Mathematical Tripos Part III 2007–08
Fundamentals of Atmosphere–Ocean Dynamics

Examples 4

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In these examples, 'BN' refers to the background lecture notes for the course.

1. A frictionless shallow-water system is initially at rest, with constant layer depth c_0^2/g, relative to a frame of reference rotating with constant nonzero angular velocity $(0, 0, \frac{1}{2}f)$. A yz-independent force per unit mass

$$F(x) = \begin{cases} F_0 \left(1 - \dfrac{x^2}{L^2}\right) , & |x| < L \\ 0 , & |x| > L \end{cases}$$

is applied to the fluid in the x direction. Show by inspection of the shallow-water equations, without solving the problem, that a response in the form of columnar disturbances is impossible.

['Columnar disturbance' means, as before, a disturbance in which the x-velocity component u and surface displacement ζ become steady and nonzero in a region continually expanding toward $x = \infty$ or $x = -\infty$.]

When F_0 is small enough to permit linearization of the equations about relative rest, show that the equations and initial conditions are satisfied by a disturbance consisting of the sum of two contributions: (a) one that satisfies the unforced shallow-water equations with zero linearized potential vorticity Q, and (b) a steady contribution whose surface elevation is

$$\zeta = \begin{cases} \dfrac{2F_0 {L_\mathrm{D}}^2 x}{gL^2} - A \sinh\left(\dfrac{x}{L_\mathrm{D}}\right) , & |x| < L , \\ \pm B \exp\left(-\dfrac{(|x| - L)}{L_\mathrm{D}}\right) , & x \gtrless L , \end{cases}$$

where

$$A = \left(\frac{2F_0 {L_\mathrm{D}}^2}{gL}\right)\left(1 + \frac{L_\mathrm{D}}{L}\right) \exp\left(-\frac{L}{L_\mathrm{D}}\right) \quad \text{and} \quad B = \frac{2F_0 {L_\mathrm{D}}^2}{gL} - A \sinh\left(\frac{L}{L_\mathrm{D}}\right) ,$$

and where g is gravity and L_D is the Rossby length of the shallow-water system. Find the corresponding velocity field for all x. In the case $L \gg L_\mathrm{D}$, sketch the profile $v(x)$ of the y-velocity v, and comment on the balance of terms in the x-momentum equation in that case.

Optional: Show by considering Fourier transforms, or otherwise, that the contribution (a) can be regarded as made up entirely of freely propagating inertia–gravity (Poincaré) waves.

2. [The following illustrates the velocity and buoyancy fields induced in an unbounded stratified, rotating model atmosphere — 'induced' in the sense of PV inversion — by an isolated PV anomaly of simple shape. The solution is a quasi-geostrophic counterpart of the two-dimensional Rankine vortex and is an idealized version of the more accurate and realistic inversion shown on BN p.145, having some though not all of the same features.]

A stably stratified, rotating, unbounded Boussinesq fluid has constant buoyancy frequency N. The frame of reference rotates with constant angular velocity $(0, 0, \frac{1}{2}f)$. The quasi-geostrophic PV anomaly Q' takes a constant positive value Q_0 within, and is zero outside, the region $r \leqslant r_0$ where

$$r^2 = x^2 + y^2 + Z^2 , \qquad Z^2 = N^2 z^2 / f^2 ,$$

with r_0 a positive constant. Show that the inversion problem for this Q distribution has the solution

$$\psi = \psi_0(r) := \begin{cases} \frac{1}{6}\, Q_0\,(r^2 - 3r_0^2) & (r < r_0), \\ -\frac{1}{3}\, Q_0\, r_0^3/r & (r > r_0), \end{cases}$$

in terms of the quasi-geostrophic streamfunction ψ. To help picture this solution, show that

(a) the associated geostrophic velocity field is continuous everywhere;

(b) the associated buoyancy-acceleration field is continuous everywhere;

(c) the fluid circulates anticlockwise, viewed from above, in each plane $z =$ constant;

(d) the velocity at given $x^2 + y^2 > r_0^2$ is greatest in the plane $z = 0$;

(e) the fluid is in rigid rotation, about a vertical axis, within $r \leqslant r_0$;

(f) the horizontal buoyancy-acceleration gradient vanishes, i.e., the stratification surfaces are horizontal, within $r \leqslant r_0$; and

(g) the radial component of velocity is zero everywhere.

How could you have deduced statement (f) immediately from statement (e), without knowing anything else about the solution?

Show that $\psi = \psi_0(r)$ is an exact steady solution of the nonlinear quasi-geostrophic equations. By considering either the horizontal momentum equations, or the buoyancy and mass-conservation equations, show that statement (g) is true not only in the leading (geostrophic) approximation but also in the next approximation for small Rossby number, assuming axisymmetry and steadiness as well as ideal (diffusionless) flow.

