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PART IB FLUID DYNAMICS

Internet version available at www.atm.damtp.cam.ac.uk/people/mem/

§0 INTRODUCTION

What is a fluid? By definition, a fluid cannot withstand any tendency for applied forces to deform it in such a way that volume is left unchanged. Such deformation may be resisted, but not prevented, by a fluid. Ordinary gases and liquids are fluids.

The course applies mathematical techniques learned in the Vector Calculus & Methods courses (e.g. vector calculus, integral theorems, solution of Laplace equation) to a description of (to mathematical models of) a large variety of physical problems involving fluid motion.

This is not just an exercise in mathematics (though it does give simple illustrations of the mathematical techniques of field theory). The course describes some mathematical models of real practical importance, applying for instance to:

- forces/energy loss in converging/diverging pipes or in jet impact
- pressure distribution near collapsing bubbles (with implications for damage to ship propellers, high-speed pumps, and other machinery using high-energy flow)
- prediction of lift force on aircraft wings
- tidal-wave (tsunami) dynamics and propagation speed
- flow rate over a weir

Beyond this course — fluid dynamics enters many important problems:

- industrial flows: fuel-efficient aircraft design, hydroelectric power, chemical processing, jet-driven cutting tools
- our fluid environment: ozone loss, climate change, ocean currents and eddies, weather and climate forecasting, volcanic eruptions, flow inside and around buildings
- geophysical and astrophysical flows: magma flow and mineral deposition, motion of continents, dynamo action in planets and stars, stellar pulsation, inflow to black holes

Books: (see Schedules).

- Acheson: Elementary Fluid Dynamics — clear, concise, covers many topics.

- Batchelor: Introduction to Fluid Dynamics — widely regarded as a 'bible' for the subject. Daunting at first sight; but lucid, thorough, and reliable. Can be read selectively.
- Guyon, Hulin & Petit: Hydrodynamique Physique — more physical viewpoint, but in French!
- Lighthill: An Informal Introduction to Theoretical Fluid Mechanics.
- Paterson: A First Course in Fluid Dynamics — careful, complete treatment
- van Dyke: Album of Fluid Motion. Beautiful photographs showing fluids in motion. [Seeing photographs or, even better, movies or videos of fluid motion, and observing real flows, gives important input to formulation of mathematical models.]

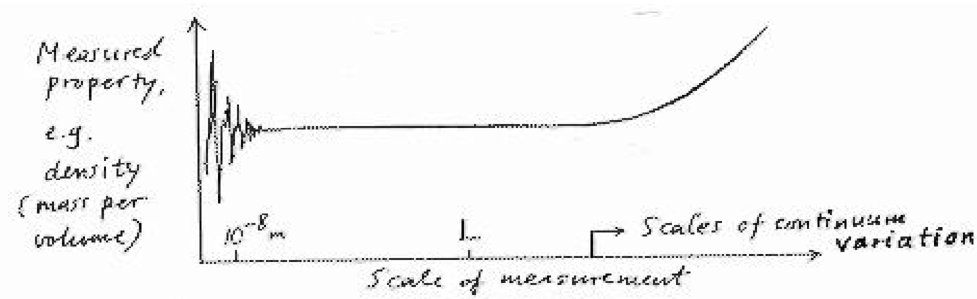
For background reading, I also like the last two chapters of the wonderful Feynman Lectures in Physics, Vol. II: *The Flow of Dry Water* and *The Flow of Wet Water*.

Multimedia: G. M. Homsy *et al*, *Multi-Media Fluid Mechanics*, CD-ROM available from CUP Bookshop, 1 Trinity St. Wonderful collection of videos; some will be shown in the lectures, starting with the shot of a jet airliner flying through two smoke plumes (near the end of the "Video Gallery"). The result is a spectacular visualization of the two trailing vortices, an essential part of how fixed-wing aircraft stay airborne.

§1 KINEMATICS

1.1 Continuum model; Lagrangian and Eulerian descriptions

Gases and liquids are made up of individual molecules, e.g. air at STP (0°C, 1 atmosphere) has 2.7×10^{25} molecules per m^3, molecules $\lesssim 10^{-9}$m long, intermolecular spacing $\lesssim 10^{-8}$m, mean free path between collisions $\sim 10^{-7}$m. On scales around 10^{-8}m measurement of density or velocity would fluctuate wildly:



On larger scales, e.g. in everyday experience, fluid appears as smooth continuum. Quantities such as density and velocity are continuous functions of position). Define continuous fields such as density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, pressure $p(\mathbf{x}, t)$, by averaging over small linear dimension L^3,

$$\text{e.g.} \qquad \rho(\mathbf{x}, t) \ = \ \frac{\text{mass in box centred at } \mathbf{x}}{\text{volume of box } (\sim L^3)}$$

with $10^{-8}\text{m} \ll L \ll$ length-scale of continuum variation (e.g. $\gtrsim 10^{-5}$ m for ordinary turbulent flow in a teacup during vigorous stirring).

The continuum approximation does not always give a good model of fluid motion. An example is a spacecraft re-entering the atmosphere: intermolecular spacings and free paths are not small in comparison with spacecraft lengthscales. But for the rest of the course we shall assume that the continuum model is adequate — which is the case under an enormous range of conditions — and view the fluid as a continuous medium described by smoothly varying fields such as $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, and $p(\mathbf{x}, t)$.

