

# Fundamentals of Atmosphere–Ocean Dynamics

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L<sup>A</sup>T<sub>E</sub>X version edited by

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## Author's preface

These are the background lecture notes for my course in Part III of the Cambridge Mathematical Tripos. They have now been converted from handwritten to L<sup>A</sup>T<sub>E</sub>X form through the herculean editorial efforts of two members of my research group, Björn Haßler and Christophe Koudella, with the very able typesetting assistance of Teresa Cronin and Sarah Shea-Simonds. My warmest thanks go to all of them for this effort, which brings new standards of legibility and attractiveness to the course material.

I'd be grateful of course to be told of any misprints or obscurities that may have survived the editing process so far.

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## Editors' preface

The handwritten lecture notes are practically free of misprints, but a large scale typesetting operation like this inevitably introduces misprints. Please bring any remaining misprints to our attention. Sets of the handwritten notes can be found in the library and in the Part III room (Dirac Graduate Centre) for consultation. Some markers are incorporated to help with finding things in the handwritten notes; see further details at the back of these printed notes.

A word of caution about the index and page cross-references. If a page is cross-referenced (e.g. "see p. 47") then the referenced material is likely to be found on that page, but you may have to look on the following page also (i.e. to look on pages 47 *and* 48). This is due to the fact that the handwritten pages and typed pages do not coincide perfectly, and to the fact that the cross-references haven't all been adjusted yet.

To improve the usefulness of these notes further, we would kindly ask you to bring any misprints, queries, additional index entries, comments, and suggestions to our attention. To do so, please e-mail

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and/or consult Professor McIntyre!

Very many thanks,  
Björn Haßler, Christophe Koudella

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# Part 0

## Preliminaries





# 0.

## Preliminaries

### §0.1 Basic equations

We summarise basic equations etc. (mainly for occasional reference). See also Appendix A. We almost always use Eulerian forms:  $\frac{\partial}{\partial t}$  is ‘at a point fixed in space’; the derivative following a fluid particle is then

$$\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

where  $\mathbf{u}$  = velocity of fluid.

**Conservation of mass:** [and *definition* of  $\mathbf{u}$ ]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (0.1)$$

where  $\rho(\mathbf{x}, t)$  is density. (Or  $\mathcal{D}\rho/\mathcal{D}t + \rho \nabla \cdot \mathbf{u} = 0$ .)

**Equation of motion:** The equation of motion relative to frame of reference rotating with constant angular velocity  $\boldsymbol{\Omega}$ , under a conservative, steady body force  $-\nabla\chi$  per unit mass is

$$\left. \begin{array}{l} \frac{D\mathbf{u}}{Dt} \\ \text{or } \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \\ \text{or } \frac{\partial \mathbf{u}}{\partial t} + \nabla(\frac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\zeta} \times \mathbf{u} \end{array} \right\} + \underbrace{-\text{Coriolis 'force'}}_{2\boldsymbol{\Omega} \times \mathbf{u}} =$$

$$= -\frac{1}{\rho} \nabla P - \nabla \left( \underbrace{\chi - \frac{1}{2}\varpi^2|\boldsymbol{\Omega}|^2}_{\text{centrifugal 'potential'}} \right) + \mathbf{F} \quad (0.2 \text{ a})$$

$= \bar{\chi}$  say, ‘effective gravitational potential’

( $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ ,  $\varpi$  distance from rotation axis).  $\mathbf{F}$  could be a viscous force per unit mass  $\mathbf{F}_{\text{visc}} = \nu \nabla^2 \mathbf{u}$  — will usually be neglected — or occasionally a hypothetical, given, non-conservative body force introduced for the purposes of a ‘thought-experiment’ on the fluid.

—0.1—



**Conservation of momentum:** if  $\mathbf{\Omega}$ ,  $\nabla\chi$ , and  $\mathbf{F}$  all zero, (0.1) and (0.2 a)  
 $\Rightarrow$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + P \mathbf{I}) = 0 \quad (\mathbf{I} = \text{identity tensor}) \quad (0.3)$$

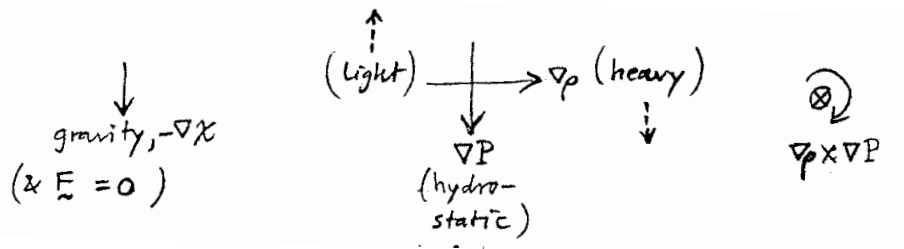
**Vorticity equation:**  $\nabla \times (0.2 a)$ ;  $\nabla \times (\boldsymbol{\zeta} \times \mathbf{u}) \equiv \mathbf{u} \cdot \nabla \boldsymbol{\zeta} - \boldsymbol{\zeta} \cdot \nabla \mathbf{u} + \boldsymbol{\zeta} \nabla \cdot \mathbf{u}$ ,  
 etc., noting  $\nabla \cdot \boldsymbol{\zeta} = 0$ :


$$\left. \begin{aligned} \frac{D\boldsymbol{\zeta}}{Dt} \\ \frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\zeta} \end{aligned} \right\} = (2\mathbf{\Omega} + \boldsymbol{\zeta}) \cdot \nabla \mathbf{u} - (2\mathbf{\Omega} + \boldsymbol{\zeta}) \nabla \cdot \mathbf{u} + \underbrace{\frac{1}{\rho^2} \nabla \rho \times \nabla P}_{(*)} + \nabla \times \mathbf{F} \quad (0.4)$$

Note: The term marked (\*) is identically zero if  $\rho = \text{func}(P)$  everywhere ('barotropic fluid'). Otherwise we say that the fluid system is 'baroclinic':  $\nabla \rho \times \nabla P$  is generically nonzero, making stratification (buoyancy) effects dynamically significant.

—0.2—

Note: the effect of the conservative body forces seems not to be present in (0.4); but since they affect  $\nabla P$  (indeed, often dominate it) via (0.2 a), they really are present, in the  $\nabla \rho \times \nabla P$  term. E.g., with fluid at rest,  $\mathbf{\Omega} = 0$ ,  $\mathbf{u} = 0$  at some instant:



In this situation,  $\partial\boldsymbol{\zeta}/\partial t$  is non-zero and has the sense  — an example of 'baroclinic generation of vorticity'.

**Ertel's potential-vorticity theorem:** Let  $\alpha$  be such that  $D\alpha/Dt = 0$ . Suppose also that  $\rho = \text{func}(P, \alpha)$  ( $\alpha$  usually specific entropy, i.e. entropy per unit mass). (Includes barotropic case  $\rho = \text{func}(P)$ , and incompressible,

heterogeneous case, where we can take  $\alpha = \rho$ .) Then performing the operations  $(\nabla\alpha) \cdot (0.4) + (2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \cdot \nabla\{\mathcal{D}\alpha/\mathcal{D}t = 0\}$  gives a conservation relation

$$\frac{\partial}{\partial t}\{(2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \cdot \nabla\alpha\} + \nabla \cdot \{\mathbf{u}(2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \cdot \nabla\alpha\} = 0, \quad (0.5\text{ a})$$

provided  $\mathbf{F} = 0$ . [It is noteworthy that the conservation form  $\frac{\partial}{\partial t}(\cdot) + \nabla \cdot \{\cdot\} = 0$  persists even if  $\mathbf{F} \neq 0$  and  $\mathcal{D}\alpha/\mathcal{D}t \neq 0$ ; consequences are discussed in P. H. Haynes and M. E. McIntyre *J. Atmos. Sci.* **44**, 828–84, **47**, 2021–31.]

Notice the pattern in (0.5 a)! It means that we can combine it with mass conservation, (0.1), to give the alternative form

$$\boxed{\frac{\mathcal{D}}{\mathcal{D}t} \left[ \frac{(2\boldsymbol{\Omega} + \boldsymbol{\zeta}) \cdot \nabla\alpha}{\rho} \right] = 0;} \quad (0.5\text{ b})$$

[ ] is called the (Rossby–Ertel) **potential vorticity**, and (0.5 b) ‘Ertel’s theorem’. For barotropic motion  $\alpha$  can be chosen to be any function of space at an initial time — think of it as a distribution of ‘dye’ (non-diffusing,  $\mathcal{D}\alpha/\mathcal{D}t = 0$ ). The statement that (0.5 b) is true for all such dye distributions is then a restatement of the Helmholtz law of constancy of vortex tube strengths. (For more history see [www.atm.damtp.cam.ac.uk/people/mem/papers/ENCYC/](http://www.atm.damtp.cam.ac.uk/people/mem/papers/ENCYC/))

**Incompressibility:** From here on we restrict attention to *incompressible* (but inhomogeneous) flow — a good approximation for many purposes in the ocean [O. M. Phillips, *The dynamics of the upper ocean*, 1966 (2nd edition e.g. pp. 25, 16, 75 1977), Cambridge University Press, eq. (2.4.5)] and in laboratory experiments. Less obviously, it is often valid, at least qualitatively, for the atmosphere, provided that density is replaced by ‘potential density’ or specific entropy [O. M. Phillips, p.13; E. A. Spiegel & G. Veronis 1960, *Astrophys. J.* **131**, 442].

We also neglect diffusion of density anomalies. Thus

$$\frac{\mathcal{D}\rho}{\mathcal{D}t} \equiv \frac{\partial\rho}{\partial t} + \mathbf{u} \cdot \nabla\rho = 0. \quad (0.2\text{ b})$$

(0.1) Conservation of mass then implies

$$\nabla \cdot \mathbf{u} = 0. \quad (0.2\text{ c})$$

Equations (0.2) are taken as the five basic equations for the five dependent variables  $\mathbf{u}(\mathbf{x}, t)$ ,  $\rho(\mathbf{x}, t)$ ,  $P(\mathbf{x}, t)$ . Note, can now take  $\alpha \equiv \rho$  in (0.5); also note the disappearance of a term  $\propto \nabla \cdot \mathbf{u}$  from (0.4). -N1-  
-0.3-  
-N2-

**Conservation of energy:** The ‘incompressible’ idealization allows evasion of thermodynamical questions (internal energy plays no role).

Take  $\rho \mathbf{u} \cdot (0.2 a) + \frac{1}{2} \rho |\mathbf{u}|^2 \cdot (0.1)$ ; use incompressibility (0.2 c), p.5 ( $\nabla \cdot \mathbf{u} = 0$ ); write, as before,  $\tilde{\chi}(\mathbf{x}) \equiv \chi - \frac{1}{2} \varpi^2 |\boldsymbol{\Omega}|^2$ ; we get

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) + \nabla \cdot \left\{ \frac{1}{2} \rho |\mathbf{u}|^2 \mathbf{u} + P \mathbf{u} \right\} = -\rho \mathbf{u} \cdot \nabla \tilde{\chi} + \rho \mathbf{u} \cdot \mathbf{F}. \quad (0.6)$$

-N3-

Take  $\tilde{\chi} \times (0.2 b)$ :

$$\frac{\partial}{\partial t} (\rho \tilde{\chi}) + \nabla \cdot \{ \rho \tilde{\chi} \mathbf{u} \} = \rho \mathbf{u} \cdot \nabla \tilde{\chi}. \quad (0.7)$$

Add, and set  $\mathbf{F} = 0$ , to get the conservation relation for energy; write  $T = \frac{1}{2} \rho |\mathbf{u}|^2$ ,  $V = \rho \tilde{\chi}$ :

$$\frac{\partial}{\partial t} (T + V) + \nabla \cdot \{ (T + V) \mathbf{u} + P \mathbf{u} \} = 0. \quad (\mathbf{F} = 0). \quad (0.8)$$

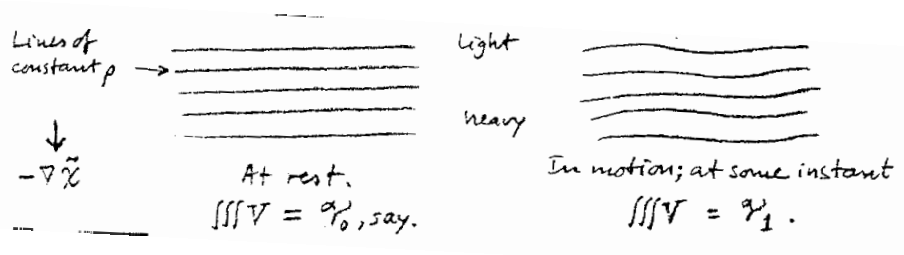
$T$  is kinetic energy/unit volume, relative to rotating frame;  $V$  is the potential energy/volume associated with the ‘effective gravitational potential’  $\tilde{\chi}$ , i.e. taking account of the centrifugal potential;  $\nabla \cdot (P \mathbf{u})$  represents the rate of working by pressure forces across an infinitesimal fluid volume, per unit volume. Use of (0.2 c) was essential in order that none of this work appear as internal energy.

**Bernoulli’s theorem:** Bernoulli’s theorem for steady motion (relative to rotating frame). Equation (0.2 b)  $\Rightarrow \rho$  const. along streamlines; then with  $\mathbf{F} = 0$ , (0.2 a) integrates (note vanishing of triple scalar products) to

$$\frac{1}{2} |\mathbf{u}|^2 + \frac{P}{\rho} + \tilde{\chi} = \text{const. along streamline.} \quad (0.9)$$

## §0.2 Available potential energy

Again, this concept is simplest under restriction (0.2 c), p.5. Many motions of interest involve only small vertical displacements of a stably-stratified fluid.



Intuitively,  $\mathcal{V}_1 > \mathcal{V}_0$  but by an amount that could be  $\ll \mathcal{V}_0$ . [Tacit assumption: the fluid is ‘contained’ in some way that prevents it from moving up or down as a whole. True of the atmosphere and oceans!] Only the difference

$$\mathcal{A} = \mathcal{V}_1 - \mathcal{V}_0$$

is available for conversion into kinetic energy, hence relevant to the dynamics of motion internal to the fluid. This is the difference between two large terms, a difference that is usually far smaller than the individual terms. We want an exact formula for  $\mathcal{A}$  *not* involving such a small difference. Here we give only the simplest form of the theory.<sup>1</sup>

We assume that *the fluid is incompressible*, restriction (0.2 c), p.5, and *contained within a fixed volume  $\mathcal{D}$  of simple shape* in the following sense. The container shape must be such that those level surfaces  $\tilde{\chi}(\mathbf{x}) = \text{const.}$  that intersect  $\mathcal{D}$  divide it into just two parts, the ‘upper’, where  $\tilde{\chi}$  is greater, and the ‘lower’, where  $\tilde{\chi}$  is less. We assume also that  $\rho$  and  $\nabla\rho$  are continuous functions of  $\mathbf{x}$  with  $|\nabla\rho| \neq 0$  almost everywhere.<sup>2</sup> More precisely, we assume that no finite volume of the fluid is homogeneous in density  $\rho$ , i.e. that no finite volume of the fluid has  $\nabla\rho = 0$ .

(1) Under the foregoing assumptions, it’s easy to show that there is a *unique*, stably-stratified ‘reference’ state of equilibrium, or ‘basic state’, with  $\mathbf{F} \equiv 0$ , to which the fluid could be brought via a hypothetical motion satisfying eqs (0.2 b) and (0.2 c). *Proof*: ‘Equilibrium’ means that all the equations, including the equation of motion (0.2 a), are satisfied with  $\mathbf{u} \equiv 0$  as well as  $\mathbf{F} \equiv 0$ ; there is no motion relative to the rotating frame. It follows that  $\rho = \text{func}(\tilde{\chi})$  alone, in order to satisfy (0.2 a). ‘Stably-stratified’ means that  $\rho$  is a *monotonically decreasing* function of  $\tilde{\chi}$ . Let  $Q(\rho)$  be the volume of fluid having density between  $\rho$  and  $\max(\rho)$ ;  $Q(\rho)$  is constant under (0.2 b) and (0.2 c). ‘Stably stratified’  $\Rightarrow$  all this fluid lies below the (unique) level surface of  $\tilde{\chi}$  that cuts off a lower part of  $\mathcal{D}$  with volume  $Q$ ; and moreover that the fluid of density  $\rho$  is at this level surface. This assigns a unique, 1 – 1 correspondence between values of  $\rho$  and values of  $\tilde{\chi}$  (since no finite volume of the fluid is homogeneous in density).

<sup>1</sup>It is related to the notions of ‘Casimir invariants’ and ‘phase-space reduction’ in abstract Hamiltonian dynamics, in which  $\mathcal{A}$  would be recognized as an ‘energy–Casimir’ invariant; but that need not concern us here.

<sup>2</sup>One is sometimes interested in cases where  $\rho$  has jump discontinuities, such as the ocean below together with the air above. The extension to such cases is not difficult, but will be omitted here.

(2) Write

$$\rho = \rho_0 + \rho'$$

where  $\rho_0$  corresponds to the reference or basic state and can therefore be expressed as a monotonically decreasing function of  $\tilde{\chi}$  alone:



Represent this by the inverse function,

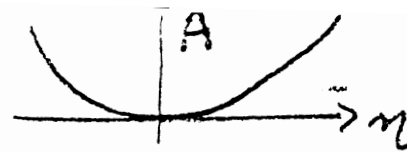
$$\tilde{\chi} = X(\rho_0),$$

also monotonically decreasing — a function known entirely in terms of the basic distribution of fluid density,  $Q(\rho)$ , and independent of motion under (0.2 b), (0.2 c).

(3) Now define

$$A(\xi, \eta) \equiv - \int_0^\eta \{X(\xi + \eta') - X(\xi)\} d\eta', \quad (0.10)$$

another function known entirely in terms of  $Q(\rho)$ . Because  $X(\cdot)$  is a monotonically decreasing function,  $A(\xi, \eta)$  is a positive definite function of  $\eta$ , for any given  $\xi$ , except that  $\eta = 0 \Rightarrow A = 0$ .



Alternative derivation in Holliday & McIntyre, *J. Fluid Mech.* **107**, 221 (1981), also referred to below.

Then if  $\mathcal{V}_0$  is the potential energy of the basic or reference state, we can show that the available potential energy  $\mathcal{A}(t)$  in the domain  $\mathcal{D}$  at time  $t$  is given by

$$\mathcal{A}(t) \equiv \mathcal{V}_1 - \mathcal{V}_0 = \iiint_{\mathcal{D}} A(\rho_0(\mathbf{x}), \rho'(\mathbf{x}, t)) dx dy dz. \quad (0.11)$$

Note that (0.11) is positive definite for any  $\rho' \neq 0$ , and zero for  $\rho' \equiv 0$  everywhere, (showing, incidentally, that under (0.2 b), (0.2 c) the potential energy

of the basic or reference state is an absolute minimum). This formula exhibits the intuitively-expected fact that the part of  $\mathcal{V}$  that matters dynamically is somehow associated with the *departure*  $\rho'$  from the reference state  $\rho_0$ .

To establish (0.11) it is sufficient to show that

$$\frac{d\mathcal{A}}{dt} = \frac{d\mathcal{V}_1}{dt}$$

for all hypothetical motions satisfying (0.2 b), (0.2 c). Since these also satisfy (0.7), we have (using the divergence theorem with  $\mathbf{u} \cdot \mathbf{n} = 0$  on the fixed boundary at  $\mathcal{D}$ )

$$\frac{d\mathcal{V}_1}{dt} = \iiint_{\mathcal{D}} \rho \mathbf{u} \cdot \nabla \tilde{\chi} \, dx \, dy \, dz . \quad (0.12)$$

From (0.11),

$$\frac{d\mathcal{A}}{dt} = \iiint_{\mathcal{D}} \left. \frac{\partial A(\xi, \eta)}{\partial \eta} \right|_{\substack{\xi = \rho_0(\mathbf{x}) \\ \eta = \rho'(\mathbf{x}, t)}} \times \frac{\partial \rho'}{\partial t} \, dx \, dy \, dz . \quad (0.13)$$

The first factor can be replaced by  $-\{X(\rho(\mathbf{x}, t)) - X(\rho_0(\mathbf{x}))\}$ , in virtue of the definition (0.10). The second factor can be replaced by  $\partial\rho/\partial t$  and therefore by  $-\nabla \cdot (\rho \mathbf{u})$ , in virtue of mass conservation (0.1). Therefore, now using incompressibility  $\nabla \cdot \mathbf{u} = 0$  to move the factor  $\mathbf{u}$  to the left or right of the  $\nabla$  operator as necessary, we have

$$\begin{aligned} \frac{d\mathcal{A}}{dt} &= \iiint_{\mathcal{D}} \underbrace{X(\rho(\mathbf{x}, t)) \nabla \cdot (\rho \mathbf{u})}_{=\nabla \cdot \{\mathbf{u} \int^{\rho} X(\bar{\rho}) \, d\bar{\rho}\}} \, dx \, dy \, dz \\ &\quad - \iiint_{\mathcal{D}} X(\rho_0(\mathbf{x})) \nabla \cdot (\rho \mathbf{u}) \, dx \, dy \, dz \end{aligned}$$

so that the first integral vanishes (again using the divergence theorem with  $\mathbf{u} \cdot \mathbf{n} = 0$  on the fixed boundary at  $\mathcal{D}$ ). The second integral is

$$\begin{aligned} &+ \iiint_{\mathcal{D}} \rho \mathbf{u} \cdot \nabla [X(\rho_0(\mathbf{u}))] \, dx \, dy \, dz \\ &= \iiint_{\mathcal{D}} \rho \mathbf{u} \cdot \nabla [\tilde{\chi}(\mathbf{x})] \, dx \, dy \, dz = (0.12) \text{ above,} \end{aligned}$$

which completes the proof.

(4) It follows also from (0.6) that when  $\mathbf{F} \equiv 0$

$$\mathcal{A} + \mathcal{T} = \text{constant} \quad \left( \mathcal{T} \equiv \iiint_{\mathcal{D}} T \, dx \, dy \, dz \right).$$

To summarize the most important aspects:  $\mathcal{A}$  has a useful physical meaning when the fluid is contained within a fixed volume  $\mathcal{D}$ ; then, if no external work is done ( $\mathbf{F} = 0$ ),  $\mathcal{A} + \mathcal{T}$  is a constant of the motion (always assuming it satisfies (0.2 a)–(0.2 c));  $\mathcal{A}$  is the *available* potential energy in the sense that  $\mathcal{A} \geq 0$  with  $\mathcal{A} = 0$  only when the distribution of fluid mass within  $\mathcal{D}$  has the lowest possible potential energy  $\mathcal{V}_0$ .

**Remark:** If the fluid is in its reference state and we cool a small portion of it locally (removing thermal energy and increasing  $\rho$ ) then we diminish  $\mathcal{V}_0$ . But  $\rho' \neq 0$  and so (0.11), being positive definite, shows that we have **increased**  $\mathcal{A}$ . Thus consideration of  $\mathcal{V}_1$  rather than  $\mathcal{A}$  could be qualitatively misleading.

