theta$$

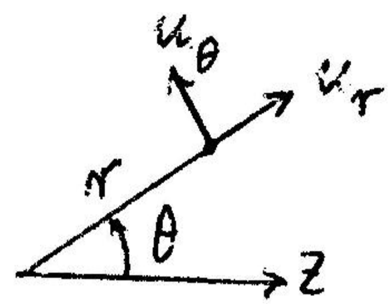
If this is to fit the boundary condition at ∞, we must choose $A_1 = U$.

If it is also to fit the boundary condition at $r = a$, then we must choose $\partial\phi/\partial r = (A_1 - 2B_1 r^{-3})\cos\theta = 0 \quad \forall\ \theta$, hence $A_1 = 2B_1 a^{-3}$, hence $B_1 = a^3 U/2$. Therefore

$$\text{solution is} \qquad \phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta\ .$$

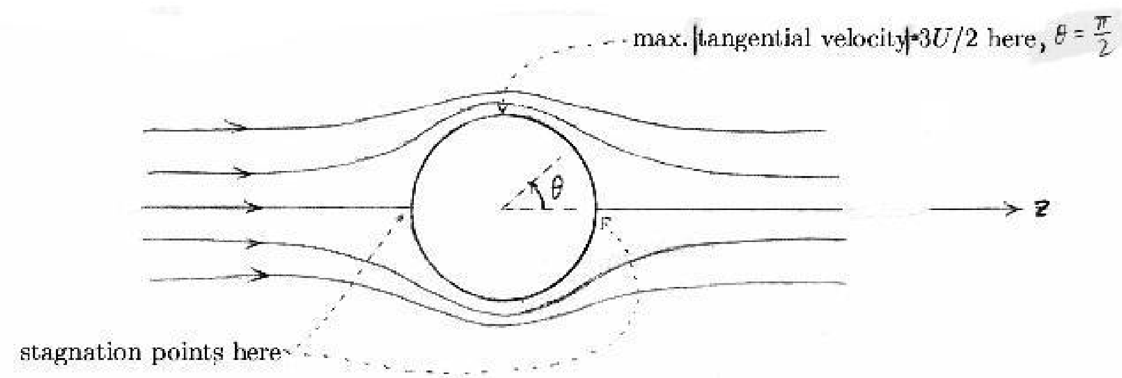
The corresponding velocity field $\nabla\phi$ — it is convenient to calculate its components in spherical polars, to check that we have satisfied the boundary condition at $r = a$ — is

$$\mathbf{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial r}\,,\ \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,,\ 0\right) = \left(U\cos\theta(1 - \frac{a^3}{r^3}),\quad -(1 + \frac{a^3}{2r^3})U\sin\theta,\ \ 0\right)$$



[As a consistency check, you can verify that these expressions for the spherical components of $\nabla\phi$ correspond to a linear combination of the velocity fields derived in (ii) and (iii) above when $A_1 = U$ and $B_1 = a^3 U/2$.]

*Experiment shows that this is *not* a good model for steady flow round a solid sphere, essentially because rotational fluid from near the surface is carried very quickly into the wake behind the sphere. But it *is* a good model when the solid sphere is replaced by a spherical bubble. It can also be a good model for a rapidly oscillating flow past a solid sphere, such as would be induced by a sound wave whose wavelength $\gg a$. This can be modelled by the above solution, unchanged except to make U a function of time that oscillates sufficiently rapidly about zero (so that fluid particles travel distances $\ll a$ during one oscillation, which will be the case if the oscillation frequency $\gg U/a$).*

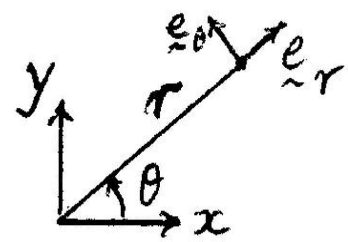


B: Cylindrical (circular 2-D) geometry

Much the same pattern as before, except that, as always in 2-D potential theory, the solutions involve logarithms as well as powers of the radial coordinate r; and potentials can be multi-valued. Also, there is no particular direction corresponding to the axis of symmetry in case **A**.

Recall the general solution obtained by separation of variables in 2-D cylindrical polars r, θ:

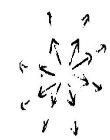
$$\phi = A_0 \log r + B_0\theta + \sum_{n=1}^{\infty} \left\{ A_n r^n \cos(n\theta + \alpha_n) + B_n r^{-n} \cos(n\theta + \beta_n) \right\}$$



[*or replace cosine terms by equivalent complex forms, terms $\propto e^{in\theta}$ times r^n or r^{-n}). Powers, or other analytic functions, of a complex variable $z = x + iy$ or its complex conjugate $\overline{z} = x - iy$ are solutions of Laplace's equation in two dimensions.*] Again, consider just the first few terms:

(i) all A's, B's zero except A_0:

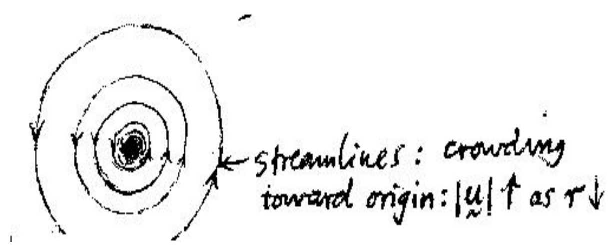
$$\phi = A_0 \log r \qquad \mathbf{u} = \nabla\phi = \frac{A_0 \mathbf{e}_r}{r} = \frac{A_0 \mathbf{x}}{r^2} \ ,$$



Represents radial source flow if $A_0 > 0$ — flow radially outward from a 2-D mass source (line source in 3-D). Total outflow across circle of radius R is $2\pi R \times (A_0/R) = 2\pi A_0$ (independent of R), 2-D version of case **A**(i) above. Radial sink flow (if $A_0 < 0$):

(ii) all A's, B's zero except B_0:

$$\phi = B_0\theta \qquad \mathbf{u} = \nabla\phi = \frac{B_0\mathbf{e}_\theta}{r} \ .$$



Note that ϕ is multi-valued — recall earlier remarks about islands in 2-D — whereas $\mathbf{u}$ is single-valued, as it must be in order to make physical sense.

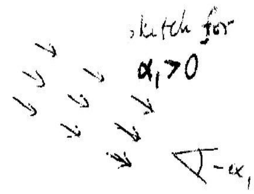
This represents circular flow, with flow speed B_0/r. The circulation $\mathcal{C} = \oint \mathbf{u}\,.\,d\mathbf{l}$ round any circle of radius R is nonzero. It is $2\pi R \times (B_0/R) = 2\pi B_0$ (independent of R), $= \kappa$, say.

This flow has $\nabla \times \mathbf{u} = 0$ *except* at $R = 0$, where there is a δ-function singularity in the vorticity, of strength κ. [This follows from Stokes' theorem applied to an arbitrarily small circle, or other closed curve, surrounding the origin.] The flow is sometimes described as a 'line vortex' — or a 'point vortex' on the understanding that we are imagining the physical domain to be a 2-D space. A superposition of such flows may serve as a simple model of more general vorticity distributions.

(iii) all A's, B's zero except A_1:

$$\phi = A_1 r\cos(\theta + \alpha_1) = A_1 r\,(\cos\theta\cos\alpha_1 - \sin\theta\sin\alpha_1) = A_1\,(x\cos\alpha_1 - y\sin\alpha_1) \ ,$$

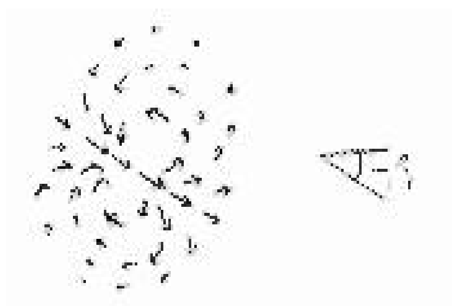
Represents uniform flow with velocity $U = A_1$ in direction $\theta = -\alpha_1$. Like spherical case **A**(ii).



(iv) all A's, B's zero except B_1:

$$\phi = B_1 \cos(\theta + \beta_1)\frac{r}{\cdot}$$

This represents a 2-D dipole pointing in the direction $\theta = -\beta_1$.



(Related to pair of mass-source solutions as before.)

(v) Uniform flow, with circulation, past cylinder ($r = a$):

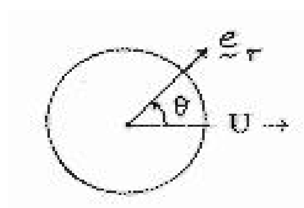
Irrotational and incompressible, so $\nabla^2\phi = 0$ in $r > a$.

Uniform flow U, plus an irrotational flow with circulation κ at ∞, so we require

$$\phi \to Ur\cos\theta + \frac{\kappa\theta}{2\pi} \qquad \text{as} \qquad r \to \infty$$

No normal flow across cylinder, so we also require

$$\frac{\partial\phi}{\partial r} = 0 \qquad \text{on} \qquad r = a\ .$$



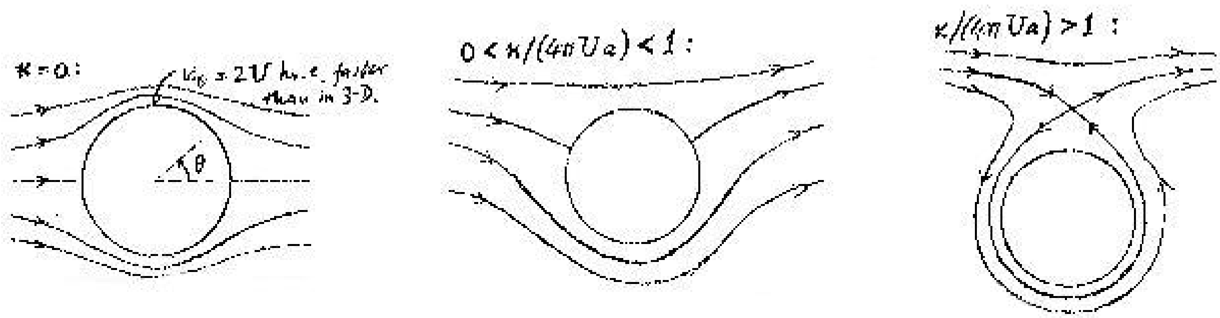
Solution:

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\kappa\theta}{2\pi}\ ; \qquad \kappa\ (= 2\pi B_0) \text{ is an arbitrary constant!}$$

Velocity field (in cylindrical polars):

$$\mathbf{u} \ = \ \left(\frac{\partial\phi}{\partial r},\ \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \ = \ \left(U\cos\theta\left(1 - \frac{a^2}{r^2}\right),\ \ -\ U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r}\right) .$$

Stagnation points (a) ($\mathbf{u} = 0$) on $r = a$ and $\sin\theta = \dfrac{\kappa}{4\pi a U}$, possible when $0 < |\kappa/(4\pi U a)| < 1$, and (b) where $\cos\theta = 0$, and r is s.t. $1 + \dfrac{a^2}{r^2} \mp \dfrac{\kappa}{2\pi U a}\dfrac{a}{r} = 0$ (quadratic equation for a/r, one real root in $r > a$ if $|\kappa/(4\pi U a)| > 1$).



§3.3 Pressure in time-dependent potential flows with conservative forces

Laplace's equation (*linear*) gives the solution for the flow, $\mathbf{u}$; but if we want to determine the pressure field p, then we must also use the momentum equation (Euler's equation, *nonlinear*):

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u}\right] = -\nabla p - \rho\nabla\Phi$$

We confine attention to conservative (potential) forces, and again use $(\mathbf{u}.\nabla)\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u} + \nabla(\tfrac{1}{2}|\mathbf{u}|^2)$, remembering that $\boldsymbol{\omega} = 0$ in potential flows, and that

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\phi = \nabla\left(\frac{\partial\phi}{\partial t}\right) .$$

Hence

$$\nabla\left\{\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi\right\} = 0$$

$\Rightarrow$

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi \ = \ \tilde{H}(t)\ , \quad \text{say,} \quad \text{independent of } \mathbf{x}.$$

So we have now found, in summary,

Two forms of Bernoulli's theorem

both applying to the flow of an inviscid, incompressible fluid under conservative (potential) forces $-\nabla\Phi$ per unit mass, and both being corollaries of the momentum equation written as

$$\frac{\partial \mathbf{u}}{\partial t} + \tfrac{1}{2}\nabla\left(|\mathbf{u}|^2\right) + \boldsymbol{\omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla\Phi\ .$$

The two forms, deduced in §2.4 and just above, respectively say that:

IF the flow is

steady, i.e. $\dfrac{\partial \mathbf{u}}{\partial t} = 0$,

irrotational, i.e. $\nabla\times\mathbf{u} = 0$,

then the quantity

$H = \tfrac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi$

$\tilde{H} = \dfrac{\partial\phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi$

obeys

$\mathbf{u}\,.\nabla(\tfrac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi) = 0$

$\nabla\left(\dfrac{\partial\phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi\right) = 0$

and hence

$\tfrac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi = H =$ constant on streamlines

$\dfrac{\partial\phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho} + \Phi = \tilde{H}(t)$, independent of spatial position $\mathbf{x}$

We may also summarize, and slightly generalize, the ways of representing the velocity field in various circumstances, as follows:

Stream function and velocity potential (§§1.8 & 3.1)

IF the flow is

incompressible, i.e. $\nabla\,.\,\mathbf{u} = 0$,

irrotational, i.e. $\nabla\times\mathbf{u} = 0$,

then there exists

*a vector potential $\mathbf{A}$ with $\mathbf{u} = \nabla\times\mathbf{A}$ *

a scalar potential ϕ with $\mathbf{u} = +\nabla\phi$

In 2-D (or axisymmetric) flow $\mathbf{A}$ has only one component, from which we define a *stream function* ψ (or Ψ)

(ϕ being called the velocity potential)

$\mathbf{u} \perp \nabla\psi$ (or $\nabla\Psi$) so
ψ (or Ψ) is constant on streamlines

$\mathbf{u} \parallel +\nabla\phi$

In a multiply connected domain in 2-D:

ψ may be multi-valued if $\exists$ islands/holes with net outflow: $\oint \mathbf{u}\,.\,\mathbf{n}\,ds \neq 0$

ϕ may be multi-valued if $\exists$ islands/holes with net circulation: $\oint \mathbf{u}\,.\,d\mathbf{l} \neq 0$

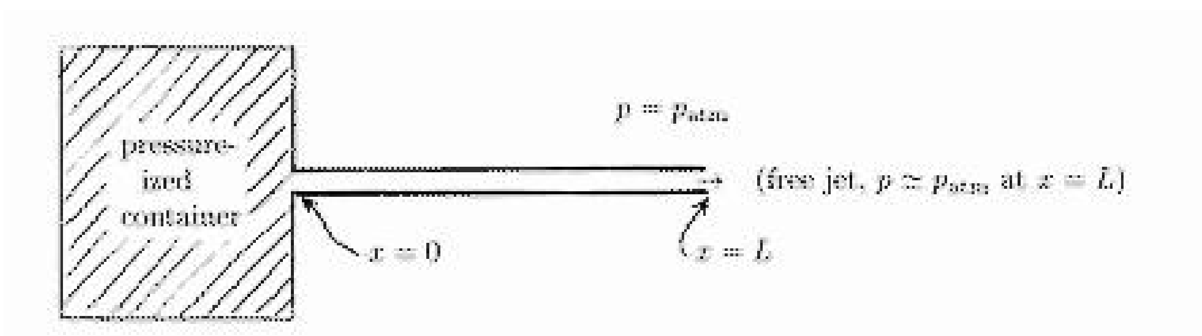
§3.4 Applications of irrotational, time-dependent Bernoulli

Inviscid, irrotational models can give useful insight into certain kinds of unsteady fluid behaviour and their timescales, relevant to laboratory and industrial fluid systems. The simplest examples include accelerating flows in tubes:

§3.4.1 Fast jet generator, version 1

In the situation sketched below we neglect gravity. The tube contains water, supplied from the container on the left. The flow starts from rest at time $t = 0$. The pressure p at $x = 0$, the left-hand end of the tube, is controlled by a feedback mechanism, not shown in the sketch; thus p at $x = 0$ is prescribed as a given function of time:

$$p\Big|_{x=0} = p_{\mathrm{atm}} + p_0(t)\;; \quad p_0(t) > 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad p_0(t) = 0 \quad \text{for} \quad t < 0\;.$$



Assume that the flow is unidirectional ($\parallel x$) in the tube and that the water emerges from the far end $x = L$ as a free jet in which the pressure $p \simeq p_{\mathrm{atm}}$, the ambient atmospheric pressure. (Not perfectly accurate, but experiment shows that it is a reasonable first-guess model of what actually happens; $p \simeq p_{\mathrm{atm}}$ in the emerging jet, because sideways accelerations, hence sideways ∇p components, are small.)