Lagrangian vs. Eulerian description. One approach to the mathematical description of fluid motion is to follow 1A Dynamics; this leads to what is called the *Lagrangian description* of the motion. Divide the fluid into limitingly small 'particles' or 'parcels', then consider the motion of each parcel in response to the forces acting on it, including forces exerted by neighbouring parcels. To keep track of which parcel is which, we label them by a variable $\mathbf{x}_0$, which could be, for instance, the position of a parcel at time $t = 0$. In this description, ρ, $\mathbf{u}$, p, etc. are functions not of $\mathbf{x}$ and t but of $\mathbf{x}_0$ and t; and the position $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ of each fluid parcel has to be determined as part of the solution.

An alternative approach is to work directly in terms of fields such as $\rho(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, and $p(\mathbf{x}, t)$, and not try to keep track of individual fluid particles or parcels. This is called the *Eulerian description.* It is easier to use than the Lagrangian description, and will be used for most of this course. Notice that if we keep $\mathbf{x}$ fixed and let t vary in $\mathbf{u}(\mathbf{x}, t)$ (and if $\mathbf{u} \neq 0$), then we are looking at the velocities of a *succession of fluid parcels* moving past a given point whose position is $\mathbf{x}$.

1.2 Flow visualization: pathlines, streamlines, and streaklines

There are three distinct types of line or curve commonly used when visualizing fluid motion in, e.g., simple laboratory experiments:

1. *Pathline*: trajectory of a single fluid parcel or particle, starting at some given point in the flow. Corresponds to long-exposure photograph of a marked particle or small blob of dye (e.g. dashed curve in plot below). This is the curve defined by the function $\mathbf{X}(\mathbf{x}_0, t)$ with $\mathbf{x}_0$ fixed and t varying.

2. *Streamline*: curve everywhere tangent to the flow direction at some instant (integral curve of the velocity field $\mathbf{u}(\mathbf{x}, t)$ at fixed t, e.g. dotted curves in plot below, p.4). First imagine taking a short-exposure photograph of densely spaced marked particles, producing short streaks, or infinitesimal trajectories, which show the local flow magnitudes and directions, i.e. visualize the vector field $\mathbf{u}(\mathbf{x}, t)$. Then, starting at some given point, draw a curve everywhere tangent to the short streaks.

3. *Streakline*: locus of dye released at some fixed point, or of parcels/particles marked as they pass that

point during some time interval — as with dye continually released from the end of a fine tube, or smoke from a chimney (e.g. solid curve in plot below).

(Both streamlines and streaklines will generally change with time.)

Each of these types of curves may be defined by a differential equation, as follows. A 2-dimensional example with a time-dependent model velocity field $\mathbf{u}(\mathbf{x}, t) = (yt, 1)$ allows simple analytic solutions of each differential equation, and is used to illustrate that the three types of curve are, in general, all different.

1. A pathline is a curve $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$, with $\mathbf{x}_0$ fixed and t varying, defined by

$$\frac{\partial}{\partial t}\mathbf{X}(\mathbf{x}_0, t) = \mathbf{u}\Big(\mathbf{X}(\mathbf{x}_0, t),\ t\Big)$$
$$\text{with } \mathbf{X} = \mathbf{x}_0 \quad \text{at} \quad t = 0\ .$$

In the example: $\dfrac{\partial X}{\partial t} = Yt, \qquad \dfrac{\partial Y}{\partial t} = 1; \qquad Y = y_0 + t$

$$X = x_0 + \tfrac{1}{2}y_0 t^2 + \tfrac{1}{3}t^3$$

Eliminate t to get shape of pathline: $\quad X = x_0 + \tfrac{1}{2}y_0(Y - y_0)^2 + \tfrac{1}{3}(Y - y_0)^3$

Plot below shows the case $\mathbf{x}_0 = (x_0, y_0) = (0, 0)$ as dashed curve, for $0 < t < 2$.

2. A streamline is a curve $\mathbf{x} = \mathbf{X}(s; \mathbf{x}_0, t)$, with $\mathbf{x}_0$ and t fixed, and the parameter s varying defined by

$$\frac{\partial}{\partial s}\mathbf{X}(s; \mathbf{x}_0, t) = \mathbf{u}\Big(\mathbf{X}(s; \mathbf{x}_0, t),\ t\Big) \qquad \text{with } \mathbf{X} = \mathbf{x}_0 \quad \text{at} \quad s = 0\ .$$

Remember, t is now fixed. Solving this problem produces a curve parameterized by s, when s is varied in the function $\mathbf{X}(s; \mathbf{x}_0, t)$. A set of such curves for different values of $\mathbf{x}_0$ gives us a snapshot of the flow at the single instant represented by the fixed value of t. The light dotted curves below illustrate this. They are streamlines for the same example $\mathbf{u} = (yt, 1)$, at the instant $t = 2$:

