Mathematical Tripos Part III 2007–08
Fundamentals of Atmosphere–Ocean Dynamics

Examples 1

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In these examples, 'BN' refers to the background lecture notes for the course.

1. (Basic properties of the Rossby–Ertel potential vorticity, BN p. 5.) Show that, even if compressibility is permitted, *material conservation* $D\chi/Dt = 0$ of a quantity whose amount per unit mass or 'mixing ratio' is χ, together with mass conservation, $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0$, is equivalent to a conservation relation in the standard general form $\partial(\ )/\partial t + \nabla\cdot\{\ \} = 0$ with density $(\ ) = \rho\chi$ and flux $\{\ \} = \rho\mathbf{u}\chi$.

   Show that when $D\alpha/Dt = 0$, and when the equation of state is any equation of the form $\mathrm{func}(\alpha, p, \rho) = 0$, then when $\nabla\times\mathbf{F} = 0$ in the vorticity equation, BN eq. (0.4), we have

$$\frac{\partial(\rho Q)}{\partial t} + \nabla\cdot\{\rho\mathbf{u}Q\} = 0$$

   where $Q = \rho^{-1}\boldsymbol{\zeta}_{\mathrm{a}}\cdot\nabla\alpha$, the Rossby–Ertel potential vorticity, and where $\boldsymbol{\zeta}_{\mathrm{a}}$ is the absolute vorticity $2\boldsymbol{\Omega} + \nabla\times\mathbf{u}$. (Note that mass conservation is not used yet.) Deduce (now using mass conservation) that $DQ/Dt = 0$ (the usual form of Ertel's theorem). If $\nabla\times\mathbf{F} \neq 0$ and $D\alpha/Dt = \mathcal{H}$ (e.g. because of heating by diffusive or infrared-radiative heat flux convergence) show that Q is still conserved, in the *general* sense, $\partial(\rho Q)/\partial t + \nabla\cdot\{\ \} = 0$, with flux $\{\ \} = \rho\mathbf{u}Q - \boldsymbol{\zeta}_{\mathrm{a}}\mathcal{H} - \mathbf{F}\times\nabla\alpha$.

2. Internal gravity waves and Rayleigh–Taylor instability (easy). An unbounded, non-rotating, Boussinesq ideal fluid has a statically unstable density profile, the associated buoyancy frequency having the constant imaginary value iM. By considering plane-wave solutions of the linearized equations, or otherwise, show that the maximum possible growth rate of small disturbances about a state of rest is M, and that the growth rate does not depend on the length scale of the disturbance.

   An unbounded, non-rotating, Boussinesq ideal fluid is either stably stratified, with constant (real) buoyancy frequency N, or unstably stratified with constant (real) M as above. A plane-wave disturbance of (real or imaginary) frequency ω and amplitude a disturbs the fluid. Show that in both cases the wave represents an exact solution of the fully nonlinear Boussinesq equations, without any restriction on a.

   In the *stably* stratified case, why would this wave be expected to become *unstable* when a is sufficiently large and $|\omega| \ll N$? [Give a qualitative argument only, based on orders of magnitude and physical reasoning. Do not try to analyze the instability of the plane wave in detail.]

3. By considering the linearized Boussinesq equation satisfied by the vertical component of vorticity, or otherwise, show that the velocity $\mathbf{u} = (u, v, w)$ in a plane internal gravity wave with wavenumber $\mathbf{k} = (k, l, m)$ obeys the relation

$$v = lu/k$$

   and deduce further that

$$w = -(k^2 + l^2)u/km\ .$$

   An infinite vertical, flexible wall executes small oscillations about $x = 0$ according to

$$x = a\sin(ly + mz - \omega t)\ ,$$

where a, l, m and ω are real, positive constants, and $al \ll 1$, $am \ll 1$. A Boussinesq, stably-stratified fluid lies on one side of the wall, $x > 0$. Find the response of the fluid (i) for $\omega > N$, (ii) for $l/(l^2+m^2)^{1/2} < \omega/N < 1$, and (iii) for $\omega/N < l/(l^2+m^2)^{1/2}$. For which, if any, of these parameter ranges does the disturbance penetrate far into the fluid?

Use your solution to illustrate the scaling arguments of BN pp. 47–50, leading to eq. (3.7), by showing that the scale-analytic relation

$$w/u \sim F_T^2 H/L \qquad (F_T = L/NHT) \qquad (T \sim \omega^{-1})$$

holds in the limit $T \to \infty$, where L and H are suitable horizontal and vertical length scales. What is the qualitative character of the motion in this limit?