Because the flow starts from rest it is irrotational, and therefore has a velocity potential. Because the flow $\parallel x$ within the tube, the velocity potential and its time derivative must take the simple forms

$$\phi = u(t)\,x + \chi(t) \qquad \text{and} \qquad \dot{\phi} = \dot{u}(t)\,x + \dot{\chi}(t)\;,$$

where $\chi(t)$ is an arbitrary function of time t alone, and where u is the flow velocity along the tube. Note that ϕ cannot depend on y or z; otherwise the flow not $\parallel x$. So u cannot depend on y or z either; nor can u depend on x, otherwise $\nabla^2\phi \neq 0$ ($\nabla.\mathbf{u} \neq 0$, mass not conserved).

Apply time-dependent Bernoulli: $\tilde{H}$ constant along tube, $\tilde{H}\big|_{x>0} = \tilde{H}\big|_{x=0}$ = constant; so

$$\dot{u}\,x + \dot{\chi} + \tfrac{1}{2}u^2 + \frac{p}{\rho} \quad = \quad \dot{u}\,0 + \dot{\chi} + \tfrac{1}{2}u^2 + \frac{p_0(t)}{\rho} \ .$$

Notice the implication that $p = p(x,t)$ and moreover that $-\nabla p = (\rho\dot{u},\, 0,\, 0)$. (Pressure varies linearly with x; pressure *gradient* is uniform along the tube, as it has to be because fluid acceleration is uniform along the tube.) Taking $p = 0$ at $x = L$, we have (with several terms cancelling or vanishing)

$$\dot{u}\,L \qquad\qquad + \frac{0}{\rho} \quad = \quad \dot{u}\,0 \qquad\qquad + \frac{p_0(t)}{\rho} \ ,$$

so

$$\dot{u} \ = \ \frac{p_0(t)}{\rho L} \ , \qquad \Rightarrow \qquad u = \frac{1}{\rho L}\int_0^t p_0(t')\,dt' \ .$$

E.g. if p_0 = positive constant for $t > 0$, then

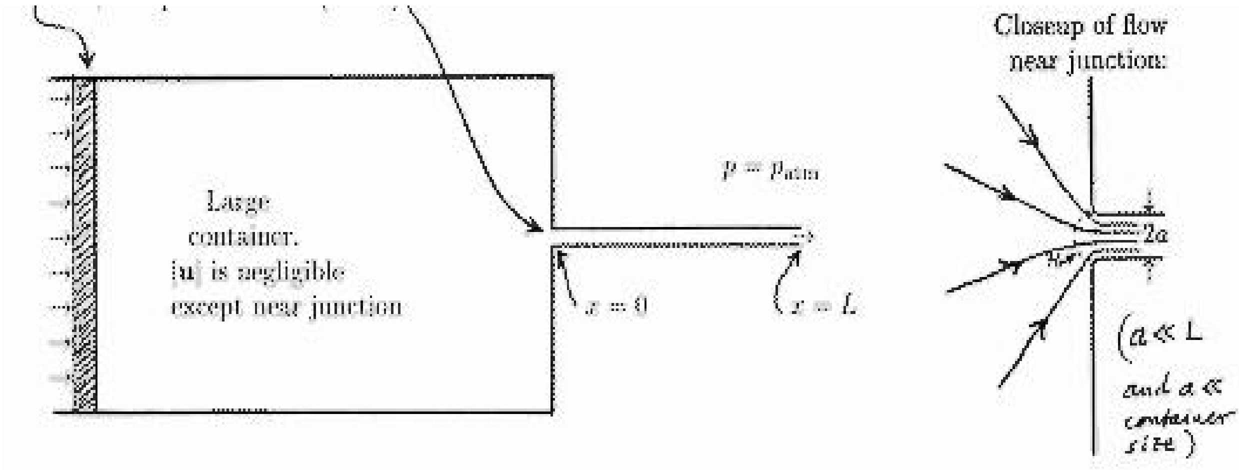
$$u \ = \ \frac{p_0}{\rho L}\, t \ .$$

So the flow keeps accelerating as long as this model remains applicable, i.e., as long as friction (viscosity) remains unimportant and as long as the excess pressure p_0 is maintained (by the pressurized container and the assumed control mechanism).

Notice that $\chi(t)$ has disappeared from the problem. Indeed we could have taken $\chi(t) = 0$ w.l.o.g. at the outset, on the grounds that only $\nabla\phi$ is of physical interest and that $\nabla\chi(t) = 0$. This would have changed the value of $\tilde{H}$, but not the thing that matters — the constancy of $\tilde{H}$.

§3.4.2 Fast jet generator, version 2

Same problem as before, except that pressure now controlled via force on piston to be $p_{\text{atm}} + p_0(t)$ *here*; so pressure *here* $(x = 0)$ is now unknown:



Assume that the flow is irrotational everywhere. Then $\mathbf{u}(\mathbf{x},t) = \nabla\phi(\mathbf{x},t)$ everywhere, $\tilde{H}$ is same constant everywhere. Within the tube, $\phi = u(t)x$ as before, now taking $\chi(t) = 0$ w.l.o.g. In the closeup view on the right, the flow outside a small hemisphere $\mathcal{H}$ of radius a (shown dotted) is approximately the same as simple mass-sink flow (**§3.2 A**, p. 25), with $\phi = \dfrac{\pi a^2 u(t)}{2\pi r} + \chi_1(t)$. (Yes, 2π not 4π: why?) Here $r^2 = x^2 + y^2 + z^2$, and $\chi_1(t)$ is *not* arbitrary. (Why? The whole flow has a single velocity potential $\phi(\mathbf{x},t)$, and we are not at liberty to introduce discontinuities!)

Indeed, solutions of $\nabla^2\phi = 0$ are very smooth. Detailed solution, beyond our scope here, (e.g. using the whole infinite series on p. 25), shows that the smooth function $\phi(\mathbf{x},t)$ is fairly well approximated if we pick $\chi_1(t)$ such that $\phi = 0$ on the hemisphere $\mathcal{H}$. Then $\chi_1(t) = -\dfrac{\pi a^2 u(t)}{2\pi a} = -\dfrac{a\,u(t)}{2}$.

In summary,

$$\phi(\mathbf{x},\,t) \quad = \quad \begin{cases} u(t)\,x & \text{within the tube, and} \\[2ex] \dfrac{\pi a^2 u(t)}{2\pi r} - \dfrac{a\,u(t)}{2} & \text{to the left of } \mathcal{H}\,. \end{cases} \qquad (*)$$

Therefore, as soon as we are far enough from the junction ($r \gg a$), $|\mathbf{u}|$ is negligible, and to good approximation ϕ is a function of time t alone: $\phi = -\frac{1}{2}a\,u(t)$. So we can use the corresponding value of $\dot{\phi}$, i.e., $-\frac{1}{2}a\,\dot{u}(t)$, when evaluating $\tilde{H}$ at the piston where the pressure is given. Now using the fact that $\tilde{H}$ has the same value at the emerging jet as it has at the piston, we have

$$\dot{u}\,L + \tfrac{1}{2}u^2 + \frac{0}{\rho} \quad = \quad -\,\tfrac{1}{2}a\,\dot{u}(t) + \;0\; + \frac{p_0(t)}{\rho}\ ,$$

Hence

$$(L + \tfrac{1}{2}a)\,\dot{u} + \tfrac{1}{2}u^2 \quad = \quad \frac{p_0(t)}{\rho}\ .$$

We can now see that the effect of the inflow at the junction is equivalent to extending the length of the tube very slightly — in fact, to a negligible extent, since we are assuming that $a \ll L$:

$$L\,\dot{u} + \tfrac{1}{2}u^2 \quad = \quad \frac{p_0(t)}{\rho}\ .(**)$$

(*We can also see that detailed, more accurate solution of $\nabla^2\phi = 0$ for the inflow near the junction can hardly change the picture: the factor $\frac{1}{2}$ multiplying $a\,\dot{u}$ will be replaced by a slightly different numerical coefficient, but this does not alter the negligible order of magnitude. Similar considerations govern the oscillatory flow near the end of an organ pipe: the pipe behaves acoustically as if it were slightly longer than it looks, by an amount roughly of the order of the pipe radius. Acousticians call this an 'end correction' to the length of the pipe.*)

Now $(**)$ is a nonlinear first-order ODE for the function $u(t)$, to be solved with initial condition $u(0) = 0$. For general $p_0(t)$ it can be solved only by numerical methods, but it can be solved analytically in the case where $p_0 =$ positive constant for $t > 0$. It is then a *separable* first-order ODE. Before solving it, let us tidy it up by defining u_0, $U(T)$ and T by

$$u_0 = \sqrt{2p_0/\rho}\ , \quad > 0 \ \text{ for } \ t > 0\ , \qquad T = u_0 t/2L\ , \quad \text{and} \quad U(t) = u(t)/u_0\ .$$

Then $(**)$ becomes

$$\frac{dU}{dT} \ = \ 1 - U^2\ ,$$

to be solved with initial condition $U(0) = 0$.

The constants u_0 and $2L/u_0$ can reasonably be called the natural velocity scale and timescale for this problem.

Solving by separation of variables then partial fractions, we have

$$\int dT = \int \frac{dU}{1-U^2} = \int \frac{dU}{2}\left(\frac{1}{1+U} + \frac{1}{1-U}\right) = \tfrac{1}{2}\log\left(\frac{1+U}{1-U}\right) + \text{const.}\ ;$$

so

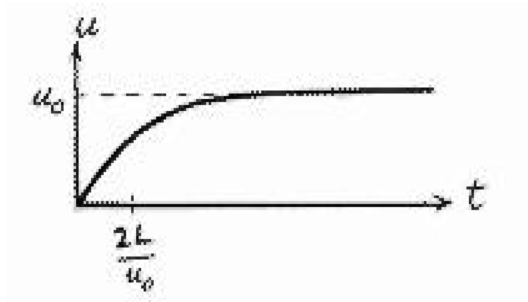
$$\frac{1+U}{1-U} \propto \text{const.} \times e^{2T}; \ \ \text{initial conditions} \Rightarrow \text{const. of proportionality} = 1\ ;$$

therefore

$$U \quad = \quad \frac{e^{2T}-1}{e^{2T}+1} \quad = \quad \tanh T\ ;$$

$$fail) \qquad u$$

$$= \quad u_0 \tanh\left(\frac{u_0 t}{2L}\right)\ .$$



So this second version of the jet problem is very different from the first version! Even in an frictionless (inviscid) model, the nonlinear term $\frac{1}{2}u^2$ in the ODE (∗∗) limits the jet speed to a finite maximum value u_0, determined entirely by the imposed pressure p_0.

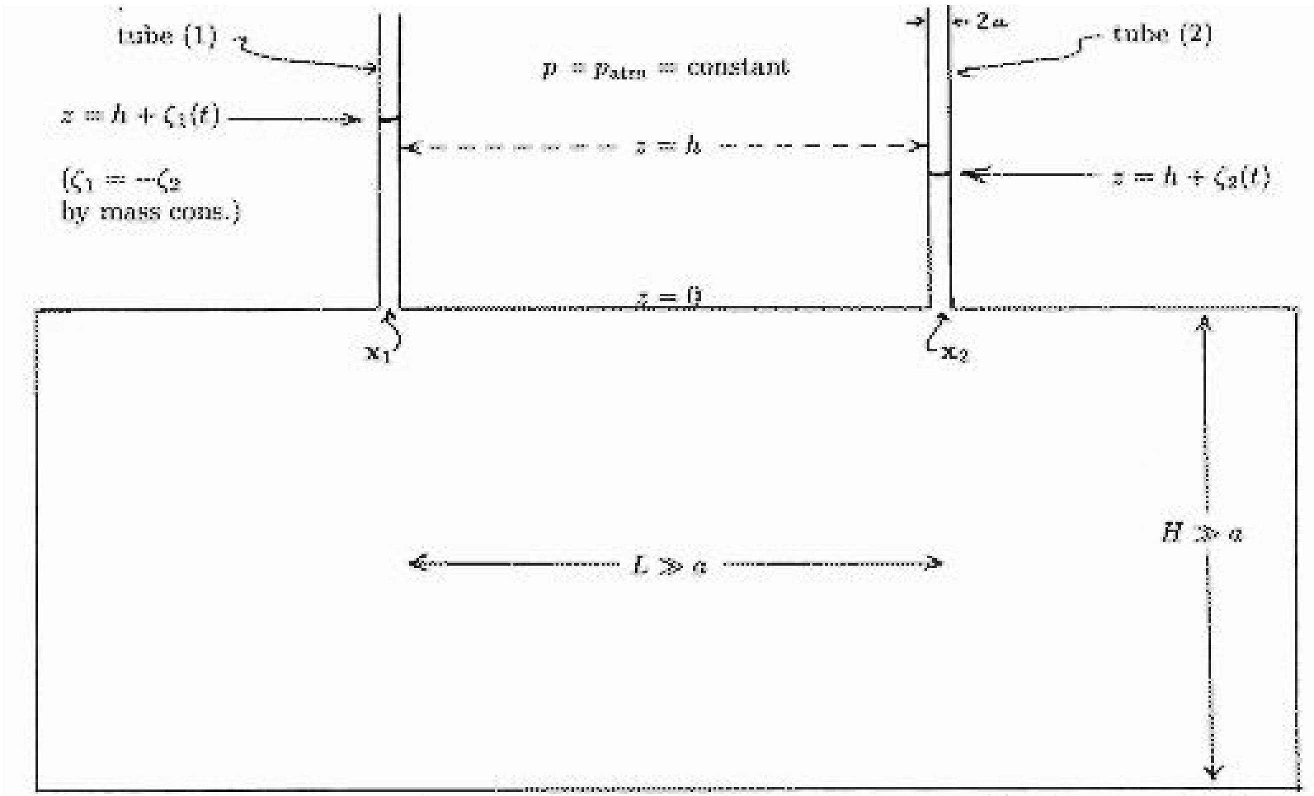
There is a similar problem in which gravity is important, and acts in place of the large piston. The water in the (large) container has a free upper surface at vertical distance h, say, above the (horizontal) tube. The answer is the same except that $u_0 = \sqrt{2gh}$ instead of $\sqrt{2p_0/\rho}$ (sheet 2 Q7).

