GEFD SUMMER SCHOOL

Some basic equations and boundary conditions

(This is mostly a summary of standard items from fluids textbooks.)

The Eulerian description is used; so the material derivative $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ — the operator for the "rate of change following the fluid". This gives the rate of change of any field $f(\mathbf{x}, t)$ not at a fixed point $\mathbf{x}$, but at a moving fluid element or "particle". Its form comes from the chain rule of differential calculus when $\mathbf{x}$ is made a function of time t.

Mass conservation, or "continuity":

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \ , \qquad \text{i.e.} \qquad \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0$$

($\rho =$ **density**, $\rho\mathbf{u} =$ **flux**, of mass). For incompressible flow, $D\rho/Dt = 0$, implying that

$$\nabla\cdot\mathbf{u} = 0 \ .$$

Associated boundary condition: Normal components of velocity must agree, if mass is conserved. That is, we must have

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{b}}\cdot\mathbf{n} \qquad \text{at the boundary (where } \mathbf{u}_{\mathrm{b}} \text{ is prescribed).}$$

Here $\mathbf{n}$ is a unit vector normal to the boundary. The right-hand side can represent e.g. the normal component of the velocity $\mathbf{u}_{\mathrm{b}}$ of the boundary material itself, if we have a moving but impermeable boundary, *or*, e.g., $\rho^{-1}\times$ mass flux across a fixed but permeable boundary. At a boundary that is both stationary and impermeable, $\mathbf{u}_{\mathrm{b}} = 0$ and so $\mathbf{u}\cdot\mathbf{n} = 0$.

Newton's second law: In an inertial frame,

$$\text{Acceleration} \ = \ \text{force per unit mass} \ .$$

In this equation, the right-hand side (RHS) and left-hand side (LHS) can be written in various ways. Defining $\boldsymbol{\zeta} = \nabla\times\mathbf{u}$, the vorticity, we can show by vector calculus that

$$\text{LHS} \ = \ \frac{D\mathbf{u}}{Dt} \ = \ \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} \ = \ \frac{\partial\mathbf{u}}{\partial t} \ + \ \boldsymbol{\zeta}\times\mathbf{u} \ + \ \nabla(\tfrac{1}{2}|\mathbf{u}|^2) \qquad (1)$$

and

$$\text{RHS} \ = \ -\frac{1}{\rho}\nabla p \ + \ \mathbf{g} \ + \ \mathbf{F} \ ,$$

where p is total pressure, including hydrostatic, and $\mathbf{g}$ is the gravitational force per unit mass, exactly the gradient of a scalar function. We take $\mathbf{g} = -\nabla\Phi$, where Φ is called the gravitational potential. The last term $\mathbf{F}$ stands for all other forces, such as friction, or, especially in large-scale atmospheric models, forces due to unresolved gravity waves.

Wave-induced forces can in a certain sense be *anti*-frictional. The most conspicuous example is the quasi-biennial oscillation of the zonal wind in the equatorial lower stratosphere (QBO). The wave-induced forces drive the stratosphere away from, not toward, solid rotation, as shown in PHH's lectures and in the QBO computer demonstration. Ordinary (molecular-viscous) fluid friction would by itself drive the atmosphere and ocean *toward* solid rotation in the absence of externally applied stresses. For further discussion see MEM's supplementary **Appendix IIIa**.

Viscous force: For ordinary (molecular-viscous) fluid friction we have

$$\mathbf{F} \;=\; \frac{\mu}{\rho}\nabla^2\mathbf{u} \;=\; \nu\,\nabla^2\mathbf{u}\;, \quad \text{say,}$$

in the simplest case of spatially uniform dynamical viscosity μ. See e.g. Batchelor's textbook for more general cases. (One might guess $\mathbf{F} = \rho^{-1}\nabla\cdot(\mu\nabla\mathbf{u})$, but that's wrong! One must replace the tensor $\nabla\mathbf{u}$ by its symmetric part. Viscous forces are achiral: they respect mirror-symmetry.)