4. A uniformly-stratifed (N constant), ideal, Boussinesq, incompressible fluid is at rest, and may be considered unbounded in all three dimensions. At time $t = 0$ a finite region of fluid near the origin is impulsively set in motion with a smooth velocity distribution

$$\mathbf{U}(\mathbf{x}) = \int \hat{\mathbf{U}}(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x})\, dkdldm\ , \qquad \mathbf{k} = (k, l, m)\ ,$$

satisfying the mass continuity equation $\nabla.\mathbf{u} = 0$. Show that $\mathbf{k}.\hat{\mathbf{U}} = 0$ and that at subsequent times the motion according to the linearized equations can be written as the sum of steady and time-dependent contributions $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_H(\mathbf{x}) + \tilde{\mathbf{u}}(\mathbf{x}, t)$, where $\mathbf{u}_H$ is horizontal and two-dimensionally nondivergent (a case of 'layerwise two-dimensional motion'), and where

$$\tilde{\mathbf{u}} = \int \left\{ \hat{\mathbf{U}} - \frac{(\hat{\mathbf{z}} \times \mathbf{k})\hat{\mathbf{U}}.(\hat{\mathbf{z}} \times \mathbf{k})}{|\hat{\mathbf{z}} \times \mathbf{k}|^2} \right\} e^{i\mathbf{k}.\mathbf{x}} \cos\omega t\ dkdldm,$$

where

$$\omega = N|\hat{\mathbf{z}} \times \mathbf{k}|/|\mathbf{k}|,$$

$\hat{\mathbf{z}}$ being a unit vertical vector. Why would you expect to find, for smooth $\mathbf{U}(\mathbf{x})$, that the oscillations persisting longest near the origin have either small scales, or frequencies close to N, or both? Give a rough order-of-magnitude estimate for how long the linearized solution remains valid as an approximate solution of the nonlinear equations. [It is usually $\mathbf{u}_H$ that ceases to be valid first; a discussion of timescales relevant to $\mathbf{u}_H$ can be found on BN pp. 47–50.]

[This initial-value problem provides the beginning of an understanding of the small-scale atmospheric motions observed to result from sudden, localized disturbances due, for instance, to thunderstorm activity. The radar data on BN p. 23, bottom panels, show the vertical motion resulting from precisely this situation. The separation into wavelike and layerwise-two-dimensional motions is crucial to a complete description of the subsequent evolution. A relevant reference is D. Lilly, J. Atmos. Sci., **40**, 749.]

5. An ideal stably-stratified Boussinesq fluid flows over a slender obstacle

$$z = h(x) = \int_{-\infty}^{\infty} \hat{h}(k) e^{ikx}\ dk \qquad [\hat{h} = (2\pi)^{-1} \int_{-\infty}^{\infty} h(x) e^{-ikx} dx].$$

The basic velocity $U > 0$ far upstream; and U and the buoyancy frequency N are both constants. Show that, according to linearized theory, the resulting steady lee waves can be described by the disturbance stream function ψ ($u = +\psi_z, w = -\psi_x$), where

$$\psi(x, z) = -2U\,\mathrm{Re} \int_0^{\infty} \hat{h}(k)\, e^{i(kx+Mz)}\, dk \qquad (x < 0 \text{ upstream}),$$

where

$$M = (K^2 - k^2)^{\frac{1}{2}}, \quad K \equiv N/U,$$

$M > 0$ for $0 \le k < K$ and $iM < 0$ when $k > K$. Show that the (horizontal) wave drag is

$$D = \int_{-\infty}^{\infty} +p\frac{dh}{dx}dx = 4\pi\rho_{00}U^2 \int_0^K kM|\hat{h}(k)|^2 dk$$

per unit spanwise distance, where ρ_{00} is the (constant) inertial density of the Boussinesq fluid. In the limiting case when the obstacle has width small compared with K^{-1}, show that $D = (3\pi U)^{-1}\rho_{00}N^3(\int_{-\infty}^{\infty} h(x)dx)^2$. (Note that this increases when U *decreases* ! Why?)

6. **Optional:** Use the method of stationary phase in the preceding example to show that, as $r^2 \equiv x^2 + z^2 \to \infty$,

$$\psi = -(\frac{8\pi K}{r})^{\frac{1}{2}} U \sin\theta \,\mathrm{Re}\left[\hat{h}(K\cos\theta)e^{iKr - \frac{1}{4}\pi i}\right] + O(r^{-1}),$$

$$0 < \theta < \tfrac{1}{2}\pi \text{ (downstream) ;}$$

$$\psi = O(r^{-1}), \qquad \tfrac{1}{2}\pi < \theta < \pi \text{ (upstream) ,}$$

where $\theta = \tan^{-1}(z/x)$, $x > 0$ being downstream. [Compare with the picture on BN p. 31; wave-crests are arcs of circles!] For a broad obstacle of width $\gg K^{-1}$, show also that

$$\psi \simeq -U\,\mathrm{Re}\left[e^{iKz}\left\{h(x) - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{h(\xi)}{\xi - x}d\xi\right\}\right] \quad \text{(Cauchy principal value) .}$$

[This solution and the picture on BN p. 31 have some relevance to severe downslope windstorms (foehns, chinooks, etc.), though far from the whole story.]