§3.4.3 Inviscid manometer oscillations

The water, mercury, or other fluid in a U-tube manometer can oscillate under gravity. Inviscid, irrotational models give useful estimates of the period of oscillation. One example is the following fluid system, in which the bottom of the U-tube is replaced by a large reservoir.

Take coordinates such that the top of the reservoir and bottom of each tube are at $z = 0$. Let the equilibrium position of the free surface in each tube, with the fluid everywhere at rest, be $z = h$. Assume $a \ll h$ and $a \ll$ size of reservoir.

Consider motion starting from an initial state of rest with one free surface higher than the other, as shown. In the real world, one could hold the system in such a state by pumping a little air into tube (2) and temporarily sealing the top; the problem is to investigate the oscillations excited when the seal is suddenly removed.



Once again, we argue that the flow starts from rest, hence remains irrotational (in this inviscid fluid model), hence may be described by a velocity potential ϕ. Just as before, we take the flow to be uniform within each tube, $\mathbf{u} = (0, 0, w)$, a function of time alone. Mass conservation requires that, in the first tube, say, $w = \dot{\zeta}_1$ where the dot denotes the time derivative, and likewise in the second tube, where $w = \dot{\zeta}_2 = -\dot{\zeta}_1$. The last relation, $\dot{\zeta}_2 = -\dot{\zeta}_1$, follows from overall mass conservation for the entire, rigidly bounded, system.

In tube (1), uniform flow $\mathbf{u} = (0, 0, \dot{\zeta}_1) \quad \Rightarrow \quad \phi = \phi_1(z, t) = \dot{\zeta}_1(t)z$;

In tube (2), uniform flow $(0, 0, \dot{\zeta}_2) \quad \Rightarrow \quad \phi = \phi_2(z, t) = \dot{\zeta}_2(t)z + \chi_1(t)$.

The additive function of time alone makes no difference to the velocity field, but as before is relevant to the Bernoulli quantity $\tilde{H}$, and so needs to be included — though we shall find, in the same way as before, that it is negligible. (Once again, we must remember, there is a single, and in this problem single-valued, potential function $\phi(\mathbf{x}, t)$ — single-valued because the fluid domain is simply connected — describing the flow in the whole system. The above expressions merely give local approximations to that single function, applicable only within tube (1) or tube (2).

Now apply the constancy of $\tilde{H}$:

$$\tilde{H} = \frac{\partial \phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + gz \quad \text{is spatially uniform (though possibly time-dependent).}$$

In particular, $\tilde{H}$ must have the same values at the two free surfaces, i.e. at $z = h + \zeta_1$ in tube (1) and $z = h + \zeta_2$ in tube (2). Therefore

$$\left(\ddot{\zeta}_1 z + \tfrac{1}{2}\dot{\zeta}_1^2 + \frac{p_{\rm atm}}{\rho} + gz\right)\Big|_{z=h+\zeta_1} = \left(\ddot{\zeta}_2 z + \dot{\chi}_1 + \tfrac{1}{2}\dot{\zeta}_2^2 + \frac{p_{\rm atm}}{\rho} + gz\right)\Big|_{z=h+\zeta_2}$$

$\Rightarrow$

$$\ddot{\zeta}_1(h+\zeta_1) + \tfrac{1}{2}\dot{\zeta}_1^2 + gh + g\zeta_1 = \ddot{\zeta}_2(h+\zeta_2) + \tfrac{1}{2}\dot{\zeta}_2^2 + gh + g\zeta_2 + \dot{\chi}_1\ ,$$

As already noted, $\zeta_1 + \zeta_2 = 0$, hence $\zeta_1\ddot{\zeta}_1 = \zeta_2\ddot{\zeta}_2$ and $\dot{\zeta}_1^2 = \dot{\zeta}_2^2$. Hence all the nonlinear terms cancel:

$$\ddot{\zeta}_1 h + g\zeta_1 = \tfrac{1}{2}\dot{\chi}_1\ .(*)$$

By considering the flow near the junctions of the tubes with the reservoir, we can show, just as before, that the right-hand side is small of order a/h relative to the first term on the left-hand side, and can therefore be neglected to a first approximation.

The equation $(*)$, with its r.h.s. replaced by zero, implies that the oscillations are sinusoidal of period $2\pi(h/g)^{1/2}$; e.g. 1 second when $h = 25\,\text{cm}$ and $g = 980\,\text{cm}\,\text{s}^{-2}$.

Note that there is no formal restriction to small amplitude, because of the cancellation of the nonlinear terms. (That cancellation, however, depends on our assumption that the two tubes have the same cross-sectional area. Ex. Sheet 2 Q8 is an example where this does not apply, and where the nonlinearities are therefore significant, just as they are in version 2 of the jet problem.)

What has just been shown can be thought of in the following way. For the purpose of understanding the manometer oscillations, we may pretend that tubes (1) and (2) are joined not by an enormous reservoir, but — surprising, isn't it? — by a very short extra length of tubing, which the above approximate analysis (p. 32) says has total length $\frac{1}{2}a$ (if it has radius a). More refined analyses say it has total length a times a modest numerical factor, not quite $\frac{1}{2}$. The main conclusion, which relies only on the assumption $a \ll h$, is unaffected.

(*But if the joining tube is *constricted* to a cross-sectional area $\ll \pi a^2$ then this *adds* significant inertia, hence equivalent to *longer* joining tube of radius a; cf. trumpet mouthpiece.*)

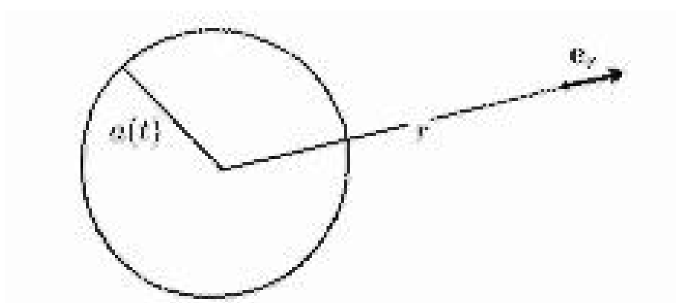
*More refined analyses (which again can be found in acoustics textbooks under the heading 'end corrections') show that equation $(*)$ and its refinements is still linear. The velocity squared terms in the Bernoulli quantity $\tilde{H}$ still cancel between the two regions where the tube flows merges into source or sink flow. Again, this cancellation holds only if the geometries of the two tubes and their outlets are the same; otherwise, if the tube geometries differ from each other, the nonlinear terms won't cancel and the theory will then apply to small oscillations only. For real fluids, there are additional reasons why the theory does not apply accurately at large amplitude (flow separation near the tube outlets, producing eddying, *rotational* flow), though it still gives a correct idea of the timescale of the oscillations.*

§3.4.4 Bubbles and cavities: oscillations and collapse

This is another practically important case of unsteady flow where inviscid, irrotational fluid models are useful, and indeed quite accurate in some circumstances. You have probably heard the musical notes, the plink, plonk sounds, that typically occur when water drips into a tank. These are due to the kind of bubble oscillations we consider here. The collapse of cavities, temporary bubbles with near-zero interior pressure, is an important consideration in the design of technologies that use high-energy liquid flow, e.g. in ship propeller design. Gravity is negligible in all these problems; and irrotational fluid models capture much of what happens.

*Such collapse can produce extraordinary concentrations of energy, and if it occurs near boundaries it can cause significant damage to the boundary material. In certain other cases (not near boundaries), local energy densities can reach such high values that atoms near the origin are excited and give off photons — a phenomenon called 'sonoluminescence'. For the latest on this, see *Nature* **409**, 782, in the Human Genome Issue of 15 February 2001.*

Consider a $\mathbf{e}_r$
spherical bubble
of radius $a(t)(t)$
centred at the origin:



Suppose that the bubble changes its radius, at rate $\dot{a}(t)$, while remaining spherical. Then the surrounding flow is the same as that due to a spherically symmetric mass source ($\dot{a} > 0$) or sink ($\dot{a} < 0$) flow in $r > a$, i.e., $\phi \propto 1/r$. Treat $r = a$ as an impermeable boundary moving with velocity $\mathbf{U} = \dot{a}\mathbf{e}_r$. The kinematic boundary condition $\mathbf{u}.\mathbf{n} = \mathbf{U}.\mathbf{n}$ at $r = a$ (with $\mathbf{n} = \mathbf{e}_r$) gives

$$\phi = -\frac{a^2\dot{a}}{r}\ ; \qquad \mathbf{u} = \nabla\phi = \frac{a^2\dot{a}}{r^2}\mathbf{e}_r\ .$$

The pressure field is important. Use time-dependent Bernoulli: values of $\tilde{H} = \dfrac{\partial\phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \dfrac{p}{\rho}$ are the same for all $r > a$ (gravity neglected). As $r \to \infty$, all the terms in $\tilde{H}$ go to zero except possibly the p term, which (because $\tilde{H}$ is constant) must tend to a finite limit $p(\infty, t)/\rho$, say; so

$$\frac{\partial}{\partial t}\left(-\frac{a^2\dot{a}}{r}\right) + \frac{a^4\dot{a}^2}{2r^4} + \frac{p(r,\,t)}{\rho} \;=\; \frac{p(\infty,\,t)}{\rho}$$

(Remember, $\partial/\partial t$ has $\mathbf{x}$, hence r, constant:) first term $= -\dfrac{a^2\ddot{a}}{r} - \dfrac{2a\dot{a}^2}{r}$

$$\Rightarrow \qquad \frac{a^2\ddot{a}}{r} + \left(\frac{4a}{r} - \frac{a^4}{r^4}\right)\frac{\dot{a}^2}{2} \;=\; \frac{p(r,\,t)}{\rho} - \frac{p(\infty,\,t)}{\rho}\,.$$

In particular, at $r = a^+$ (i.e. just within the incompressible fluid, just outside the bubble; i.e. in the limit as $r \downarrow a$ through values $> a$),

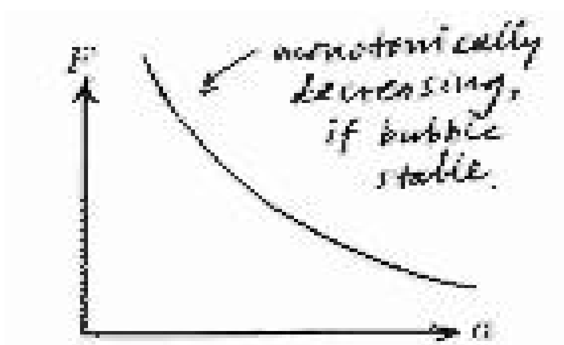
$$a\ddot{a} + \tfrac{3}{2}\dot{a}^2 \;=\; \frac{p(a^+,t)}{\rho} - \frac{p(\infty,\,t)}{\rho}\,.(\bullet)$$

(The pressure $p(a^-,t)$ just *inside* the bubble can differ from that just outside, $p(a^+,t)$, because of surface tension.)

To be able to solve for the motion, we need information about $p(\infty,\,t)$ (which may be given, as a constant or as a known function of t) and about $p(a^+,t)$. This last has to come from the equation of state of the gas in the bubble, together with information about surface tension on $r = a$. To excellent approximation, $p(a^+,t)$ is a function of a alone:

F

$$p(a^+,t) \;=\; F(a)\,, \qquad \text{say.}$$



This is essentially because the time for a sound wave to cross the interior of the bubble is far shorter than the timescales of the bubble dynamics; see below.

a

The function $F(a)$ is very simple in cases where the bubble is made of perfect gas undergoing adiabatic (thermodynamically reversible) compression or dilatation, and where surface tension is negligible (all of which is fairly accurate for air bubbles of millimetre size in water, typically within a few percent). Then

$p(a^+,t) = p(a^-,t) \propto (\text{bubble volume})^{-\gamma} \propto (a^3)^{-\gamma} \propto a^{-3\gamma}$, where γ is the ratio of specific heats. For air under ordinary conditions,

$$\gamma = 7/5 = 1.4 \quad \text{to good approximation ,}$$

with fractional error a percent or so. (You can take this on faith, or check it out in standard physics textbooks.) So in all such cases

$$F(a) \;=\; p_0 \left(\frac{a_0}{a}\right)^{3\gamma} \;, (\bullet\bullet\bullet)$$

where p_0 is the pressure in the bubble when it has radius a_0.

Frequency of small oscillations:

Put radius $a(t) = a_0 + \delta a$, with $\delta a \ll a_0$, and assume constant background pressure at infinity, $p(\infty,\, t) = p_\infty$, say.