Associated boundary conditions: For viscous fluid motion we need an extra boundary condition on $\mathbf{u}$. The commonest cases are of two kinds. **First**, if the boundary is solid, impermeable, and again moving with velocity $\mathbf{u}_\mathrm{b}$, then

$$\mathbf{u} \;=\; \mathbf{u}_\mathrm{b} \qquad \text{at the boundary;}$$

i.e., the fluid at the boundary must move entirely with the boundary. Tangential as well as normal components must agree. (There are a few special cases where this fails, e.g. two-fluid "contact lines" where continuum mechanics itself breaks down.) The agreement of tangential components is called the *no-slip condition* and may still apply when the boundary is solid but permeable. **Second**, in a thought-experiment in which the tangential stress $\boldsymbol{\tau}$ on the fluid (friction force per unit area) is prescribed at a plane boundary — e.g. at the top of a model ocean strongly constrained by gravity — then $\boldsymbol{\tau}$ controls the shear at the boundary:

$$\mu\frac{\partial(\mathbf{u}\cdot\mathbf{s})}{\partial n} \;=\; \boldsymbol{\tau}\cdot\mathbf{s} \qquad \text{at the boundary,}$$

where n is distance in the $\mathbf{n}$ direction, outward from the fluid, and $\mathbf{s}$ is any fixed unit vector normal to $\mathbf{n}$, i.e. lying in the boundary. (In more general cases with curved boundaries, we would need to use a more complicated expression for the total stress due to viscosity and pressure. Batchelor's textbook gives a clear discussion: some knowledge of tensors is required.)

Boussinesq approximation: This refers to a set of approximations for flows that feel buoyancy forces, valid in the asymptotic limit $\Delta\rho/\rho \ll 1$ where $\Delta\rho$ typifies the range of density variations, with $g\Delta\rho/\rho$ finite ($g = |\mathbf{g}| = |\nabla\Phi|$). Define For consistency we need to assume that the motion has height scales $\ll {c_\mathrm{s}}^2/g$ and that $|\mathbf{u}| \ll c_\mathrm{s}$, where c_s is the speed of sound. Mass continuity then reduces to the incompressible case $\nabla\cdot\mathbf{u} = 0$, and Newton's second law simplifies to

$$\frac{D\mathbf{u}}{Dt} \;=\; -\frac{1}{\rho}\nabla p' \;+\; \boldsymbol{\sigma} \;+\; \mathbf{F} \tag{2}$$

where the total pressure p has been replaced by the pressure *anomaly* p', in the sense of departure from a background pressure. The background pressure is defined to be in hydrostatic balance with a given background density, both background quantities being functions of Φ only. The effects of density anomalies, departures ρ' from the background density, are represented solely by the upward "buoyancy acceleration" $\boldsymbol{\sigma} = -\mathbf{g}\rho'/\rho$. In the pressure-gradient term, ρ can be taken to be constant. In summary, $\rho =$ constant when it measures mass density and inertia but not when it measures buoyancy.

Equation for buoyancy, and associated boundary conditions: Generally these involve diffusion of heat or of solutes, with gradients of one thing influencing fluxes of another. There are great simplifications in the Boussinesq case with constant $\mathbf{g}$; then

$$D\sigma/Dt \;=\; \nabla\cdot(\kappa\nabla\sigma) \qquad (\sigma = |\,\boldsymbol{\sigma}\,|)$$

may suffice. Here σ represents density anomalies relative to a strictly constant background density. That is, σ describes all of the stratification, and the background none: if the fluid is at rest then the buoyancy frequency (Brunt–Hesselberg–Milch–Schwarzschild–Väisälä frequency) is just $N^2(z) = \partial\sigma/\partial z \;\; (dz = d\Phi/g)$. If the buoyancy diffusivity κ is spatially uniform then, even more simply, $D\sigma/Dt = \kappa\nabla^2\sigma$. We usually specify a boundary condition on the buoyancy flux $\kappa\,\partial\sigma/\partial n$, or on σ itself, or on some linear combination of σ and $\partial\sigma/\partial n$. It is still consistent to take $\nabla\cdot\mathbf{u} = 0$ approximately.