Show without making detailed calculations that, in any steady two-dimensional lee-wave pattern due to the flow of an ideal stably-stratified fluid over an obstacle of finite length, the fluid must on average be moving faster on the downstream slope of the obstacle than on the upstream slope. [Assume that the wave drag D is positive and make use of a suitable version of Bernoulli's theorem.]

7. It is given that two-dimensional internal gravity waves of wavenumber (k, m) and frequency ω in a shear flow $\bar{u} = \Lambda z$ satisfy the dispersion relation

$$\omega = \Omega(k, m, z) \equiv \Lambda z k \pm Nk(k^2 + m^2)^{-\frac{1}{2}} \ ,$$

where Λ and N are positive constants, the first term giving the Doppler shift. The ray-tracing equations may be assumed to apply, namely

$$\frac{dx}{dt} = \frac{\partial\Omega}{\partial k} \ , \quad \frac{dz}{dt} = \frac{\partial\Omega}{\partial m} \ ,$$

$$\frac{dk}{dt} = -\frac{\partial\Omega}{\partial x} \ , \quad \frac{dm}{dt} = -\frac{\partial\Omega}{\partial z} \ ,$$

where d/dt means the rate of change at a point $P = \{(x(t), z(t)\}$ moving with the group velocity, and the ∂'s indicate partial derivatives of the function $\Omega(k, m, x, z) = \Omega(k, m, z)$. What do these equations imply about the behaviour of k and ω following the point P? Show that the corresponding behaviour of m is given by $m = m_0 - \Lambda kt$, where m_0 is a constant. By inspection of the dispersion relation, show that, for any given ω, there exist three special values $z_{\min} < z_0 < z_{\max}$ of z, to be determined, such that if $t > m_0/\Lambda k$ then the point P moves towards z_0, and takes an infinite time to get there, whereas if $t < m_0/\Lambda k$ then P moves either towards $z_{\min}$ or towards $z_{\max}$, and takes a finite time to get there. Can P ever go outside the interval $[z_{\min}, z_{\max}]$?

What would you expect to happen near z_0 if the waves were subject to a very slight dissipation, for instance by Newtonian cooling, giving fixed dissipative time scale, or by diffusive processes, giving time scale $\propto (k^2 + m^2)^{-1}$?

Show that, along the path of the point P, $dx = \Lambda z_0 dt + (k/m)dz$. Integrate this relation, after expressing m as a function of z, or vice versa, to obtain $x = \text{constant} + \Lambda z_0 t - (z - z_0)m/k$. Hence or otherwise show that

$$x = \text{constant} + \Lambda z_0 t \pm N\Lambda^{-1}k^{-1}m(k^2 + m^2)^{-\frac{1}{2}} \ ,$$

$$z = z_0 \mp N\Lambda^{-1}(k^2 + m^2)^{-\frac{1}{2}} \ .$$

(Note that this determines x and z explicitly as functions of t, in virtue of the relation $m = m_0 - \Lambda k t$.)

8. **Optional:** An unbounded, Boussinesq liquid is stably stratified with constant buoyancy frequency N. A force $F = \epsilon\delta(x)\delta(z)H(t)$ per unit mass (ϵ infinitesimal) is applied at the origin in the x-direction, where $\delta(\cdot)$ is the Dirac delta function, and $H(t)$ the Heaviside step function (defined to take values 1 for $t \geq 0$ and 0 for $t < 0$). *Assuming the hydrostatic approximation* (even though this may turn out not to be self-consistent everywhere) show that according to linearized theory the resulting two-dimensional velocity field has horizontal component of the form

$$u(x, z, t) = \tfrac{1}{2}\epsilon\pi^{-1}Nt^2x^{-2}\phi(Ntz/x) \ ,$$

where the function $\phi(\cdot)$ is given by a certain integral.
[Start with the fact that $\delta(z) = (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(imz)\,dm$, then apply the principle of superposition to the solution given in BN pp. 37–41, ignoring the fact that the time dependence is not gradual.]

By evaluating the integral you have found, show further that $\phi(\cdot)$ is given by

$$\phi(q) = q^{-1}\sin q + q^{-2}(\cos q - 1) \ .$$

Partially check your result by showing directly from one of the governing equations that the solution must satisfy $\int_{-\infty}^{\infty} u(x, z, t)\,dz = 0$, for fixed, nonzero x and t, and then showing that the function $\phi(\cdot)$ has the corresponding property $\int_{-\infty}^{\infty}\phi(q)\,dq = 0$. What pattern do the lines $u = 0$ make in two-dimensional space at a given instant, and how does this pattern change in time? (The pattern, reminiscent of a fan closing, is similar to the starting-transient pattern seen in the laboratory movies shown in lectures.) Why is this behaviour of the pattern plausible from the elementary dispersion properties of internal gravity waves? In particular, why is it plausible that the vertical scale of the velocity profile at given $x \neq 0$ and, say, at $z = 0$, should shrink proportionately to t^{-1} as time goes on?

In what locations is the calculated velocity field most nearly consistent with the hydrostatic approximation made originally?

Please email any corrections/comments to mem@damtp.cam.ac.uk