Assume system is in equilibrium, $\mathbf{u} = 0$ everywhere, when $a = a_0$,

$\Rightarrow$

$$p(a^+) = p_0 = p_\infty \qquad \text{when in equilibrium.}$$

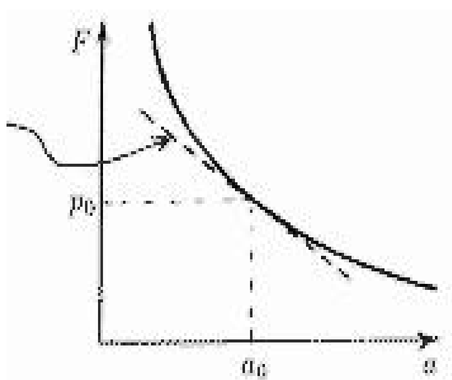
Pressure *fluctuation* δp at $r = a^+$, i.e., $\delta p = p(a^+,t) - p_0$, is related to δa, the fluctuation in a, by $(\bullet\bullet\bullet)$, which together with $p(a^+,t) = F(a)$ linearizes to

F

$$\delta p \;=\; -\frac{3\gamma p_0}{a_0}\delta a$$

p_0 Therefore, also linearizing ($\bullet$) above (noticing that the quadratic term $\frac{3}{2}\delta\dot{a}^2$ linearizes to zero), we have simply

$$a_0\,\delta\ddot{a} a_0 = a\frac{\delta p}{\rho} \;=\; -\frac{3\gamma p_0}{\rho a_0}\delta a \;.$$



This represents simple harmonic motion $\delta a(t) \propto \sin(\sigma t + \text{const.})$ with (radian) frequency

$$\sigma \quad = \quad \sqrt{\frac{3\gamma p_0}{\rho a_0^2}} \ .$$

Putting in typical numbers for air bubbles in water, under ordinary conditions ($p_0 = 10^5\,\text{Pa}$, $\rho = 10^3\,\text{kg}\,\text{m}^{-3}$) we find that the frequency in hertz or cycles per second, i.e. $\sigma/2\pi$, $\approx 650\,\text{Hz cm}/2a_0 = 6.5\,\text{kHz mm}/2a_0$. E.g. to get 1 kHz, the pitch of the radio time-pips, need a_0 to be about 3.3 mm.

We can now check, in this case, our original assumption that the time a_0/c_{air} for a sound wave to cross the bubble is far shorter than the bubble oscillation timescale σ^{-1}. The sound speed in air is, under ordinary conditions, $c_{\text{air}} = \sqrt{(\gamma p_0/\rho_{\text{air}})} \approx 340\text{ms}^{-1}$, ρ_{air} being the air density, $\sim 10^{-3}\rho$, where $\rho =$ density of water, $10^3\,\text{kg}\,\text{m}^{-3} = 1$ tonne/m^3. We can now rewrite the formula for σ as $(c_{\text{air}}/a_0)\sqrt{(3\rho_{\text{air}}/\rho)}$, which evidently $\ll (c_{\text{air}}/a_0)$, as assumed.

To analyze the collapse problem, and other nonlinear problems, we now need to develop the general theory a little further:

Energy relation: Notice the pattern of time derivatives on l.h.s. (•). This suggests further simplification to a single term $\propto \dot{a}^{-1}(d/dt)(\dot{a}^2 a^n)$ for some power n; and $n = 3$ does the trick: $\dot{a}^{-1}(d/dt)(\dot{a}^2 a^3) = 2\ddot{a}a^3 + 3\dot{a}^2 a^2 = 2a^2(a\ddot{a} + \frac{3}{2}\dot{a}^2)$, so we have, multiplying both sides by $a^2\dot{a}$,

$$\frac{d}{dt}\left(\tfrac{1}{2}a^3\dot{a}^2\right) \;=\; a^2\dot{a}\left(\frac{p(a^+,t)}{\rho} - \frac{p(\infty,\,t)}{\rho}\right) \quad .(\bullet\bullet)$$

This is the energy equation for the fluid in $r > a$. We might have guessed this from the extra factor $\dot{a}$; multiplying (•) by $a^2\dot{a}$ is a bit like its counterpart in Newtonian particle dynamics, i.e., scalarly multiplying Newton's law of motion $m\ddot{\mathbf{x}} = ...$ by $\dot{\mathbf{x}}$ to get $(d/dt)\left(\frac{1}{2}m|\dot{\mathbf{x}}|^2\right) = ...$, the equation for the rate of change of kinetic energy. We can check this out for the fluid problem by calculating the total kinetic energy K of flow in $r > a$, i.e. the volume integral of $\frac{1}{2}\rho|\mathbf{u}|^2$:

$$\begin{aligned} K \;&=\; \int_a^\infty \tfrac{1}{2}\rho|\mathbf{u}|^2\,4\pi r^2\,dr \\ &=\; \int_a^\infty \tfrac{1}{2}\rho\left(\frac{a^2\dot{a}}{r^2}\right)^2 4\pi\,r^2\,dr \;=\; 4\pi\rho\,\tfrac{1}{2}\,a^4\dot{a}^2\int_a^\infty \frac{dr}{r^2} \\ &=\; 4\pi\rho\,\tfrac{1}{2}\,a^3\dot{a}^2 \ . \end{aligned}$$

Integral is convergent and K is finite! Its rate of change $\dot{K}$ must equal the rate of working by the pressure forces on the fluid in $r > a$. The force per unit area exerted by the bubble on the surrounding fluid is $p(a^+,t)$; the rate of working of that force is $\dot{a}\,p(a^+,t)$ per unit area, which sums to $4\pi a^2\dot{a}\,p(a^+,t)$ for the whole bubble. Similarly, the rate of working by the fluid in $r < R$, say, on the fluid beyond, in $r > R$, is

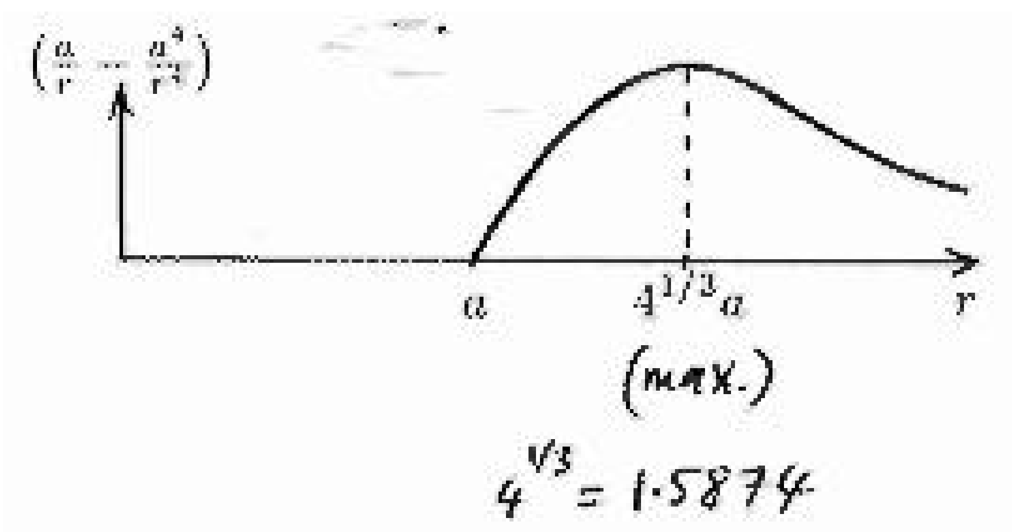
$$4\pi R^2\left(\frac{a^2\dot{a}}{R^2}\right)p(R,\,t) \;\to\; 4\pi a^2\dot{a}\,p(\infty,\,t) \quad \text{as} \quad R \to \infty \ .$$

So the net rate of working on the whole fluid is $4\pi a^2 \dot{a}\big\{p(a^+,\,t) - p(\infty,\,t)\big\}$. This must be equal to $\dot{K}$, which is what (●●) says, after multiplying it by $4\pi\rho$. (Notice that, in this model, strict incompressibility has the peculiar consequence that nonzero work is done 'at infinity'.)

Pressure field in $r > a$: If we know $a(t)$ and $p(\infty,\,t)$ then $p(r,\,t)$ is determined by the equation displayed before (●). Alternatively, we can eliminate $\ddot{a}$ from that equation, using (●) itself, to give

$$p(r,\,t) - p(\infty,\,t) = \Big(p(a^+,t) - p(\infty,\,t)\Big)\frac{a}{r} + \tfrac{1}{2}\rho\,\dot{a}^2\left(\frac{a}{r} - \frac{a^4}{r^4}\right) \;.(\bullet\bullet\bullet\bullet)$$

The term $\propto \dot{a}^2$ is always positive (because $r > a$). $\left(\frac{a}{r} - \frac{a^4}{r^4}\right)$
This is helpful in understanding the collapse problem, to be considered next:



Collapse of cavity: A cavity is a bubble with very small interior pressures, usually formed as a result of 'cavitation' in high-energy flows of liquids around convex solid boundaries, where relative flow speeds can become large and pressures low, as suggested by Bernoulli's theorem, even to the point of becoming negative. The classic case is that of ships' propellers.

Liquids under ordinary conditions cannot withstand tension, i.e. negative pressure; so when the pressure is reduced sufficiently, cavities will form and grow. When such a cavity is carried into surroundings where the pressure is positive again, it tends to collapse violently. The limiting case of zero interior pressure, with $p(\infty,\,t)$ a positive constant, is relevant as a simple model of this situation:

Consider a spherical cavity of radius $a(t)$, with $p(a^+,\,t) = 0$ and with the motion starting from rest: initial conditions are $\boxed{\;a = a_0 \quad \text{and} \quad \dot{a} = 0 \quad \text{at} \quad t = 0\;.}$

Background pressure $p(\infty,\,t)$ = constant = p_∞. Use (●●) above (p. 38) but now taking $p(a^+,t) = 0$ and $p(\infty,\,t) = p_\infty$ = constant we have simply

$$\frac{d}{dt}\left(\tfrac{1}{2}a^3\dot{a}^2 + \frac{p_\infty}{3\rho}a^3\right) \;=\; 0\;.$$

So, using the initial conditions, we have

$$a^3\dot{a}^2 = \tfrac{2}{3}\,\frac{p_\infty}{\rho}(a_0^3 - a^3)\;.$$

Taking the appropriate branch of the square root (anticipating that $\dot{a} < 0$ during collapse):

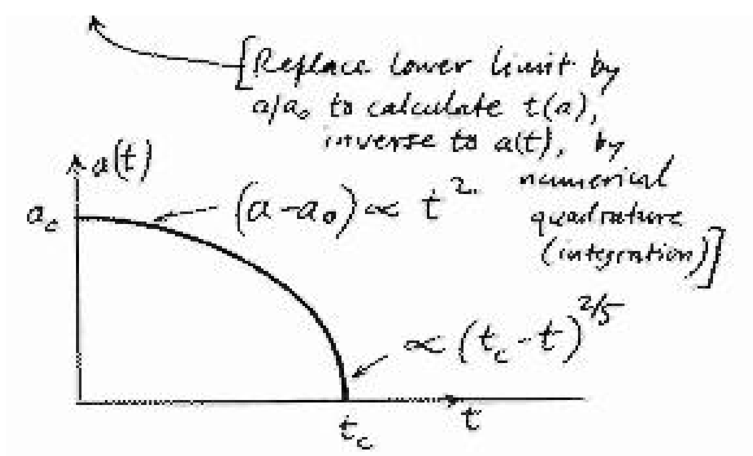
$$-\,\dot{a} = \left(\frac{2}{3}\frac{p_\infty}{\rho}\right)^{1/2}\left(\frac{a_0^3}{a^3} - 1\right)^{1/2} \;>\; 0\;.$$

If $a \to 0$, then as soon as $a \ll a_0$ we have approximately $\;-\,\dot{a} \approx \left(\dfrac{2}{3}\dfrac{p_\infty}{\rho}\right)^{1/2}\left(\dfrac{a_0}{a}\right)^{3/2}$, suggesting that collapse occurs in finite time t_c. (In this approximation, $dt \propto \int a^{3/2}da$: integral is convergent, showing that the singularity at $a = 0$ is integrable giving a finite time interval $t - t_\mathrm{c}$, and that $t - t_\mathrm{c} \propto a^{5/2}$ hence $a \propto (t - t_\mathrm{c})^{2/5}$.) More precisely, and confirming the finiteness of t_c,

$$t_\mathrm{c} = \int_0^{a_0}\frac{da}{\left[\dfrac{2}{3}\dfrac{p_\infty}{\rho}\left(\dfrac{a_0^3}{a^3} - 1\right)\right]^{1/2}} \quad=\quad \left(\frac{3\rho a_0^2}{2p_\infty}\right)^{1/2}\int_0^1\frac{d\alpha}{\{(1/\alpha)^3 - 1\}^{1/2}} \qquad (\alpha = a/a_0)$$

$$=\quad 0.92\left(\frac{\rho a_0^2}{p_\infty}\right)^{1/2}\,,$$

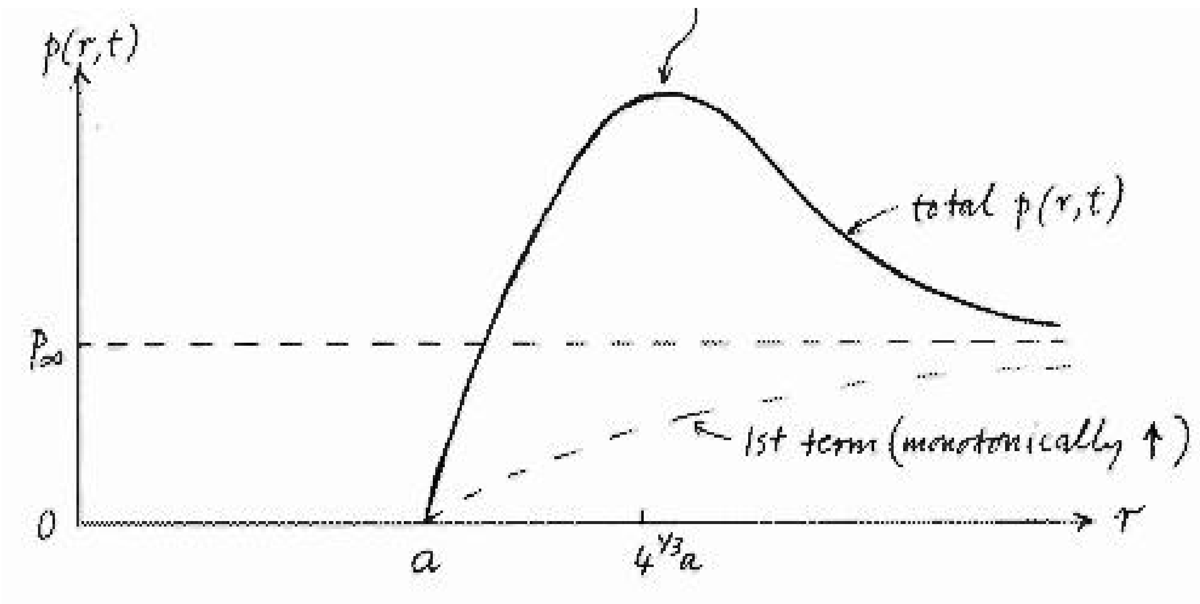
from numerical integration.