Rotating reference frames: To the RHS of Newton's second law we must add

$$-\,2\boldsymbol{\Omega}\times\mathbf{u} \;-\; \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

(see **Supplement 1**, p. 5 below, for a derivation). Here $\boldsymbol{\Omega}$ is the angular velocity of the reference frame, assumed constant, $\mathbf{r}$ is position relative to any point on the rotation axis, and $\mathbf{u}$ is now velocity relative to the rotating frame. The first and second terms are respectively the Coriolis and centrifugal forces per unit mass — "fictitious forces" felt e.g. by an observer sitting in a rotating room or in a spinning aircraft. It is convenient, and conventional, to recognize that not only $\mathbf{g}$ but also $-\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$ is the gradient of a potential and that there exists, therefore, an "effective gravitational potential"

$$\Phi_{\text{eff}} = \Phi - \tfrac{1}{2}|\boldsymbol{\Omega}|^2{r_\perp}^2$$

such that $\mathbf{g} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) \;=\; -\nabla\Phi_{\text{eff}}$, where Φ is again the gravitational potential in the ordinary sense, and $r_\perp$ is the shortest, i.e. perpendicular, distance to the rotation axis.

On the rotating Earth, the level surfaces $\Phi_{\text{eff}} =$ const. are only slightly different from the surfaces $\Phi =$ const. The Φ_{eff} and Φ surfaces tangent to each other at the north pole are only about 11 km apart at the equator. *Brain-teaser*: why does this differ from the equatorial bulge of the actual figure of the Earth, about 21 km? (This is a good exercise in an important transferable skill, that of spotting unconscious assumptions.)

So in the rotating frame Newton's second law can be written

$$\frac{D\mathbf{u}}{Dt} \ + \ 2\boldsymbol{\Omega}\times\mathbf{u} \ = \ -\frac{1}{\rho}\nabla p \ - \ \nabla\Phi_{\rm eff} \ + \ \mathbf{F}\ . \tag{3}$$

It is traditional to display the Coriolis force per unit mass on the LHS, even though in the rotating frame it has the role of force rather than acceleration. Whether terms are written on the RHS or LHS, with appropriate sign changes, is of course entirely a matter of convention. Mathematical equations may look like, but are of course different from, computer code! (There are some myths in the research literature that might come from forgetting this. One example is the misleading, though persistent, idea that the stratospheric Brewer–Dobson circulation is driven by solar heating. The heating term is usually written on the RHS, unhelpfully suggesting that it be thought of as a known forcing, even though, in reality, it is more like Newtonian relaxation toward a radiative-equilibrium temperature. See MEM's **Appendices III, IIIa**, e.g. "polar-cooling thought-experiment", **III** p. 14.)

The Boussinesq approximation can again be introduced, with $-\nabla\Phi_{\rm eff}$ replacing $\mathbf{g}$.

Vorticity equations in a rotating frame: Same as in a inertial frame except for just one thing: replace the relative vorticity $\boldsymbol{\zeta} = \nabla\times\mathbf{u}$ by the absolute vorticity $\boldsymbol{\zeta}^{\rm a} = 2\boldsymbol{\Omega} + \boldsymbol{\zeta}$.

Incompressible but not Boussinesq:

$$D\boldsymbol{\zeta}/Dt \ = \ \boldsymbol{\zeta}^{\rm a}\cdot\nabla\mathbf{u} \ - \ \frac{1}{\rho^2}\nabla p\times\nabla\rho \ + \ \nabla\times\mathbf{F}$$

Boussinesq:

$$D\boldsymbol{\zeta}/Dt \ = \ \boldsymbol{\zeta}^{\rm a}\cdot\nabla\mathbf{u} \ + \ \nabla\times\boldsymbol{\sigma} \ + \ \nabla\times\mathbf{F}$$

(with $\boldsymbol{\Omega}$ constant, so that $D\boldsymbol{\zeta}^{\rm a}/Dt = D\boldsymbol{\zeta}/Dt$). If $g_{\rm eff} = |\nabla\Phi_{\rm eff}|$ can be taken as constant on each level surface $\Phi_{\rm eff} =$ const., then $\nabla\times\boldsymbol{\sigma}$ simplifies to $-\hat{\mathbf{z}}\times\nabla\sigma$ where $\hat{\mathbf{z}}$ is a vertical unit vector, i.e. parallel to $\nabla\Phi_{\rm eff}$. These equations can be derived by taking the curl of Newton's second law, in the per-unit-mass form (3) or its Boussinesq counterpart.