In the example: $\dfrac{\partial X}{\partial s} = Yt, \qquad \dfrac{\partial Y}{\partial s} = 1; \qquad Y(s; \mathbf{x}_0, t) = y_0 + s$

$$X(s; \mathbf{x}_0, t) = x_0 + y_0 ts + \tfrac{1}{2}ts^2$$

Eliminate s to get shape of streamline at time t: $\quad X = x_0 + y_0 t(Y - y_0) + \tfrac{1}{2}t(Y - y_0)^2$

(a parabola) $\qquad$ i.e., $\quad X = \tfrac{1}{2}tY^2 + (x_0 - \tfrac{1}{2}ty_0^2)$

The light dotted curves (below) show the cases $t = 2,\ y_0 = 0,\ x_0 = -3.75,\ -3.5, ..., 3.5,\ 3.75,\ 4.$

3. A streakline is like the plume from a chimney. A streakline at some instant $\tilde{t}$ is a curve $\mathbf{x} = \mathbf{X}$ defined by fixing $\tilde{t}$ and varying s, over some range of $s \leqslant \tilde{t}$, in another function $\mathbf{X}(s; \mathbf{x}_0, \tilde{t})$. *This* function is defined by solving the following *set* of initial-value problems. Each is like problem 1 above, except that the initial condition is imposed $\boxed{\text{at time } t = s}$ rather than at $t = 0$ (and $\mathbf{x}_0$ still fixed throughout — "chimney-top position"):

$$\frac{\partial}{\partial t}\mathbf{X}(s; \mathbf{x}_0, t) = \mathbf{u}\Big(\mathbf{X}(s; \mathbf{x}_0, t),\ t\Big)$$

$$\text{with } \mathbf{X} = \mathbf{x}_0 \quad \boxed{\text{at} \quad t = s}$$

After solving this (for general s) set $t = \tilde{t}$.

This expresses in symbols the fact that the streakline can be thought of as being made of dye emitted from $\mathbf{x} = \mathbf{x}_0$ during a time interval equal to the range of s.

In the example: $\dfrac{\partial X}{\partial t} = Yt, \quad \dfrac{\partial Y}{\partial t} = 1, \quad \text{with} \quad \begin{cases} X = x_0 \text{ at } t = s \\ Y = y_0 \text{ at } t = s \end{cases}$

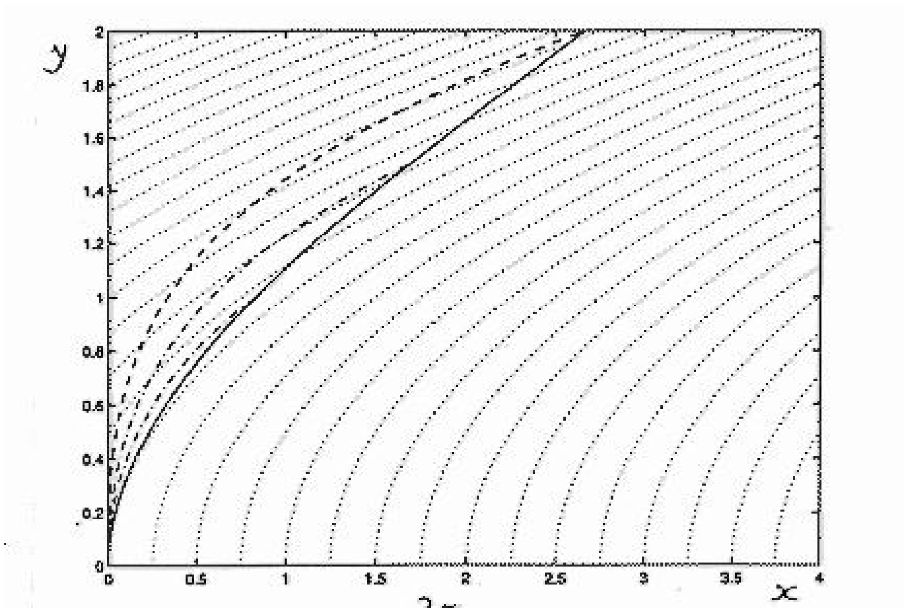
so $Y = y_0 + t - s;\ X \quad = \quad x_0 + (\tfrac{1}{2}y_0t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{2}st^2) - (\tfrac{1}{2}y_0s^2 - \tfrac{1}{6}s^3)$

Eliminate s, set $t = \tilde{t}$, and simplify to get shape at time $\tilde{t} > s$:

$$X \quad = \quad -\tfrac{1}{6}Y^3 + \tfrac{1}{2}\tilde{t}Y^2 + \tfrac{1}{2}y_0^2Y + (x_0 - \tfrac{1}{2}\tilde{t}y_0^2 - \tfrac{1}{3}y_0^3)$$

(Several lines of calculation — check it!) For $\mathbf{x}_0 = 0$ this is the solid curve in the plot below. It traces out the ends of a succession of pathlines initiated at the succession of times $t = s$, as s varies over the range 0 to 2. Three of these pathlines are shown: the dashed one already mentioned, corresponding to $s = 0$, and two more shown dash-dotted, corresponding to $s = 0.5$ and $s = 1$.