### Approximate formula for $\mathcal{A}$ when $\rho' \ll \rho_0$

Unlike the exact formula (0.11), the following depends on Taylor expansion and so cannot be extended beyond cases in which reference profile  $X(\rho_0)$  is a smooth, well-behaved function. Taylor-expanding  $A$  for small  $\eta$ , i.e. approximating  $A$  by the parabola, we have

$$A(\xi, \eta) = -\frac{dX(\xi)}{d\xi} \times \frac{1}{2} \eta^2 + O(\eta^3) \quad \text{as } \eta \rightarrow 0. \quad (0.14)$$

Then (0.11) becomes

$$\mathcal{A} = -\iiint_{\mathcal{D}} \left. \frac{dX}{d\xi} \right|_{\xi=\rho_0} \times \frac{1}{2} \{\rho'(\mathbf{x}, t)\}^2 \, d\mathbf{x} + O(\rho'^3) \quad \text{as } \rho' \rightarrow 0. \quad (0.15)$$

In terms of the local gravity acceleration  $g \equiv |\nabla \tilde{\chi}|$ , and ‘buoyancy frequency’ (to be discussed in next section) of the reference state, defined as

$$N^2(\mathbf{x}) \equiv \frac{-\{g(\mathbf{x})\}^2}{\rho_0 (\partial X / \partial \xi)_{\xi=\rho_0}},$$

(0.15) can be rewritten

$$\mathcal{A} = + \iiint_{\mathcal{D}} \frac{\frac{1}{2} g^2 \rho'^2}{\rho_0 N^2} \, d\mathbf{x} + O(\rho'^3) \quad \text{as } \rho' \rightarrow 0. \quad (0.16)$$



This equation is exact if (0.14) is exact — i.e. if the stratification of the reference state is linear in the sense that  $\rho_0 \propto \tilde{\chi}$ .

The concept of ‘available potential energy’ is due to Margules and was given its modern form by E. N. Lorenz, *Tellus* **7**, 157 (1955) — or see chap. V of his monograph),<sup>3</sup> who develops approximate formulae of the type (0.16) for the case of a perfect gas. Exact formulae of the type (0.11) are given by

Holliday & McIntyre, *J. Fluid Mech.* **107**, 221 (1981) and D. G. Andrews, *J. Fluid Mech.* **107**, 227 (1981). Andrews shows how (0.11) generalizes to compressible fluids. Note incidentally that an assertion of applicability to mixing processes in the Holliday & McIntyre paper is **wrong** and should be ignored.

—0.8—

### §0.3 Wavcrest kinematics

The stationary phase approximation (the standard theory for waves in homogeneous media, plural)<sup>4</sup> shows that, for  $t$  large enough, a dispersing wave disturbance can usually be represented as a sum of terms of the form

$$f(x, t) e^{i\theta(x, t)} \quad (0.17)$$

where  $f(x, t)$  is *slowly varying* (‘SV’ for short) over distances and times of the order of one wavelength or period. That is, for  $t$  large enough,  $\partial f/\partial x \ll k_0 f$  and  $\partial f/\partial t \ll \omega_0 f$  where the local wavenumber  $k_0$  and frequency  $\omega_0$  are given by

$$k_0 = k_0(x, t) = \partial\theta/\partial x , \quad (0.18)$$

$$\omega_0 = \omega_0(x, t) = -\partial\theta/\partial t . \quad (0.19)$$

Similarly,  $\partial^2 f/\partial x^2 \ll k_0^2 f$ , etc. Also,  $k_0, \omega_0$  satisfy the relevant branch of the dispersion relation,

$$\omega_0 = \Omega(k_0) , \quad (0.20)$$

and  $k_0, \omega_0$  are themselves SV functions of  $x$  and  $t$ .  $\Omega(\cdot)$  may be called the ‘dispersion function’.

(1) Note that we need only the statements (0.18)–(0.19) to deduce that

$$\partial k_0/\partial t + \partial\omega_0/\partial x = 0 \quad (0.21)$$

<sup>3</sup>E. N. Lorenz (1967), *The Nature and Theory of the General Circulation*, Geneva, World Met. Organization.

<sup>4</sup>See e.g. Lighthill’s *Waves in Fluids*, 1978, Cambridge University Press

and hence, by (0.20), that [writing  $c_g(k_0) \equiv \Omega'(k_0)$ ]

$$\left( \frac{\partial}{\partial t} + c_g(k_0) \frac{\partial}{\partial x} \right) k_0 = 0 \quad (0.22 \text{ a})$$

$$\left( \frac{\partial}{\partial t} + c_g(k_0) \frac{\partial}{\partial x} \right) \omega_0 = 0, \quad (0.22 \text{ b})$$

which *rederives* the result already found from the stationary-phase approximation, that an observer moving with velocity  $c_g(k_0)$  will continue to observe waves of length  $2\pi/k_0$  and period  $2\pi/\omega_0$ . We may say that “values of wavenumber and frequency propagate with the group velocity”.

-N10-

(2) The above restatement of the stationary-phase results suggests a very important *generalization to inhomogeneous media, plural*, in 1, 2 or 3D. Suppose (0.20) replaced by (dropping subscript 0's)

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t). \quad (0.23)$$

[It is plausible that this gives a good approximation describing locally almost-sinusoidal waves  $f(\mathbf{x}, t) e^{i\theta(\mathbf{x}, t)}$ , provided that  $\Omega(\mathbf{k}, \mathbf{x}, t)$  is SV w.r.t.  $\mathbf{x}, t$ .] Then (0.21) and (0.22) are evidently replaced by

$$\partial \mathbf{k} / \partial t + \nabla \omega = 0 \quad [\text{also } \nabla \times \mathbf{k} = 0] \quad (0.24)$$

so by the chain rule

$$\left( \frac{\partial}{\partial t} + \mathbf{c}_g(\mathbf{k}) \cdot \nabla \right) \mathbf{k} = (-\nabla_{\mathbf{x}} \Omega)_{\mathbf{k}, t \text{ const.}} \quad (0.25 \text{ a})$$

$$\left( \frac{\partial}{\partial t} + \mathbf{c}_g(\mathbf{k}) \cdot \nabla \right) \omega = (+\partial \Omega / \partial t)_{\mathbf{k}, \mathbf{x} \text{ const.}} \quad (0.25 \text{ b})$$

-0.9-

where  $\mathbf{k}(\mathbf{x}, t) \equiv \nabla \theta$ ,  $\omega(x, t) \equiv -\partial \theta / \partial t$ ,  $\mathbf{c}_g(\mathbf{k}) = (\nabla_{\mathbf{k}} \Omega)_{\mathbf{x}, t \text{ const.}}$ .

## The ray-tracing equations

(or, “how to use equations (0.25) in practice”.)

Let  $\mathbf{x}(t)$  represent the path of an observer moving with the group velocity, and write  $\dot{x}_i(t) \equiv dx_i(t)/dt$ . Then since  $(\mathbf{c}_g)_i = \partial \Omega / \partial k_i$ ,

$$\dot{x}_i(t) = \partial \Omega(\mathbf{k}, \mathbf{x}, t) / \partial k_i \quad (0.26 \text{ a})$$

and if  $\mathbf{k}(t), \omega(t)$  represents the values of  $\mathbf{k}, \omega$  seen by this observer at time  $t$  [i.e.  $k(t)$  is shorthand for  $\mathbf{k}\{\mathbf{x}(t), t\}$ ,  $\omega(t)$  for  $\omega\{\mathbf{x}(t), t\}$ ] then (0.25) is equivalent (again by the chain rule) to

$$\dot{k}_i(t) = -\partial\Omega(\mathbf{k}, \mathbf{x}, t)/\partial x_i \quad (0.26 \text{ b})$$

$$\dot{\omega}(t) = \partial\Omega(\mathbf{k}, \mathbf{x}, t)/\partial t. \quad (0.26 \text{ c})$$

These are the *same* results (0.25) as before, rewritten to show that they may be regarded as a set of *ordinary differential equations*. This is very useful in practice since such equations may easily be solved numerically by standard computer routines. A solution  $[\mathbf{x}(t), \mathbf{k}(t), \omega(t)]$  of the set (0.26 a,b,c) of ODE's is said to trace out a *ray* (the path  $\mathbf{x}(t)$ ) as well as giving the values of  $\mathbf{k}$  and  $\omega$  along the ray. The moving point  $\mathbf{x}(t)$  may be termed a 'ray point' [not standard terminology]. The values of  $\theta$  along the ray may be found by integrating

$$\dot{\theta}(t) = -\Omega(\mathbf{k}, \mathbf{x}, t) + k_i \partial\Omega(\mathbf{k}, \mathbf{x}, t)/\partial k_i, \quad (0.27)$$

where again  $\theta(t)$  is shorthand for  $\theta\{\mathbf{x}(t), t\}$  so that

$$\dot{\theta} \equiv d\theta/dt = \frac{\partial\theta}{\partial x_i} \dot{x}_i(t) + \frac{\partial\theta(\mathbf{x}, t)}{\partial t} = k_i \dot{x}_i - \omega$$

by definition of  $\mathbf{k}$  and  $\dot{\omega}$ ; this and (0.26 a)  $\Rightarrow$  (0.27).

We have of course been assuming that (0.26) are to be integrated with *initial conditions* satisfying the relevant branch of the dispersion relation, (0.23), i.e.  $\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$ , which was the starting point of the analysis. Then (0.23) will be satisfied by the resulting solution for *all*  $t$ ; and this is useful in practice as a check on the solution. Alternatively, (0.23) can be used in place of one of (0.26), reducing the order by 1.

## Wavecrests

By definition, the surfaces (in 3D) or lines (in 2D)  $\theta(\mathbf{x}, t) = \text{constant}$ , the wavecrests, are known once we have solved (8) for a large enough number of rays. *The crests need not be orthogonal to rays* (because  $\mathbf{k} = \nabla\theta$ , not  $\parallel \mathbf{c}_g$ ) except in the case of isotropic propagation  $\Omega = \Omega(|\mathbf{k}|, \mathbf{x}, t)$  (which does of course give  $\mathbf{k} \parallel \mathbf{c}_g$ ). Sometimes  $\theta(\mathbf{x}, t)$  can be found analytically in simple cases, e.g. *ship-wave pattern*:

$$\Omega(\mathbf{k}, \mathbf{x}, t) = \Omega(\mathbf{k}) = Uk + g^{\frac{1}{2}}(k^2 + l^2)^{\frac{1}{4}} \quad \text{or} \quad Uk - g^{\frac{1}{2}}(k^2 + l^2)^{\frac{1}{4}}$$

(2D, anisotropic,  $U, g$  const.).<sup>5</sup>

<sup>5</sup>Assume rays all come from origin. RHS(0.26 b,c) zero; rays are straight lines; (0.27) becomes  $\dot{\theta} = -\Omega(\mathbf{k}) + \mathbf{k} \cdot \mathbf{c}_g (= \text{const.})$  on a ray. If ray point is at origin then  $t = t_0$ ,

## Remark

This mathematics of wave dynamics is precisely the same as the mathematics of particle dynamics, in the Hamiltonian description. In this sense Sir William Rowan Hamilton anticipated quantum mechanics by nearly a century!

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then  $\mathbf{x}(t) = (t - t_0)\mathbf{c}_g$ ;  $\theta(\mathbf{x}, t) = \theta(\mathbf{0}, t_0) + (t - t_0)\{-\Omega(\mathbf{k}) + \mathbf{k} \cdot \mathbf{c}_g\}$ . For ship waves  $\omega = \Omega(\mathbf{k}) = 0$  [selects set  $\mathcal{K}$  of possible  $\mathbf{k}$ ]; since  $t - t_0 = |\mathbf{x}|/|\mathbf{c}_g|$ , we get finally  $\theta(\mathbf{x}, t) = \theta(\mathbf{x}) = \theta(\mathbf{0}) + \mathbf{k} \cdot \hat{\mathbf{c}}_g|\mathbf{x}|$ ,  $\hat{\mathbf{c}}_g \equiv \mathbf{c}_g/|\mathbf{c}_g| = \mathbf{x}/|\mathbf{x}|$ , hence ‘reciprocal polar’ construction for wavefront  $\theta = \text{const.}$

# Part I

## Stratified, non-rotating flow under constant gravity $g$



# 1.

## Boussinesq approximation (for incompressible fluid)

In this first part of the course, we neglect rotation:  $\boldsymbol{\Omega} = 0$ .

–1.1–

### §1.1 (Oberbeck–)Boussinesq equations

The (Oberbeck–)Boussinesq equations represent the mathematically simplest model that captures the essential effects of buoyancy and stratification on fluid motion.

The equations apply in a parameter limit that is often an excellent approximation, especially for oceanic and laboratory cases. As well as incompressibility (sound speed infinite) — already assumed above — we assume further that gravity  $g$  is infinitely large and fractional density variations  $\Delta\rho$  infinitesimally small, such that the product  $g\Delta\rho$  remains finite in the limit  $g \rightarrow \infty$ ,  $\Delta\rho \rightarrow 0$ . In this limit, usually called the ‘Boussinesq limit’, we can treat the density  $\rho$  as constant where it represents inertia in Newton’s second law, even though not constant where it represents buoyancy effects.

To make this explicit, we introduce some new notation, as follows. In equation (0.2 a), set  $\boldsymbol{\Omega} = 0$ ,  $\chi = gz$  and (for the moment)  $\mathbf{F} = 0$ ;  $z$  is an upward-directed Cartesian coordinate. Substitute

$$\rho = \rho_{00} + \rho_1(\mathbf{x}, t) \quad (\rho_{00} = \text{const.}) \quad (1.1)$$

$$P = -g\rho_{00}z + p_1(\mathbf{x}, t). \quad (1.2)$$

Then (0.2 a) becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_{00} + \rho_1} \nabla p_1 - \frac{\rho_1}{\rho_{00} + \rho_1} g \hat{\mathbf{z}},$$

where  $\hat{\mathbf{z}}$  is the unit vertical vector  $\{0, 0, 1\}$ . Take the limiting case  $\frac{\rho_1}{\rho_{00}} \rightarrow 0$  with  $\frac{g\rho_1}{\rho_{00}}$  finite [i.e.

$$\left. \begin{aligned} \text{“ } \frac{\rho_1}{\rho_{00}} \ll 1 \quad \text{but} \quad \frac{g\rho_1/\rho_{00}}{\text{(typical acceleration)}} \approx 1 \text{ ”} \end{aligned} \right] \quad (1.3)$$
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_{00}} \nabla p_1 + \sigma_1 \hat{\mathbf{z}},$$

where  $\sigma_1(\mathbf{x}, t) \equiv -g \rho_1 / \rho_{00}$ , the **buoyancy acceleration** (positive for a fluid element *less* dense than its surroundings).

For stratified fluid systems it is convenient to split  $\sigma_1$  into contributions  $\bar{\sigma}(z)$  and  $\sigma(\mathbf{x}, t)$  representing background stratification and departures therefrom associated with fluid motion:

$$\begin{aligned}\sigma_1 &= \bar{\sigma}(z) + \sigma(\mathbf{x}, t) \\ p_1 &= \rho_{00} \int^z \bar{\sigma}(z') dz' + p(\mathbf{x}, t)\end{aligned}$$

Thus, if the fluid is at rest we have  $\sigma(\mathbf{x}, t) \equiv 0$  and  $p \equiv 0$  everywhere.

The background stratification  $\bar{\sigma}(z)$  is often chosen as a horizontal average of  $\sigma_1$ . The strength of the stratification is naturally measured by the quantity  $N^2(z)$  defined by  $N^2(z) \equiv \frac{d\bar{\sigma}}{dz} \left( = \frac{-g}{\rho_{00}} \frac{d\bar{\rho}}{dz} \right)$  where  $\bar{\rho}$  is defined analogously to  $\bar{\sigma}$ .

We assume  $N^2 > 0$ : the stratification is *stable*, with heavy fluid below light.

In summary, the limiting process has produced the following simplified set of equations, the (Oberbeck–)Boussinesq equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_{00}} \nabla p + \sigma \hat{\mathbf{z}}; \quad (1.4 a)$$

$$(0.2 b) \Rightarrow \frac{\partial \sigma}{\partial t} + \mathbf{u} \cdot \nabla \sigma + \hat{\mathbf{z}} \cdot \mathbf{u} N^2(z) = 0 \quad (1.4 b)$$

-N11- and (0.2 c) is

$$\nabla \cdot \mathbf{u} = 0 \quad (1.4 c)$$

If density diffuses, say with constant diffusivity  $\kappa$ , then terms  $\propto \kappa$  appear in the last two equations; however, we shall usually neglect  $\kappa$  altogether.

[\*Optional exercise: Show that if  $\kappa \neq 0$  then (1.4 b) is replaced by  $\partial \sigma_1 / t + \mathbf{u} \cdot \nabla \sigma_1 = \kappa \nabla^2 \sigma_1$  and (1.4 c) by  $\nabla \cdot \mathbf{u} = -\kappa \rho^{-1} \nabla^2 \rho$ . To see why the small extra  $\kappa$  term is needed in (1.4 c), consider the fact that the velocity field  $\mathbf{u}$  is *defined* in terms of mass conservation: it must be such that the flux of mass is  $\rho \mathbf{u}$  in the mass-conservation equation (0.1). So  $\mathbf{u}$  can't quite be zero if dense fluid diffuses into less dense. Imagine for instance what happens to the fluid's centre of mass when density diffusion causes the stable stratification to diminish over time, in a body of fluid otherwise at rest.\*]

Equations (1.4) are, as already indicated, the **(Oberbeck–)Boussinesq equations for an ideal incompressible fluid**.  $N(z)$  is called the buoyancy frequency [\*or Brunt–Väisälä frequency, or Brunt–Väisälä–Schwartzchild–Milch–Hesselberg frequency\*] and is the sole property of the resting medium  $\mathbf{u} = p = \sigma = 0$ , given the Boussinesq approximation (and ideal fluid, inviscid with  $\kappa = 0$ ).



The physical meaning of  $N$  can be seen very simply from the fact that

$$\mathbf{u} = \{0, 0, \hat{w}(x, y) e^{\pm i N t}\}, \quad p \equiv 0, \quad \sigma = \pm i N \hat{w} e^{\pm i N t}$$

solves (1.4) formally, when  $N = \text{const.}$ . “A long thin vertical column of fluid displaced vertically oscillates with frequency  $N$ ” (*not* a fluid element of arbitrary shape, as sometimes said).

Typical values of  $N$ :

**Centrally-heated room:**  $1^\circ\text{K m}^{-1}$ :  $N \simeq 0.18\text{s}^{-1}$ ;  $\frac{2\pi}{N} \simeq 34$  sec.

**Troposphere:**  $10^{-2}\text{s}^{-1}$ ;  $\frac{2\pi}{N} \simeq 10$  min; lower stratosphere  $2 \times 10^{-2}\text{s}^{-1}$ ,  $\frac{2\pi}{N} \simeq 5$  min.

**Ocean:**  $\frac{2\pi}{N}$  varies from minutes to hours; see O. M. Phillips fig. 5.4 (5.9 in 2nd edition) reproduced at bottom right of figure 1.1 on page 21.

**Boussinesq vorticity equation:**  $\nabla \times$  (1.4 a) gives, with (1.4 c),

$$\frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\zeta} = \boldsymbol{\zeta} \cdot \nabla \mathbf{u} - \hat{\mathbf{z}} \times \nabla \sigma. \quad (1.5)$$

Alternatively, note that in equation (0.4), to be consistent with the Boussinesq approximation, we must take  $\rho^{-2} \nabla P = -\rho_{00} g \hat{\mathbf{z}}$ , and that only the horizontal part of  $\nabla \rho$  then matters. (Figure on p. 4 directly relevant.)

—1.2.a—



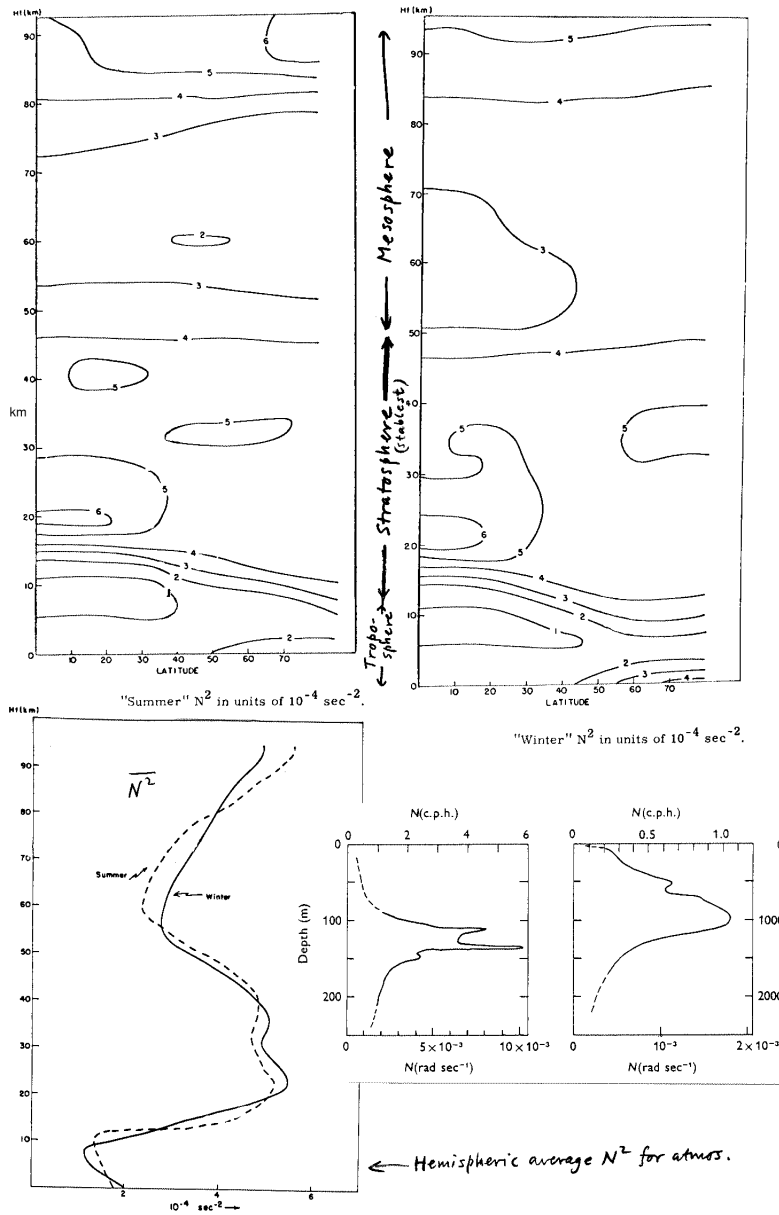
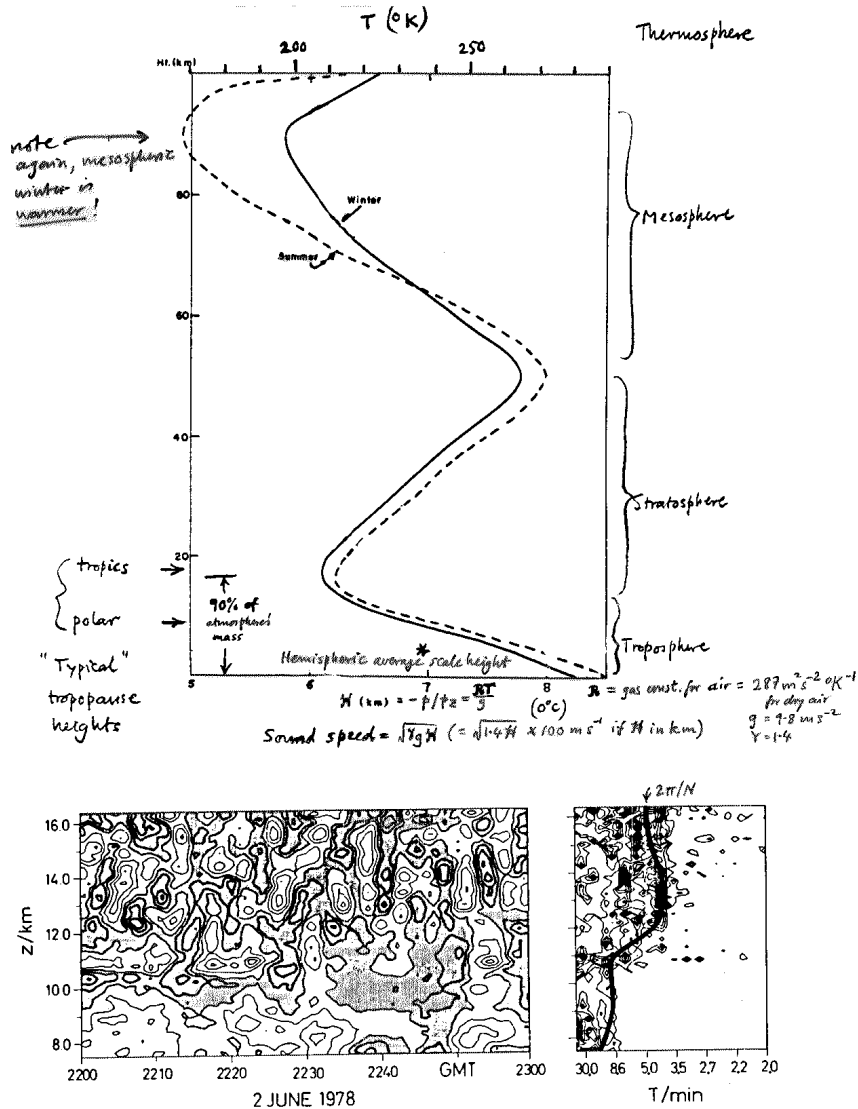


Figure 1.1: **Top panels and bottom left panel:** Buoyancy or Brunt-Väisälä frequency in the summer, winter, and summer-winter comparison. **Bottom right panels:** Some representative distributions of the buoyancy or Brunt-Väisälä frequency  $N(z)$  measured in the ocean. A multiple shallow thermocline structure, found by Montgomery and Stroup (1962, p. 21) near the equator in the Pacific Ocean is shown on the left. The thermocline on the right is deep and diffuse; it was measured by Iselin (1936, fig. 6) as part of a section between Chesapeake Bay and Bermuda. From Phillips (1966) *Dynamics of the Upper Ocean*. **Overleaf:** Typical globally-averaged temperature profiles, expressed as density scale height  $\mathcal{H} = \mathcal{R}T/g$ , where  $T$  is temperature and  $\mathcal{R}$  is the gas constant for dry air.