3. [Rossby-wave resonance in a stratified flow] Rossby waves generated by constant basic flow $(U, 0, 0)$ past a lower boundary $z = \mathrm{Re}\ (\epsilon\, H\, e^{ikx}\ \sin(ly)$ $(\epsilon \ll 1)$ are trapped in a waveguide by the lower boundary and a rigid lid at $z = H$. Linearize the Boussinesq, quasi-geostrophic equations, assuming constant $\beta = df/dy$ and constant buoyancy frequency N and putting $\psi = -Uy + \epsilon\psi'$. Consider the parameter range $k^2 + l^2 < \beta/U$. Let

$$m = \frac{N}{f_0}\left(\frac{\beta}{U} - k^2 - l^2\right)^{1/2} .$$

When $\sin(mH) \neq 0$, show that the equations admit the steady solution

$$\psi' = \ \mathrm{Re}\ \left[\frac{-\,N^2 H}{f_0\, m\, \sin(mH)}\ e^{ikx}\ \sin(ly)\ \cos\{m(z-H)\}\right] .$$

When $\sin(mH) = 0$, show that the equations admit a solution of the form

$$\psi' = \ \mathrm{Re}\ \left[\,e^{ikx}\ \sin(ly)\big\{B\,(t-t_0)\cos(mz)\ +\ C\,(z-H)\sin(mz)\big\}\right] ,$$

where B and C are constants to be found, and t_0 is an arbitrary constant.

[Answer: $B = 2ikf_0U^2/\beta$; $C = N^2/(f_0 m)$. Such 'resonant growth' may contain our first clues as to the dynamical nature of the Antarctic 'final spring warming'. This is a phenomenon in which the stratospheric polar vortex becomes highly disturbed two or three months later, often with large amplitudes of zonal wavenumber 1.]

4. From the quasi-geostrophic equations for unstratified, low-Rossby-number flow between nearly-horizontal boundaries such that

$$2\Omega\, h^{-1}\nabla h = (0, \beta) \qquad (\beta \text{ const.})$$

(as in BN §10) derive the linearized equation for the case of small disturbances about a nonzero mean flow $\{U(y), 0\}$, in the form

$$D_t \nabla^2 \psi' + (\beta - U_{yy})\psi'_x = 0\ , \tag{1}$$

where the linearized material derivative $D_t = \partial/\partial t + U\partial/\partial x$, and where $\psi'(x, y, t)$ is the disturbance streamfunction, so that the disturbance velocities $u' = -\psi'_y\,,\ v' = \psi'_x$. Assume that the mean velocity $U(y) = \Lambda y$, with constant shear Λ, in an unbounded domain. Show that a 'sheared disturbance', of the form $\psi' = \hat{\psi}(t)\exp\{ik(x - \Lambda y t)\}$, where k is a real constant, is a solution provided that the function $\hat{\psi}(t)$ satisfies the ordinary differential equation

$$\frac{d}{dt}\Big((1 + \Lambda^2 t^2)\hat{\psi}(t)\Big) \;-\; i\frac{\beta}{k}\,\hat{\psi}(t) \;=\; 0\ .$$

[Notice that the operator D_t annihilates the exponential factor.] Show that the last equation has solutions

$$\hat{\psi}(t) \;\propto\; \frac{\exp\left\{i\beta(k\Lambda)^{-1}\arctan(\Lambda t)\right\}}{1 \;+\; \Lambda^2 t^2}\ .$$

Deduce that the relative vorticity field behaves like a passive tracer for large $t \gg \Lambda^{-1}$. Comment on why this could have been anticipated at the outset, from a knowledge of the general properties of Rossby waves and/or PV inversion, together with the idea that background shear tends to rotate the wavecrests like the elements of a Venetian blind (an idea already encountered in the ray-theoretic problem on Sheet 1). Show that the sheared-disturbance solution is valid at finite amplitude, i.e. that it satisfies the full nonlinear vorticity equation.

[* This is the simplest example of a fundamental and ubiquitous process, in which the length scale of PV anomalies shrinks irreversibly; see also BN pp. 99–104. Such processes — discussed in the literature under various headings, e.g. 'enstrophy cascade', 'critical-layer absorption', 'wave capture', 'Rossby-wave breaking', depending on the context — lead to **irreversible rearrangements** of PV distributions. Such rearrangement, in turn, gives rise to the irreversible transport of momentum by Rossby waves, as illustrated in sheet 3 question 5(c) and 6, gyroscopically pumping the stratospheric Brewer–Dobson circulation and other global-scale atmospheric circulations; again see www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/ *]

5. A stratified, rotating, Boussinesq fluid has constant angular velocity $(0, 0, \Omega)$ and constant buoyancy frequency N. With the help of suitably-oriented axes, or otherwise, derive the dispersion relation for plane inertio-gravity waves in the form

$$\omega^2 = N^2\cos^2\theta + 4\Omega^2\sin^2\theta\ ,$$

where θ is the angle that the wavenumber vector $\mathbf{k}$ makes with the horizontal. Deduce that when $\Omega = 0$ the group velocity has magnitude $N|\mathbf{k}|^{-1}\sin\theta$. Hence or otherwise deduce that if $N^2 > 4\Omega^2$ then the direction of the group velocity must be the same as when $\Omega = 0$. What if $2\Omega = N$?