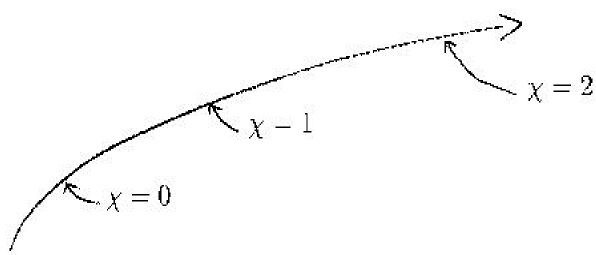
So pathlines, streamlines, and streaklines are all different in general. They would coincide only if the flow were *steady*, $\partial\mathbf{u}(\mathbf{x}, t)/\partial t = 0$. So if $\mathbf{u} = (yt, 1)$ were replaced by $\mathbf{u} = (2y, 1)$, then the pathlines, streamlines, and streaklines would all coincide with the light dotted curves.



1.3 Material derivative

Consider a field $\chi(\mathbf{x}, t)$. An observer moving with the flow may see changing values of χ even if at each location χ is independent of time, i.e. $\partial\chi/\partial t = 0$.

Observer follows pathline, i.e. parcel trajectory:



Let the position of the observer at time t be $\mathbf{x} = \mathbf{X}(t)$. If the observer is moving with the fluid then $d\mathbf{X}/dt = \mathbf{u}(\mathbf{X}, t)$. The rate of change of χ with time seen by such an observer is, by the chain rule,

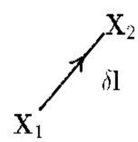
$$\frac{d}{dt}\chi\Big(\mathbf{X}(t), t\Big) \qquad = \frac{d\mathbf{X}}{dt}.\nabla\chi\big|_{\mathbf{x}=\mathbf{X}} + \frac{\partial\chi}{\partial t}$$

$$= \frac{\partial\chi}{\partial t} + \mathbf{u}.\nabla\chi \equiv \frac{D\chi}{Dt} \qquad \text{(material derivative)}$$

(all evaluated at $\mathbf{x} = \mathbf{X}$). The *material derivative* D/Dt occurs naturally in many fluids problems, e.g. if $\chi(\mathbf{x}, t)$ is the concentration of an inert substance ('dye'), measured as amount per unit mass of fluid, then $D\chi/Dt = 0$ if we ignore molecular diffusion. When $D\chi/Dt = 0$, we can have nonzero values of $\partial\chi/\partial t$ if, e.g., blobs of dye are being carried past a fixed point $\mathbf{x}$.

1.4 Equation for material line element

A material line element is a small line element marked in the fluid, i.e., made up of fluid parcels, with end points $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ moving with the flow. (Full Lagrangian labelling not needed here.)



We write $\delta\mathbf{l}(t) = \mathbf{X}_2(t) - \mathbf{X}_1(t)$. Then the rate of change of $\delta\mathbf{l}(t)$ is

$$\begin{aligned} \frac{d}{dt}\delta\mathbf{l} &= \frac{d\mathbf{X}_2}{dt} - \frac{d\mathbf{X}_1}{dt} = \mathbf{u}(\mathbf{X}_2, t) - \mathbf{u}(\mathbf{X}_1, t) \\ &= \mathbf{u}(\mathbf{X}_1 + \delta\mathbf{l}, t) - \mathbf{u}(\mathbf{X}_1, t) \\ &= \mathbf{u}(\mathbf{X}_1, t) + (\delta\mathbf{l}.\nabla)\mathbf{u}|_{\mathbf{x}=\mathbf{X}_1} - \mathbf{u}(\mathbf{X}_1, t) + O(|\delta\mathbf{l}|^2) \\ &= (\delta\mathbf{l}.\nabla)\mathbf{u}|_{\mathbf{x}=\mathbf{X}_1} + O(|\delta\mathbf{l}|^2) \end{aligned}$$

In the limit of infinitesimal $|\delta\mathbf{l}|$,

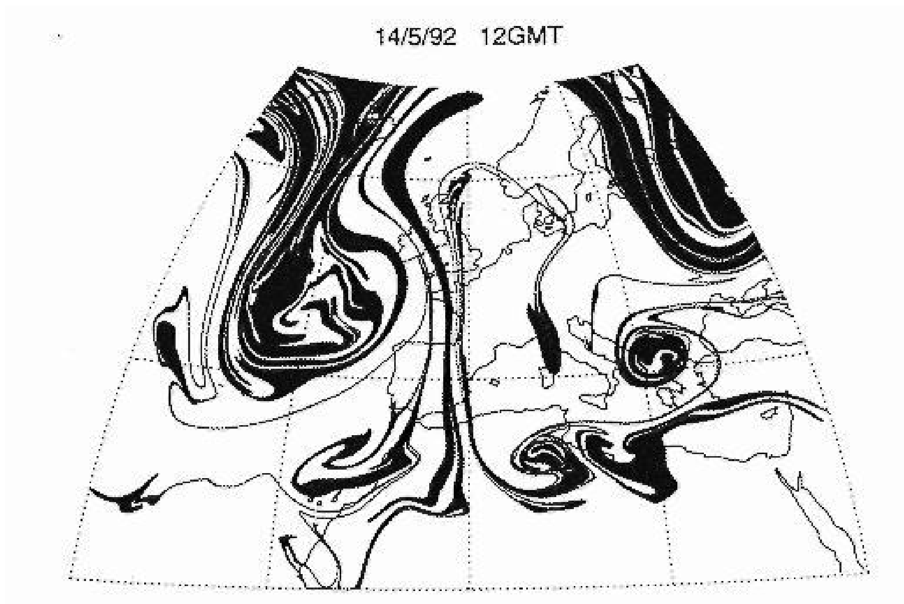
$$\frac{d}{dt}\delta\mathbf{l} = (\delta\mathbf{l}.\nabla)\mathbf{u}|_{\mathbf{x}=\mathbf{X}_1} ,$$