-I.2.b- -N12-



\* (Variation with latitude is about  $\pm \frac{1}{2}$  km, up to 70 km.)

Figure 1.2: (bottom panels): Observational evidence for internal gravity waves from a high-powered Doppler radar, from Röttger (1980) *Pure & Appl. Geophys.* **118**, 510. Note observed frequencies  $\leq N$  (buoyancy or Brunt-Väisälä frequency of the stable stratification). Author's caption reads as follows: Left (a): Contour plot of vertical velocity measured with the SOUSY-VHF-Radar after the passage of a thunderstorm. The grey-shaded and non-grey-shaded parts denote upward and downward velocity. Velocity difference between contour lines is  $0.125 \text{ m s}^{-1}$ . The velocity time series are smoothed with a Hamming filter with cut-off period 3 min. Right (b): Spectrogram: Velocity power spectra (deduced from unfiltered velocity data) plotted in form of contour lines. The peaks in the spectrogram correspond to a velocity power density of  $1.1 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}$ . The dotted curve [replaced above by a **heavy curve**] shows the height profile of the mean Brunt-Väisälä period calculated from radiosonde data taken at noon on 2 and 3 June 1978 in Berlin.



## 2.

### Small disturbances, constant $N$ :

#### §2.1 The simplest example of internal gravity waves

[Derivation 1, using vector analysis:]

Boussinesq equations, linearized about rest:

$$\mathbf{u}_t = -\frac{1}{\rho_{00}} \nabla p + \sigma \hat{\mathbf{z}} \quad (2.1 \text{ a})$$

$$\sigma_t = -N^2 \hat{\mathbf{z}} \cdot \mathbf{u} \quad (2.1 \text{ b})$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2.1 \text{ c})$$

Eq. (1.5) gives

$$\boldsymbol{\zeta}_t = -\hat{\mathbf{z}} \times \nabla \sigma. \quad (2.2)$$

$(\hat{\mathbf{z}} \times \nabla)^2$  (2.1 b) —  $(\hat{\mathbf{z}} \times \nabla) \cdot \frac{\partial}{\partial t}$  (2.2) gives, with (2.1 c):

$$\nabla^2(\hat{\mathbf{z}} \cdot \mathbf{u}_{tt}) + N^2(\hat{\mathbf{z}} \times \nabla)^2 \hat{\mathbf{z}} \cdot \mathbf{u} = 0,$$

or in cartesians, with  $\mathbf{u} = (u, v, w)$ :

$$\nabla^2 w_{tt} + N^2(w_{xx} + w_{yy}) = 0 \quad (2.3)$$

[Note, valid also when  $N = N(z)$ .] We have thus found a DE for the vertical component  $w$  of the velocity  $\mathbf{u}$ .

[Derivation 2, avoiding vector analysis:]

We may also derive (2.3) without vector analysis. Use cartesian components, with axes s.t. gravity  $\mathbf{g} = (\mathbf{0}, \mathbf{0}, -g)$ . Write the components of vorticity as  $\boldsymbol{\zeta} = (\xi, \eta, \zeta)$ ; (2.1 b) is  $\sigma_t = -N^2 w$  with  $\mathbf{u} = (u, v, w)$ , cpts. of velocity the components of (2.2) are

$$\xi_t = \sigma_y \quad (2.4 \text{ a})$$

$$\eta_t = -\sigma_x \quad (2.4 \text{ b})$$

$$\zeta_t = 0 \quad (2.4 \text{ c})$$

$$N^2 = N^2(z)$$

$$\begin{cases} \xi = -v_z + w_y \\ \eta = -w_x + u_z \end{cases}$$

so we take

$$\underbrace{\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}_{\nabla_H^2 \text{ say}} (2.1 b) + \frac{\partial}{\partial y} \frac{\partial}{\partial t} (2.4 a) - \frac{\partial}{\partial x} \frac{\partial}{\partial t} (2.4 b)$$

giving

$$\nabla_H^2 \sigma_t + \underbrace{(\xi_y - \eta_x)_{tt}}_{-v_{zy} + w_{yy} + w_{xx} - u_{zx}} = -N^2 \nabla_H^2 w + \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) \sigma_t.$$

But (2.1 c) is

$$u_x + v_y + w_z = 0 \Rightarrow -v_{zy} - u_{zx} = w_{zz}.$$

Therefore

$$(w_{xx} + w_{yy} + w_{zz})_{tt} + N^2 \nabla_H^2 w = 0,$$

which is (2.3).

[The reason this works so neatly is that  $\nabla \times \boldsymbol{\zeta} = \nabla \times (\nabla \times \mathbf{u}) = -\nabla^2 \mathbf{u}$ , in virtue of  $\nabla \cdot \mathbf{u} = 0$ ; note also that  $\hat{\mathbf{z}} \cdot \nabla \times (\nabla \times \mathbf{u}) = -\nabla^2 w$ .]

From (2.3) we see that the fundamental plane-wave solutions are

$$w = \text{Re}[\hat{w} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}], \quad \mathbf{k} = (k, l, m) \quad (2.5)$$

[ $\partial_t \rightarrow -i\omega$ ,  $\partial_x \rightarrow ik$  etc.] where  $(k^2 + l^2 + m^2)\omega^2 - N^2(k^2 + l^2) = 0$ .

(2.6) is traditionally called the “dispersion relation”. Equivalently, in vector notation,

$$\omega = \pm \frac{|\hat{\mathbf{z}} \times \mathbf{k}|}{|\mathbf{k}|} N = \frac{\pm(k^2 + l^2)^{1/2}}{(k^2 + l^2 + m^2)^{1/2}} N. \quad (2.6)$$

Group velocity  $\mathbf{c}_g$

$$\begin{aligned} \mathbf{c}_g &\equiv \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right) = \pm \frac{\{(\hat{\mathbf{z}} \cdot \mathbf{k}) \mathbf{k} - |\mathbf{k}|^2 \hat{\mathbf{z}}\} \hat{\mathbf{z}} \cdot \mathbf{k}}{|\mathbf{k}|^3 |\hat{\mathbf{z}} \times \mathbf{k}|} N \\ &= \frac{\pm \{k m, l m, -(k^2 + l^2)\}}{(k^2 + l^2 + m^2)^{3/2} (k^2 + l^2)^{1/2}} m N. \end{aligned} \quad (2.7)$$

These results can be summarized geometrically, see figure 2.1.



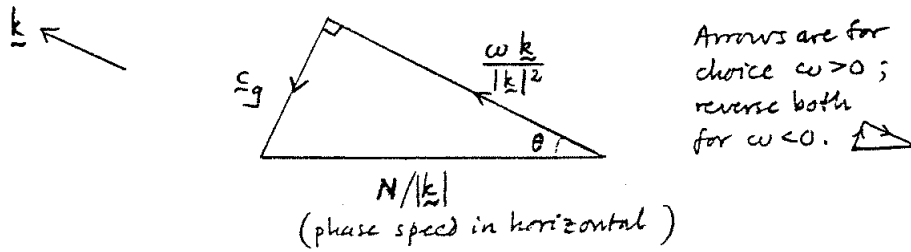


Figure 2.1:

Note  $\mathbf{c}_g \perp \mathbf{k}$  and that  $\omega \hat{\mathbf{z}} \cdot \mathbf{k}$  has opposite sign to  $\hat{\mathbf{z}} \cdot \mathbf{c}_g$  ('energy upward  $\Leftrightarrow$  phase downward' and vice versa). ( $\omega$ 's independence of  $|\mathbf{k}|$  could have been anticipated on dimensional grounds: there is no length scale in the equations, i.e. no length scale in the problem apart from the inverse wavenumber  $|\mathbf{k}|^{-1}$ ; thus frequency can depend only on the direction, not the magnitude, of  $\mathbf{k}$ .)

-N15-

**Wave-energy density and flux:**  $\rho_{00} \times [\mathbf{u} \cdot (2.1 a) + N^{-2} \sigma (2.1 b)]$  gives

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_{00} |\mathbf{u}|^2 + \frac{1}{2} \frac{\rho_{00} \sigma^2}{N^2} \right) + \nabla \cdot (p \mathbf{u}) = 0 \quad (2.8)$$

[\*again valid when  $N = N(z)$ ; but then, smallness of  $\sigma$  (compared with  $N^2 \times$  height scale on which  $N$  varies) is necessary in order for  $\frac{1}{2} \rho_{00} \sigma^2 / N^2$  to be a consistent approximation to the exact "available potential energy"; see (0.11)–(0.16)\*].

Write

$$E \equiv \left( \frac{1}{2} \rho_{00} \overline{|\mathbf{u}|^2} + \frac{1}{2} N^{-2} \rho_{00} \overline{\sigma^2} \right)_{\text{plane wave}}$$

$$\mathbf{F} \equiv (\overline{p \mathbf{u}})_{\text{plane wave}} \quad [\text{not the same } \mathbf{F} \text{ as in (0.2)!}]$$

where  $\overline{(\quad)}$  means the average over a wavelength or period [also true *without*  $\overline{(\quad)}$ , in this case].

-N16-

We find (e.g. from (2.12) below) [note  $|\hat{w}|^2 = 2\overline{w^2}$ , as  $\overline{\cos^2} = \overline{\sin^2} = \frac{1}{2}$ ],

-I.4-

$$E = \frac{1}{2} \rho_{00} \frac{|\mathbf{k}|^2}{|\hat{\mathbf{z}} \times \mathbf{k}|^2} |\hat{w}|^2 = \frac{1}{2} \rho_{00} \frac{N^2}{\omega^2} |\hat{w}|^2 \quad (2.9)$$

$$\mathbf{F} = E \mathbf{c}_g \quad (\text{see below fig. 2.2})$$

( $|\hat{w}|$  is the *maximum* vertical particle speed over a cycle) where moreover the kinetic and potential contributions to  $E$  are equal (but we shall find later on that 'equipartition of energy' in this sense does *not* usually hold, once the Coriolis effect is introduced).

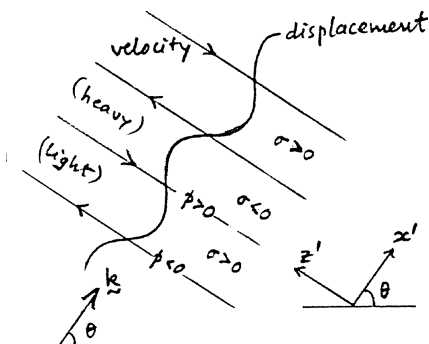
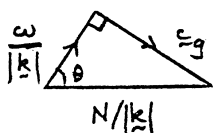


Figure 2.2:  $\omega > 0$ ; phase propagates upward.

Note downward particle velocity is correlated with high  $p$ ;  $\overline{p\mathbf{u}}$  is directed downwards (see below left). From (2.12 c) and (2.12 a):

$$|p\mathbf{u}| = \frac{1}{2} \rho_{00} \frac{|\hat{\sigma} \sin \theta|}{|\mathbf{k}|} |\hat{w}'| = \underbrace{\frac{1}{2} \rho_{00} |\hat{w}'|^2}_E \underbrace{\frac{|\omega|}{|\mathbf{k}|} \left| \frac{\sin \theta}{\cos \theta} \right|}_{c_g}; \quad (2.10)$$



$\hat{\sigma}$  is the complex amplitude of  $\sigma$  (see just below); so  $|p\mathbf{u}| \propto c_g$ , see fig 2.1.

**Typical flow field:** A typical flow field is sketched in fig. 2.2. The dynamics is succinctly described in equations (2.1) expressed in the  $(x', y', z')$  system (axes chosen with  $x'$ -axis  $\parallel \mathbf{k}$ ; thus  $\mathbf{k} = (k', 0, 0)$ ). We assume  $(\mathbf{u}, p, \sigma) = (\hat{u}', \hat{v}', \hat{w}', \hat{p}, \hat{\sigma}) e^{i(k' x' - \omega t)}$ ; note  $y' = y$ ,  $\partial/\partial y' = \partial/\partial y = 0$ ,  $\partial/\partial z' = 0$ ; thus

$$\begin{aligned} y' \text{ cpt of (2.1 a)} &\Rightarrow \hat{v}' = 0 && (\text{if } \omega \neq 0) \\ (2.1 \text{ c}) &\Rightarrow \hat{u}' = 0 \end{aligned} \quad (2.11)$$

Notice also the ‘trivial’ steady solutions  $\omega = 0$ ,  $\mathbf{u} \propto (l, -k, 0)$  (old axes, horizontal flow) — important later. The  $z'$  cpt of (2.1 a) is simply

$$-i\omega \hat{w}' = \hat{\sigma} \cos \theta \quad (2.12 \text{ a})$$

and (2.1 b) reduces to

$$-i\omega \hat{\sigma} = -N^2 \hat{w}' \cos \theta \quad (2.12 \text{ b})$$

whence the dispersion relation (2.6) is immediately recovered in the form

$$\omega^2 = N^2 \cos^2 \theta.$$

Finally, the  $x'$  cpt of (2.1 a) describes how the pressure fluctuations fit in:

$$0 = -\frac{1}{\rho_{00}} \frac{\partial p}{\partial x'} + \sigma \sin \theta \quad (2.12 c)$$

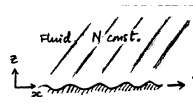
(hence  $\hat{p} = (ik')^{-1} \rho_{00} \hat{\sigma} \sin \theta$ ). The picture makes it obvious that

$$u' = v' = 0, \quad w' = e^{-i\omega t} \times \{\text{any func.}(\mathbf{k} \cdot \mathbf{x})\} \quad (2.12 d)$$

will still give a solution; also that a single such solution has  $\mathbf{u} \cdot \nabla \mathbf{u} = 0$ ,  $\mathbf{u} \cdot \nabla \sigma = 0$ , and so at *finite amplitude* is still a solution of the Boussinesq equations. Also the physical reason why  $|\mathbf{c}_g| \propto \text{wavelength}$  is seen from (2.12 c) to be that, for given  $E$ ,  $\hat{\sigma}$ ,  $\theta$ ,  $\hat{w}$ , the pressure fluctuations and therefore  $|\overline{p\mathbf{u}}| \propto \text{wavelength}$ .

—1.5—

## Example 1: Disturbance due to steadily-translating boundary



$$z = h(x, t) \equiv \epsilon \sin\{k(x - Ut)\}; \quad k > 0; \quad \epsilon \ll 1.$$

Linearized boundary condition:  $w = -U \partial h / \partial x$ , i.e.

Taylor-expand  
about  $z = 0$ 

$$w = -\epsilon U k \cos\{k(x - Ut)\}, \quad \text{at } z = 0.$$

Use this with the plane-wave solution (2.5) and equation (2.3) to determine the solution

**Case 1:**  $\omega \equiv U k > N$ :

$$w = -\epsilon \omega e^{-m_0 z} \cos(kx - \omega t); \quad m_0 = \left| k \left( 1 - \frac{N^2}{\omega^2} \right)^{1/2} \right| \quad (2.13)$$

(rejecting  $e^{+m_0 z}$ ). Quasi-irrotational flow ( $\omega \gg N$  is classical, irrotational limiting case).

**Case 2:**  $0 < \omega \equiv U k < N$ :

$$w = -\epsilon \omega \cos(kx - \omega t \pm m_0 z); \quad m_0 = \left| k \left( \frac{N^2}{\omega^2} - 1 \right)^{1/2} \right|. \quad (2.14)$$

Horizontal mean force exerted by boundary on fluid

$$= -\overline{p \frac{\partial h}{\partial x}} \quad \text{per unit area}$$

-N18-

As before,  $\overline{(\quad)}$  means the average over a wavelength or period; the  $x$ -component of (2.1 a) gives  $p = \frac{\omega}{k} u = \mp \frac{m_0 \omega}{k^2} w$  using  $\nabla \cdot \mathbf{u} = 0$ ; so horizontal force/area on fluid

$$= \mp \frac{1}{2} \rho_{00} m_0 k^{-1} \epsilon^2 \omega^2 = \mp \frac{m_0}{k} \frac{\omega^2}{N^2} E = \mp m_0 k E / (k^2 + m_0^2) . \quad (2.15)$$

From (2.7),

$$\mathbf{c}_g \cdot \hat{\mathbf{z}} = \mp \frac{k m_0 N}{(k^2 + m_0^2)^{3/2}} = \mp \frac{m_0 \omega^3}{k^2 N^2}; \quad \left[ \omega^2 = \frac{N^2 k^2}{k^2 + m_0^2} \right];$$

from (2.9),

$$E = \frac{1}{2} \rho_{00} N^2 \epsilon^2;$$

so

$$E \mathbf{c}_g \cdot \hat{\mathbf{z}} = \mp \frac{1}{2} \rho_{00} m_0 k^{-2} \epsilon^2 \omega^3 = (2.15) \times U .$$

That is, the upward energy flux is equal to the rate of working by the boundary per unit area.

Strictly speaking, the condition for validity of linearization,  $\epsilon \ll 1$ , should be replaced by  $\epsilon m \ll 1$ . (Why?)

The physically-plausible requirement that both are positive can also be deduced from the Sommerfeld radiation condition,  $\mathbf{c}_g \cdot \hat{\mathbf{z}} > 0$  in this problem. The radiation condition selects the lower sign in (2.14), corresponding to the wave crests tilting forward rather than trailing behind. The same feature can be seen in more complicated problems of stratified flow over obstacles, constructed by Fourier superposition, in which energy is radiated vertically. The radiation condition then says that *each* Fourier component has  $\mathbf{c}_g \cdot \hat{\mathbf{z}} > 0$ ; see section §2.2. Here's an example of such a problem:

-I.6-

The figure shows streamlines of steady flow over a semi-elliptical obstacle (solution obtained by Huppert & Miles (1969) *J. Fluid Mech.* **35**, 494). Because the flow is steady (and buoyancy diffusion zero, as above) the streamlines are also lines of const.  $\sigma_1 = N_z^2 + \sigma$ . Here we are in a frame of reference in which the obstacle has been brought to rest. The waves are *lee* waves (i.e., they appear in the lee of the obstacle, i.e. downstream of the obstacle) because the horizontal phase velocity ( $\omega/k$ ) exceeds the horizontal component of group velocity  $\mathbf{c}_g$ , as can be checked from the formulae above. This is actually a finite-amplitude, exact solution; see §3 below. The sixth

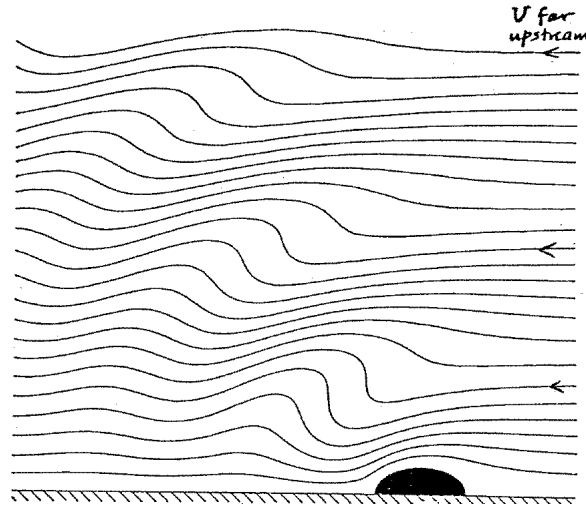


Figure 2.3: Semi-elliptical obstacle,  $\frac{\text{height}}{\text{half-width}} = 0.6$ ;  $\frac{N}{U} \times$  (height of semi-ellipse) = 1.12. [see Bernoulli's Theorem (0.9)] [streamlines closer on lee side]

streamline up is vertical at just one point. For some laboratory schlieren pictures of similar wave patterns see Stevenson 1968: Some two-dimensional internal waves in a stratified fluid, *J. Fluid Mech.* **33**, 720.)

The force can *increase*(!) as  $U$  decreases, owing to a decrease in the relevant values of  $k$ . [This kind of force is a significant contribution to the force between the atmosphere and mountainous parts of the earth, and has to be taken into account in climate modelling. One of the pioneering papers was that of Bretherton 1969: Momentum transport by gravity waves, *Quart. J. Roy. Met. Soc.* **95**, 213–243; there's now a vast literature.]

## §2.2 Justification of the radiation condition

Consider a linear wave problem in which you have a source of waves, generating a wave pattern that can be regarded as a superposition of plane waves propagating through a uniform medium. We want the wave pattern to correspond to a thought-experiment in which the wave motion is set up from an

initial state of rest. That is, we are interested in the ‘physically relevant’ or ‘causal’ solutions, the solutions that could have been set up from a state of rest. We may reasonably expect such solutions to have group velocities away from the source: the  $\mathbf{c}_g \cdot \hat{\mathbf{z}} > 0$  for each plane wave or Fourier component. This is the radiation condition. We now justify this in the context of the previous example, §2.1.

We first need a way of representing the thought-experiment mathematically. The simplest way is as follows. Let

$$h(x, t) = \text{Re}[-i \epsilon F(T) e^{i(kx - \omega t)}] \quad (2.16)$$

where  $T = \mu t$  ( $\mu \ll 1$ )

and  $F$  is a smooth, real-valued ‘fade-in’ function, such that  $F \rightarrow 0, 1$  as  $T \rightarrow -\infty, \infty$  (see sketch). The small parameter  $\mu$  expresses the idea of a wave source turned on *gradually*. A formal asymptotic solution, as  $\mu \rightarrow 0$ , of the linearized equations and boundary conditions is now easy to find correct to  $O(\mu)$ .