The expression (• • ••) for $p(r,\,t)$ tell us something interesting about the way the pressure $p(r)$ varies within the liquid; (• • ••) simplifies to

$$p(r,\,t) \;=\; p_\infty\left(1-\frac{a}{r}\right)+\tfrac{1}{2}\rho\,\dot a^2\left(\frac{a}{r}-\frac{a^4}{r^4}\right)\ .$$

(Remember that $p(a^+,t)=0$ now.) As remarked before, the term $\propto \dot a^2$ is always positive (because $r>a$); see graph above (and note maximum value is at $r/a = 4^{1/3} = 1.5874$). Since $\dot a$ increases without bound, an interior pressure maximum must form:



This contrasts with the earliest stages of collapse, with $(a_0-a)\propto t^2$ and $\dot a \propto t$, for $t \ll t_{\rm c}$. (See sketch graph of $a(t)$ at bottom of previous page, and check the behaviour $(a-a_0)\propto t^2$ by considering $\int_{1-\epsilon}^{1}\{(1/\alpha)^3-1\}^{-1/2}\,d\alpha$, where $\epsilon \ll 1$.) Then the first term in the expression $p_\infty\left(1-\frac{a}{r}\right)+\frac{1}{2}\rho\,\dot a^2\left(\frac{a}{r}-\frac{a^4}{r^4}\right)$ dominates the second; there is no pressure maximum, implying that ∇p is directed radially outward everywhere. Hence particle accelerations $D\mathbf{u}/Dt$ are radially inward everywhere, at this early stage.

In the latest stages, after formation of the pressure maximum, with $a\to 0$ like $(t-t_{\rm c})^{2/5}$, and $\dot a^2\to\infty$, the second term, the term $\propto \dot a^2$, dominates the first. The pressure maximum has formed and is moving inward, and, in the limit $a\to 0$, its value asymptotically approaches the value given by the second term alone, at $r\approx 4^{1/3}a$. That value can be seen, from

substituting $\dot a \approx \left(\dfrac{2}{3}\dfrac{p_\infty}{\rho}\right)^{1/2}\left(\dfrac{a_0}{a}\right)^{3/2}$, to be asymptotically $\dfrac{p_\infty}{4^{4/3}}\,\dfrac{a_0^3}{a^3}$.

E.g. at $\dfrac{a}{a_0} = \dfrac{1}{10}$, find $p_\infty = 1$ atm $\Rightarrow p_{\max} \sim 160$ atm

and $\dot{a} \sim 260$ ms^{-1}

(approaching pressures that can melt some metals!)

(Remember 1 atm = 10^5Pa = 10^5N m^{-2}; common liquids, e.g. water, have $\rho \approx 10^3$kg m^{-3}; so $p_\infty/\rho \approx 10^2$m s^{-2}.)

Again, at $\dfrac{a}{a_0} = \dfrac{1}{100}$, find $p_\infty = 1$ atm $\Rightarrow p_{\max} \sim 160\,000$ atm

(But now the theory has well and truly predicted its own breakdown: 8 000 ms^{-1} is well over the sound speed in most liquids, e.g. for water, sound speed $\sim$ 1500 m s^{-1}, and incompressibility won't be a good approximation.)

The pressure maximum forms because, in our incompressible model, the inward flow at any fixed position $r > 0$ outside the cavity *decelerates*, toward zero velocity, as the cavity volume becomes vanishingly small ($a \ll r$). (You should check that $|\mathbf{u}| \to 0$ at fixed r; recall, e.g., that $\phi = -a^2\dot{a}/r$.) Inward *de*celeration or outward *ac*celeration corresponds to inward-directed ∇p.

*When departures from spherical symmetry are allowed for, it turns out that the extreme velocities and pressures tend, unfortunately, to be directed *toward* the nearest solid surface in the form of tiny but powerful jets within the bubble at its final stage of collapse. This can cause what is called 'cavitation damage'.*

§3.5 Translating sphere, and inertial reaction to acceleration

First consider steady motion:

Recall the velocity potential for uniform flow past fixed sphere (§3.2):

$$\phi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$

Velocity in spherical polars: $\mathbf{u} = \nabla\phi = \left[U\cos\theta\left(1 - \frac{a^3}{r^3}\right),\ -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right),\ 0\right]$.

Calculate pressure force on sphere, using time-dependent Bernoulli:

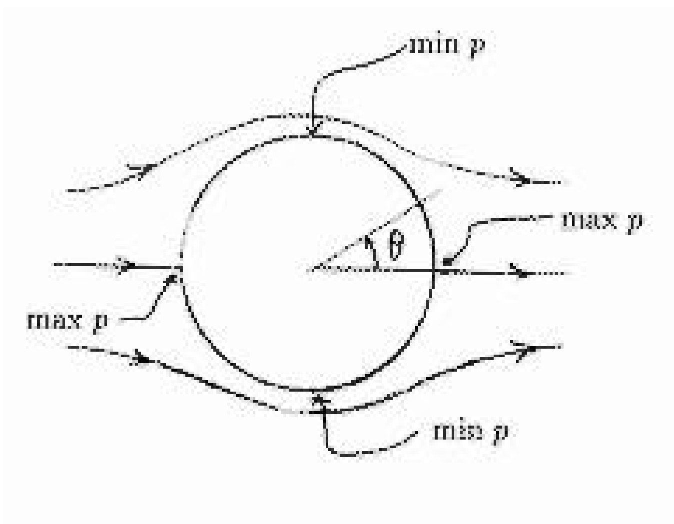
$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + \Phi \quad = \quad \tilde{H}(t) \quad = \quad p_\infty + \tfrac{1}{2}U^2$$

(steady) (no body forces)

So $p(a,\theta) \quad = \quad p_\infty - \tfrac{1}{2}|\mathbf{u}|^2\big|_{r=a} + \tfrac{1}{2}U^2$.

On $r = a$, $\mathbf{u} = (0,\ -\frac{3}{2}U\sin\theta,\ 0)$

$$p(a,\theta) \ = \ p_\infty - \frac{9}{8}U^2\sin^2\theta + \tfrac{1}{2}U^2$$



Pressure force $\parallel$ to stream, i.e. 'drag':

$$F_\parallel = \int_0^\pi p(a,\theta)\ \cos\theta\, 2\pi a^2 \sin\theta d\theta \qquad \text{(component of } \mathbf{n} \parallel \text{stream)}$$

$$= 2\pi a^2 \int_0^\pi \left(p_\infty + \tfrac{1}{2}U^2 - \frac{9}{8}U^2\sin^2\theta\right)\sin\theta\cos\theta d\theta = 0$$

by symmetry — no need to do the integral! Similarly, force $\perp$ stream is zero. (Remember the symmetry of the flow field, which was sketched in §3.2 **A**.)

Pressure has two maxima (at the front and rear stagnation points), and a minimum on the bisecting plane $\perp$ stream.

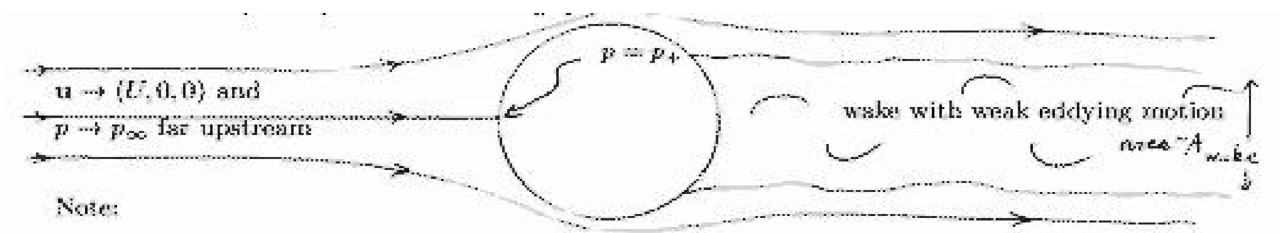
Energy considerations (§3.7 below) are enough to show that the drag, or component of net force parallel to the stream, must vanish for potential flow round any 3-D body (§3.7 below). This is called d'Alembert's paradox — 'paradox' because experience is that 3-D bodies moving steadily in real fluids do experience nonzero drag.

(*It can also be shown, though not from energy considerations, that the $\perp$ force vanishes as well — not just for the sphere, but for any 3-D body in potential flow: Batchelor p. 405.*)

Effects of friction (explanation of the 'paradox')

No-force result is correct for inviscid (frictionless) fluids, but often, in real fluids, the effect of friction cannot be neglected, even when a naive estimate says that it is likely to be small. Potential flow is often a bad approximation, for flow past solid bodies.

Observed flow typically has the following qualitative character:



Note:

Upstream flow is irrotational to good approximation, with effects of friction confined to thin boundary layer: $p_+ = p_\infty + \frac{1}{2}\rho U^2$

As the flow over the back half slows toward the rear stagnation point, the frictional boundary layer 'separates', taking rotational fluid elements away from the boundary and into a nearly-stagnant wake behind body, bounded by (highly rotational) shear layers.

Pressure in wake $\simeq p_\infty$, otherwise there would be stronger lateral acceleration of wake fluid than observed.

hence $\exists$ pressure difference between front and back $\sim \frac{1}{2}\rho U^2$

In rough order-of-magnitude terms, therefore, we expect the drag to be ρU^2 times the cross-sectional area A_{wake} of the wake times some dimensionless number of order unity. Because A_{wake} is not a precisely defined quantity, but usually comparable to the projected area A of the body viewed from far upstream — which *is* precisely defined — it is conventional to define a dimensionless number C_D such that

$$\text{Drag} = \tfrac{1}{2}\rho\, U^2\, A\; C_D\ .$$

C_D is by convention called the *drag coefficient*; it is typically a modest fraction of unity, though the precise value depends very much on circumstances (e.g. on just how small the viscosity is, and whether the body surface is rough or smooth). As the rough argument just given suggests, C_D is numerically close to unity if the wake is 'fat' in the sense that $A \sim A_{\text{wake}}$.

It is the cross-sectional area of the wake that is sensitive to circumstance. With a thin wake, conditions are closer to ideal potential-flow conditions, and C_D can be a fairly small fraction of unity. This can occur for a solid sphere in certain (complicated) circumstances (to do with very delicate properties of turbulent boundary layers).

Summary so far: Potential flow is a *bad* model for steady flow past solid bodies like spheres, indeed any such body that is not highly streamlined, like an aircraft. Potential flow is a *good* model for:

- Flow past or around bubbles
- Acceleration of rigid bodies, for short time intervals such that there is insufficient time for vorticity to escape from boundaries
- Small amplitude oscillations.

§3.6 Accelerating sphere

We continue to neglect gravity, and solve in a frame such that fluid is at rest at ∞ (otherwise it is necessary to take account of non-inertial effects, i.e. 'fictitious forces' in a non-inertial reference frame). Consider a sphere of radius a, centre $\mathbf{x}_0(t)$, with $d\mathbf{x}_0/dt = \mathbf{U}(t)$. Write $r = |\mathbf{x} - \mathbf{x}_0|$. Outward normal $\mathbf{n} = (\mathbf{x} - \mathbf{x}_0)/r$. Potential problem to solve is

$$\begin{aligned} \nabla^2 \phi &= 0 \\ \nabla \phi &\to 0 \qquad \text{as} \quad r \to \infty \\ \mathbf{n}.\nabla\phi &= \mathbf{n}.\mathbf{U}(t) \quad \text{on} \quad r = a\ . \end{aligned}$$

Rather than solving from scratch, we can use the solution constructed previously, on p. 27, with $\cos\theta = \mathbf{n}.\mathbf{U}/|\mathbf{U}|$, after subtracting the uniform flow at infinity. (The superposition principle for the linear equation $\nabla^2\phi = 0$ says that we still have a potential flow. But the sphere is now moving in the $-z$ direction; to reverse this we replace U on p. 27 by $-|\mathbf{U}|$ here.) The result is

$$\phi = -\frac{|\mathbf{U}|a^3}{2r^2}\cos\theta = -\frac{\mathbf{U}.(\mathbf{x} - \mathbf{x}_0)a^3}{2|\mathbf{x} - \mathbf{x}_0|^3}\ .$$

Rewriting the solution in vector notation frees us from any particular coordinate system. (*Exercise:* check that this does solve the above boundary-value problem!)

Note that there is no memory in the problem — the solution at any instant depends only on boundary conditions at that instant. So solving for ϕ is indifferent to whether or not $\mathbf{U}$ and $\mathbf{x}_0$ are functions of t.

Taking $\partial/\partial t$, we have

$$\frac{\partial\phi}{\partial t} = -\frac{\dot{\mathbf{U}}.(\mathbf{x}-\mathbf{x}_0)a^3}{2|\mathbf{x}-\mathbf{x}_0|^3} + \mathbf{U}.\nabla\left\{\frac{\mathbf{U}.(\mathbf{x}-\mathbf{x}_0)a^3}{2|\mathbf{x}-\mathbf{x}_0|^3}\right\} \qquad = \mathbf{U}.\nabla\phi = \mathbf{U}.\mathbf{u}$$

(The second term comes from the time-dependence of $\mathbf{x}_0$, using the chain rule, along with the fact that the gradient with respect to $\mathbf{x}_0$ is minus the gradient with respect to $\mathbf{x}$, of any function of $\mathbf{x}-\mathbf{x}_0$ alone.)

Now use time-dependent Bernoulli, $\tilde{H} = \tilde{H}_\infty$,

$$p \;=\; p_\infty - \rho\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho|\mathbf{u}|^2 \;=\; p_\infty + \rho\dot{\mathbf{U}}.\frac{(\mathbf{x}-\mathbf{x}_0)a^3}{2|\mathbf{x}-\mathbf{x}_0|^3} - \tfrac{1}{2}\rho|\mathbf{u}|^2 - \rho\mathbf{U}.\mathbf{u}\;.$$

Force $\mathbf{F}$ on sphere (now taking $\mathbf{x}_0 = 0$ w.l.o.g. — no more time differentiation):

$$\begin{aligned}\mathbf{F} \;&=\; -\int_{|\mathbf{x}|=a} p\,\mathbf{n}\,dS = -\int_{r=a} p\,\frac{\mathbf{x}}{r}\,dS \\ &=\; -\rho\int_{r=a}\frac{\dot{\mathbf{U}}.\mathbf{x}\,a^3}{2r^3}\,\frac{\mathbf{x}}{r}\,dS - (p_\infty + \tfrac{1}{2}\rho|\mathbf{U}|^2)\int_{r=a}\mathbf{n}\,dS + \rho\int_{r=a}\tfrac{1}{2}|\mathbf{u}+\mathbf{U}|^2\mathbf{n}\,dS\end{aligned}$$

In components, $\quad F_i = -M^*_{ij}\dot{U}_j$ where

$$M^*_{ij} = \rho\int_{r=a}\frac{a^3x_j}{2a^3}\frac{x_i}{a}dS = \tfrac{1}{2}\rho\int_{r=a}\frac{x_ix_j}{a}dS = \tfrac{1}{2}\rho\;4\pi a^3\;\tfrac{1}{3}\delta_{ij}\;.$$

(Easiest proof of last step: note that the integral must be an isotropic tensor, hence $\propto \delta_{ij}$, so it is enough to evaluate the trace of the integral, i.e., to set $i=j$ and sum, noting that $x_ix_i = a^2$ and noting also that $\delta_{jj} = 3$.) So:

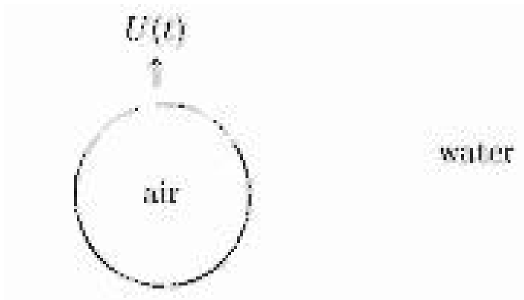
$$M^*_{ij} = \frac{2\pi\rho a^3}{3}\delta_{ij} = \tfrac{1}{2}\rho V\delta_{ij} = M^*\delta_{ij}\;,\quad\text{say;}\qquad\text{so}\quad \mathbf{F} = -\tfrac{1}{2}\rho V\dot{\mathbf{U}}.$$

So for the sphere, $\mathbf{F} = -M^*\dot{\mathbf{U}} = -\tfrac{1}{2}\rho V\dot{\mathbf{U}}$. It is natural to call $M^* = \tfrac{2}{3}\pi\rho\,a^3 = \tfrac{1}{2}\rho V$ the *added mass*, or *virtual mass*, for a sphere.