In the Boussinesq case, $\rho =$ constant and so $\mu/\rho = \nu$ is spatially uniform whenever μ is. Then when $\mathbf{F}$ is viscous,

$$\nabla\times\mathbf{F} \ = \ \nu\nabla^2\boldsymbol{\zeta}\ .$$

So vorticity behaves diffusively in this case — though not in all other cases, as the example of a viscous jet in inviscid surroundings reminds us. (There, all the vorticity ends up on the interface: diffusive behaviour is countermanded by terms involving $\nabla\mu \neq 0$.)

Ertel's potential-vorticity theorem: This is a cornerstone of today's understanding of atmosphere–ocean dynamics (MEM's and PHH's lectures) and the dynamics of stratified, rotating flow in other naturally occurring bodies of fluid e.g. the Sun's interior (http://www.atm.damtp.cam.ac.uk/people/mem/). The theorem says that if (a) there exists a thermodynamical variable θ that is a function of pressure p and density ρ alone *and* is materially conserved, $D\theta/Dt = 0$, and (b) $\mathbf{F} = 0$ or is the gradient of a scalar, then

$$DQ/Dt = 0 \quad \text{where} \quad Q = \rho^{-1}\boldsymbol{\zeta}^{\mathrm{a}}\cdot\nabla\theta, \quad \text{the "Rossby–Ertel potential vorticity".}$$

For adiabatic motion of a simple fluid (thermodynamic state definable by p and ρ alone), θ can be taken as specific entropy S (**Supplement 2**, p. 6 below) or any function of S. Adiabatic motion means that fluid elements neither receive nor give up heat, implying that S is materially conserved. In the atmospheric sciences it is conventional to take θ to be the potential temperature, which is a function of S: for a perfect gas, $\theta \propto \exp(S/c_p)$.

In the Boussinesq approximation: As above but with ρ = constant (implying that the factor ρ^{-1} can be omitted from the definition of Q), and with θ replaced by ρ', or by $\sigma = -g_{\mathrm{eff}}\rho'/\rho$ if $g_{\mathrm{eff}} = |\nabla\Phi_{\mathrm{eff}}|$ can be taken as constant on each level surface Φ_{eff} = const.

Bernoulli's (streamline) theorem for steady, frictionless, adiabatic motion: First, steady and adiabatic $\Rightarrow$ $\partial/\partial t = 0$ and $D\theta/Dt = 0$. Therefore $\mathbf{u}\cdot\nabla\theta = 0$, i.e., θ = constant along any streamline. Second, the right-hand-most expression in eq. (1) on page 1 allows us to replace $D\mathbf{u}/Dt + 2\boldsymbol{\Omega}\times\mathbf{u}$ by $\boldsymbol{\zeta}^{\mathrm{a}}\times\mathbf{u} + \nabla(\frac{1}{2}|\mathbf{u}|^2)$. With $\mathbf{F} = 0$ (or $\mathbf{F}$ workless, $\mathbf{u}\cdot\mathbf{F} = 0$) we can take $\mathbf{u}\cdot$(3), noting that $\mathbf{u}\cdot\boldsymbol{\zeta}^{\mathrm{a}}\times\mathbf{u} = 0$, to give simply

$$\mathbf{u}\cdot\nabla\left(\tfrac{1}{2}|\mathbf{u}|^2 + \Phi_{\mathrm{eff}}\right) = -\frac{1}{\rho}\,\mathbf{u}\cdot\nabla p\,, \quad \Rightarrow \quad \mathbf{u}\cdot\nabla\left(\tfrac{1}{2}|\mathbf{u}|^2 + \Phi_{\mathrm{eff}} + H\right) = 0\,.$$