As usual this is shorthand for saying that the solution has been shown to satisfy the equations and boundary conditions except for a remainder  $O(\mu^2)$  as  $\mu \rightarrow 0$ . Substitute into (2.3) the following trial solution:

$$w = \text{Re}[\epsilon \omega G(Z, T) e^{i(kx - \omega t \pm m_0 z)}]; \quad (2.17)$$

where  $Z = \mu z$ .

–1.7–

Apply the chain and Leibniz rules to expressions like (2.17). Using rules like  $\nabla^2 = -k^2 - m_0^2 \pm 2im_0 \mu \frac{\partial}{\partial Z} + O(\mu^2)$  where  $\partial/\partial Z$  is defined to operate only on slowly-varying functions like  $G$ , and similarly  $\frac{\partial^2}{\partial t^2} = -\omega^2 -$

–N19–

$2i\omega \mu \frac{\partial}{\partial T} + O(\mu^2)$ , we find that (2.3) is already satisfied at zero<sup>th</sup> order in  $\mu$  by previous choice of  $m_0$ , eq. (2.14), and is satisfied at first order if

$$\frac{\partial G}{\partial T} + \mathbf{c}_g \cdot \hat{\mathbf{z}} \frac{\partial G}{\partial Z} = 0 \quad (2.18)$$

because, taking  $l = 0$  in (2.6) and (2.7), we have  $-\mathbf{c}_g \cdot \hat{\mathbf{z}} = \pm m_0 \omega / (k^2 + m_0^2)$ . Thus our solution has

$$G(Z, T) = -F\left(T - \frac{Z}{\mathbf{c}_g \cdot \hat{\mathbf{z}}}\right),$$

which can satisfy the initial condition of no disturbance *only if*  $\mathbf{c}_g \cdot \hat{\mathbf{z}} > 0$ , i.e. if we take the *lower* of the two signs, given the assumed ‘fade-in’ form of  $F(T)$ .

A particular case of special interest is  $F(T) = e^{\mu t}$  ( $\mu > 0$ ) since then (2.17) is clearly an exact solution of the linearized problem, with

$$G(Z, T) = -e^{\mu(t+\alpha z)} \quad (\alpha = \text{const.})$$

— not merely a formal asymptotic solution as  $\mu \rightarrow 0$ . But the previous algebra evidently implies that when  $\mu \rightarrow 0$ , the number  $\alpha$  has the behaviour

$$\alpha = -\frac{1}{\mathbf{c}_g \cdot \hat{\mathbf{z}}} + O(\mu),$$

which is all we need to know.

**Generalisation of (2.18):** Equation(2.18) is a special case of a more general way of looking at group velocity. For a linear partial differential equation with constant coefficients like (2.3)

$$P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = 0,$$

where  $P$  is a polynomial s.t. the dispersion relation  $P(-i\omega, ik, il, im) = 0$  gives  $\omega$  real when  $\mathbf{k}$  real (*not* like the diffusion equation!), we have that

$$P\left(-i\omega + \mu \frac{\partial}{\partial T}, ik + \mu \frac{\partial}{\partial X}, il + \mu \frac{\partial}{\partial Y}, im + \mu \frac{\partial}{\partial Z}\right) G(\mathbf{X}, T) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)},$$

where  $\mathbf{X} = (X, Y, Z) = \mu \mathbf{x}$ , is evidently zero at first order in  $\mu$  if

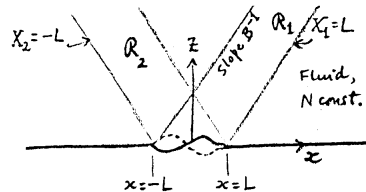
$$\frac{\partial G}{\partial T} + \mathbf{c}_g \cdot \nabla_{\mathbf{x}} G = 0$$

where

$$\mathbf{c}_g = + \frac{\{\partial P / \partial(ik), \partial P / \partial(il), \partial P / \partial(im)\}}{\partial P / \partial(-i\omega)}.$$

This coincides with the usual definition of  $\mathbf{c}_g$  when  $P(-i\omega, ik, il, im) = 0$ . —1.8—

## Example 2: 2-D disturbance due to a harmonically oscillating piston



Take the ‘boundary + piston’ in the form

$$z = h(x) \omega_0^{-1} \sin(\omega_0 t), \quad \text{with } \omega_0 \text{ const. and}$$

$$h(x) = \int_0^\infty (\hat{h}(k) e^{ikx} + \hat{h}^*(k) e^{-ikx}) dk \quad (\text{real}).$$

-N20-

For descriptive purposes take  $h = 0$  for  $|x| > L$ .

Pose solution of form  $w = \text{Re}\{\tilde{w}(x, z) e^{-i\omega_0 t}\}$ . Then from (2.3) and boundary condition,  $\tilde{w}$  satisfies, with  $B^2$  defined as  $(N^2/\omega_0^2) - 1$ ,

$$\tilde{w}_{zz} - B^2 \tilde{w}_{xx} = 0 \quad (2.19)$$

$$\tilde{w} = h(x) \quad (\text{real}) \quad \text{at } z = 0. \quad (2.20)$$

[When  $\omega_0^2 > N^2$ , (2.19) is elliptic ( $B^2 < 0$ ) and we have another example of quasi-irrotational flow; here  $\tilde{w} = O(r^{-1})$  ( $r^2 = x^2 + z^2 \rightarrow \infty$ ) and  $O(r^{-2})$  if  $\int h dx = 0$ , etc. etc.]

Now assume  $\omega_0^2 < N^2$ ; then (2.19) is hyperbolic.<sup>1</sup> (But the problem is not a Cauchy problem and one should not, e.g., jump to the conclusion that only the regions of ‘influence’  $\mathcal{R}_1$  and  $\mathcal{R}_2$  respond to the boundary motion; see below.) The following expression satisfies (2.3) or (2.19), and boundary condition:

$$w(x, z, t) = \text{Re} \int_0^\infty \{\hat{h}(k) e^{ik(x \pm Bz)} + \hat{h}^*(k) e^{-ik(x \pm Bz)}\} e^{-i\omega_0 t} dk.$$

Any choice of signs gives a solution of (2.3) and (2.20); but we now assume that the solution is such as could have been set up from rest, and this implies that *each Fourier component* must have  $\mathbf{c}_g$  directed away from the source. This is plausible from the last example together with considerations of superposition, and can be justified by solving various initial-value problems (which we shall not do here). This is the appropriate form of the ‘radiation condition’, and implies the lower sign in the first term, and the upper in the second, with the conventions

$$\begin{cases} B > 0 \\ \omega_0 > 0. \end{cases}$$

Write

$$f(X) \equiv \int_0^\infty \hat{h}(k) e^{ikX} dk \quad (\text{N.B.: complex-valued}) \quad (2.21)$$



and

$$\begin{aligned} X_1 &= x - Bz \quad (\text{So } \mathcal{R}_1 \text{ is the region } |X_1| < L \text{ in picture}) \\ X_2 &= x + Bz. \end{aligned}$$

Then the unique solution satisfying the radiation condition can be written

$$w = \operatorname{Re}\{f(X_1)e^{-i\omega_0 t} + f(X_2)e^{+i\omega_0 t}\} \quad (2.22 \text{ a})$$

using the fact that  $\operatorname{Re}(a^*) = \operatorname{Re}(a)$ , in the second term. I.e.

$$\begin{aligned} w &= \operatorname{Re} f(X_1) \cdot \cos \omega_0 t + \operatorname{Im} f(X_1) \cdot \sin \omega_0 t \\ &+ \operatorname{Re} f(X_2) \cdot \cos \omega_0 t - \operatorname{Im} f(X_2) \cdot \sin \omega_0 t. \end{aligned} \quad (2.22 \text{ b})$$

Now  $\operatorname{Re} f(X) = \frac{1}{2} h(X)$ ; thus *when the boundary is instantaneously flat* (and moving fastest) (e.g. at  $t = 0$ ), the velocity profile is precisely that of the boundary, and the fluid outside  $\mathcal{R}_1$  and  $\mathcal{R}_2$  is instantaneously at rest.<sup>2</sup> At all other times this fluid is not at rest but undergoes a ‘standing’ oscillation, described by the ‘Im $f$ ’ terms.

That  $\operatorname{Im} f(X)$  is indeed  $\neq 0$  for  $|X| > L$  can be verified as follows.  $\hat{h}(k)$  is the Fourier transform of the real function  $h(X)$  provided we take  $\hat{h}(-k) = \hat{h}^*(k)$ ; then

$$h(X) = \int_{-\infty}^{\infty} \hat{h}(k) e^{ikX} dk.$$

But  $\operatorname{Im} f(X)$  has the Fourier transform  $(2i)^{-1} \operatorname{sgn}(k) \cdot \hat{h}(k)$ , since

$$\begin{aligned} \operatorname{Im} f(X) &= \frac{1}{2i} \int_0^{\infty} (\hat{h}(k) e^{ikX} - \hat{h}^*(k) e^{-ikX}) dk \\ &= \frac{1}{2i} \int_{-\infty}^{\infty} \operatorname{sgn}(k) \cdot \hat{h}(k) e^{ikX} dk. \end{aligned}$$

It follows that  $\operatorname{Im} f(X) = \frac{1}{2i} X$  convolution of  $h(X)$  with the

Fourier transform of  $\operatorname{sgn}(k)$ , which is  $2i/X$ , with the convention that the Cauchy principal value is understood both in the relation

$$\operatorname{sgn}(k) = \frac{1}{2\pi} i \int_{-\infty}^{\infty} \frac{2i}{X} \cdot e^{-ikX} dX$$

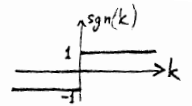
and in the convolution itself. Thus

$$\operatorname{Im} f(X) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(X') \frac{dX'}{X - X'}. \quad (2.22 \text{ c})$$

<sup>1</sup>What happens when  $\omega_0^2 = N^2$  requires a full ‘transient’ analysis, not pursued here.

<sup>2</sup>In this respect there is a qualitative difference from an analogous problem described in Greenspan’s book Greenspan, fig. 4.4, p.202. In that problem there are no times at which *all* the fluid outside  $\mathcal{R}_1$  and  $\mathcal{R}_2$  is instantaneously at rest.

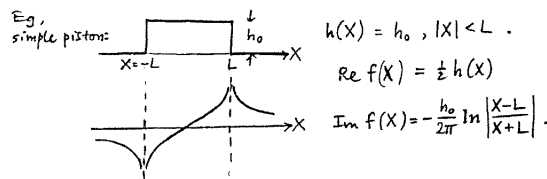
-N21-



-I.10-

This is obviously nonzero for general  $|X| > L$ . It is also evident that wherever  $h(X)$  has a finite discontinuity,  $\text{Im}f(X)$  has a (logarithmic) *infinite discontinuity*: not merely do finite discontinuities persist along characteristics as in the Cauchy problem for (2.19) — they turn into infinite ones! (but still integrable). [This is typical of such problems; see also the picture on p. 40; not unrelated to the large-time limit of the problem of q.2, example sheet 1. Note that viscosity and density diffusion have been neglected, and that an infinite time has been allowed for a steadily-oscillating state to be reached. The corresponding initial-value problem contains no such singularities at finite time, as would be anticipated from the fact that an arbitrarily small-scale component takes an arbitrarily long time to propagate from the source of the disturbance to any interior point, inasmuch as  $|\mathbf{c}_g| \propto |\mathbf{k}|^{-1}$ .]

E.g., simple piston:



The resulting motion (2.22 b) gives the qualitative appearance of a downward ‘phase’ propagation, as one might guess — but only within  $\mathcal{R}_1$  and  $\mathcal{R}_2$ . This feature appears in laboratory experiments, e.g. that of Mowbray & Rarity (1967) A theoretical and experimental investigation of the phase configuration of internal waves of small amplitude in a density stratified liquid. *J. Fluid Mech.* **28**, 1–16.

Essentially the same problem has been discussed, in great detail, by Baines (1969) PhD thesis, Cambridge, pp.45–55. (Cf. *internal tides* generated at continental shelves.)

—I.11—

### Example 3: concerning low frequency motions in 2 dimensions

When  $\omega^2 \ll N^2$  for a plane wave,  $\mathbf{k}$  is nearly vertical. That is, the wave crests and  $\mathbf{u}$  are nearly horizontal.<sup>3</sup>

<sup>3</sup>In (2.6),  $m^2 \gg k^2 + l^2$ . It should be noted that neglect of  $k^2 + l^2$  in the denominator of (2.6) corresponds to neglect of  $\partial w / \partial t$  in the vertical component of (2.1 a) — i.e. to

This is the simplest example of a very general tendency of stable stratification to ‘flatten out’ the motion, whenever the kinematics involves time scales all  $\gg N^{-1}$ . (See §3.2 below for further discussion.)

Suppose now that we do the following thought-experiment on a uniformly-stratified Boussinesq fluid initially at rest. *Gradually* apply the following horizontal body-force/unit mass

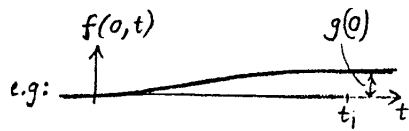
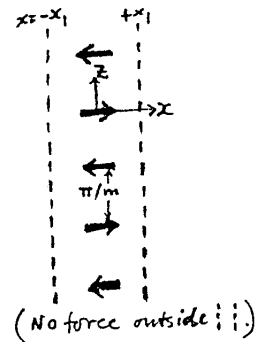
$$\mathbf{F} = (F, 0, 0); F = f(x, t) e^{imz} \quad \left[ \begin{array}{l} \text{real part understood} \\ f \text{ real} \end{array} \right.$$

where

$$f = 0 \text{ for } |x| > x_1, \text{ say, and for } t < 0;$$

$$f = g(x) \text{ for } t > t_1, \text{ say;}$$

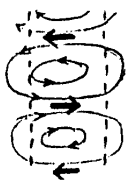
e.g.



and ‘gradually’ will mean

$$\left| \frac{\partial^2 f}{\partial x \partial t} \right| \ll |m N f_{\max}| \tag{2.23}$$

— so if  $f$  varies spatially on horizontal scale  $m^{-1}$ , its time scale  $\gg N^{-1}$ .



If the fluid were not stratified, it would respond as shown at left (continually accelerating). (We mean the simplest solution ( $\downarrow$  exponentially in  $|x|$ ) with the given forcing, i.e. we ignore possible hydrodynamical instabilities). The influence of the forcing does not extend far outside the forcing region  $\ddagger$ . With ‘strong stratification’ (‘strong’ is another way of verbalizing (2.23)) we might expect the force at each level to accelerate the fluid mainly at

that level  $\longrightarrow \ddagger \longrightarrow$  over a large horizontal extent. This does happen, in a way that will become clear from the following solution of the *linearized problem* for sufficiently small  $\mathbf{F}$ :

hydrostatic balance for the *disturbance*, eq. (2.25) below.

Linearized momentum equations: ( $v, \partial/\partial y = 0$  :)

$$\begin{aligned} u_t &= -\frac{1}{\rho_{00}} p_x + f(x, t) e^{imz} \\ w_t &= -\frac{1}{\rho_{00}} p_z + \sigma ; \quad [\text{we are going to neglect the LHS here}] \end{aligned}$$

these imply that

$$\eta_t = -\sigma_x + i m f e^{imz} \quad (2.24 \text{ i})$$

where  $\eta = u_z - w_x$  vorticity [ $y$ -component of (1.5)]. The linearized equation for the buoyancy is

$$\sigma_t = -N^2 w \quad (2.24 \text{ ii})$$

Continuity:

$$\begin{aligned} u_x + w_z = 0 &\Rightarrow \exists \psi(x, z) \quad \text{s.t.} \quad u = \psi_z, \quad w = -\psi_x \\ \eta = u_z - w_x &= \nabla^2 \psi = \psi_{xx} + \psi_{zz} \end{aligned}$$

—1.12—

The mathematics is enormously simplified if we now *assume* (subject to checking for self-consistency afterwards) that the disturbance is hydrostatic, to a first approximation, i.e. that

$$0 = -\frac{1}{\rho_{00}} p_z + \sigma \quad [w_t \text{ now neglected}] . \quad (2.25)$$

This corresponds to neglect of the first term in  $\eta_t = \partial_t(\psi_{xx} + \psi_{zz})$ , the rate of change of  $y$ -vorticity — which in turn is consistent with the intuition that hydrostatic balance will hold when horizontal length scales  $\gg$  vertical length scales. [But see footnote 4 below.]

Under the assumptions  $\eta_t = \psi_{zzt}$  and  $\psi \propto e^{imz}$ , quations (2.24 i) and (2.24 ii) become

$$\begin{aligned} -m^2 \psi_t + \sigma_x &= i m f(x, t) e^{imz} \\ \sigma_t - N^2 \psi_x &= 0 ; \end{aligned} \quad (2.26 \text{ a})$$

$\sigma$  can be eliminated to give

$$\psi_{tt} - c^2 \psi_{xx} = \frac{-i}{m} f_t e^{imz}; \quad c = N/|m|. \quad (2.26 \text{ b})$$

Equation is of classical form for forced nondispersive waves (e.g. stretched string);  $c$  = (horizontal) phase and group speed. (Note that (2.6) and (2.7) give  $\omega/k = \partial\omega/\partial k = \pm N/m = \pm c$ , when  $l = 0$ ,  $k \ll m$ .)

$c$  will be denoted  $c_h$  in section §3.1

For arbitrary  $f(x, t)$  it can be verified that a solution is

$$\left. \begin{aligned} \psi &= \frac{-i e^{imz}}{2m} \int_{-\infty}^t \left[ f(x - c(t' - t), t') + f(x + c(t' - t), t') \right] dt' \\ \sigma &= \frac{-icm e^{imz}}{2} \int_{-\infty}^t \left[ f(x - c(t' - t), t') - f(x + c(t' - t), t') \right] dt' \end{aligned} \right\} \quad (2.27)$$

[It is the only *causal* solution — if  $f(x, t)$  is zero for  $t <$  some given time,  $t = 0$ , say, then (2.27) is the only solution satisfying  $\psi \equiv \sigma \equiv 0$  for  $t < 0$  and for  $|x|$  large enough at each  $t > 0$ .]

Note  $|cm| = N$

We can now verify that (2.27)  $\Rightarrow w_t \ll \sigma$ , i.e. that our use of the hydrostatic approximation was self-consistent — provided that (2.23) holds.<sup>4</sup> Write  $\tau = t' - t$  in (2.27):

$$\psi = \frac{-i e^{imz}}{2m} \int_{-\infty}^0 \left[ f(x - c\tau, t + \tau) + f(x + c\tau, t + \tau) \right] d\tau$$

—I.13—

$$\therefore w_t = -\psi_{xt} = \frac{i e^{imz}}{2m} \int_{-\infty}^0 \left[ \frac{\partial^2 f(x - c\tau, t + \tau)}{\partial x \partial t} + \frac{\partial^2 f(x + c\tau, t + \tau)}{\partial x \partial t} \right] d\tau$$

$$\begin{aligned} \therefore |w_t| &< \frac{1}{2|m|} \int_{-\infty}^0 \left| \frac{\partial^2 f(x - c\tau, t + \tau)}{\partial x \partial t} \right| d\tau + \int_{-\infty}^0 \left| \frac{\partial^2 f(x + c\tau, t + \tau)}{\partial x \partial t} \right| d\tau \\ &\ll \frac{1}{2|m|} |2m N f_{\max}| \frac{2x_1}{c} \quad \text{from (2.23); range of integration } 2x_1/c, \end{aligned}$$

which from (2.27) is a typical magnitude for  $\sigma$ . So it was, indeed, consistent to neglect vertical acceleration  $w_t$ .

For the particular form of  $f$  assumed, (2.27)  $\Rightarrow$

$$\left. \begin{aligned} \psi &= \sigma = 0 \quad \text{for } |x| > ct + x_1, \quad \text{and} \\ \psi &= \frac{-i e^{imz}}{2N} G, \quad \sigma = im \left\{ -\frac{1}{2} G + \int_{-\infty}^x g(x') dx' \right\} e^{imz} \\ &\quad \text{for } |x| < c(t - t_1) - x_1 \end{aligned} \right\} \quad (2.28)$$

(so  $u = (2N)^{-1} m e^{imz} G$ ) where  $G \equiv \int_{-\infty}^{\infty} g(x) dx$ .

So the motion is *steady* and (therefore) horizontal (by (2.24 ii)) throughout a region expanding with constant speed  $c$  to either side, including the

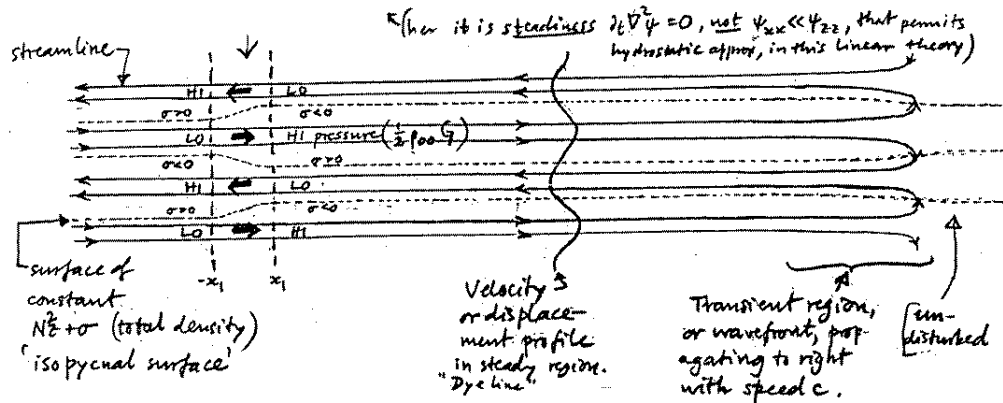
<sup>4</sup> It is interesting that the horizontal scale of variation of  $f$  does *not* have to be  $\gg m^{-1}$ . This is because  $\partial/\partial t \ll N$  in the forcing region.

forcing region  $|x| < x_1$ . Within the latter (i.e. inside : :), the applied force is balanced by a pressure gradient associated with a tilting of the constant-density surfaces:

$$\left\{ \begin{array}{l} 0 = -\frac{1}{\rho_{00}} p_x + g(x) e^{imz} \\ 0 = -\frac{1}{\rho_{00}} p_z + \sigma \end{array} \right\} \quad \text{or} \quad \sigma_x = im g(x) e^{imz} \quad (2.29)$$

-N22-

(here it is steadiness ... see top of image)



The horizontal pressure gradient across the wavefront is precisely that required to accelerate a fluid particle from rest up to its steady value

$$u = \psi_z = \frac{e^{imz}}{2c} G$$

during the passage of the wavefront. Continuity requires  $w$  such that the lifting up or down of the isopycnal surfaces produces  $\sigma = \frac{1}{2} im G e^{imz}$ . Hydrostatic compatibility of this with the pressure then determines the value of  $c$ .