(*For a general, non-isotropic, body the added mass becomes a general, non-isotropic tensor quantity, M^*_{ij}, not $\propto \delta_{ij}$, because the integral involving $x_i x_j$ will emphasize some directions more than others.*)

As the sphere is accelerated, a body of fluid is accelerated with it. Because the fluid exerts a force on the sphere, the sphere must exert a force on the fluid, in the opposite direction (Newton's third law) — it is of course given by the same calculation as above, with the sign of $\mathbf{n}$ reversed.

Application to rising spherical bubble (e.g. air bubble in water under gravity, kept spherical by surface tension if the radius a is small enough):



Newton's second law for bubble of mass m: (Archimedes: recall p. 22)

$$\begin{aligned} m\dot{U} &= \text{buoyancy force} - \text{weight of bubble} - \text{added-mass force} \\ &= \frac{4\pi a^3}{3}\rho\, g - m\, g - \tfrac{1}{2}\frac{4\pi a^3}{3}\rho\, \dot{U} \qquad \Rightarrow \qquad \dot{U} = \frac{2M^* - m}{M^* + m}\, g\ . \end{aligned}$$

In the limit as $m \to 0$, $\dot{U} \to 2g$ (upward). In this limit, the added mass is the only relevant mass. It is the mass of just *half* the volume of water displaced by the bubble, and so the bubble accelerates upward at just *twice* the gravity acceleration. (*It will continue to do so until viscosity/turbulence/wake effects are no longer negligible.*)

§3.7 Kinetic Energy

This section gives a different view of problems considered previously, and in particular provides a fairly general proof of d'Alembert's paradox. Consider fluid density ρ, velocity $\mathbf{u}$ in volume V.

$$\text{Kinetic energy} \qquad K = \int_V \tfrac{1}{2}\rho|\mathbf{u}|^2 dV$$

Assuming potential flow, with $\mathbf{u} = \nabla\phi$, we have

$$\begin{aligned} K = \tfrac{1}{2}\rho \int_V (\nabla\phi)^2\, dV &= \tfrac{1}{2}\rho \int_V \big(\nabla.(\phi\nabla\phi) - \phi\nabla^2\phi\big) dV = \tfrac{1}{2}\rho \int_S \mathbf{n}.(\phi\nabla\phi)\, dS \\ &= \tfrac{1}{2}\rho \int_S \phi\, \mathbf{u}.\mathbf{n}\, dS \end{aligned}$$

where S is the surface bounding V, and $\mathbf{n}$ is now the unit normal vector pointing *outward* from the fluid.

If the velocity of points on the boundary is denoted by $\mathbf{U}^{\text{boundary}}$ then, by the kinematic boundary condition, $\mathbf{U}^{\text{boundary}}.\mathbf{n} = \mathbf{u}.\mathbf{n} = \mathbf{n}.\nabla\phi$ on S and hence

$$K = \tfrac{1}{2}\rho \int \phi\, \mathbf{U}^{\text{boundary}}.\mathbf{n}\, dS.$$

(You should check that this agrees with what was calculated in the special case of p. 38.)

Now apply this to our standard case of the translating sphere in a fluid at rest far from sphere. Take the origin to be instantaneously coincident with the centre of the sphere, $\mathbf{x}_0 = 0$, and take the volume V of integration to be the volume $a < |\mathbf{x}| < R$, where $R \gg a$. Then, by above,

$$K = \tfrac{1}{2}\rho \int_{|\mathbf{x}|=R} \phi\, \mathbf{u}.\mathbf{e}_r\, dS - \tfrac{1}{2}\rho \int_{|\mathbf{x}|=a} \phi\, \mathbf{U}.\mathbf{e}_r\, dS,$$

where $\mathbf{e}_r$ is the unit vector in the radial direction.

The first integral $\to 0$ as $R \to \infty$. For we have $\phi = -Ua^3\cos\theta/(2r^2)$, so on $|\mathbf{x}| = R$, $\phi \sim R^{-2}$, and $\mathbf{u}.\mathbf{e}_r \sim R^{-3}$, and since $dS \sim R^2$ the first integral $\sim R^{-3}, \to 0$ as $R \to \infty$.

Hence

$$\begin{aligned} K &= -\tfrac{1}{2}\rho \int_{\theta=0}^{\pi} -\frac{Ua^3\cos\theta}{2a^2}\, U\cos\theta\; 2\pi a^2 \sin\theta\, d\theta \qquad \left(-\textstyle\int_0^\pi \cos^2\theta\, d(\cos\theta) = \tfrac{2}{3}\right) \\ &= \tfrac{1}{2}\rho U^2\, \frac{2\pi a^3}{3} = \tfrac{1}{2}(\tfrac{1}{2}\,\rho V)U^2 = \tfrac{1}{2}M^*\, U^2 \end{aligned}$$

So the effective mass M^* of fluid moving with the sphere, and giving rise to kinetic energy (K.E.) of the fluid, is the same as that accelerating with the sphere, and giving rise to the force on it.

This must be the case; we are merely checking that the whole picture is self-consistent. If the sphere accelerates, then

$$\begin{aligned} \text{rate of change of K.E.} &= \text{rate of working of sphere on fluid} \\ \text{i.e.,} \quad M^*\,\mathbf{U}.\dot{\mathbf{U}} &= -\,\mathbf{F}.\mathbf{U}\ . \end{aligned}$$

(There is no pressure-working at ∞ because $\mathbf{u}.\mathbf{e}_r \sim R^{-3}$, cf. R^{-2} on p. 38.)

So $\mathbf{F} = -M^*\dot{\mathbf{U}} + \mathbf{F}_\perp$ where $\mathbf{F}_\perp$ is a possible force perpendicular to the direction of motion.

Note that the energy argument cannot tell us anything about $\mathbf{F}_\perp$, because $\mathbf{F}_\perp$ does no work.

*But the energy argument, applied to a body of arbitrary fixed shape, does, now, lead to a general proof of d'Alembert's paradox if we are prepared to assume — this can be proven, but there is no room here! — that the first integral, the integral over $|\mathbf{x}| = R$ in the above expression for K, still vanishes as $R \to \infty$ even if the body is no longer a sphere.

(The proof takes the general separation-of-variables solution for ϕ, and shows that all the terms $O(r^{-1})$ vanish because mass is conserved and the body has constant volume, so that in the far field we have the same magnitude as above, $O(R^{-3})$, for the integral over $|\mathbf{x}| = R$, plus smaller terms $O(R^{-4})$, $O(R^{-5},)$,... Batchelor's book covers this point thoroughly. The argument is like that to be given on p. 49: the body has fixed volume and so there are no r^{-1} terms in the general solution on p. 25 and its 3-D generalization. Nor, of course, are there any positive powers of r.)*

The energy argument now says that (for a body of arbitrary shape) $F_i = -M^*_{ij}\dot{U}_j + F_{\perp\, j}$, and so the drag, $\mathbf{F} - \mathbf{F}_\perp$, must vanish when $\dot{\mathbf{U}} = 0$.

(*$\mathbf{F}_\perp$ can also be shown to be zero for a three-dimensional body of *any* shape — essentially by applying the momentum integral over the same volume V, but integrands evanesce more slowly and the argument about $R \to \infty$ is more delicate! Notice the implication: a glider or other fixed-wing aircraft could not stay airborne if the flow past it were everywhere a potential flow. Yet efficient aircraft, especially high performance gliders, are designed so that the flow past them is as close to potential/irrotational as can be managed. Fixed-wing aircraft and gliding birds stay up because, occupying a relatively small volume, there is a pair of trailing vortices coming off the wings; and $|\nabla \times \mathbf{u}|$ is anything but small at the centre of any such vortex. Trailing vortices, in other words, are not incidental features; they are indispensable to staying up! (See p. 50.)*)

§3.8 Steady flow past cylinder with circulation: 2-D lift forces

Two-dimensional flow is conceptually important because for the first time we get a simple potential-flow model *in which there is a lift force*, that is, a nonzero force component $\mathbf{F}_\perp$ at right angles to the stream $\mathbf{U}$. This was already implied by the crowding of streamlines, solutions on p. 29.

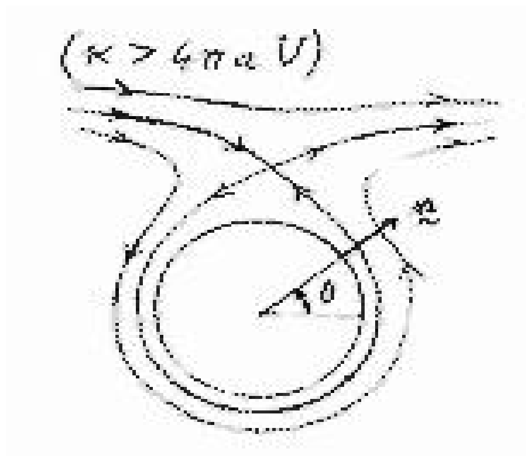
We revert to considering steady flow only; it is again convenient to use the frame in which the obstacle — an infinitely long cylinder — is stationary and the flow velocity at infinity is $\mathbf{U}$.

Now recall the solution for potential flow round the cylinder with circulation κ (last item in §3.2B), p. 29:

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\kappa\theta}{2\pi} \qquad \big(U = |\mathbf{U}|\big)$$

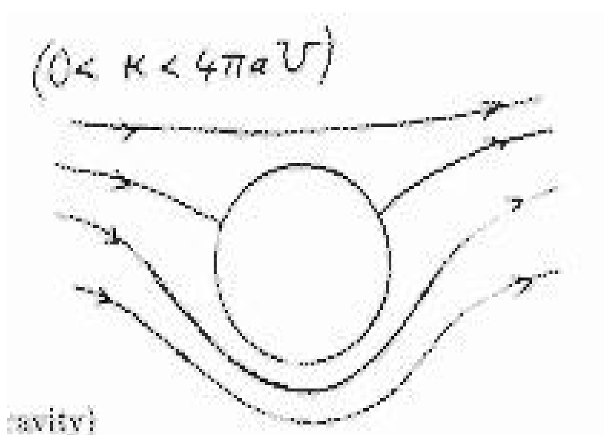
$$\mathbf{u} = \nabla\phi = \left(U\cos\theta\left(1 - \frac{a^2}{r^2}\right),\ \ -U\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r}\right)$$

(components in 2-D polars)



Hence, on surface of cylinder $r = a$,

$$|\mathbf{u}| = \left| -2U\sin\theta + \frac{\kappa}{2\pi a} \right| \, .$$



Bernoulli for steady, irrotational flow (now $\partial\phi/\partial t = 0$) $\Rightarrow$

$$\tfrac{1}{2}\rho\left(\frac{\kappa}{2\pi a} - 2U\sin\theta\right)^2 + p\big(a,\theta\big) \quad = \quad \tfrac{1}{2}\rho U^2 + p_\infty$$

pressure on cylinder — pressure at ∞ (neglecting gravity)

Force on cylinder, per unit length (let $ds = a\,d\theta$ = element of arc length):

$$\mathbf{F} \;=\; -\oint_{\text{cylinder}} p\,\mathbf{n}\,ds \;=\; -\int_{\theta=0}^{2\pi}\left\{\tfrac{1}{2}\rho U^2 + p_\infty - \tfrac{1}{2}\rho\left(\frac{\kappa}{2\pi a} - 2U\sin\theta\right)^2\right\}(\cos\theta,\ \sin\theta)a\,d\theta$$

$$= \tfrac{1}{2}\rho a \int_0^{2\pi} \left(A\,U^2 \sin^2\theta - \frac{2\kappa U \sin\theta}{\pi a} \right) (\cos\theta,\ \sin\theta)\, d\theta \qquad \text{where } A \text{ is a constant,}$$

about whose value we don't care because its contribution to the integral is zero, by inspection: note that $\sin^2\theta\cos\theta$, $\sin^3\theta$ and $\sin\theta\cos\theta$ all integrate to zero, being odd functions about some point in $(0, 2\pi)$. Recall too that $\int_0^{2\pi} \sin^2\theta\, d\theta = \pi$.

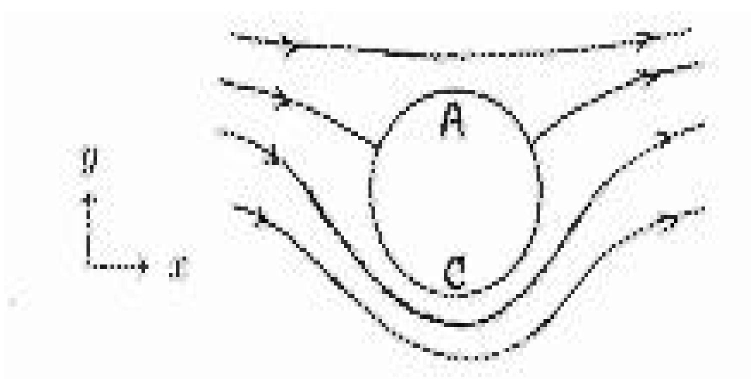
$$= \frac{-\rho\kappa U}{\pi} \int_{\theta=0}^{2\pi} (0, \sin^2\theta) d\theta$$

$$= (0,\ -\rho\kappa U) \quad \text{per unit length .}$$

So there is a *lift force* $\mathbf{F}_\perp$ perpendicular to the oncoming flow, of magnitude $|\mathbf{F}_\perp| = |\rho\kappa U|$.