or, in Cartesian tensor (suffix) notation ($\partial u_i/\partial x_j$ is the velocity gradient tensor),

$$\frac{d}{dt}\delta l_i = \delta l_j \left(\frac{\partial u_i}{\partial x_j}\right)\bigg|_{\mathbf{x}=\mathbf{X}_1} .$$

In the limit, we are talking about *one* position $\mathbf{x} = \mathbf{X}_1$ in space, and so can now, if we wish, think of the d/dt as D/Dt (of $\delta\mathbf{l}$ regarded as a field, i.e. expressed as a function of $\mathbf{x}$ and t).

Line elements are generally stretched and rotated by the flow, similarly area elements and volume elements. This often leads to amazing patterns (cf. also cream on coffee):

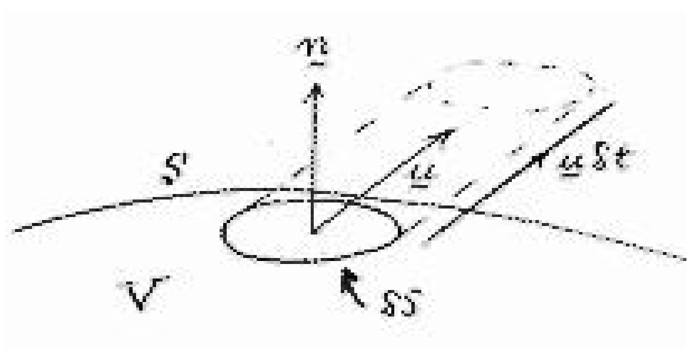


The picture shows the stretching and rotation of lines of fluid parcels, over a time interval of 4 days, by a velocity field that approximates the real (nearly-horizontal) velocity field at jetliner cruise altitudes.

1.5 Conservation of mass

We assume that fluid is neither created nor destroyed, i.e., that fluid mass is conserved. Look at what this means mathematically, for a general flow field.

Consider an arbitrary finite volume V fixed in space, bounded by surface S, with outward normal $\mathbf{n}$:



Consider fluid flowing through a small portion of S,

i.e. through area δS in time δt $\quad$ (δS, δt small)

Volume of 'slug' or slanted cylinder of fluid that has passed out through δS is $(\mathbf{n}\,\delta S).(\mathbf{u}\,\delta t)$

mass is $\qquad \rho(\mathbf{u}.\mathbf{n})\delta S\delta t$; $\qquad$ mass/unit time is $\rho(\mathbf{u}.\mathbf{n})\delta S$.

Now integrate over whole surface S, and express the fact that the only way mass can get in or out is to cross the surface S. Mass flows out of V (through S) at rate

$$\int_S \rho(\mathbf{u}.\mathbf{n}) dS \ ;$$

this must be equal to the rate of change of the total mass occupying V, $\int_V \rho dV$, implying that

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho(\mathbf{u}.\mathbf{n}) dS \ .$$

Now the volume V is fixed in space, so $\qquad \text{LHS} = \int \frac{\partial \rho}{\partial t}\, dV$.

Use divergence theorem: $\qquad \text{RHS} = -\int_V \nabla.(\rho\mathbf{u})\, dV$.

Equation holds for arbitrary V, hence, assuming that integrands are continuous functions,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{u}) = 0 \ , \tag{1}$$

the (local) equation expressing mass conservation. This is one case of a situation repeatedly encountered in theoretical physics, when dealing with an additive quantity described by a continuous field (mass in this case — but it could equally well be energy, momentum, chemical substance, etc.). If the additive quantity (the 'stuff', if you like) is conserved in any finite volume V, in the above sense (i.e., if the total amount of stuff within V can change only because stuff gets in or out across the boundary V), then one always has a local conservation equation of the form $\dfrac{\partial(\quad)}{\partial t} + \nabla.\{\quad\} = 0$, i.e., a four-dimensional divergence vanishes, derivable an analysis like the above. We call ( ) the density (amount of stuff per unit volume), and { } the flux (amount of stuff crossing unit surface area per unit time).

In the present case (mass conservation), we have $\nabla.(\rho\mathbf{u}) = \mathbf{u}.\nabla\rho + \rho\nabla.\mathbf{u}$, and so equation (1) above can be rewritten as

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} = 0 \ . \tag{2}$$

1.6 Incompressibility

From here on, in this course, we restrict attention to fluid behaviour for which ρ can be taken to be constant. Consider for convenience the second, D/Dt form of the mass conservation equation just derived.