-I.14-

This continually-forced, ever-lengthening disturbance may be called a 'columnar disturbance'. [Note same solution applies when horizontal boundaries  $\frac{z}{H}$  present, if choose  $m = \text{integer} \times \pi/H$ . Columnar disturbances in laboratory wind or water tunnels for stratified flow can evidently penetrate far upstream when a criterion like  $U/NH < \pi^{-1}$  is satisfied (which is the parameter regime of greatest interest for all stratification effects); this 'upstream influence' prevents one from being able to prescribe the upstream

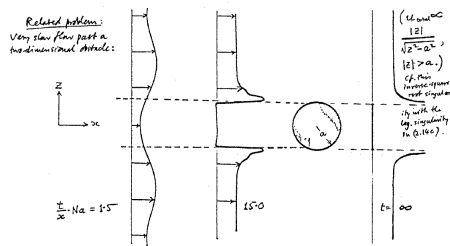
velocity and density profiles independently of what is going on in the ‘test section’ of the tunnel — e.g. Odell & Kovasznay (1971) A new type of water channel with density stratification. *J. Fluid Mech.* **50**, 535–543 (fig.5).]<sup>5</sup>

-N23-

The response to a horizontally *and* vertically localized weak horizontal force can be found by Fourier superposition of the above solution for all  $m$ . Corresponding velocity profiles are typically very wiggly; at a given place the wiggles become finer as time passes and higher wave numbers arrive. (A simple example is that of Sheet 1, last question.) (Cf. the ‘jets’ seen in Long’s film on stratified flow, ca. 16–17’ in; <http://web.mit.edu/fluids/www/Shapiro/ncfmf.html> or websearch “National Committee for Fluid Mechanics”.)



**Related problem:** very slow flow past a two-dimensional obstacle. Velocity profiles for impulsively-started flow  $U$  of ideal Boussinesq stratified fluid past cylinder, on linearized theory for ‘small  $U$ ’ — actually  $(U/Na) \ll (Nt)^3$ . (After Bretherton (1967) The time-dependent motion due to a cylinder moving in an unbounded rotating stratified fluid. *J. Fluid Mech.* **28**, 545–570; see also chapter 6 on page 113).



-N24-

The peculiar peaks in the profiles for  $tNa/x \geq 15$  are ‘due to’ the fluid trying to get over or under the cylinder with minimal vertical excursion; the resulting jet-like profiles, again because of the stratification, then tend to be found at greater and greater horizontal distances to either side of the cylinder.

<sup>5</sup>Reproduced on p. 52 below.





# 3.

## Finite-amplitude motions

—I.15—

[We ignore phenomena, such as ‘flow separation’ and turbulence, which occur in homogeneous (viscous) fluids and are merely modified by stratification. Another excuse is that the problem of describing these phenomena theoretically is largely unsolved, especially in the (albeit important) case of turbulent flow — in which (by definition<sup>1</sup>) strong nonlinearity, 3-dimensionality, unsteadiness, viscosity, and diffusion of  $\sigma$  are all essential ingredients.]

### §3.1 Some exact solutions of the Boussinesq, constant- $N$ equations (all 2-dimensional)

We note these briefly and then move on; simple exact, finite-amplitude solutions often miss the phenomena of greatest physical interest (cf. §3.3 ff.). The known finite-amplitude solutions can be classified as follows:

- (i) A single plane wave, or any superposition of plane waves with parallel  $\mathbf{k}$ 's, as in (2.12d) above. Obviously a finite-amplitude solution because, as already mentioned on page 29,  $\mathbf{u}$  is perpendicular to  $\nabla\sigma, \nabla u, \nabla v, \nabla w$ ; so  $\mathbf{u} \cdot \nabla\sigma = 0, \mathbf{u} \cdot \nabla\mathbf{u} = 0$ .
- (ii) Less obviously, any superposition with all  $\mathbf{k}$ 's in the same vertical plane (the  $xz$  plane, say) and such that all the horizontal phase ‘velocities’  $\omega/k$  have the same value  $c_h$ . This is a particular case of:
- (iii) More generally, any real solution of the linearized equations such that  $\mathbf{u}$  and  $\sigma$  are of form  $\text{func}(x - c_h t, z)$ . This is an exact solution of the full equations *provided also* that there is some  $x_0(z, t)$  such that

$$\text{and } \left. \begin{array}{l} (\nabla^2\psi - c_h^{-1}\sigma)|_{x=x_0(z,t)} = 0 \\ (\sigma + N^2c_h^{-1}\psi)|_{x=x_0(z,t)} = 0 \end{array} \right\} \forall z, t, \quad (3.1)$$

---

<sup>1</sup>By definition of ‘turbulent’, that is, in the sense of classical 3-dimensional fluid dynamics — not in the dynamical-systems sense, meaning ‘anything chaotic’.

where  $\psi$  is a streamfunction for the two-dimensional motion, defined by  $u = \psi_z$ ,  $w = -\psi_x$ . It is *not* true that  $\mathbf{u} \cdot \nabla \mathbf{u} = 0$  for such solutions; but the full  $y$ -vorticity and buoyancy equations are, from (1.5), (1.4 b) respectively,

$$\left. \begin{aligned} \partial_t \nabla^2 \psi + \sigma_x &= \psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x \\ \partial_t \sigma - N^2 \psi_x &= \psi_x \sigma_z - \psi_z \sigma_x, \end{aligned} \right\} \quad (3.2)$$

-1.16-

and the nonlinear terms on the right are zero! For the left-hand sides are zero (linearized equations satisfied) and  $\partial_t = -c_h \partial_x$ , whence the left-hand sides of (3.1) are zero for all  $x$  if zero for  $x = \text{some } x_0$ . Since  $N$  and  $c_h$  are constant, we therefore have  $\nabla^2 \psi \propto \psi \propto \sigma$ , so that the Jacobians that make up the right-hand sides of (3.2) are zero. It follows that  $\mathbf{u} \cdot \nabla \mathbf{u}$ , though nonzero, is *irrotational* and balanced by  $-\rho_{00}^{-1} \nabla p$ .

It is easily verified that the plane-wave superposition (ii) satisfies (3.1). Another case, which has received much attention in the literature, is that of *lee waves* generated by an obstacle, e.g. as in figure 2.3 on page 31. The usual linearized solutions for constant-speed, constant- $N$  basic flow, valid for an infinitesimally slender obstacle (e.g. question 5 on example sheet 1) have the property that  $\psi, \sigma$  and derivatives  $\rightarrow 0$  far upstream. Hence these solutions satisfy (3.1), with  $|x_0| = \infty$ , and hence they are also solutions of the full nonlinear *equations* at any amplitude.<sup>2</sup> (But only the linearized *boundary condition* at the obstacle is satisfied; so such a finite-amplitude solution will satisfy an exact boundary condition only for an obstacle of *different shape*. Solving the finite-amplitude boundary-value problem for a *prescribed* obstacle, as in the case of figure 2.3 on page 31, requires additional mathematical ingenuity.)

[\*Finite-amplitude solutions of this type have been found also for a number of special cases of non-Boussinesq basic flow,<sup>3</sup> the simplest of which is  $\rho = \rho(z)$ ,  $U = U(z)$ ,  $\rho U^2 = \text{const.}$ , and  $d\rho/dz = \text{const.}$  (*not*  $N^2 = \text{const.}$ ) (but not necessarily small compared with  $\rho \div$  height scales of interest).\*]

[\*Whether these particular solutions represent an asymptotically steady state, reached a long time after introduction of a finite obstacle into an initially-uniform, inviscid flow, is a separate question. In some cases, columnar disturbances (p. 39) arise and penetrate far upstream, altering the velocity and density profiles (by amounts of the order of the square of the obstacle amplitude  $a$  and violating (3.1) even for very shallow obstacles.<sup>4 5</sup> \*]

<sup>2</sup>Long (1953) *Tellus* **5**, 42.

<sup>3</sup>Yih (1965) *Dyn. Inhomog. Fluids*, Chap. 3, §3.

<sup>4</sup>McIntyre, M. E. (1972) *J. Fluid Mech.* **52**, 209; see also **60**, 808, fig. 2, and **106**, 335, fig. 1.

<sup>5</sup>For fast enough switch-on, and  $NH/\pi U$  in range  $(n, n + \epsilon)$  where  $n = \text{integer}$  and  $0 < \epsilon < 1$ ,  $\epsilon$  depending on height of obstacle *and* rapidity of switch-on, one gets columnar

Furthermore, most finite-amplitude solutions, of either type (i) or (ii) and (iii), are likely to be *unstable* on time scales generally of the order  $N^{-1} \times (\text{amplitude})^{-1}$ ; see §3.5 below.

—1.17—

If the amplitude is *sufficiently* large, fluid can become *unstably stratified* locally (i.e. density increases upwards) — and therefore probably unstable on time scales generally  $\approx N^{-1}$ . (Sheet 1 question 2.)

*Exercise:* Show that for *plane* internal gravity waves the local unstable stratification begins to occur when the maximum horizontal fluid-velocity fluctuation  $|\hat{u}|$  just exceeds the horizontal phase speed  $|\omega/k|$ . (E.g. use (2.12b) with  $|\hat{u}| = |\hat{w}' \sin \theta|$ .) In figure 2.3 on page 31, conditions are just critical in this sense: if the obstacle amplitude were to be increased any more, the fluid would be statically unstable just above and to left of the obstacle. (Note again that the flow is steady in picture's frame of reference, so streamlines are isopycnal (constant-density) lines, by (0.2b)). Of course  $\omega/k = \hat{c}$  say, is the phase speed *relative* to the (now-moving) background flow.

Laboratory experiments usually show patches of turbulence approximately in the same places; the associated nonlinear effects (e.g. Reynolds stress) can (e.g.) be equivalent to a force distribution  $\rightleftharpoons$  and thus could generate strong columnar disturbances, again violating (3.1) upstream.<sup>6</sup> (Again cf. the 'jets' observed by Long and shown in his film, URL on p. 41 — these are columnar disturbances of high vertical wave numbers, probably modified by viscosity and perhaps also by density diffusion.)

Yet other kinds of instability can be increasingly important (as discussed in §3.5 below) as wave amplitude, defined as  $|\hat{u}k/\omega|$ , approaches order-unity values. But what is important in practice is that wave amplitudes can seldom greatly exceed unity without dissipation becoming very strong. In practice, amplitudes<sup>7</sup> tend to 'saturate' at values around unity. Ocean waves breaking on a beach are merely the most visible example. Much the same thing happens when internal gravity waves propagate up into the mesosphere — amplitude unity is typically reached at altitudes  $\approx 50$  km in winter, 80 km in summer. This has important consequences for strengths of prevailing eastward or westward winds in the upper mesosphere and as the solution to the sometime enigma of the cold summer mesopause — for reasons to emerge in §3.3.

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disturbances  $0(a)$ , not  $0(a^2)$ , generated by intrinsically transient nonlinear effects near the obstacle. Experiments by Baines (1929) *Tellus* **31**, 383; also his book, Baines, P. G., 1995: *Topographic effects in stratified flows*, Cambridge University Press, xvi+482pp; paperback with corrections, 1998.

<sup>6</sup>For some computer simulations bearing on these points see Pierrehumbert & Wyman (1985) *J. Atmos. Sci.* **42**, 986–987.

<sup>7</sup>of most kinds of dispersive waves

### §3.2 ‘Slow’ motions (low Froude number)

Without solving the equations we can obtain useful information about the probable general character of stratified flows in the limiting case

$$F_U \equiv \frac{U}{NH} \ll 1, \quad (3.3 \text{ a})$$

$$F_T \equiv \frac{L}{NH} \cdot \frac{1}{T} \ll 1, \quad (3.3 \text{ b})$$

$$\frac{H}{L} \lesssim 1, \quad (3.3 \text{ c})$$

assuming in the usual way that the motion is sufficiently simple, in its functional dependence on space and time, that there exists single ‘scales’  $U$ ,  $L$ ,  $H$ ,  $T$  for the magnitude of  $\mathbf{u}$  and the lengths, heights and times over which  $\mathbf{u}$  and  $\sigma$ , in general, vary significantly. That is, we suppose for example that  $(L/U)\partial u/\partial x$  is finite in the limit  $F_U, F_T \rightarrow 0$ , and generally bounded away from 0 in that limit: we say  $\partial u/\partial x$  is generally or typically “of order  $U/L$ ”, and write this as

$$\frac{\partial u}{\partial x} \approx \frac{U}{L}, \quad \text{or} \quad \frac{L}{U} \frac{\partial u}{\partial x} \approx 1. \quad (3.4)$$

(We use  $\approx$  rather than  $\sim$  so that the latter remains available for its usual, and stronger, mathematical meaning of asymptotic *equality*.) [Similarly, the notation

$$a \lesssim b$$

is defined to mean that  $a/b$  is bounded in the limit (including the possibility that  $a/b \rightarrow 0$  everywhere, for all motions in the general class of interest. Equivalently to  $a \lesssim b$  we can also write ‘ $a = O(b)$ ’ (Lighthill, 1958, *Fourier Transforms and Generalized Functions*, CUP, p.2), pronounced “a equals Oh b”, or “a is Oh b”. To a mathematician,  $a = O(b)$  does not mean ‘a is of order b’ in the sense of (3.4); it corresponds, rather, to  $a \lesssim b$ . Finally,  $a \ll b$  and  $a = o(b)$  both mean that  $a/b$  *does* tend to 0.]

Back to our problem of slow motion. Take the equations to be as follows, assuming no external forces but not, now, linearizing. These are the full nonlinear (Boussinesq) equations, starting with the *vorticity equation*, as is

often convenient:

$$\boldsymbol{\zeta}_t + \mathbf{u} \cdot \nabla \boldsymbol{\zeta} = \boldsymbol{\zeta} \cdot \nabla \mathbf{u} - \hat{\mathbf{z}} \times \nabla \sigma \quad (\text{Boussinesq vorticity}) \quad (3.5 \text{ i})$$

$$\nabla \cdot \mathbf{u} \equiv u_x + v_y + w_z = 0 \quad (3.5 \text{ ii})$$

$$\sigma_t + \mathbf{u} \cdot \nabla \sigma = -N^2 w. \quad (3.5 \text{ iii})$$

Postulate scales

$$L \text{ for } x, y, \text{ and } H \text{ for } z \quad (3.5 \text{ iv})$$

$$T \text{ for } t \quad (3.5 \text{ v})$$

$$U \text{ for } (u^2 + v^2)^{\frac{1}{2}} \quad (\text{horizontal cpt. of velocity}) \quad (3.5 \text{ vi})$$

$$\Sigma \text{ for (horizontal and temporal variation of) } \sigma \quad (3.5 \text{ vii})$$

$$(\text{For } \partial \sigma_1 / \partial z \text{ we shall assume } N^2 + \sigma_z \approx N^2) \quad (3.5 \text{ viii})$$

Then (3.5 ii) and (3.5 iv)  $\Rightarrow w \lesssim UH/L$ . (' $\lesssim$ ' because we do not wish to exclude the possibility that  $w_z$  be generally of smaller order than  $u_x$  and  $v_y$  — a possibility which, moreover, turns out to be true!). Thus |horizontal cpt. of  $\boldsymbol{\zeta}$ |  $\lesssim (u_z^2 + v_z^2)^{\frac{1}{2}} \approx U/H$ . So using (3.5 i) and (3.5 vii),

—1.19—

$$\frac{\Sigma}{L} \lesssim \max\left(\frac{1}{T}, \frac{U}{L}\right) \frac{U}{H}. \quad (3.6)$$

—N25—

Equation (3.5 iii) (using (3.5 viii)) now implies a severe restriction on  $|w|$ :

$$w \lesssim \max\left(\frac{1}{T}, \frac{U}{L}\right) \frac{\Sigma}{N^2} \lesssim \max\left(\frac{1}{T}, \frac{U}{L}\right)^2 \frac{1}{N^2} \cdot \frac{UL}{H}$$

i.e.

$$\boxed{w \lesssim \epsilon^2 \frac{UH}{L}}, \quad (3.7)$$

where  $\epsilon \equiv \max(F_U, F_T) \ll 1$ .

—N26—

This restriction expresses in a very general way the 'flattening-out' effect of stratification — even for general  $N(z)$ , and for fully nonlinear, time-dependent motions — subject only to assumed constraint of low generalised Froude number. Fluid particles move nearly horizontally, on trajectories whose slopes are  $O(\epsilon^2)H/L$  or less. [The steady part of the columnar-type motion described by the linearized theories of pp. 36–40 has this property,

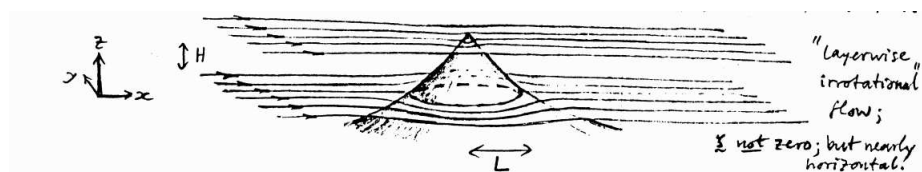


Figure 3.1: ‘layerwise 2-dimensional and irrotational’ flow around a Fujiyama-shaped obstacle;  $\zeta$  not zero; but nearly horizontal.

trivially, because  $w = 0$ . (But note that the transient wavefronts have  $F_T \lesssim 1$  if *their* horizontal scale is  $L$  and the vertical scale is  $H = m^{-1}$ .)]

An immediate consequence of (3.7), (3.5 iv), and (3.5 vi) is that  $w_z \ll u_x, v_y$  and  $w \partial_z \ll u \partial_x, v \partial_y$ . Therefore the limiting forms of the continuity and horizontal momentum equations are (with typical relative error  $O(\epsilon^2)$ ):

-N27-

$$\nabla_h \cdot \mathbf{u}_h \equiv u_x + v_y = 0 \quad (3.8 a)$$

$$\partial_t \mathbf{u}_h + \mathbf{u}_h \cdot \nabla_h \mathbf{u}_h = -\rho_{00}^{-1} \nabla_h p \quad (3.8 b)$$

where  $\mathbf{u}_h$  and  $\nabla_h$  are the two-dimensional vectors  $(u, v)$ ,  $(\partial_x, \partial_y)$ . These are three equations for three dependent variables  $u, v, p$  — a *closed system*. This has the striking implication that *the motion is determined at each level  $z = \text{const}$ , independently of the other levels*, to leading order. It is “layerwise two-dimensional” — the simplest example of a very typical situation in atmosphere–ocean dynamics.

—1.20—

Moreover equations (3.8) are identical to the equations of *classical inviscid hydrodynamics in two dimensions*, and so the general character of the solutions is a corollary of the classical theory. E.g. *if initially  $v_x - u_y$ , the vertical component of  $\zeta$  is zero it remains zero* (‘persistence of irrotationality’) (unless we generalize by adding a rotational force to the r.h.s. of (3.8 b)). E.g. slow flow past a Fujiyama-shaped obstacle, see figure 3.1.

[\*And for a *slightly-viscous* fluid we might expect flow separation and unsteadiness, but ‘layerwise two-dimensional’ flow nevertheless. Note that a blunt-topped obstacle (more realistic Mt. Fuji shape) would tend to give rise to a flow with infinite<sup>8</sup> shear  $\partial u/\partial z, \partial v/\partial z$  at the corresponding level.



-N28-

This violates the assumptions on which the initial approximations were based;<sup>9</sup> nevertheless for real fluids it correctly suggests the

<sup>8</sup>i.e. “ $H = 0$ ”

<sup>9</sup>Further discussed by P. G. Drazin 1961, *Tellus* **13**, 239 — also work by e.g. Lin & Pao (see Lilly 1983 *J. Atmos. Sci.* **40**, 751).

probable existence of an ‘internal boundary layer’ or ‘detached shear layer’, centred on that level — viz., a thin transition region in which viscosity and density diffusivity are important no matter how small. The possibility of such regions *well away from boundaries* is characteristic of stratified flow, and also of rotating homogeneous flow (as will be seen later).\*]

The vorticity equation already having been taken into account, it follows that the vertical momentum equation must be able to be satisfied with whatever  $p$  field has already been determined by (3.8) (up to an additive  $f(z)$ ). The way in this is achieved is via *hydrostatic balance*;  $\sigma$  can likewise contain an arbitrary function of  $z$ :

$$0 = -\frac{1}{\rho_{00}} p_z + \sigma + O(\delta) \quad (3.9)$$

whose error  $\delta$  is bounded by the scale for the vertical acceleration,

$$\delta = \epsilon^2 \max\left(\frac{1}{T}, \frac{U}{L}\right) U \frac{H}{L} \quad \left(\gtrsim \frac{Dw}{Dt}\right),$$

which generally is smaller than  $\sigma$  by a factor  $\epsilon^2 H^2/L^2$ ; recall (3.6). In other words, assuming  $\epsilon^2 H^2/L^2 \ll 1$ , the stratification plays the essential role of *hydrostatically supporting whatever pressure field is required by (3.8)*;  $\epsilon^2 \frac{H^2}{L^2} \ll 1$  means that the stratification can do this via small vertical material displacements ( $\approx \Sigma/N^2 \lesssim \epsilon^2 H$ , from (3.3a,b) and (3.6), or directly from  $L \times$  trajectory slope), consistent with the approximate horizontal non-divergence, (3.8 a). In general we do, of course, have that (3.6) is in fact true with  $\approx$  instead of  $\lesssim$ . We certainly can’t have  $\ll$  in general; the  $\sigma$  term in (3.5 i) is essential to avoid having a near-balance  $D\zeta/Dt \sim \zeta \cdot \nabla \mathbf{u}$  in (3.5 i), implying classical (Helmholtz) vorticity dynamics in the limit — obviously ruled out, except for trivial special cases, by the restriction (3.7) on the magnitude of  $w$ .

The kind of procedure leading to (3.7) and (3.8) is often called **scale analysis**.<sup>10</sup> If the resulting approximate equations have a solution consistent with the postulated scales, this solution evidently comprises the leading order terms in a **formal asymptotic expansion** in powers of  $\epsilon$ . The higher terms could also be obtained by iteration, first considering corrections of relative order  $\epsilon$ , and so on. The same expansion could be constructed, alternatively, by (a) nondimensionalizing the equations using the scales, (b) posing expansions of the dependent variables in powers of  $\epsilon$ , and (c) equating like powers of  $\epsilon$ . (Try it!) (Actually, the first non-zero correction will be

<sup>10</sup>Other examples of its use will appear later in the course. It is an indispensable research tool, especially when first exploring an unfamiliar research problem.

$O(\epsilon^2)$  in the present case.) [It is sometimes said that the latter expansion procedure is ‘more rigorous’. This is nonsense, because the two procedures are precisely equivalent.<sup>11</sup> But note that ‘scale analysis’ has the practical advantage, especially when used as an exploratory tool in an unfamiliar problem, of greater flexibility. (There is then no need to decide on all the scales for nondimensionalization before one has any idea as to how they might be inter-related.) It usually becomes evident in the course of the argument that some formally-possible scale relationships are physically absurd, enabling one to cut down the number of possibilities considered.]

In practice one wants to know how to interpret the scale relationships numerically — e.g. “*how* small must  $\epsilon$  be for a given solution of (3.8) to be a qualitatively good approximation?” There is no simple answer to this, but obviously a rough rule is to say “numerically small compared with 1”, *provided* the scales are chosen so that  $\partial u/\partial x$  or  $\partial v/\partial x$  is typically numerically close to  $U/L$ , and so on. E.g. if  $u$  is sinusoidal in  $x$  with wavelength  $\lambda$ , then  $L = \lambda/2\pi$  would be a better choice than  $L = \lambda$ . [In the (viscous conducting fluid layer heated from below) the scale relationships for convective instability to be just able to win against diffusion are  $\sigma \approx \nu \nabla^2 w$ ,  $N^2 w \approx \kappa \nabla^2 \sigma$  ( $\kappa =$  density diffusivity,  $N^2 < 0$ ), implying that the critical Rayleigh number

$$R \equiv |N^2| H^4 / \nu \kappa \approx 1. \quad \nu = \text{kinematic viscosity}$$

Because the height scale  $H$  appears to the fourth power,  $R_{crit}$  will be *numerically* nowhere near 1 unless  $H$  is chosen rather carefully to be close to the true scale of variation of  $w$  — actually about  $1/6 \times$  depth of layer if the boundaries are no-slip, in which case the typical variation of  $w$  with  $z$  is qualitatively like one wavelength of a sine wave (see fig. 3.2). If  $H$  is taken as the whole depth,  $R_{crit}$  is numerically about  $6^4 \approx 10^3$ !]