E.g., for the case with two stagnation points, $0 < \kappa < 4\pi a U$ (again recall end of §3.2),

$\Rightarrow p_A > p_C$, $\Rightarrow$ Force in negative y direction

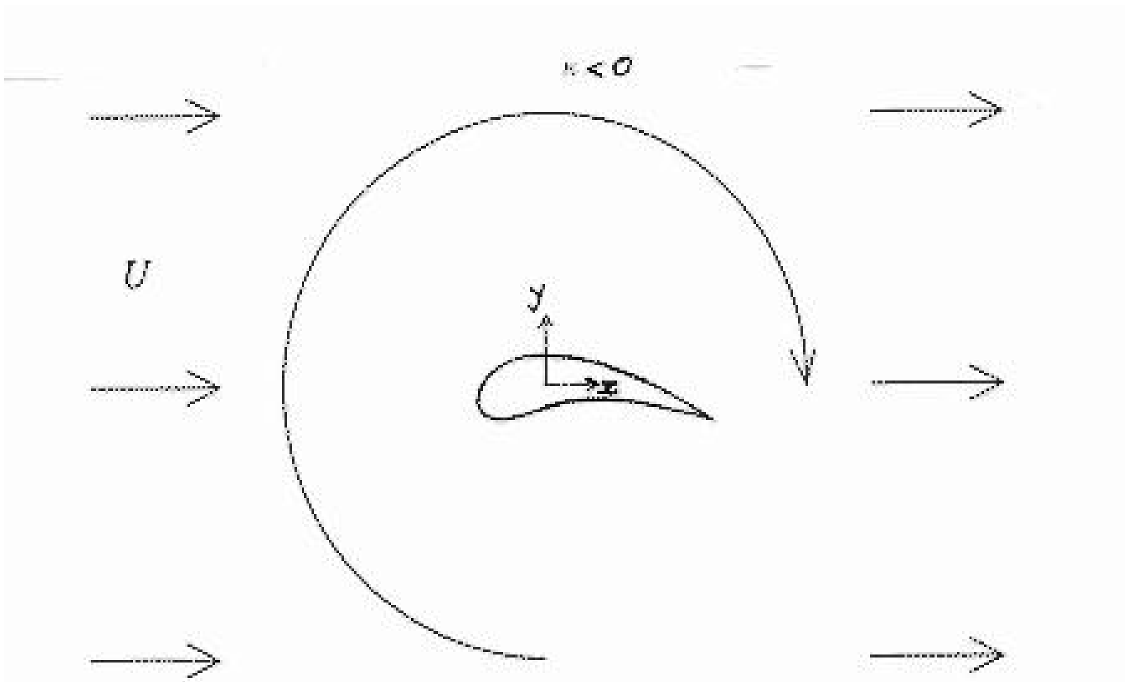


*See Batchelor §6.6 for discussion of why this irrotational flow may be relevant to real flows, around *rapidly spinning* circular cylinders, for which $\dfrac{\kappa}{Ua} \gg 1$.*

Lift on an arbitrary aerofoil with circulation κ in uniform flow U

The simple formula $|\mathbf{F}_\perp| = |\rho\kappa U|$ for the lift force holds more generally than for flow around a cylinder. This is a consequence of the momentum integral.

Consider uniform flow U in the positive x-direction with circulation κ in a *clockwise* sense, around an aerofoil that includes the origin; this means $\kappa < 0$:



The far-field velocity will be the same as that in the corresponding problem for a cylinder, to sufficient approximation. This means sufficient to be able to apply the momentum integral, as follows. We apply it to the volume (i.e. area in 2-D, or volume per unit z-distance) between the surface of the aerofoil, $S_{\rm a}$ say, and a large circle S_R of radius R and centred at the origin, then consider the limit $R \to \infty$. The flow is assumed steady, implying zero rate of change of the momentum (per unit z-distance). Therefore

$$-\int_{S_{\rm a}} \big(p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}.\mathbf{n})\big)\, ds = -\int_{S_R} \big(p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}.\mathbf{n})\big)\, ds \qquad (s = \text{arc length}).$$

We assume that the boundary of the aerofoil is impermeable. Therefore the second term in the left-hand integral is zero. So the left-hand integral is equal to the force $\mathbf{F}$ on the aerofoil:

$$\mathbf{F} = -\int_{S_R} \big(\rho(\mathbf{u}.\mathbf{n})\mathbf{u} + p\mathbf{n}\big)\, ds\ .$$

Now using Bernoulli for irrotational flow ($\tilde{H}$ uniform everywhere), continuing to neglect gravity, and noting that constant contributions to p in the integrand integrate to zero on the right, we get

$$\mathbf{F} = -\rho\int_{S_R} \big((\mathbf{u}.\mathbf{n})\mathbf{u} - \tfrac{1}{2}|\mathbf{u}|^2\mathbf{n}\big) ds\ .$$

Key point: when the limit $R \to \infty$ is taken, any contribution to the integrand that is $O(r^{-2})$ as $r \to \infty$ will vanish. Now recall the general form of the velocity potential ϕ in the separation-of-variables solution for Laplace's equation in 2-D polars (p. 27). We may ignore all the terms in r^n for $n \geqslant 2$ because they all give unbounded velocities $\mathbf{u} = \nabla\phi$ at large distances from the origin. We may also ignore the $\log r$ term, because if that term were nonzero then there would be a finite mass flux to or from infinity, hence into or out of the aerofoil, by mass conservation (contradicting impermeability). Therefore, in the notation of p. 27, with $\alpha_1 = 0$ and $A_1 = U$ to match the uniform flow at infinity, and $B_0 = \kappa/2\pi$ as on p. 28:

$$\phi = Ur\cos\theta \;+\; \frac{\kappa}{2\pi}\,\theta \;+\; \sum_{n=1}^{\infty}\left\{B_n r^{-n}\cos(n\theta+\beta_n)\right\} \;.$$

Taking the gradient and noting that the resulting contribution from $\sum_{n=1}^{\infty}$ is $O(r^{-2})$, we have $\mathbf{u}$ in polar components:

$$\mathbf{u} = \Big(U\cos\theta + O(r^{-2})\;,\quad -\,U\sin\theta + \frac{\kappa}{2\pi r} + O(r^{-2})\Big) \qquad (\text{as } r \to \infty).$$

Substituting this far-field expression into $\mathbf{u}.\mathbf{n}$ and $\frac{1}{2}|\mathbf{u}|^2$ and noting that $\mathbf{n} = \mathbf{e}_r$ and hence $\mathbf{u}.\mathbf{n} = \mathbf{u}.\mathbf{e}_r = U\cos\theta + O(R^{-2})$, we see that the *Cartesian components* of $\mathbf{F}$ are

$$\mathbf{F} = -\rho\int_0^{2\pi}\left(U\cos\theta + O(R^{-2})\right)\left(U - \frac{\kappa\sin\theta}{2\pi R} + O(R^{-2}),\; \frac{\kappa\cos\theta}{2\pi R} + O(R^{-2})\right) R\,d\theta$$

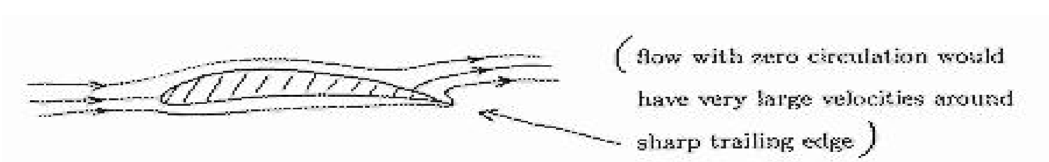
$$+\rho\int_0^{2\pi}\tfrac{1}{2}\left(U^2 - \frac{\kappa U\sin\theta}{\pi R} + O(R^{-2})\right)(\cos\theta,\;\sin\theta)R\,d\theta$$

(because $\mathbf{n} = \mathbf{e}_r = (\cos\theta,\;\sin\theta)$). Evaluating the integrals in the limit $R \to \infty$, and again using $\int\cos\theta\,d\theta = \int\sin\theta\cos\theta\,d\theta = 0$, and $\int\cos^2\theta\,d\theta = \int\sin^2\theta\,d\theta = \pi$, gives (in Cartesian components)

$$\mathbf{F} \;=\; (0,\;\, -\,\rho\kappa U)\;.$$

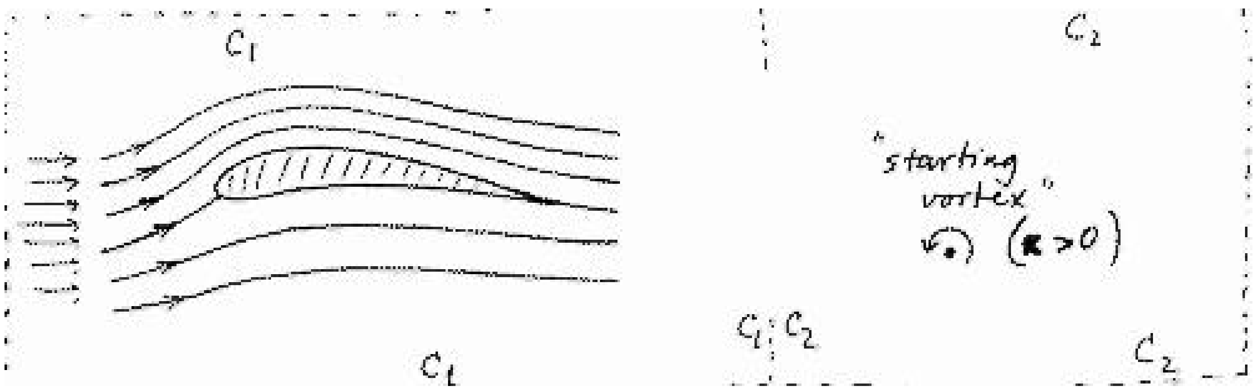
*__Generation of circulation__ (to give the lift $\rho\kappa U$)

To predict lift we need only to predict κ. Setting up flow from rest must involve friction:



(see Batchelor §6.7)

Small (even vanishingly small) friction doesn't allow this as a steady flow. We need to add circulation *such that trailing edge velocity is zero;* it can be shown that there is just one such value κ of the circulation, when the aerofoil is oriented at a given (shallow) angle:



(see Batchelor)

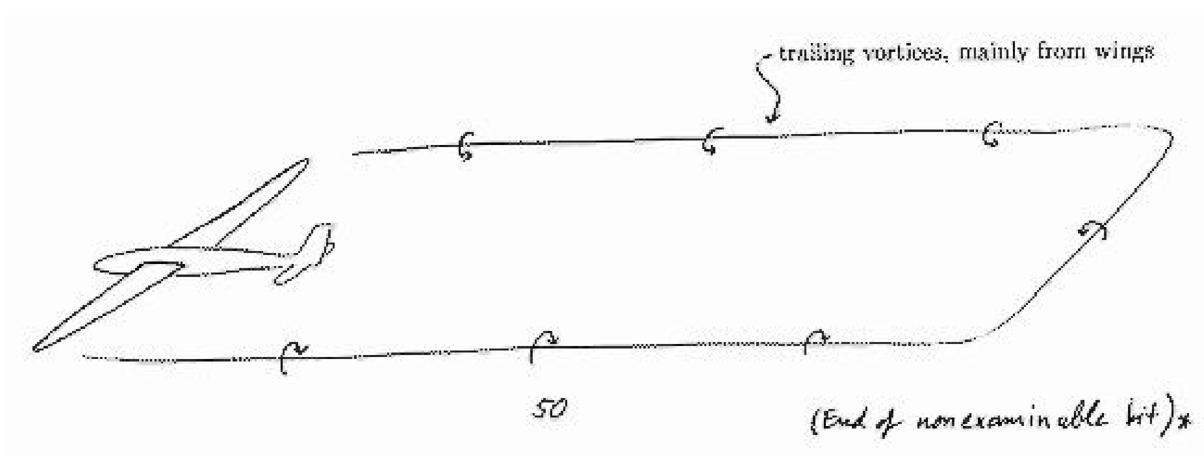
Circulation around material circuit C_1 has the nonzero value $-\kappa$.

Circulation around the outer material circuit $C_1 + C_2$, which surrounded the body at $t = 0$, is zero by Kelvin's circulation theorem.

Hence there is a nonzero circulation, $+\kappa$ about C_2. There is a *starting vortex* with circulation $+\kappa$ that is generated around the wing by shedding vorticity from trailing edge when the aircraft starts to move forward. *Cf. 'teaspoon vortices', done next lecture.*

What then of trailing vortices — how is lift generated in 3-D?

In 3-D system $\nabla . \boldsymbol{\omega} = 0$, so vortex lines cannot end in the fluid. (To a rough approximation, they can be thought of as ending at the wingtips; this idea is used in certain idealized models of aircraft lift. What really happens, though, with fixed-wing aircraft, is that the trailing vortex lines enter the boundary layer on the wings.)



(End of nonexaminable bit)*

§3.9 Vortex motion: the point-vortex model

We now develop the simplest possible model in which the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is nonzero and has a nontrivial role. This is a first step toward being able to describe, understand, and as far as possible predict, a vast variety of fluid flows in the real world — flows having the complicated eddies and vortices that are so conspicuously absent from irrotational or potential flow.

As already hinted in several ways, departures from irrotational or potential flow, i.e. nonzero vorticity $\boldsymbol{\omega}$, can be highly significant even if confined to a small part of the fluid domain — essentially because vorticity can be carried by the fluid motion from one place to another: as noted earlier, 'vortex lines move with the fluid'. This is a profoundly different situation from that of irrotational flow, which has no vortex lines to be carried around.

Vortex lines can also diffuse through the fluid, if viscosity is important; and they can move through the fluid in other ways, too, if density stratification is important, as in the earth's atmosphere and oceans and in the Sun's interior. But that is another story.

When the vorticity distribution $\boldsymbol{\omega}(\mathbf{x}, t)$ is changed, from whatever cause, there have to be consequent changes in the velocity field. Otherwise, $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ would fail to hold. The velocity field in turn affects how vorticity is carried around and otherwise changed. There results a nontrivial dynamical behaviour, very often unsteady and chaotic — and very different indeed from flow that is irrotational everywhere.

In the abstract language of dynamical systems, fluid systems are nonlinear dynamical systems with infinite-dimensional phase spaces. Because their general character is familar and because they are also easy to observe, they are a uniquely important example. There are other, more abstract reasons (including Poincaré's notion of 'flow' in phase space itself) why concepts from fluid dynamics are of interest in more general dyamical-systems studies.