With ρ constant we have $D\rho/Dt = 0$ trivially, therefore

$$\nabla.\mathbf{u} = 0\ . \tag{3}$$

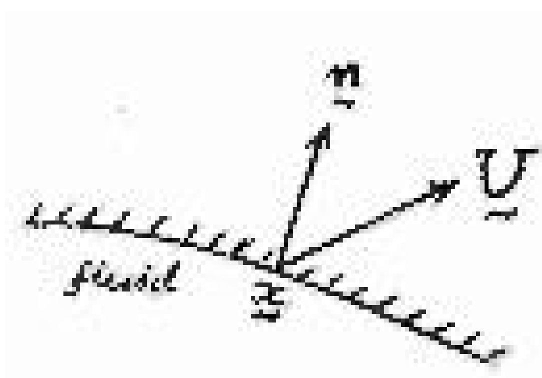
The velocity field is nondivergent, i.e. solenoidal. This expresses mass conservation for incompressible flow.

Notice that the assumption of constant ρ excludes sound waves: the assumption can be shown to be consistent when fluid velocities $|\mathbf{u}| \ll$ sound speed, 340 m s^{-1} in air at 1 bar and 15°C; $\sim$ 1500 m s^{-1} in water. Constant ρ also implies that we are excluding density-driven flows such as thermal convection.

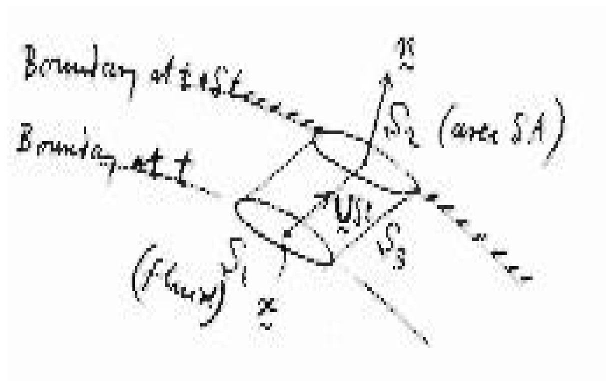
1.7 Boundary condition at an impermeable boundary

We continue to assume that fluid is neither created nor destroyed, and restrict attention to the case of an impermeable boundary: no fluid can get across it (nor be absorbed nor expelled from it). We assume that the fluid stays in contact with boundary. If the boundary is stationary, then impermeability $\Rightarrow \mathbf{u}.\mathbf{n} = 0$, where $\mathbf{n}$ is normal to the boundary. If the boundary is moving, then one can derive the relevant boundary condition as follows.

Consider what happens near a fixed point $\mathbf{x}$ that coincides with some point on the boundary at a particular instant t. Let $\mathbf{n}$ be the outward normal at $\mathbf{x}$. Suppose first that the boundary is outward-moving at $\mathbf{x}$, with velocity $\mathbf{U}$, so that $\mathbf{U}.\mathbf{n} > 0$:



Consider the motion, between time t and time $t + \delta t$, of a small area element δA on the boundary, initially a neighbourhood of the point $\mathbf{x}$. In much the same way as before, δA sweeps out a 'slug' or slanted cylinder with volume $(\mathbf{U}\delta t)\,.\,(\mathbf{n}\delta A)$ as the boundary moves. At time $t + \delta t$ this volume is filled with fluid of density ρ, and this fluid has mass $\rho\,(\mathbf{U}\delta t)\,.\,(\mathbf{n}\delta A) = \rho\,\delta A\,\delta t\,\mathbf{U}.\mathbf{n}$.



In the notation of the diagram, S_1 is fixed in space and coincides with the area δA as it was at time t. The amount of mass that has crossed S_1 is $\rho\,\delta A\,\delta t\,\mathbf{u}.\mathbf{n}$, where $\mathbf{u}(\mathbf{x},t)$ is the fluid velocity field as before.

Therefore

$$\rho\,\delta A\,\delta t\,\mathbf{U}.\mathbf{n} = \rho\,\delta A\,\delta t\,\mathbf{u}.\mathbf{n} + M_3\ , (*)$$

where M_3 is the mass that has entered through the side S_3. M_3 need not be zero, because $\mathbf{u}$ need not be parallel to $\mathbf{U}$. But M_3 becomes relatively negligible if we take the limit $\delta A \to 0$, $\delta t \to 0$ with $\delta t/(\delta A)^{1/2} \to 0$, holding everything else constant including $\mathbf{U}$. Then the length of the side S_3, $|\mathbf{U}|\delta t$, multiplied by the circumference of S_1 or S_2, which $\propto (\delta A)^{1/2}$, gives the area of S_3, showing that the maximum area of S_3 (attained at time $t+\delta t$) is $O\{\delta t(\delta A)^{1/2}\}$ as $\delta A \to 0$, $\delta t \to 0$, so that

$$M_3 \le \rho \times (\text{max. area of } S_3) \times |\mathbf{u}|\delta t = O\{\delta t^2\,(\delta A)^{1/2}\}$$

as $\delta A \to 0$, $\delta t \to 0$, which becomes negligible relative to the LHS of $(*)$ because

$$\frac{\delta t^2\,(\delta A)^{1/2}}{\delta A\,\delta t} = \frac{\delta t}{(\delta A)^{1/2}} \to 0$$

in the specified limit. Hence $(*)$ implies that $(\rho\,\delta A\,\delta t\,\mathbf{U}.\mathbf{n})/(\rho\,\delta A\,\delta t\,\mathbf{u}.\mathbf{n}) \to 1$ in the limit, and therefore that

$$\mathbf{u}.\mathbf{n} = \mathbf{U}.\mathbf{n}\ .$$

In other words, the normal component of velocity relative to the boundary, i.e. the normal component of $\mathbf{u}-\mathbf{U}$, is zero.