[Remark: The ‘persistence of layerwise irrotationality’ is an *exact* consequence of equations (3.5 i)–(3.5 iii) if we redefine ‘layers’ to mean isopycnal surfaces (whose slopes are  $\sigma_x/N^2 = O(\epsilon^2)$  for small  $\epsilon$ ). This is an immediate consequence of Ertel’s theorem: equation (0.5 b), with  $\alpha = e^p$  and  $\mathbf{\Omega} = 0$ ; it implies that if  $\boldsymbol{\zeta} \cdot \nabla \rho$  is zero at some time, it remains zero.]

<sup>11</sup>and *neither* is ‘rigorous’ in the mathematical sense, that a solution of the ‘full (un-approximated) equations’ — whatever they are taken to be — has been shown to exist and to possess the expansion in question. What is shown is that the full equations are satisfied apart from a residual which is  $O(\epsilon^p)$  times [largest term], when  $p$  terms are taken in the expansion — a *necessary* condition for the first statement to be true. (“Formal asymptotics”, as before.) There is often good reason to believe that it is also a sufficient condition for it to be true over a finite domain in spacetime, in many problems of practical interest — but proof is usually unattainable (and even if attainable, not worth the likely years of effort!). We are dealing with nonlinear partial differential equations. See also ‘On Approximations’ below.



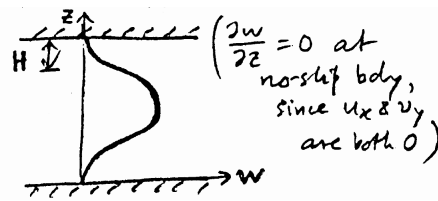


Figure 3.2:  $\partial w/\partial z = 0$  at no-slip boundary since  $u_x$  and  $v_y$  are both 0.

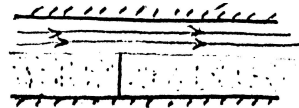


Figure 3.3: Some jargon: “blocking”.

[For stratified flow in a *rotating* frame of reference, it is evident that the potential vorticity  $(2\mathbf{\Omega} + \mathbf{\zeta}) \cdot \nabla\rho \neq 0$  in general. And even when  $\mathbf{\Omega} = 0$  we can often have  $\mathbf{\zeta} \cdot \nabla\rho \neq 0$  in (layerwise rotational) flows (e.g. vortex shedding in a more realistic version of the “Fujiyama” flow hypothesized on p. 48, rather like the laboratory flow shown in the well-known paper by D. K. Lilly (1983, Stratified turbulence and the mesoscale variability of the atmosphere, *J. Atmos. Sci.*, **40**, 749–761)). For any such layerwise-2D rotational flow the distribution of potential vorticity on each constant density surface — or for more general, compressible flow the distribution of  $\rho^{-1}(2\mathbf{\Omega} + \mathbf{\zeta}) \cdot \nabla\alpha$  ( $\alpha = \theta$  for perfect gas) on constant- $\alpha$  surfaces — plays a key role in the dynamical evolution just as in ordinary 2D flow the distribution of ordinary vorticity does (e.g. *Q. J. Roy. Met. Soc.* **111**, 877 and **113**, 402, Hoskins *et al.*)]

-N30-

-N31-

—I.23—

**2-dimensional motion** If  $\partial/\partial y = v = 0$ :

$$(3.8 \text{ a}) \text{ (continuity) says } \frac{\partial u}{\partial x} = 0 \tag{3.10}$$

$$(3.8 \text{ b}) \text{ says } u_t = -\frac{1}{\rho_{00}} p_x .$$

Equations are trying to tell us that in 2D “slow motion”,  $u$  is independent of  $x$  (or that  $x$ -scale  $\gg L$ , equivalently.) (fig. 3.3). The solution given on pp. 36–40 provides an explicit example of how a flow like this could be set up from rest.

**Case of large  $\epsilon$ :** Exercise: show by scale analysis that if  $\epsilon \gg 1$  than all stratification effects may be *neglected*. [“Fast motion does not feel the stratification” (actually it may *destroy* it).]

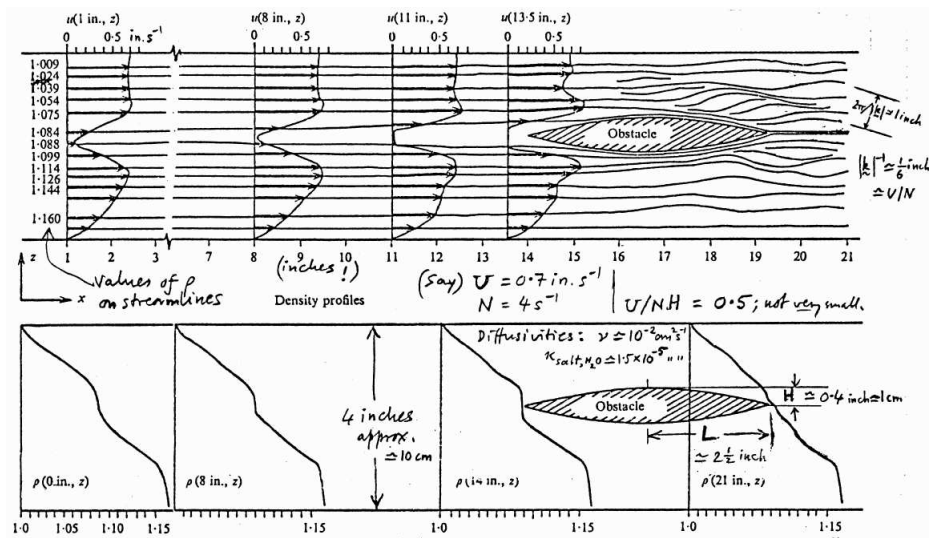


Figure 3.4: Odell-Kovaszny experiment

[Small  $\epsilon$  again:] An example of steady flow past a 2-dimensional obstacle in a salt-stratified water tunnel: Odell & Kovaszny 1971, JFM **50**, 535. Note the lee waves behind the obstacle <sup>12</sup>, and that the blocking is upstream only (cf. 40). This is characteristic of steady stratified flows in which *viscosity* is important but density diffusivity much less so. The streamlines are also isopycnals, to close approximation; note that the pair grazing the obstacle converge together at its rear. There the buoyancy force *pulls* the fluid against viscous retardation; in terms of vorticity, the boundary generated vorticity is balanced by that due to  $\sigma_x$ , i.e. to the isopycnal slope.

—I.23.a—  
-N32-

### On Approximations (some philosophical remarks)

(illustrating the notion of ‘hierarchy of models’ that is so important throughout science — and indeed fundamental to how human perception and cognition work. See the *Lucidity* papers, copies in the Part III room & also via [www.atm.damtp.cam.ac.uk/people/mem/lucidity-principles-in-brief/](http://www.atm.damtp.cam.ac.uk/people/mem/lucidity-principles-in-brief/))

NOTATION:

$\longleftrightarrow$  ‘empirical comparison’, i.e. comparison of a finite number of *cases* at finite accuracy.

- - -  $\rightarrow$  formal asymptotics, i.e. error in *equations* made limitingly small, but

<sup>12</sup>and that they are about the right length;  $|\mathbf{k}|^{-1} = U/N \approx 0.17$  inch (perpendicular distance  $2\pi/|\mathbf{k}|$  between wavecrests  $\approx 1$  inch, which fits as well as one can tell from the experimental picture).

no proof of corresponding behaviour of *solutions*. (As in para 3.2 in  $\lim \epsilon \rightarrow 0$ )

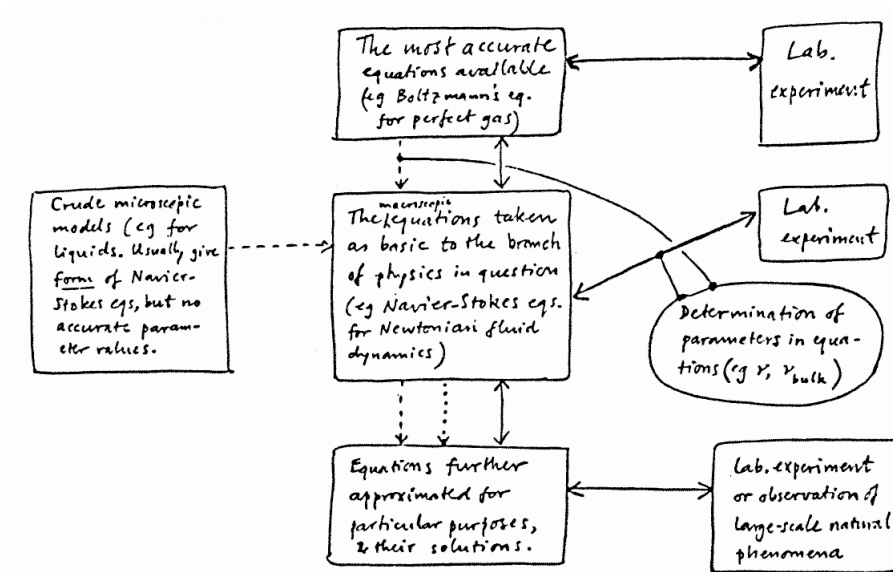
$\implies$  Mathematically rigorous asymptotics.

$\dots >$  ad-hoc approximations (their *only* justification is ' $\longleftrightarrow$ ')

BASIC REMARK:

'The exact equations'  $\equiv$  Nature (unknown)

We always work with approximate sets of equations. E.g. in fluid dynamics:



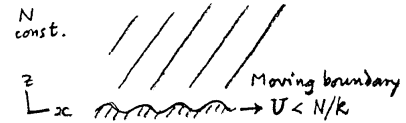
Notes:

1. The term 'the exact equations' is often used to mean the middle box, re 'macroscopic equations taken as basic... e.g. Navier-Stokes equations...' (It is convenient to call these 'exact', but such loose usage may lead to the desirability of ' $\implies$ ' being overestimated).
2. ' $\implies$ ' does not appear, and plays no important role simply because it is not available in practice (because it is too difficult) except in *very* simple cases.
3. In astrophysical fluid dynamics we usually work only in the bottom two boxes. A sufficient reason is the vast range of scales of motion. We need equations averaged over 'small' scales that we cannot describe explicitly. If the mean effects of the small-scale motions is negligible, we may claim the relation  $\dots >$  to the middle box; if not, then the vertical link is usually no stronger than  $\dots >$ , if only because of the intractability of the problem of 'turbulence'.

4. Working in the bottom two boxes still provides a very powerful check on the self-consistency of any set of ideas being used to try to understand observed phenomena and make predictions about them. That's why it's important!

### §3.3 Weakly nonlinear effects: an example of second order mean flow change due to gravity waves

This is our first glimpse of the radiation-stress-dominated atmosphere. A very simple example illustrates most of the important points. Take the 2-dimensional internal wave pattern (2.14) studied on p. 29 with



$$\text{boundary } z = h(x, t) = \epsilon \sin(kx - \omega t); \quad \omega = Uk, \quad U < Nk,$$

$$k > 0, \quad \epsilon m \ll 1 \quad (\text{to apply linearized boundary condition}),$$

$$m_0 = k \left( \frac{N^2}{\omega^2} - 1 \right)^{1/2} > 0, \quad \text{real}$$

$$\begin{aligned} w &= -\epsilon \omega \cos \vartheta, \\ u &= \frac{-\epsilon m_0 \omega}{k} \cos \vartheta \end{aligned} \tag{3.11}$$

(since  $u_x + w_z = 0$ ), where

$$\vartheta \equiv kx - m_0 z - \omega t$$

and where  $-m_0$  is chosen to satisfy the radiation condition.

Average  $x$ -component of equation ((1.4)a) w.r.t.  $x$ . Take (1.2a)

$$u_t + u u_x + w u_z = \frac{-1}{\rho_{00}} p_x \Rightarrow \bar{u}_t = -\overline{(uu)_x} - \overline{(wu)_z} = -\frac{\partial}{\partial z} \overline{uw} \tag{3.12}$$

again using  $u_x + w_z = 0$ , and *assuming* that  $p$  and  $uu = u^2$  are finite at  $x = \pm\infty$ . Notation:  $\overline{(\ )} \equiv \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L (\ ) dx$

[Finiteness of  $p$  at  $x = \pm\infty$  is not the only physically-reasonable possibility — but the resulting problem will, for instance, correctly idealize the geophysically interesting one in which  $x$  corresponds to longitude and  $y$  to latitude (thus *periodic* in  $x$ )].

( $\vartheta$  is not to be confused with the earlier  $\theta$  denoting wavecrest and wavenumber orientation)

-N34-

Notice that  $-\overline{uw}$ ,  $-\overline{uw}$  and  $-\overline{w\overline{u}}$  are cpts. of the (symmetric) ‘Reynolds stress’ tensor

Now

$$(3.11) \quad \Rightarrow \quad \overline{uw} = \frac{\epsilon^2 \omega^2 m_0}{k} \frac{1}{\cos^2 \vartheta} = \frac{1}{2} \frac{\epsilon^2 \omega^2 m_0}{k}. \quad (3.13)$$

This is *constant*, and (3.12) merely says  $\bar{u}_t = 0$ , as we might have anticipated from the fact that (3.11) is already a finite-amplitude solution of the equations, see (§3.1(i)) on page 43! But this does not mean that the waves never affect the mean flow, as can be seen by asking, again, what happens if the disturbance is *set up from the rest*. The simplest such transient solution is that in §2.2 on page 31; the disturbance is given by

$$(3.11) \times F(T - Z/w_g),$$

in the notation of section §2.2 except that we have written  $T, Z$  in place of  $\tau, \zeta$ , the slow variables, and

$$w_g = \mathbf{c}_g \cdot \hat{\mathbf{z}} = m_0 \omega^3 / k^2 N^2,$$

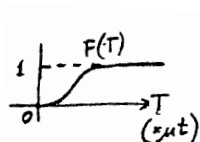
the vertical component of  $\mathbf{c}_g$ . Thus (3.13) is multiplied by  $|F(T - Z/w_g)|^2$  and (3.12) becomes (using  $\partial/\partial z = \mu \partial/\partial Z$ )

$$\bar{u}_t = -\frac{1}{2} \epsilon^2 \frac{\omega^2 m_0}{k} \mu \frac{\partial}{\partial Z} \{|F(T - Z/w_g)|^2\}. \quad (3.14)$$

Note that it is a formally self-consistent procedure to evaluate the r.h.s. of (3.12) from the linearized solution for  $\bar{u} \equiv 0$ , since an  $O(\epsilon^2)$  contribution to  $\bar{u}$  is negligible in the linearized problem. Again this is equivalent to taking the leading terms in a formal (double) asymptotic expansion in powers of  $\epsilon$  (as well as  $\mu$ ).

In (3.14),  $\rho_{00}$  times the coefficient in front of  $\frac{\partial}{\partial Z}$  is equal to minus the wave drag per unit area, (2.15). Thus (3.14) states that *the mean flow feels the horizontal component of the force exerted by the boundary, not at the boundary, but at the other ‘end’ of the wavetrain*. This feature is characteristic of more complicated, but similar, problems (some further discussion, and references, are given below) and obviously has far-reaching consequences for the problem of correctly describing, e.g., the response of the large, ‘meteorological’ scales of atmospheric motion to the wave drag due to flow over mountains [which has been estimated to be a significant part of the total drag force exerted by the earth on the atmosphere].<sup>13</sup> Using

<sup>13</sup>This is now routinely, though still not very accurately, taken account of in weather-forecasting and climate models.



$\mu \frac{\partial}{\partial Z} = -\frac{\mu}{w_g} \frac{\partial}{\partial T} = -\frac{1}{w_g} \frac{\partial}{\partial t}$ , we may integrate (3.14) at once:

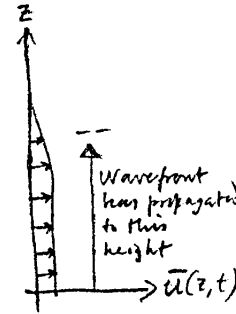
$$\bar{u} = -\frac{1}{2} \frac{\epsilon^2 \omega^2 m_0}{k w_g} [F(T - Z/w_g)]^2,$$

assuming  $\bar{u} \equiv 0$  initially.

Writing  $E = \frac{1}{2} \rho_{00} N^2 \epsilon^2$  as before (page 30), we find

$$\rho_{00} \bar{u} = \frac{E}{U} [F(T - Z/w_g)]^2. \quad (U = c_h = \omega/k) \quad (3.15)$$

[And the expression (2.15) for wave drag/area =  $\frac{1}{2} \rho_{00} \epsilon^2 \omega^2 m_0/k$  =  $w_g E/U$ .] In this problem and some others of similar simplicity, all the wave drag, then, is realized as actual momentum (this is because of the assumption that  $\bar{p}_x = 0$ ). This momentum, according to (3.15), appears to travel with the wave. Such results have led some authors to suggest an analogy between waves in fluids and photons *in vacuo* — which latter do possess momentum equal to energy/phase speed — but such an analogy is in fact wrong in general. In the present case it depends on the special circumstance that  $\partial/\partial x$  (all mean quantities) = 0). [For further discussion and references see McIntyre, M. E. 1981, *J. Fluid Mech.* **106**, 331.]



—I.26—

**Energetics** We noted on page 30 that the rate of working of the wave-drag force needed to move the boundary at speed  $U$  is equal to the rate of increase of wave-energy,  $w_g E$ , per unit  $xy$  area.<sup>14</sup> So the energy of the waves comes from the boundary, in the present frame of reference. However this cannot be so in the frame in which the boundary is stationary and the fluid moves past it towards the left; in that frame, *no* work can be done via the boundary. But we can now see where the energy comes from — namely from a reduction in the kinetic energy of the mean flow, which is now significant since

$$\begin{aligned} \int_0^\infty \frac{\partial}{\partial t} \frac{1}{2} \rho_{00} (-U + \bar{u})^2 dz &= - \int_0^\infty \rho_{00} U \frac{\partial \bar{u}}{\partial t} dz + O(\epsilon^4) \quad [\bar{u} = O(\epsilon^2)] \\ &= -2E \int_0^\infty F' \left( T - \frac{Z}{w_g} \right) F(T - Z/w_g) \cdot \underbrace{\mu}_{dz} dz, \quad \text{by (3.15)} \\ &= -w_g E, \text{ provided that } F(T) \text{ has reached its final value of unity.} \end{aligned}$$

<sup>14</sup>both being equal to the upward flux of wave-energy,  $\bar{p}\bar{w} = w_g E$ ; see (2.8) and (2.9).

Further discussion in: *Theoretical and Applied Mechanics 1992*, Proc. XVIII Int. Congr. Theor. Appl. Mech., Haifa, ed. S. R. Bodner, J. Singer, A. Solan, Z. Hashin, 459 pp; Amsterdam, New York, Elsevier, 281–304 (published 1993) (available on my website ftp as the files beginning “airsea”); see also Bühler & McI. 2003, *J. Fluid Mech* **492**, 207–230.

Thus the ‘energy budget’ — in this case whether the wave-energy appears to come from the boundary, or from the mean flow (or some mixture of the two) — *depends entirely on one’s frame of reference*. (This of course is a rather elementary dynamical point — but one that is forgotten about surprisingly often in the literature!)

—I.27—

**Effect of wave dissipation** Clearly RHS (3.12) can be nonzero, *after* the transient wavefront has passed, if the waves are being attenuated by any of the dissipative processes we have neglected. The mean flow can then accelerate, *persistently*, at a given level, for much longer than the time for the transient wavefront to go past; substantial cumulative alterations to the mean flow can result. The parameter  $\epsilon^2$  may (perhaps) be small, but  $\epsilon^2 t$  certainly need not be, if we wait long enough.

There is a kind of positive feedback in that these alterations tend to *reduce* the phase speed  $U - \bar{u}$  relative to the local mean flow, which tends in turn to enhance the effectiveness of wave dissipation, and so on. This is part of why the QBO exists (recall the video of the Plumb–McEwan experiment, and see also the .avi movie of the same experiment repeated, on the website [http://www.gfd-dennou.org/library/gfd\\_exp/](http://www.gfd-dennou.org/library/gfd_exp/)) (Remember, this illustrates the generic point that dynamically-organized fluctuations about the mean can lead to **anti-frictional behaviour**)

### The simplest example of vacillation due to wave, mean-flow interaction: Plumb and McEwan’s laboratory analogue of the ‘quasi-biennial oscillation’ QBO

If the waves on p. 55 are dissipating *throughout* the depth of the fluid, then the height scale  $D$  for wave attenuation (“height scale” measured in terms  $e$ -folding attenuation of  $E$  or  $\epsilon^2$ ) tends to be proportional to the vertical component  $w_g$  of the group velocity. For uniform dissipation the mean flow will initially develop as in Fig. 3.5a below [ $\bar{u} \propto \exp(-z/D)$ ], with the biggest change near the boundary  $z = 0$ .



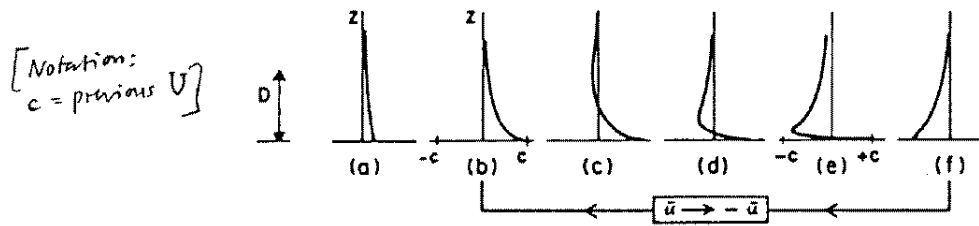


Figure 3.5: (a), (b): Effect on the mean-flow profile  $\bar{u}(z)$  of a single internal gravity wave with phase speed  $+c$  at two successive times. (c)–(f): effect of two waves with phase speeds  $\pm c$ , after Plumb (1977).

We can think of the problem as composed of two sub-problems: sub-problem (i) is the one just discussed, that of cumulative mean flow evolution due to wave dissipation. Sub-problem (ii) is something we haven't discussed yet, namely the effect of mean shear, once it becomes large enough, upon wave propagation and dissipation. (A phenomenon describable in this way is often referred to in the literature as “wave, mean-flow interaction”, or “wave–mean interaction” for brevity.)

The main point about sub-problem (ii) is that the feedback of the mean-flow change onto the waves affects  $D$ . When the intrinsic phase speed  $c - \bar{u}$  gets small enough,  $w_g$  becomes small too (a fact that we shall use again, and which is easily verified from the dispersion properties of plane internal gravity waves); therefore  $D$  decreases and also becomes a function of  $z$  — we may still speak of it as the *local* height scale for wave dissipation — and it is smallest of all near  $z = 0$ . Clearly this cannot go on forever since there is a limiting situation, shown schematically in Fig. 3.5b, in which  $\bar{u} = c$  at  $z = 0$ , and no more waves are generated and no more wave-induced mean-flow change takes place. Actually, linear theory must break down near  $z = 0$  before this situation is reached, but the idea is qualitatively right. We are tacitly assuming that viscosity has a negligible effect on the mean flow, especially near  $z = 0$ .