Our simplest possible model can be used to illustrate all the features just mentioned. We confine attention to 2-D inviscid flow, in the xy plane, say, so that the vorticity equation $D\boldsymbol{\omega}/Dt = (\boldsymbol{\omega}.\nabla)\mathbf{u}$ becomes simply

$$D\boldsymbol{\omega}/Dt = 0 \qquad \text{with} \quad \boldsymbol{\omega} = \omega \mathbf{e}_z \ ,$$

where $\mathbf{e}_z \perp xy$ plane. We take the regions of nonzero vorticity to be *infinitesimally small.* More precisely, we assume that all such regions take the form of what are called *line vortices* $\parallel$ z-axis, also called *point vortices* when thinking within the xy plane. Thus $\omega(x, y, t)$ at any given moment t is zero everywhere except for a finite number of points $\mathbf{x}_i(t) = (x_i(t),\, y_i(t))$ in the xy plane.

We can think of each such point vortex as a limiting case in which the vorticity ω is confined to a small region that shrinks to zero. In the limit, the area of the region is reduced to zero while increasing the average value of the vorticity in the region, in such a way that the circulation remains finite. Thus the circulation κ of such a vortex is a natural measure of its strength, and by convention is called 'the strength' of the vortex.

Mathematically, the corresponding $\omega(\mathbf{x})$ is a delta function, corresponding also to the idea of a 'point charge' in electromagnetic problems, or a 'point mass' in particle-dynamical problems like the Newtonian theory of the solar system. Here the volume containing the charge or mass is reduced to zero while the total charge or total mass is held constant.

Recall from §3.2a the velocity potential for a point vortex at the origin, with circulation κ, i.e. with 'strength' κ (and going back to standard sign convention):

$$\phi = \kappa\theta/2\pi,$$

and recall that this satisfies Laplace's equation everywhere except at the origin. The corresponding velocity field is

$$\mathbf{u} = \nabla\phi = \frac{\kappa}{2\pi r}\mathbf{e}_\theta = \frac{\kappa}{2\pi}\frac{\mathbf{e}_z \times \mathbf{x}}{|\mathbf{x}|^2} = \frac{\kappa}{2\pi}\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

where the components in the last expression are referred to Cartesian axes.

Laplace's equation is linear, so we can linearly superpose a finite number N of point vortices with different strengths and positions $\mathbf{x}_i$ $(i = 1, ...N)$, thus

$$\phi(\mathbf{x}) = \sum_{i=1}^{N} \frac{\kappa_i\theta_i}{2\pi} \quad \text{and} \quad \mathbf{u}(\mathbf{x}) = \sum_{i=1}^{N} \frac{\kappa_i}{2\pi}\frac{\mathbf{e}_z \times (\mathbf{x}-\mathbf{x}_i)}{|\mathbf{x}-\mathbf{x}_i|^2} \qquad (\mathbf{x} \neq \mathbf{x}_i \ \ \forall i).(*)$$

This, by linearity, satisfies Laplace's equation everywhere except at $\mathbf{x} = \mathbf{x}_i$. The crucial point now is that we satisfy $D\boldsymbol{\omega}/Dt = 0$ by requiring that the $\mathbf{x} = \mathbf{x}_i$ are functions of time t such that:

Each vortex is moved by the velocity field due to all the other vortices.$(**)$

This is consistent! The velocity field of one vortex by itself cannot move that vortex. An isolated vortex just sits in one place, spinning but doing nothing else (as is obvious by symmetry; no particular direction is distinguished). The dynamical system thus defined consists of a set of N first-order nonlinear ordinary differential equations for the N functions $\mathbf{x}_i(t)$:

$$\dot{\mathbf{x}}_i(t) = \sum_{j\neq i} \frac{\kappa_j}{2\pi}\frac{\mathbf{e}_z \times (\mathbf{x}_i-\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^2} \qquad (i = 1, ...N)\ , (***)$$

where $\dot{\mathbf{x}}_i$ means the time derivative, as usual, $d\mathbf{x}_i(t)/dt$, and the summation is from $j = 1$ to N but omitting the term for which $j = i$.

Being nonlinear, these differential equations must usually be solved numerically, which is easy these days on any personal computer for modest values of N. It is known that when $N \geqslant 4$ the system behaves chaotically, in most cases, implying an exponential blowup in sensitivity to initial conditions as time increases. Such sensitivity is liable to be important whenever the equations are integrated over timescales much longer than $t_{\rm vort} \sim 2\pi L^2/\kappa$, if κ is a typical vortex strength and L a typical separation between vortices so that $|\dot{\mathbf{x}}_i|$ has typical values of order $2\pi\kappa/L$. This is fundamentally like the initial-condition sensitivity that limits the timespan of weather forecasting. It can be shown that cyclones and anticyclones in the real atmosphere behave in some respects like the vortices in our 2-D point-vortex model, with $t_{\rm vort}$ values of the order of a day or two. (The real behaviour is closer to 2-D than 3-D because of the stable density stratification, but that is another story.) The expectation of chaotic behaviour for $N \geqslant 4$ correctly suggests why it is impracticable to forecast the weather in detail many days ahead.

There are a few simple cases of $(***)$ that are both soluble analytically and practically important.

The simplest is what is called a *vortex pair*, the case $N = 2$ and $\kappa_1 = -\kappa_2 = \kappa$, say. By inspection of $(***)$ we see that $\dot{\mathbf{x}}_2(t) = \dot{\mathbf{x}}_1(t), = \mathbf{U}$ say, from which it follows at once that $d(\mathbf{x}_1 - \mathbf{x}_2)/dt = 0$, so that $(\mathbf{x}_1 - \mathbf{x}_2)$ is a constant of the motion. The line segment joining the two vortices is constant in length and orientation.

Because of the vector products in $(***)$, we see also that $\mathbf{U} \perp (\mathbf{x}_1 - \mathbf{x}_2)$. The two vortices move as one, with the same constant velocity $\mathbf{U}$ having magnitude

$$|\mathbf{U}| = \frac{|\kappa|}{2\pi|\mathbf{x}_1 - \mathbf{x}_2|}$$

and directed at right angles to the line joining the two vortices.

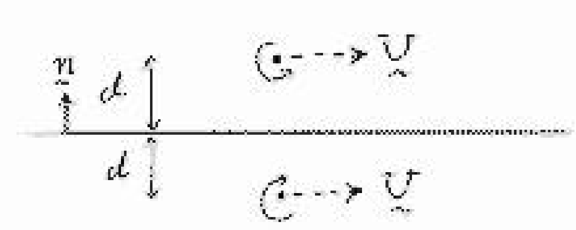
This property is the 2-D counterpart of the way a circular vortex ring in 3-D, sometimes made visible as a smoke ring, translates through a fluid in the direction at right angles to its plane. The vortex-pair solution also adds to our understanding of how aircraft stay up: the trailing vortices extend a long way behind the aircraft and behave very like our simple 2-D vortex pair, with $\mathbf{U}$ directed downward. One can show that the associated velocity field, $(*)$ with $N = 2$, carries nearby air predominantly downward as well, as might be expected from Newton's third law. To stay up, the aircraft has to push air down.

Use of 'images'

The analysis just given solves another problem, that of a single point vortex of strength κ at a point $\mathbf{x}_1$ at distance d from a rigid straight boundary — a plane boundary if we want to think in 3-D — on which $\mathbf{u}.\mathbf{n} = 0$. We need only compare this boundary-value problem with the problem in which:

(a) an 'image vortex' of strength $-\kappa$ is placed at a point $\mathbf{x}_2$ on the opposite side of the boundary, such that the boundary bisects the line segment $(\mathbf{x}_1 - \mathbf{x}_2)$, and

(b) the boundary is then removed.



This last problem is the *same* problem as that just solved, the vortex-pair problem; so we already know the solution. The points $\mathbf{x}_1$ and $\mathbf{x}_2$ move with the constant velocity $\dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_2 = \mathbf{U}$ parallel to the former boundary, with $|\mathbf{U}| = |\kappa|/4\pi d$. Moreover, by symmetry, $\mathbf{u}.\mathbf{n} = 0$ where the boundary was located; so we can put the boundary back without changing anything!

More specifically, and taking the boundary to be $y = 0$ — the yz plane if we want to think in 3-D — we have solved the following boundary value problem. We have satisfied Laplace's equation $\nabla^2\phi = 0$ in $y > 0$ with boundary conditions of evanescence at infinity together with

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \qquad \text{i.e.,} \quad \mathbf{u}.\mathbf{n} = 0 \;\text{ with }\; \mathbf{n} = (0,1)$$

and

$$\phi(\mathbf{x}) \to \frac{\kappa\theta_1}{2\pi} \quad \text{as} \quad |\mathbf{x} - \mathbf{x}_1| \to 0\ ,$$

where θ_1 is the angle giving the direction of $(\mathbf{x}-\mathbf{x}_1)$, that is, $\theta_1 = \arctan\{(y - y_1)/(x - x_1)\}$. Defining also $(x_2,\, y_2) = (x_1,\, -y_1)$, the image point, and $\theta_2 = \arctan\{(y - y_2)/(x - x_2)\} = \arctan\{(y + y_1)/(x - x_1)\}$, we see that the velocity potential and the corresponding velocity field are, for all $\mathbf{x} \neq \mathbf{x}_1,\ \mathbf{x}_2$,

$$\phi(\mathbf{x}) = \frac{\kappa\theta_1}{2\pi} - \frac{\kappa\theta_2}{2\pi} \qquad \Rightarrow \qquad \mathbf{u}(\mathbf{x}) = \frac{\kappa\mathbf{e}_z}{2\pi} \times \left\{ \frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^2} - \frac{(\mathbf{x}-\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_2|^2} \right\}\ .$$

2-vortex problems in general

The properties that $\dot{\mathbf{x}}_1(t) \perp (\mathbf{x}_1 - \mathbf{x}_2)$ and $\dot{\mathbf{x}}_2(t) \perp (\mathbf{x}_1 - \mathbf{x}_2)$ can be seen, again by inspection of $(***)$, to hold for all point-vortex problems with $N = 2$, including, that is, cases where $\kappa_1 \neq \kappa_2$. All these 2-vortex problems share the property that $|\mathbf{x}_1 - \mathbf{x}_2|$ is a constant of the motion, making them all analytically soluble. The motion is always on arcs of circles, if we include straight lines by regarding them as circles of infinite radius.

In the case $\kappa_1 = \kappa_2 = \kappa$, for instance, the two vortices move around each other at speed $|\mathbf{U}| = |\kappa|/4\pi a$ in the same circle, of radius a say, centred on the midpoint $\mathbf{x}_{\text{mid}} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$. The midpoint is, indeed, stationary: $\dot{\mathbf{x}}_1 = -\dot{\mathbf{x}}_2$, so that $\dot{\mathbf{x}}_{\text{mid}} = 0$. For further variations on this theme, try Q1 on Ex. Sheet 3.)

More complicated boundaries:

The method of images can handle more elaborate boundary configurations. The next simplest is two boundaries forming a right-angled corner, say the positive x and y axes. With this configuration, a single point vortex of strength κ at $(x_1 > 0,\ y_1 > 0)$ has three images, at $(x_1,\ -y_1)$, $(-x_1,\ y_1)$, $(-x_1,\ -y_1)$ with strengths respectively $-\kappa$, $-\kappa$, κ. From $(***)$ we have

$$\dot{x}_1 \;=\; \frac{\kappa}{2\pi}\left\{\frac{1}{2y_1} - \frac{y_1}{2(x_1^2 + y_1^2)}\right\} \;=\; \frac{\kappa}{4\pi}\,\frac{x_1^2}{y_1(x_1^2 + y_1^2)}$$

and

$$\dot{y}_1 \;=\; \frac{\kappa}{2\pi}\left\{-\frac{1}{2x_1}+\frac{x_1}{2(x_1^2+y_1^2)}\right\} \;=\; -\;\frac{\kappa}{4\pi}\,\frac{y_1^2}{x_1(x_1^2+y_1^2)}\;.$$

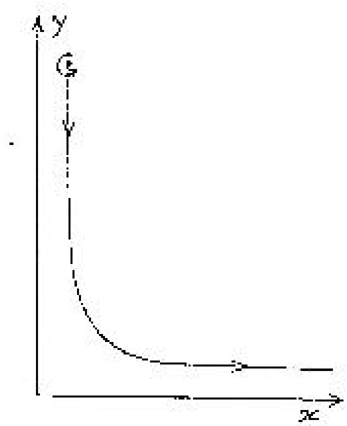
It is easy to deduce the path of the vortex, because

$$\frac{dy_1}{dx_1} \;=\; \frac{\dot{y}_1}{\dot{x}_1} \;=\; -\left(\frac{y_1}{x_1}\right)^3\;,$$

which can be integrated to give

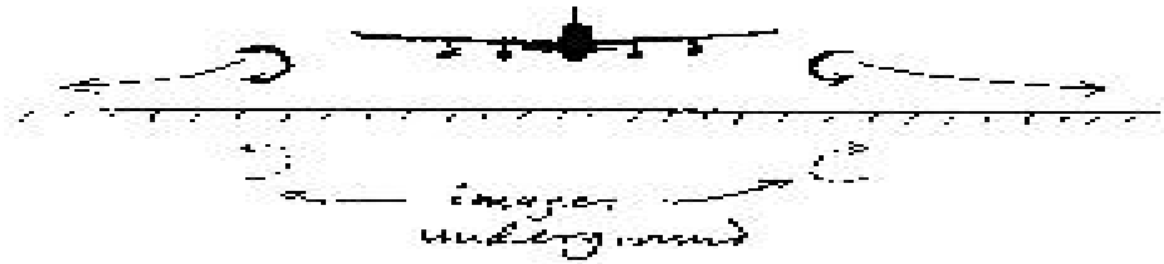
$$\frac{1}{x_1^2}+\frac{1}{y_1^2} \;=\; \text{constant}\;.$$

The path is sketched below.



By removing one boundary, we see that the same solution describes a vortex pair approaching a single boundary. The two vortices move apart.

Aircraft trailing vortices behave like this when the aircraft lands or takes off: the wingtip vortices migrate away to either side of the runway. These vortices can be formidably strong for a large aircraft like a jumbo jet: smaller aircraft beware! You can see qualitatively the same thing in miniature by spoon-dipping in a teacup. With a little practice you can generate a half-vortex-ring, made visible by a pair of dimples on the surface, moving toward the side of the cup. The dimples move apart as they approach the boundary.



By removing the remaining boundary, we see that the same solution solves the vortex 'elopement' or 'ménage a quatre' problem. Two equal and opposite vortex pairs approach each other, from $x = \pm\infty$ say, exchange partners, then recede toward $y = \pm\infty$ (sketch below). The above expressions apply if we take $\kappa < 0$. Notice, for instance, that the above expression for $\dot{y}_1 \to |\kappa|/(4\pi x_1)$ as $y_1 \to \infty$ and $x_1 \to$ const., agreeing with the earlier result for the translation speed of an isolated vortex pair.