The fluid occupies the half-space $z > 0$, and is subjected to a given vertical velocity $w_0 = w(x, 0, t)$ at $z = 0$, assumed small enough for linear theory to apply. Find the velocity field throughout $z > 0$ when $w_0 = \epsilon\sin kx\cos\omega t$ (ϵ, k, ω constant) and $4\Omega^2 < \omega^2 < N^2$. Indicate briefly how this solution could be generalized to the case of an arbitrary prescribed velocity $w_0(x, y, t)$.

Extend your solution to the case $\omega^2 < 4\Omega^2$ (when $w_0 = \epsilon\sin kx\cos\omega t$). Derive the corresponding solution of the Boussinesq quasi-geostrophic equations [BN §17], and show that the two solutions agree in the limit $\omega^2 \ll 4\Omega^2$. Why was this agreement to be expected?

6. An infinite rectangular channel $0 < y < L$, $0 < z < H$ contains a Boussinesq, constant-N, inviscid stratified fluid initially at rest, relative to a frame of reference rotating with angular velocity $\{0, 0, \frac{1}{2}f\}$. On the assumption that all quantities are independent of x, and that appropriate geostrophic and hydrostatic balance conditions hold, analyse the early response of the fluid to a hypothetical weak distributed force per unit mass

$$F = F_0 \sin(\pi y/L) \sin^2(\pi z/H) \quad (F_0 \text{ constant })$$

in the x-direction, and show in particular that the actual acceleration $\partial u/\partial t$ in the x-direction is *not* equal to F. Sketch streamlines of the motion (v, w) in the yz plane, as well as graphs showing the variation of the along-channel flow component u with y, z and t. Explain why the meridional motion (v, w) is inevitable if the thermal-wind equation is to continue to be satisfied. On the basis of your solution, estimate how weak F_0 must be for self-consistency, i.e. for the approximate validity of the assumed geostrophic and hydrostatic balance, and comment briefly on what other fluid motions, neglected in your solution, might be excited by the switching-on of the force F.

[*This is an idealized form of the early (switch-on) stages of the gyroscopic pumping problem. To understand the later stages, one must introduce thermal relaxation; the simplest model is Newtonian cooling. See 'downward control' in www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/ *]

7. Consider quasi-geostrophic motion of a stably stratified, unbounded Boussinesq fluid with Coriolis parameter $f_0 + \beta y$, where the coordinates are taken as Cartesian and where f_0 and β are constants. The buoyancy frequency N is constant. Let overbars denote averages with respect to the 'eastward' direction x. Let $\overline{\psi}(y, z)$ represent a background flow with horizontal and vertical shear, and $\psi'(x, y, z, t)$ a disturbance to that flow, not necessarily of small amplitude. Define the *Eliassen–Palm flux*, a two-dimensional vector in the yz plane, by

$$(F, G) \ := \ \left(-\overline{u'v'}, \ \frac{f_0}{N^2} \overline{v'\sigma'} \right)$$

where as usual $u' = -\psi'_y$, $v' = \psi'_x$ and $\sigma' = f_0 \psi'_z$. With Q' denoting the disturbance part of the quasi-geostrophic potential vorticity, derive the *Taylor identity*

$$\nabla \cdot (F, G) \ = \ \overline{v'Q'}$$

where ∇ is in the yz plane. If furthermore the disturbance amplitude is small, deduce with the help of the linearized potential-vorticity equation that the quantity $\mathcal{A} := \frac{1}{2}\overline{Q'^2}/\overline{Q}_y$ is the density in a conservation relation whose flux is the Eliassen–Palm flux,

$$\frac{\partial \mathcal{A}}{\partial t} \ + \ \nabla \cdot (F, G) \ = \ 0 \ .$$

For a plane Rossby wave with $\overline{\psi} = 0$ show that the vector $\mathcal{A}\,(F, G)$ is equal to the projection of the group velocity on to the yz plane.

[*These relations, and the PV-mixing scenarios of Sheet 3 qq. 5(c) and 6 (note that $\overline{v'Q'} < 0$ during mixing) are key to understanding how the large-scale 'wave–turbulence jigsaw' fits together and gives rise, for instance, to the stratosphere's 'gyroscopic pump' (BN p.176). They are also key to understanding the famously persistent narrowness of jet streams (e.g. *Science* **315**, 467, 26 January 2007) — what used to be called 'negative viscosity'. The planet Jupiter provides a spectacular example (http://www.atm.damtp.cam.ac.uk/people/mem/#chapmanconf, link at end of the paragraph). The vertical flux component G is equal to the mean stress across an undulating stratification surface $\sigma_1 =$ const., arising from the dynamical organization of fluctuations — more precisely, from the correlation of surface slope with disturbance pressure. Oceanographers call G the *form stress* as a way to remember its association with the undulating shape, or form, of the stratification surface — not to be confused with the 'form drag' or nonlinear force due to flow past a bluff body. The Taylor identity is named after pioneering work by G. I. Taylor published in 1915.*]

Please email any corrections/comments to mem@damtp.cam.ac.uk