If we now add to the input of waves at  $z = 0$  a component travelling with equal and opposite phase speed  $-c$ , something very interesting happens. (The first theory demonstrating the effect was that of Holton and Lindzen, (Holton and Lindzen (1972) *J. Atmos. Sci.* **29**, 1076) and our understanding of it was greatly improved by the work of Plumb (1977) *J. Atmos. Sci.* **34**, 1847.) Suppose for simplicity that the two waves, with phase speeds  $\pm c$ , have equal amplitudes so that the boundary is now executing a *standing* wave —1.28—

$$\begin{aligned}
 z = h(x, t) &\equiv a \sin k(x - ct) + a \sin k(x + ct) \\
 &= 2a \sin kx \cos kct;
 \end{aligned}
 \tag{3.16}$$

and suppose moreover that  $2kc$  is less than 0.816 times the buoyancy frequency  $N$  of the stratification. Then not only can the leftward-travelling component propagate even if  $\bar{u} = +c$ , but it can also be shown that the relation between  $w_g$  and intrinsic horizontal phase speed is strictly monotonic,<sup>15</sup> so that  $w_g$  and therefore  $D$  for the leftward-travelling component is necessarily *larger*, for all values of  $x$ , than it was for the rightward-travelling component before the mean flow developed. Thus, it is easy to see that the leftward-travelling wave will now induce a negative acceleration  $\partial\bar{u}/\partial t$  throughout a comparatively *deep* layer, leading to the appearance of a downward-moving zero in the mean velocity profile as shown in Figs. 3.5c–e, and as we saw in the video of the Plumb–McEwan experiment.

In Fig. 3.5e,  $D$  for the leftward-travelling wave has become small just *above* the narrow shear layer at the bottom; however, the leftward-travelling wave cannot by itself quite destroy the shear layer because if  $\bar{u}$  were to become slightly different from  $+c$  at  $z = 0$  the effect of the rightward-travelling wave would reassert itself in a very shallow layer near  $z = 0$ . The shear layer must nevertheless get destroyed sooner or later, either because mean viscous effects become dominant (Plumb (1977) J. Atmos. Sci. **34**, 1847) or, more likely in a real fluid, because the Richardson number  $\text{Ri} = N^2/(\bar{u}_z)^2$  becomes small (see ‘Miles–Howard theorem’, etc., page 94) and the shear layer goes turbulent. This will quickly wipe out the shear layer and leave us all of a sudden with something like the profile of Plumb Fig. 3.5f — i.e. qualitatively like Fig. 3.5b, but with the sign changed. (Plumb (1977) J. Atmos. Sci. **34**, 1847) refers to this transition between the profiles of Figs. (3.5)e and f as ‘switching’.

At this point, we can see that the same sequence of events will take place all over again, with the signs changed. The double feedback loop, sub-problems (i) and (ii), between the mean flow and the dissipating waves, has led to a vacillation cycle in which the mean flow reverses again and again, entirely because of the constant input of waves. Figures 3.5b–f qualitatively depict just half this vacillation cycle.

The Plumb–McEwan experiment (Plumb & McEwan (1978) J. Atmos. Sci. **35**, 1827) beautifully demonstrates this behaviour — which as already emphasized is a clearcut example of “anti-frictional” behaviour in the sense of

<sup>15</sup>Exercise: Check this! (Show first that  $w_g^2 \propto N^2 \hat{c}^4 - k^2 \hat{c}^6$  ( $\hat{c} = c - \bar{u}$ ), assuming that  $|c| = N/\sqrt{k^2 + m^2}$ , dispersion relation in frame moving locally with mean flow.)

driving the system away from solid rotation. The mean motion is *entirely* the result of the fluctuations introduced by the wavemaker at the boundary!

As we saw in the opening lecture, Plumb and McEwan took an annulus of salt-stratified fluid (*not* rotating) and introduced a standing wave via the motion of an elastic membrane at the bottom, so that equal amounts of clockwise and anticlockwise-travelling waves with periods of a fraction of a minute were generated. The initial conditions involved no mean flow — an almost completely symmetrical situation — yet, sooner or later, substantial mean flows would appear, going through a vacillation cycle just as in Figs. 3.5b–f. (In modern parlance, this is an example of **spontaneous symmetry-breaking**.) The initial state is unstable to the vacillation cycle (Plumb 1977). The period of the cycle depended of course on the wave amplitude  $a$ , but was typically an hour or so. No mean flow developed if the wave amplitude  $a$  was too small. That is because of the stabilizing effect of viscous forces on the mean flow.

It is overwhelmingly likely that an essentially similar mechanism underlies the real atmosphere's QBO: the quasi-biennial reversal, every 27 months or so, of the zonal (= east–west) mean winds in the equatorial lower stratosphere — a spectacular example of **order out of chaos** in the real world. The 27 months or so is thus to do with the amplitudes that the relevant (troposphericly-generated) waves happen to have — and nothing to do with any 'obvious' periodicity such as the annual cycle as was thought in the 1950s, when only the first two cycles had been observed. Plumb and McEwan's finding that the mechanism is rather easily killed off by viscous diffusion of the mean flow immediately suggests one of the reasons why the large atmospheric models run on supercomputers, mainly for weather and climate prediction, have only very recently managed to produce anything resembling the QBO. Because of their (still quite coarse) spatial resolution, these models have artificial viscosities that can kill a QBO-type phenomenon, and in any case cannot resolve all the waves that drive it.

—1.29—

Figure 3.6 shows a well known picture of the real QBO, from ongoing work at the Free University of Berlin ( $\bar{u}(z, t)$  in metres/sec). It is clear that the zonal wind reversals do not phase-lock with the annual, seasonal cycle. The average period, defined as the total time divided by the number of cycles in Figure 3.6, is around 27–28 months and nowhere near 24 months. Wave-induced momentum transport is the only mechanism that can produce the real QBO; this is very strongly arguable from the evidence, including negative evidence from early attempts to model the QBO without invoking such momentum transport. These early negative results, especially the groundbreaking work of Wallace and Holton (1968, *J. Atmos. Sci.*, **25**, 280–292) illustrate the “Michelson–Morley principle” in science — the importance of

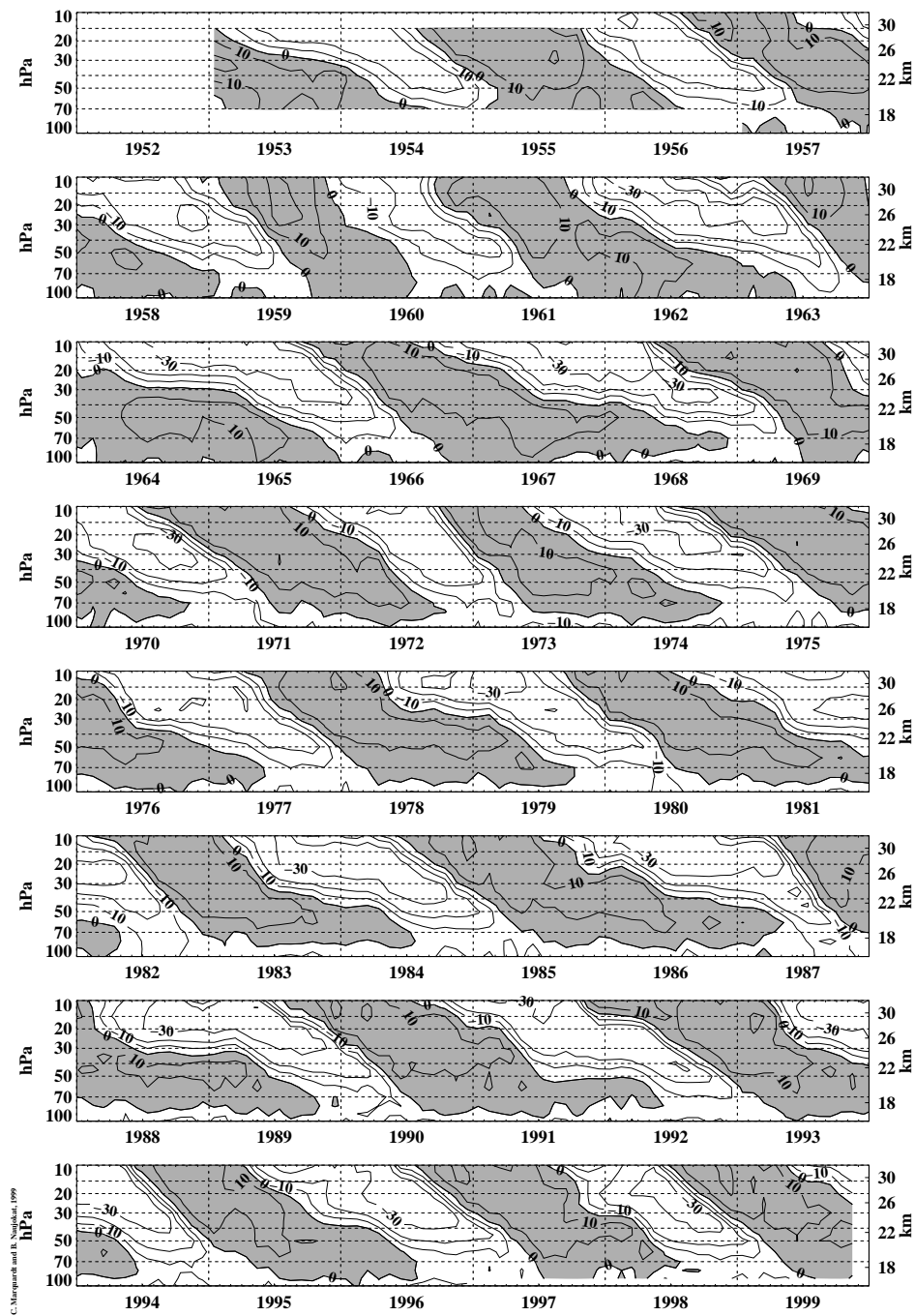


Figure 3.6: Time-height section of the zonal wind near  $9^{\circ}\text{N}$  with the 15-year average of the monthly means subtracted to remove annual and semiannual cycles. Solid isotachs are placed at intervals of  $10 \text{ m s}^{-1}$ . Shaded areas indicate ‘westerlies’, i.e. **eastward** winds. Courtesy Dr Barbara Naujokat, Free University of Berlin.

negative results (something of which today's bureaucrats need constant reminders). For recent reviews see Baldwin et al. 2001: *Revs. Geophys.* **39**, 179–229. and my Millennium review in *Perspectives in Fluid Dynamics*, ed. G. K. Batchelor, H. K. Moffatt, M. G. Worster; Cambridge, University Press, 557–624 (please read wedges as crosses — corrected in the 2003 paperback edition). Not all the waves that drive the real QBO are well observed, but they are thought to be mostly generated in the troposphere below — in a highly nonlinear and very poorly-understood way, e.g. by tropical thunderstorms. Two effects not included in our simple model are important here: (1) the decrease with height of the atmosphere's density (e-folding scale height 6–7 km) and (2) Coriolis effects on the waves, which (i) trap some of them in an equatorial waveguide  $\sim \pm 10^\circ$  latitude — see examples sheet 3 — and (ii) in any case constrain the mean-flow evolution in such a way as to confine the signal in  $\bar{u}_t$  to latitudes within  $\pm 10$ –20 degrees of the equator (Haynes, P. H., 1998, *Q. J. Roy. Meteorol. Soc.*, **124**, 2645–2670).

—1.30—

-N35-

## Digression on ‘eddy diffusion’ and ‘eddy viscosity’

What's involved in the issue of friction versus anti-friction? Why are *molecular transport processes* usually equivalent to a diffusion (macroscopically), but *eddy transport processes* (contributions like  $\bar{\rho} \overline{u'_i u'_j}$  to fluxes in the mean equations, arising from the fluctuating parts of the total fields in the nonlinear terms) *not*, except in very special circumstances? ‘Diffusion’ of a mean or macroscopic field  $\bar{Q}(x, t)$  means that  $\bar{Q}$  is conserved, with an eddy contribution  $\mathcal{F}_{\text{fluct}}$  to its flux which is *linearly dependent on (proportional to) the local gradient of  $\bar{Q}$*  and vanishes when  $\nabla \bar{Q} = 0$ . [The coefficients of the linear relation, or ‘coefficient’ when  $\bar{Q}$  is a scalar in the isotropic case, may be constant or may depend on local scalar functions of the mean fields — e.g. temperature. Their signs are such that no upgradient  $\mathcal{F}_{\text{fluct}}$  can occur.]

The following conditions seem to be characteristic of the ‘molecular’ case [and they are satisfied for air and water under terrestrial conditions]:

1. The fluctuating motions do not depend for their *existence* upon particular mean-field configurations. Also, fluctuation energies (thermal energies) are *enormous* compared with typical mean-flow kinetic energies (in any sensible frame).
2. Moreover the relevant properties of the fluctuations are little affected by the mean fields, except perhaps for a local dependence on mean scalars: the *dependence on mean-field nonuniformities is weak*.

3. The time and space scales of the mean fields are all much larger than those of the fluctuating motions.
4. Moreover changes in the properties of the fluctuations at one place *do not propagate* a significant distance away, on the scale of the mean fields; thus the (weak) dependence on the mean fields is also *local*. Another way of saying this is that the transport of conservable quantities by the fluctuations is a *short-range* process. By contrast, wave-induced momentum transport, being very much a *long-range* process, spectacularly violates this condition!

[It is 3 and 4 together with considerations of symmetry that seem to underlie the (observed) linear dependence on the local  $\nabla\bar{Q}$  alone, and the vanishing of  $\mathcal{F}_{\text{fluct}}$  when  $\nabla\bar{Q} = 0$ .]

*In the ‘eddy’ case*, some or all of 1–4 are typically untrue, if only because of the ubiquity of wave propagation mechanisms. E.g. 3 and 4 are untrue for classical turbulent shear flow (with  $(\ )$  an ensemble or a ‘downstream’ average). When  $\bar{Q}$  is momentum, 1 and 2 are *almost always untrue* in practice. When the fluid is stably stratified, true turbulence (involving cascade of fluctuation energy down to Kolmogorov microscale) typically occurs very intermittently in space and time, reflecting extreme *sensitivity* of the properties of the fluctuating motions to mean-flow configuration. Moreover, 4 is then violated because of the existence (and essential role in contributing to  $\mathcal{F}_{\text{fluct}}$  for mean momentum) of wave propagation mechanisms, giving rise to **systematic correlations among the fluctuating fields** and hence to (long-range) momentum transport. (It is this violation of 4 that permits contributions to  $\mathcal{F}_{\text{fluct}}$  that have nothing to do with the local mean flow, and e.g. can be nonzero even if  $\nabla\bar{Q} = 0$ , and often with ‘anti-frictional’ sign when  $\nabla\bar{Q} \neq 0$ .)

## The temperature of the mesopause

Wave-induced mean forces of the kind under discussion also seem certain to be the correct explanation for the one-time enigma of the ‘cold summer mesopause’ (and are also important for the ‘warm winter mesopause’). At altitudes around 80–90 km, temperatures seem inexplicable in terms of infrared cooling, photochemical reactions, etc. Dynamics is the only remotely viable explanation. The temperature distribution is related to  $\bar{u}$  (because the associated Coriolis force affects the mean pressure field, which in turn is related hydrostatically to temperature — see ‘thermal wind’ in third part of course (stratification and rotation), page 169. And  $\bar{u}$  is *strongly* affected, at these altitudes, by the  $\overline{u'w'}$  stress associated with internal gravity waves (e.g. Houghton (1978) *Q. J. Roy. Met. Soc.*), most of them propagating all the way from the troposphere below and reaching their ‘breaking’ or ‘saturation’ amplitudes ( $u' \approx$  horizontal phase speed relative to mean flow) anywhere upwards of 50 km or so in winter, and 80 km or so in summer. For the pioneering modelling efforts see Holton, J. R. (1982) ‘The role of gravity wave induced drag and diffusion in the momentum budget of the mesosphere’, *J. Atmos. Sci.* **39**, 741, building on the ideas of Lindzen (1981) *J. Geophys. Res.* **86**, 9707.

We have so far had only a glimpse of why wave dissipation should be crucial, in general, to these phenomena (just as it is to their classical analogue, ‘acoustic streaming’), even when  $\bar{u}$  becomes a function of  $z$  — in which case it is no longer obvious whether  $\partial(\overline{u'w'})/\partial z$  should be zero for steady waves in the absence of dissipation. Nor have we demonstrated in complete generality the (admittedly plausible) qualitative dependence of the dissipation height scale  $D$  upon  $w_g$ . These matters can be investigated using standard techniques — take disturbances of form  $f(z)e^{i(kx-\omega t)}$  and study the ordinary differential equation governing  $f(z)$  [which can be quite complicated for some dissipation mechanisms, although JWKB techniques, related to our simple example using the ‘slow’ scale  $Z$ , can help to get approximate solutions].

However, a more powerful procedure is available (Andrews & McIntyre (1976) *J. Atmos. Sci.* **33**, 2031 and **35**, 175). It works essentially because of the existence of three distinct but interrelated conserved quantities: momentum; pseudomomentum; and Kelvin’s circulation. The conserved quantity called pseudomomentum is, in turn, closely related to a more general one called wave-action.

### §3.4 A more general theoretical approach

—1.32—

(But still neglecting  $t$ -dep'ce of  $\bar{u}(z, t)$  in the linearized problem for  $\mathbf{u}'$ , so treating  $\bar{u}(z, t)$  as  $\bar{u}(z)$  in the linearized problem.)

Let  $\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$ , where  $\overline{(\quad)}$  means an average over  $x$  as before, and

$$\bar{\mathbf{u}} = \{\bar{u}(z, t), 0, 0\}, \quad \mathbf{u}' = \mathbf{u}'(x, z, t), \quad \overline{\mathbf{u}'} = 0 \quad [\text{take } \overline{(\quad)} \text{ of } \mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}']$$

So  $\mathbf{u}'$  represents a disturbance on a more general mean flow  $\bar{u}(z)$ ; we also allow  $N = N(z)$ . Now linearize the 2D Boussinesq equations (1.4) about this mean state; write  $\partial/\partial t + \bar{u}\partial/\partial x \equiv \mathcal{D}_t$ ; assume  $\bar{\sigma}$  and  $\bar{p}$  both zero so  $\sigma = \sigma'$ ,  $p = p'$ . (Consistent, within linear theory.)

$$\mathcal{D}_t u' + \bar{u}_z w' = -p'_x - X' \quad (\text{set } \rho_{00} = 1) \quad (3.17 \text{ a})$$

$$\mathcal{D}_t w' = -p'_z + \sigma' - Z' \quad (3.17 \text{ b})$$

$$\mathcal{D}_t \sigma' + N^2 w' = -Q' \quad (3.17 \text{ c})$$

$$u'_x + w'_z = 0 \quad (3.17 \text{ d})$$

Here the extra terms  $X'$ ,  $Z'$ ,  $Q'$  have been inserted to allow for possible wave dissipation mechanisms. E.g. the fluctuating force  $\mathbf{F}' = (-X', 0, -Z')$  could be a viscous force  $\nu \nabla^2 \mathbf{u}'$ , and  $-Q'$  a heat conduction or buoyancy diffusion term  $\kappa \nabla^2 \sigma'$ , or an effect of radiative heat transfer (known to be important, e.g. for the actual stratospheric waves generating the QBO). Then again,  $\mathbf{F}'$  or  $Q'$  could also represent *given* fluctuating forces or heating, *generating* the waves (which happens e.g. with solar heating in the stratosphere of Venus, and similarly with Earth atmospheric tides).

Now define *particle displacements*  $\boldsymbol{\xi}(x, z, t) = \{\xi(x, z, t), 0, \zeta(x, z, t)\}$  by<sup>16</sup>

$$\mathcal{D}_t \xi = u' + \bar{u}_z \zeta \quad (3.18 \text{ a})$$

$$\mathcal{D}_t \zeta = w' \quad (3.18 \text{ b})$$

and  $\xi = \zeta = 0$  everywhere, at some initial time when (it is assumed) there was no disturbance ( $u', w', \sigma', p', X', Z', Q'$  all zero). We can now verify that

$$\bar{\xi} = \bar{\zeta} = 0, \quad (3.19)$$

$$\nabla \cdot \bar{\boldsymbol{\xi}} = \xi_x + \zeta_z = 0, \quad (3.20)$$

by respectively taking  $\overline{(\quad)}$  and  $\nabla \cdot$  of (3.18) [e.g.  $w'_z = \frac{\partial}{\partial z}(\mathcal{D}_t \zeta) = \mathcal{D}_t \zeta_z + \bar{u}_z \zeta_x$ ;  $u'_x = \frac{\partial}{\partial x}(\mathcal{D}_t \xi - \bar{u}_z \zeta) = \mathcal{D}_t \xi_x - \bar{u}_z \zeta_x$ ; so  $0 = u'_x + w'_z = \mathcal{D}_t(\xi_x + \zeta_z)$ ].

<sup>16</sup>RHS's (3.18) are "Lagrangian disturbance velocities" of a fluid particle, relative to the mean flow *at the height where the particle originally came from*.



Now use *initial conditions* to get (3.20), noting that, for each fixed  $z$ ,  $\mathcal{D}_t$  is the directional derivative in  $xt$ -space along characteristic lines with slopes  $dt/dx = \bar{u}^{-1}$ . So if  $\xi_x + \zeta_z = 0$  at some initial  $t$  and all  $x$ , then we can integrate along each characteristic to deduce that  $\xi_x + \zeta_z = 0$  for all  $t$  as well as all  $x$ .]

Now take  $\overline{\xi_x \times (3.17a)} + \overline{\zeta_x \times (3.17b)}$  [like forming the kinetic energy equation, **except** that the  $x$ -derivative of  $\boldsymbol{\xi}$  is used in place of  $\mathbf{u}'$ ]. Define

$$u^l \equiv u' + \bar{u}_z \zeta, \quad [= \text{rhs of (3.18 a)}] \quad (3.21)$$

(horizontal component of ‘Lagrangian disturbance velocity’). Define also

$$-q \equiv \sigma' + N^2(z) \zeta, \quad (3.22)$$

same as the ‘Lagrangian disturbance buoyancy’  $\sigma_1^l$  ( $\equiv 0$  when  $Q' \equiv 0$ ); note

$$\bar{q} = 0, \quad \mathcal{D}_t q = Q', \quad \text{by (3.19), (3.18 b) and (3.17 c)}. \quad (3.23)$$

Assuming, as before, that any term of the form  $\overline{\partial(\ )/\partial x} = 0$  (true if we continue to assume either  $x$ -periodicity, or that everything is bounded as  $|x| \rightarrow \infty$ , as in eqn (3.12) on page 55. Noting now that LHS (3.17 a) =  $\mathcal{D}_t u^l$  and that

$$\begin{aligned} \overline{\xi_x \mathcal{D}_t u^l} &= \overline{\mathcal{D}_t(\xi_x u^l)} - \overline{u^l u_x^l} \quad \text{by (3.18 a) and (3.21)} \\ &= \mathcal{D}_t(\overline{\xi_x u^l}), \end{aligned}$$

and similarly that  $\overline{\zeta_x \mathcal{D}_t w'} = \mathcal{D}_t \overline{\zeta_x w'}$ , we get

$$\boxed{\frac{\partial}{\partial t} (\overline{\xi_x u^l} + \overline{\zeta_x w'}) + \frac{\partial}{\partial z} \{\overline{\zeta_x p'}\} = -\overline{\xi_x X'} - \overline{\zeta_x Z'} - \overline{\zeta_x q}} \quad (3.24)$$

where  $\sigma'$  has been eliminated using (3.22). Also, (3.20) has been used to write  $\xi_x p_x + \zeta_x p_z = (\xi_x p)_x + (\zeta_x p)_z$ . The relation (3.24) can be used to compute how wave amplitude varies with height, as will be shown below.