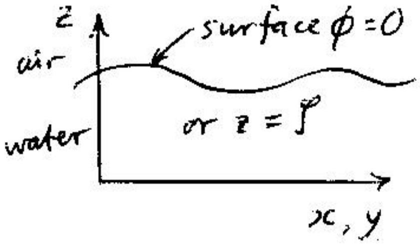
When $\mathbf{U}.\mathbf{n} < 0$ at $\mathbf{x}$, essentially the same derivation applies, including the above diagram, if we take $\delta t < 0$.

*The above would need modification, of course, when the boundary is *not* impermeable, i.e. when there is mass flux across fluid boundary for some reason, e.g. due to evaporation of water at its free surface.*

Remark: The set of points that mark the boundary (each point moving with an appropriate velocity $\mathbf{U}$) can be replaced by another set of points moving with velocities $\mathbf{U} + \mathbf{U}_{\rm tang}$ for any $\mathbf{U}_{\rm tang} \perp \mathbf{n}$. This evidently makes no difference to $\mathbf{U}.\mathbf{n}$.

Application to free surface: (relevant to wave theory later in the course):

Let the shape of the surface be $z = \zeta(x, y, t)$ (well behaved if the slope of the surface stays finite; this applies e.g. to waves that are not breaking).



To apply the condition $\mathbf{U}.\mathbf{n} = \mathbf{u}.\mathbf{n}$, we need to find $\mathbf{n}$ and $\mathbf{U}$ in terms of the function $\zeta(x, y, t)$.

Recall that the normal to a surface $\phi(\mathbf{x}) =$ constant is $\nabla\phi/|\nabla\phi|$.

Use $\phi = z - \zeta(x, y, t) = 0$; so $\quad \nabla\phi = \left(-\dfrac{\partial\zeta}{\partial x}, -\dfrac{\partial\zeta}{\partial y}, 1\right)$

In virtue of the remark about $\mathbf{U}_{\rm tang}$, we can take $\mathbf{U}$ to be simply $\left(0, 0, \dfrac{\partial\zeta}{\partial t}\right)$,
so $\mathbf{u}.\mathbf{n} = \mathbf{U}.\mathbf{n}$ becomes

$$\mathbf{u}\,.\,\frac{\left(-\dfrac{\partial\zeta}{\partial x}, -\dfrac{\partial\zeta}{\partial y}, 1\right)}{\left(1 + \left(\dfrac{\partial\zeta}{\partial x}\right)^2 + \left(\dfrac{\partial\zeta}{\partial y}\right)^2\right)^{1/2}} = \frac{\left(0, 0, \dfrac{\partial\zeta}{\partial t}\right) . \left(-\dfrac{\partial\zeta}{\partial x}, -\dfrac{\partial\zeta}{\partial y}, 1\right)}{\left(1 + \left(\dfrac{\partial\zeta}{\partial x}\right)^2 + \left(\dfrac{\partial\zeta}{\partial y}\right)^2\right)^{1/2}}$$

i.e. $\quad \dfrac{\partial\zeta}{\partial t} + u\dfrac{\partial\zeta}{\partial x} + v\dfrac{\partial\zeta}{\partial y} - w = 0 \quad$ is the appropriate boundary condition, i.e.,

$$\frac{D}{Dt}\zeta(x, y, t) - \frac{Dz}{Dt} = 0$$

i.e.,

$$\frac{D}{Dt}(\zeta(x, y, t) - z) = 0\ .$$

So if $\zeta(x, y, t) - z$ is zero initially, it remains zero: fluid on the surface stays on the surface.

1.8 Stream functions

A stream function is a useful — and frequently used — way of describing a two-dimensional (2-D) incompressible velocity field. The velocity field, when written in terms of a stream function, automatically satisfies the incompressibility constraint $\nabla.\mathbf{u} = 0$.

For 2-D incompressible flow we have

$$\nabla.\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Now recall the standard result from calculus (re 'exact differentials' etc.) that

$$\exists\ \psi(x, y) \text{ such that } d\psi = P(x, y)\, dx + Q(x, y)\, dy \quad \Leftrightarrow \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\ .$$

This applies with $Q = u$ and $P = -v$ at any given time t, because then

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = -\nabla.\mathbf{u} = 0\ .$$

In other words,

$\exists\ \psi(x, y, t)$ such that $d\psi = -v\, dx + u\, dy$ at fixed t, i.e. such that

$$u = +\frac{\partial \psi}{\partial y}\ , \qquad v = -\frac{\partial \psi}{\partial x}\ .$$

(Remember what ∂ means in this context: for instance $\partial\psi/\partial x$ means, by definition, the rate of change of ψ as x varies, with y and t both held fixed.)