The last step introduces the specific enthalpy H (**Supplement 2**, p. 6 below) and uses the constancy of θ, hence S, on the streamline. In the notation of Supplement 2, $dH = V\,dp = \rho^{-1}dp$ on the streamline. So the Bernoulli quantity $\frac{1}{2}|\mathbf{u}|^2 + \Phi_{\mathrm{eff}} + H$ is constant on a streamline, even for stratified, rotating flow, in the circumstances assumed.

Boussinesq Bernoulli quantity: this can be taken as $\frac{1}{2}|\mathbf{u}|^2 + \rho'\Phi_{\mathrm{eff}}/\rho + p'/\rho$ with ρ = constant. If $\nabla\Phi_{\mathrm{eff}}$ = constant then $\rho'\Phi_{\mathrm{eff}}/\rho$ may be replaced by $-\sigma z$ where z is vertical distance, $z = \Phi_{\mathrm{eff}}/|\nabla\Phi_{\mathrm{eff}}|$. So then $\frac{1}{2}|\mathbf{u}|^2 - \sigma z + p'/\rho$ is constant on a streamline.

Supplement 1: Coriolis and centrifugal "forces"

This part of the problem is just the same for continuum mechanics as for particle dynamics. The best tactic is to view everything from an *inertial* (sidereal) frame of reference.

Let $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, be an orthogonal triad of unit vectors that rotate rigidly with constant angular velocity $\boldsymbol{\Omega}$. (The triad can be right handed too, but that's not essential.) Thus $\hat{\mathbf{x}}$ is time dependent, with time derivative

$$\dot{\hat{\mathbf{x}}} = \boldsymbol{\Omega}\times\hat{\mathbf{x}}; \quad \text{similarly} \quad \dot{\hat{\mathbf{y}}} = \boldsymbol{\Omega}\times\hat{\mathbf{y}}, \quad \dot{\hat{\mathbf{z}}} = \boldsymbol{\Omega}\times\hat{\mathbf{z}}. \tag{S1}$$

Consider the position $\mathbf{X}(t)$ of a single particle (viewed, as always, in the inertial frame). Let $\mathbf{X}\cdot\hat{\mathbf{x}} = X(t)$, $\mathbf{X}\cdot\hat{\mathbf{y}} = Y(t)$, $\mathbf{X}\cdot\hat{\mathbf{z}} = Z(t)$; thus (by orthogonality)

$$\mathbf{X}(t) = X(t)\,\hat{\mathbf{x}}(t) + Y(t)\,\hat{\mathbf{y}}(t) + Z(t)\,\hat{\mathbf{z}}(t)\,. \tag{S2}$$

Take the first time derivative, using (S1) and the properties of of vector multiplication:

$$\dot{\mathbf{X}}(t) \;=\; \dot{X}(t)\,\hat{\mathbf{x}}(t) \;+\; \dot{Y}(t)\,\hat{\mathbf{y}}(t) \;+\; \dot{Z}(t)\,\hat{\mathbf{z}}(t) \;+\; \boldsymbol{\Omega}\times\mathbf{X}(t) \tag{S3}$$

$$\;=\; \dot{\mathbf{X}}_{\mathbf{rel}}(t) \;+\; \boldsymbol{\Omega}\times\mathbf{X}(t), \quad \text{say.} \tag{S4}$$