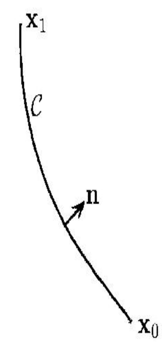
Conversely, such a velocity field always satisfies $\nabla.\mathbf{u} = 0$, as follows from the equality of mixed partials, $\partial^2\psi/\partial x\partial y = \partial^2\psi/\partial y\partial x$.

The function $\psi(x, y, t)$ is called a *stream function* for the 2-D flow.

Key properties:

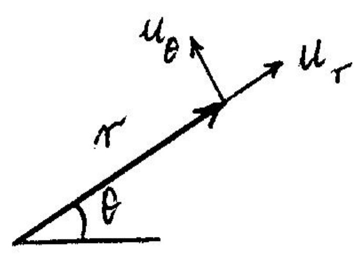
(1) The curves ψ = constant are tangential to $\mathbf{u}$ at each point (x, y), i.e., they are streamlines (because $\nabla\psi \perp \mathbf{u}$), i.e. integral curves of the vector field $\mathbf{u}$ at fixed t;

(2) $|\mathbf{u}| = |\nabla\psi|$; therefore, fluid moves *faster* where streamlines are *closer*; (3) Consider two points $\mathbf{x}_0$ and $\mathbf{x}_1$ joined by some curve $\mathcal{C}$ on which the line element is $d\mathbf{l}$. Then $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$ can be expressed as an integral along $\mathcal{C}$:



$$
\begin{aligned}
\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0) &= \int_{\mathbf{x}_0}^{\mathbf{x}_1} d\psi = \int_{\mathbf{x}_0}^{\mathbf{x}_1} \left(\frac{\partial \psi}{dx} dx + \frac{\partial \psi}{dy} dy \right) = \int_{\mathbf{x}_0}^{\mathbf{x}_1} \nabla \psi \,.\, d\mathbf{l} \\
&= \int_{\mathbf{x}_0}^{\mathbf{x}_1} (-v\, dx + u\, dy) \\
&= \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u}.\mathbf{n}\, ds \qquad ds = |d\mathbf{l}| = d(\text{arc length});\ \ \mathbf{n}\, ds = (dy, -dx) \\
&= \ (\text{volume flux across } \mathcal{C}), \quad \text{i.e.} \quad (\text{mass flux})/\rho\ ,
\end{aligned}
$$

reckoning fluxes per unit z-distance. Note that the right hand side, the volume flux across $\mathcal{C}$ (per unit z-distance), is independent of which curve $\mathcal{C}$ you pick between any given pair of points $\mathbf{x}_0$ and $\mathbf{x}_1$. This is immediate from $\int d\psi$ in the first line, and again reflects mass conservation. (Draw a separate curve $\mathcal{C}'$ between $\mathbf{x}_0$ and $\mathbf{x}_1$ and consider the mass of fluid per unit z-distance enclosed between $\mathcal{C}$ and $\mathcal{C}'$. Or just use (3) above and the divergence theorem.)

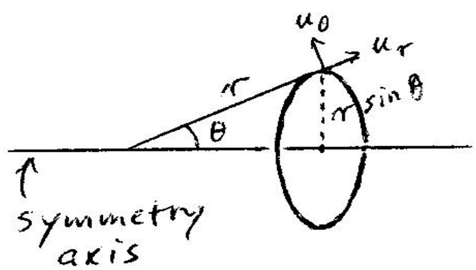


Note that $\mathbf{u} = -\mathbf{k} \times \nabla\psi$, where $\mathbf{k} = (0, 0, 1)$, a unit vector perpendicular to the xy plane. Because $\mathbf{u} = -\mathbf{k} \times \nabla\psi$ is a vector equation, we can immediately deduce the relation between ψ and components of $\mathbf{u}$ in other coordinate systems. E.g., in 2-D polar coordinates (r, θ):

$$\nabla.\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) =$$

$$\Rightarrow \exists\, \psi \text{ s.t.} \qquad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}\,, \qquad u_\theta = -\frac{\partial \psi}{\partial r}\,, \qquad (u_r, u_\theta) = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}\right)\,.$$

3-D axisymmetric case:



We can use the same approach for axisymmetric flow. In spherical polars (r, θ, φ), the flow is independent of the azimuthal coordinate φ by definition of axisymmetry. So the standard formula for the divergence of a vector in spherical coordinates gives, with $\mathbf{u} = (u_r, u_\theta, 0)$,

$$\nabla.\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\; u_\theta) = 0$$

$$\Rightarrow \exists\, \psi \quad \text{s.t.} \qquad u_r = \frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}\,, \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}\,.$$

Stream functions for axisymmetric flow are sometimes called *Stokes stream functions.*

In the above picture, $\mathcal{C}$ becomes a surface of revolution, and $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0) = \int_{\mathbf{x}_0}^{\mathbf{x}_1} r\;\sin\theta\,\mathbf{u}.\mathbf{n}\,ds$ (note $r\,\sin\theta$ = distance to the symmetry axis, the r of Q7). This is the volume flux across $\mathcal{C}$ per radian of azimuth.

For axisymmetric flows and Stokes stream functions described in cylindrical polars, see Q7 on Ex. Sheet I.

(See also Acheson §§4.2, 5.5 or Paterson §§3.3, 3.6.)