(Notice now that $\dot{\mathbf{X}}_{\mathbf{rel}} \;=\; \dot{X}\hat{\mathbf{x}}+\dot{Y}\hat{\mathbf{y}}+\dot{Z}\hat{\mathbf{z}}$ is, by definition, the rate of change that $\mathbf{X}$ would appear to have if it were viewed from a reference frame rotating with angular velocity $\boldsymbol{\Omega}$. That is, $\dot{\mathbf{X}}_{\mathbf{rel}}$ is the particle's velocity relative to that rotating frame.) Remembering that $\boldsymbol{\Omega}$ = constant, we can differentiate (S3) to get the second time derivative of $\mathbf{X}(t)$, the (absolute) acceleration:

$$\ddot{\mathbf{X}}(t) \;=\; \ddot{X}\,\hat{\mathbf{x}} \;+\; \ddot{Y}\,\hat{\mathbf{y}} \;+\; \ddot{Z}\,\hat{\mathbf{z}} \;+\; \boldsymbol{\Omega}\times\dot{\mathbf{X}}_{\mathbf{rel}} \;+\; \boldsymbol{\Omega}\times\dot{\mathbf{X}}\;,$$

using (S1) again. The sum of the first three terms, $=\ddot{\mathbf{X}}_{\mathbf{rel}}$, say, give, by definition, the particle's acceleration relative to the rotating frame. Using (S4) in the last term, we have

$$\ddot{\mathbf{X}}(t) \;=\; \ddot{\mathbf{X}}_{\mathbf{rel}} \;+\; 2\boldsymbol{\Omega}\times\dot{\mathbf{X}}_{\mathbf{rel}} \;+\; \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{X}), \tag{S5}$$

the standard result showing that $\ddot{\mathbf{X}}$ can be equated to $\ddot{\mathbf{X}}_{\mathbf{rel}}$ plus, respectively, a Coriolis and a centripetal acceleration. Centripetal = inward (from Latin *centrum*, centre, + *petere*, to seek): note that $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{X}) \;=\; -|\boldsymbol{\Omega}|^2\mathbf{X} \;+\; \boldsymbol{\Omega}\cdot\mathbf{X}\,\boldsymbol{\Omega} \;=\; -|\boldsymbol{\Omega}|^2\mathbf{r}_\perp$ ($\mathbf{r}_\perp$ outward).

If we were now to go into the rotating frame, we could regard $\ddot{\mathbf{X}}_{\mathbf{rel}}$ as the acceleration entering Newton's laws provided that we also regard the forces acting as including a Coriolis force $-2\boldsymbol{\Omega}\times\dot{\mathbf{X}}_{\mathbf{rel}}$ and a centri*fugal* (outward) force $-\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{X})$ per unit mass.

Supplement 2: Thermodynamic relations

There is a mnemonic trick, not widely known, for remembering the thermodynamic relations for a simple compressible fluid. They can all be read off from the single array

$$\begin{array}{ccc} V & A & T \\ E & \swarrow\!\!\!\!\searrow & G \\ S & H & p \end{array}$$

which itself can be remembered aurally: say *"vatg, veshp"* to yourself, with a New York accent if it helps. The notation is conventional. On the corners, read T as temperature, p as total pressure (including hydrostatic background), S as specific entropy ($= c_p \ln\theta +$ const. in the case of a perfect gas with c_p the specific heat at constant p), V as $1/\rho$, the specific volume (volume of a unit mass of fluid), and, in between, read A as Helmholtz free energy, G as Gibbs free energy, H as enthalpy, and E as internal energy, all 'specific' in the per-unit-mass sense. The diagram contains all the standard thermodynamic relations and Legendre transformations, of which the most important for fluid dynamics are

$$dE = -p\,dV + T\,dS\;, \qquad dH = \;V\,dp + T\,dS \qquad (H = E + Vp)\;.$$

The first of these, scanned left to right, corresponds to the pattern $\vdots\;\vdots$ with E at mid-left representing dE, etc., and with the diagonal arrows giving the sign: thus $p\,dV$ has a minus (against the arrow) and $T\,dS$ has a plus (with the arrow). The four Maxwell relations $(\partial T/\partial V)_S = -(\partial p/\partial S)_V$, $(\partial V/\partial S)_p = (\partial T/\partial p)_S$, etc., can be read off in a similar way.

The speed of sound is $c_{\rm s} = \sqrt{(\partial p/\partial\rho)_S} = -V\sqrt{(\partial p/\partial V)_S}$.