Note that if  $X', Z', Q'$  all  $\equiv 0, \forall t$ , then  $q \equiv 0$  by (3.23) and the initial conditions, and so RHS (3.24)  $\equiv 0$ . **Then the relation (3.24) is a conservation relation.** That is,  $\overline{\xi_x u^l} + \overline{\zeta_x w'}$  is the density (remember  $\rho_{00} = 1$ ) of a *conserved quantity*, with vertical flux  $\overline{\zeta_x p'}$  (and in fact horizontal flux  $\overline{\xi_x p}$  which, however, makes no contribution since under our present assumptions it is independent of  $x$ ). This conserved quantity is closely related to a more generally conserved quantity, the **wave-action** (got by replacing  $x$ -derivatives and the averaging with respect to  $x$  by derivatives with respect

to, and averaging with respect to, an ensemble or *phase-shift* parameter  $\alpha$ , usually in the range  $0 \leq \alpha < 2\pi$ ).

The present conserved quantity, associated with  $\partial/\partial x$  rather than  $\partial/\partial\alpha$ , should strictly speaking be called minus the **pseudomomentum** or **quasi-momentum**. In our Boussinesq system with  $\rho_{00} = 1$  the density of the  $x$ -component of pseudomomentum is

$$\mathbf{p} \equiv -\overline{\xi_x u^l} - \overline{\zeta_x w^l}. \quad (3.25)$$

From a theoretical-physics viewpoint, conservation of  $\mathbf{p}$  is associated with invariance of the *mean flow* with respect to translations in the  $x$  direction — just as conservation of *momentum* is associated with a different translational invariance, that of the *whole physical problem* including gravitational potentials. The translational invariance encountered here is expressed through the way in which  $\partial/\partial x$  appears throughout the calculations, and the vanishing of  $\overline{\partial(\ )/\partial x}$ . [This is analogous to quantum-mechanical momentum and pseudomomentum operators. Momentum and pseudomomentum are often confused in the literature; for a celebrated controversy attributable to this confusion, see Peierls, R. E. (1991) *More Surprises in Theoretical Physics*, Princeton University Press.]

We now use (3.24), with right-hand side nonzero, **to relate wave dissipation to mean-flow changes**. (We don't actually have to worry about the wider significance of (3.24) and its relatives just now.) First we find a relation between the Reynolds stress  $-\overline{u' w'}$ , (minus the vertical flux of horizontal momentum) and the vertical flux  $-\overline{\zeta_x p'}$  of pseudomomentum. (We expect that there should be a relation, because we have already noted that  $-\overline{\zeta_x p'}$  is also minus the *wave drag* — cf. below equation (2.14) — on a surface displaced by  $\zeta(x, t)$  about a horizontal plane.)

(Remember the traditional but perverse sign convention that defines 'stress' with sign opposite to that of 'momentum flux'.)

To find a relation between  $\overline{u' w'}$  and  $\overline{\zeta_x p'}$ , multiply equation (3.17 a) by  $\zeta$  and note that

$$-\overline{\zeta p'_x} = +\overline{\zeta_x p'} \quad (3.26)$$

and that

$$\begin{aligned} \overline{\zeta \mathcal{D}_t u'} &= -\overline{u' \mathcal{D}_t \zeta} + \mathcal{D}_t(\overline{u' \zeta}) \\ &= -\overline{u' w'} + \frac{\partial}{\partial t} \overline{u' \zeta}; \end{aligned} \quad (3.27 \text{ a})$$

$$\overline{\zeta w'} = \overline{\zeta \mathcal{D}_t \zeta} = \frac{1}{2} \frac{\partial}{\partial t} (\overline{\zeta^2}). \quad (3.27 \text{ b})$$

Hence we get

$$-\overline{u' w'} = +\overline{\zeta_x p'} - \overline{\zeta X'} - \frac{\partial}{\partial t} \left[ \frac{1}{2} \overline{u_z \zeta^2} + \overline{u' \zeta} \right], \quad (3.28)$$

correct to second order in small quantities, anticipating that  $\bar{u}_t$  and therefore  $\bar{u}_{zt}$  will be small of second order in wave amplitude (the latter true *a fortiori* in the ‘slowly-varying’ example (3.15)).

Finally, we have (confirming that  $\bar{u}_t$  is indeed generally second order)

$$\bar{u}_t = -(\overline{u'w'})_z \quad (\text{neglecting } \bar{X} \text{ provisionally}) \quad (3.29)$$

in the same way as before, (3.12). Substituting from (3.28) and then eliminating  $\overline{\zeta_x p'}$  via (3.24) we see that, correct to second order,

$$\boxed{\begin{aligned} \bar{u}_t = & -(\overline{\zeta X'})_z - \overline{\xi_x X'} - \overline{\zeta_x Z'} - \overline{\zeta_x q} \\ & + \frac{\partial}{\partial t} \left[ -(\overline{\zeta u'})_z - \overline{\xi_x u'} - \overline{\zeta_x w'} - \frac{1}{2}(\overline{u_z \zeta^2})_z \right]. \end{aligned}} \quad (3.30)$$

*This expresses the mean acceleration  $\bar{u}_t$  directly in terms of  $X', Z', q$  i.e. wave dissipation, or other departures from conservative wave motion (first line), and departures from steadiness of the wave pattern (second line) such as would occur with the upward-propagating wave envelope envisaged earlier. It is actually a disguised generalization of Kelvin’s circulation theorem, though to see this one has to rework the theory using Lagrangian averaging since undulating material contours are involved.*

[*Exercise:* verify that (3.15) may be rederived directly from (3.30) by setting  $X', Z', q$  to zero and using the ‘slowly-varying’ approximation (which makes the first and last terms within the square brackets negligible in comparison with the other two. Recall also (3.18), in which  $\bar{u}_z \zeta$  is negligible, and (2.9) together with the remark below it concerning ‘equipartition of energy’. Note again, however, that (3.30) is much more general: **it applies to any mean profiles  $\bar{u}(z), N(z)$ .**]

## Simplification when disturbances have the form

$$f(x - ct, z, T) \quad (T = \mu t, \mu \ll 1), \quad (3.31)$$

where  $c = \text{constant}$  — i.e. time-dependence slowly-varying apart from phase propagation. (Slight generalization of earlier problem — includes it, because  $z$ -dependence *could* be  $e^{-im_0 z} \times \text{func}(Z_1)$  — but  $\bar{u}(z), N(z)$  and therefore disturbance now allowed to depend on  $z$  in a more complicated way.) Then

$$(\bar{u} - c)\{\xi_x, \zeta_x, q_x\} = \{u^l, w', Q'\} + O(\mu) \quad (3.32)$$

and (3.30) becomes

$$\begin{aligned} \bar{u}_t = & -(\overline{\zeta X'})_z + \frac{1}{(c - \bar{u})} \{ \overline{u^l X'} + \overline{w^l Z'} - \overline{\zeta Q'} \} \\ & + \frac{\partial}{\partial t} [ -(\overline{\zeta u^l} + \frac{1}{2} \bar{u}_z \overline{\zeta^2})_z + \frac{1}{(c - \bar{u})} \{ \overline{u^{l2}} + \overline{w^{l2}} \} ]. \end{aligned} \quad (3.33)$$

(You can use *this* result as an easier stepping stone to the Exercise above.)

In the simplest, dissipation-dominated, feedback situation (mid. p. 58),  $\frac{\partial}{\partial t} [ \ ]$  is negligible, as is  $-(\overline{\zeta X'})_z$  if ‘slowly-varying’ holds for  $z$ -dependence. The term  $\{ \}$  in the first line is *positive* for almost-plane waves if  $X'$ ,  $Z'$  and  $Q'$  are *dissipative*. **Hence** tendency for  $\text{sgn } \bar{u}_t = \text{sgn } (c - \bar{u})$ , even when  $\bar{u}$  and  $N$  depend on  $z$ .

The foregoing theory leads to a more elegant (and generalizable) derivation of the result that the dissipation scale height  $D \propto w_g$ . For instance:

**Exercise:** Height scale for dissipation-induced mean-flow acceleration, for a single model of wave dissipation. Let  $X' = 0$ ,  $Z' = 0$ ,  $Q' = \alpha \sigma'$  (‘Newtonian cooling’) and  $\alpha$  (+ve const.) *small enough* and  $\bar{u}(z)$ ,  $N(z)$  *vary slowly enough* that  $\overline{w^l p^l} \simeq w_g E$  as on p. 27, and (3.32) holds, with  $u^l = u'$ . Show that  $\overline{\zeta_x p^l} \simeq w_g E / (\bar{u} - c)$ , and (noting that (3.17c) and (3.18b)  $\Rightarrow \zeta = -\sigma' / N^2 + O(\alpha)$ ) that  $-\overline{\zeta_x q} = \overline{\zeta q_x} = -\alpha \overline{\sigma'^2} / N^2 (\bar{u} - c) + O(\alpha^2) = -\alpha E / (\bar{u} - c)$  by ‘equipartition’, p. 27. Deduce that (3.24) and (3.30) with  $\partial/\partial t$  terms neglected on r.h.s. imply a height scale  $D = w_g / \alpha$  for  $\overline{\zeta_x q}$  and therefore for  $\bar{u}_t$ . [This confirms what was assumed on p. 59 ff. (where  $U = c$ ). Note  $w_g \downarrow 0$  as  $\omega = k(c - \bar{u}) \downarrow 0$ , by equation before (3.14).]

Connection with **Bretherton–Garrett formula** for ‘wave-action’: By similar manipulations, again involving ‘equipartition’ between kinetic and potential energy, we can show that for slowly-varying plane waves the pseudomomentum  $\mathbf{p}$  appearing in (3.24) — recall definition (3.25) — is given to leading order by

$$\mathbf{p} = E / (c - \bar{u}) = E k / \hat{\omega} \quad (E \text{ defined on p. 27; } \rho_{00} = 1)$$

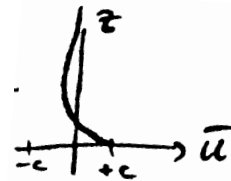
where  $\hat{\omega} = k(c - \bar{u}) = \omega - \bar{u} k$ , the frequency relative to the mean state, sometimes called the ‘**intrinsic frequency**’;  $E / \hat{\omega}$  is often called the (density of) ‘wave-action’. (When applying this as an approximation for slowly-varying  $\bar{u}(z)$ ,  $N(z)$  we could think of  $\hat{\omega}$  as the frequency “intrinsic to the local wave dynamics”.) So multiplying (3.24) by  $k$  we see that  $E / \hat{\omega}$  is conserved (Bretherton & Garrett (1968) *Proc. Roy. Soc. A* **302**, 529). This is a *nice*

form of the result, because it also applies, as is easily shown (details omitted here), when variation of waves and mean state is allowed in  $x, y, t$  as well as  $z$ . The proof is similar to that of (3.24) — the only trick is to replace averaging and differentiation with respect to  $x$  by averaging and differentiation with respect to the phase shift  $\alpha$ , noting that  $\overline{\partial(\ )/\partial\alpha} = 0$  is *always* true.

[An echo of our digression re ‘eddy viscosity’, ‘anti-friction’ etc: The situation depicted in the middle of p. 58 is an example of the ‘peculiar’ behaviour of ‘eddy viscosity’, the appropriate definition of which is

$$-\overline{u'w'}/(\partial\bar{u}/\partial z)$$

in the present context. Notice that this is *infinite* at the turning point in the  $\bar{u}(z)$  profile, and *negative* below it (where left-travelling wave dominates). Of course there is nothing surprising about this (now that we understand what’s going on) but it is worth keeping in mind when encountering the numerous references to ‘eddy viscosities’, tacitly assumed well-behaved and, in particular, positive, in the literature. Once again we see how the *long-range* momentum transport associated with a wave propagation mechanism (through systematic correlations between the fluctuating fields, such as  $u'$  and  $w'$ ) gives rise to the ‘anti-frictional’ behaviour already mentioned, as distinct from the ‘frictional’ behaviour associated with the ‘eddy-viscosity’ predicted by classical turbulence theories. In all those turbulence theories, be they simple or sophisticated, wave propagation mechanisms are wholly neglected, so that momentum transport becomes a short-range effect — over lengthscales of the order of material particle displacements.



-N37-

-I.36-

### §3.5 Resonant interactions among internal gravity waves of small but finite amplitude; resonant-interactive instability

The finite-amplitude solution (2.12) is unstable (*for all* amplitudes) because of a mechanism ubiquitous in nonlinearized dispersive wave problems — ‘resonant interaction’. It can most simply be described by a formal asymptotic solution correct to second order in amplitude. The algebraic details are complicated, even for the case of plane internal gravity waves in an unbounded Boussinesq fluid with constant  $N$ ; but this complication is inessential for appreciating the main points.

If we replace equations (2.1) and (2.2) by their nonlinear counterparts (1.4) and (1.5), (2.3) is evidently replaced by

$$\begin{aligned} \nabla^2(\mathbf{n} \cdot \mathbf{u}_{tt}) + N^2(\mathbf{n} \times \nabla)^2 \mathbf{n} \cdot \mathbf{u} \\ = -(\mathbf{n} \times \nabla)^2(\mathbf{u} \cdot \nabla \sigma) + (\mathbf{n} \times \nabla) \cdot \frac{\partial}{\partial t} (\mathbf{u} \cdot \nabla \boldsymbol{\zeta} - \boldsymbol{\zeta} \cdot \nabla \mathbf{u}). \end{aligned}$$

A superposition of 2 plane waves with constant  $w$ -amplitudes  $a_1, a_2$  (complex) — they will be treated as ‘small but finite’ —

$$w \equiv \mathbf{n} \cdot \mathbf{u} = \operatorname{Re}\{a_1 e^{i(\mathbf{k}_1 \cdot \mathbf{x} - \omega_1 t)} + a_2 e^{i(\mathbf{k}_2 \cdot \mathbf{x} - \omega_2 t)}\}, \quad (3.34)$$

satisfies the linearized equations (LHS (3.17) = 0) if each  $\mathbf{k}_j, \omega_j$  satisfies the dispersion relation (2.6). A second-order correction to (3.34) is obtained by substituting (3.34), and the corresponding  $\mathbf{u}, \boldsymbol{\zeta}, \sigma$  fields, known from (2.1) or (2.12), with RHS (3.34), and then finding a solution of the resulting inhomogeneous (linear) equation for  $w$ .

The RHS of (3.34) is quadratic in the dependent variables, and so without carrying out the substitution we see that its result must consist of a sum of terms of the form  $\operatorname{Re}\{\gamma e^{i(\mathbf{k}_3 \cdot \mathbf{x} - \omega_3 t)}\}$  where  $\gamma = O(a_1 a_2)$  as  $a_1, a_2 \rightarrow 0$  and

$$\mathbf{k}_3 = \pm \mathbf{k}_1 \pm \mathbf{k}_2, \quad (3.35 \text{ a})$$

$$\omega_3 = \pm \omega_1 \pm \omega_2. \quad (3.35 \text{ b})$$

Here the same set of signs must be taken for  $\omega_3$  as for  $\mathbf{k}_3$ . (There are no terms with factors like  $e^{2i(\mathbf{k}_1 \cdot \mathbf{x}_1 - \omega_1 t)}$ ,  $e^0$  (“there are no self-interactions”) because a single plane wave is an exact solution of (3.34). In general, the response to such a forcing will also be  $O(a_1 a_2)$ ,  $\forall t$ , since, in general,  $\mathbf{k}_3$  and  $\omega_3$  will not satisfy the dispersion relation (2.6). But if a possible  $\{\mathbf{k}_3, \omega_3\}$  does satisfy (2.6), then the response will be resonant. I.e., it will contain a contribution

$\operatorname{Re}\{a_3 e^{i(\mathbf{k}_3 \cdot \mathbf{x} - \omega_3 t)}\}$  where  $|a_3|$  increases with time according to

$$\frac{d|a_3|}{dt} = B|a_1 a_2| \quad (3.36)$$

where  $B$  is a positive constant for a given set of  $k_i, \omega_i$ . (Cf. resonantly-forced oscillator  $\ddot{u} + u = \epsilon e^{it} : u = a e^{it} + O(1)$  ( $a = \frac{-i}{2} \epsilon t$ ) as  $t \rightarrow \infty$  — essentially the *same thing*.)

The only source of energy available for the growth of this third wave is the energy of the other two. (3.36) suggests that they will have transferred a substantial portion of their energy to the third wave<sup>17</sup> after a (long) time

<sup>17</sup>More precisely, that  $|a_3|$  will have reached a value of order  $|a_1 a_2|^{1/2}$ .

$t_i \approx B^{-1}|a_1 a_2|^{-1/2}$ . A set of three waves, which can interchange substantial energy this way despite smallness of amplitude and consequent weakness of their nonlinear interaction, because (3.35) and the dispersion relation for all three are satisfied, is called a ‘resonant triad’.

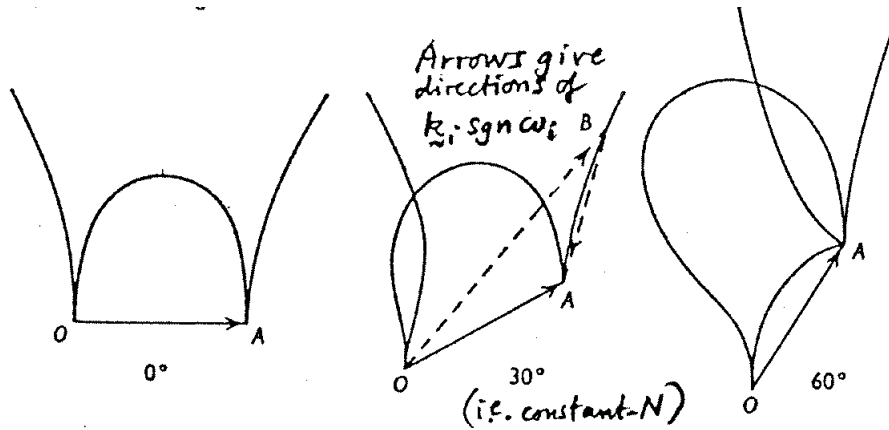


Figure 3.7: Resonant interaction diagrams for short (i.e. constant- $N$ ) internal gravity waves, when  $\theta_2 = 0, 30^\circ$  and  $60^\circ$ . Any point  $B$  on a branch of the curve specifies a resonant triad  $OB, BA, OA$ ; the wave-number with the least slope being always the vector sum of the other two. Only wave-numbers in the vertical plane are shown. The wave-number scale is arbitrary. (Phillips, O. M. (1966) 177; (1977) 230).

The dispersion relation (2.6) admits many resonant triads. [E.g: show that any 3  $\mathbf{k}$ 's forming an equilateral triangle in a vertical plane qualify.] for each given wave number,  $OA$  on the right, the diagrams define a single-infinite family of triads. In three dimensions the corresponding family is doubly infinite.

The description using constant  $a_1, a_2$  clearly is valid only for times  $\ll$  the interaction time  $t_i$ . We can get an approximate solution which is self-consistent for all  $t$  if we allow  $a_1$  and  $a_2$ , as well as  $a_3$ , to vary slowly in time, cf. scale  $N^{-1}$ . Let  $\alpha_j \equiv i a_j / \omega_j$  (complex amplitude for vertical *displacements*). Since  $\text{Re}\{\alpha e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\} = \text{Re}\{\alpha^* e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\}$ ,  $*$  = complex conjugate) we can suppose without loss of generality, and for the sake of symmetry, that the triad satisfies (3.35) with *all signs chosen minus*. Then after much algebra<sup>18</sup> it can be shown that

-N38-

<sup>18</sup>(but the basic mathematical trick is simply the one used at top of p. 32, applied to  $\frac{\partial}{\partial t}$ )

$$\frac{1}{\omega_3} \frac{d\alpha_3}{dt} = C \alpha_1^* \alpha_2^* \quad (3.37)$$

and two similar equations obtained merely by permuting the indices; here  $C$  is the *same* complex constant<sup>19</sup> in all three equations; its value depends (symmetrically) on  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ . [This symmetry of (3.37) is not obvious from (3.34), but can be shown to be a consequence of the fact that the dynamics of the system can be derived from Hamilton's principle (Hasselmann (1967) *J. Fluid Mech.* **30** 737; Bretherton (1969) *Radio Sci.* **4** 1279; Simmons (1969) *Proc. Roy. Soc. A* **3-9** 562; Olbers (1976) *J. Fluid Mech.* **74** 375) using the Lagrangian (particle-following) description of the fluid motion. [Note that (3.37) contains (3.36); take  $\alpha_3 \ll \alpha_1, \alpha_2$ ;  $\alpha_1, \alpha_2 \approx \text{const.}$ ]

We now see that (3.37) implies the above-mentioned instability. Any single plane wave  $\alpha_3 e^{i(\mathbf{k}_3 \cdot \mathbf{x} - \omega_3 t)}$  is unstable to any two other triad members of initially infinitesimal amplitude such that  $\omega_3$  is the frequency of largest magnitude ("Hasselmann's theorem"). For then  $\omega_1 \omega_2 > 0$  in (3.35)<sub>minus</sub> and, linearizing in  $\alpha_1, \alpha_2$  ( $\alpha_3$  const):

$$\frac{1}{\omega_1 \omega_2} \frac{d^2 \alpha_1}{dt^2} = C \alpha_3^* \frac{1}{\omega_2} \frac{d\alpha_2^*}{dt} = C \alpha_3^* \cdot C^* \alpha_3 \alpha_1$$

i.e.

$$\frac{d^2 \alpha_1}{dt^2} = \underbrace{\omega_1 \omega_2 |C|^2 |\alpha_3|^2}_{> 0 \text{ by assumption}} \quad (3.38)$$

so  $\alpha_1$  (and also  $\alpha_2$ ) will in general grow like  $\exp\{(\omega_1 \omega_2)^{1/2} |C \alpha_3| t\}$  initially.

This kind of instability has been strikingly demonstrated in laboratory experiments by McEwan (1971) *J. Fluid Mech.* **50**, 431, Martin, Simmons & Wunsch (1972) *J. Fluid Mech.* **53**, 17, and McEwan & Robinson (1975) *J. Fluid Mech.* **67**, 667. the instability, and resonant-triad interactions in general, are almost certainly significant in most internal-wave fields occurring in nature; the oceanic main thermocline (away from continental shelves) being the most studied case (e.g. Müller & McComas (1981) *J. Phys. Oc.* **11**, 970; recent updates in Polzin, K. (2004) *J. Phys. Oc.* **34**, 214 & refs.).

The standard theory considers a continuum of triads and assumes a random phase relation between  $\alpha_1, \alpha_2, \alpha_3$  leading to a Boltzmann equation. There is an analogous theory for surface gravity waves, and an analogous

<sup>19</sup>And  $C$  may be taken real, without loss of generality. For if  $C = |C|e^{i\delta}$ , replace  $\alpha_j$  by  $\alpha_j \exp(\frac{1}{3}i\delta)$ . Then  $\frac{d}{dt}(\alpha_1 \alpha_2 \alpha_3)$  is real, so if  $\alpha_1 \alpha_2 \alpha_3$  initially real it remains real; this says that waves 1 and 2 are resonantly generating a third wave *in phase* with the initial wave 3, etc.



instability, ‘Benjamin–Feir instability’, the details being more complicated since (3.35) has no solutions when the dispersion relation connecting  $\mathbf{k}$  and  $\omega$  is that of surface gravity waves,  $\omega^2 = g|\mathbf{k}|$  (two-dimensional), and so the leading-order resonant effects come from quartets of plane waves, not triads. (Discussed in Phillips’ book.)

Note incidentally that the wave-action density  $E/\hat{\omega}$  comes into this theory too. Let  $E_j = \frac{1}{2}\rho_{00}N^2|\alpha_j|^2$  (wave-energy density for  $j$ th triad member). Then multiply (3.37) by  $\alpha_1^*$ , and add the result to its complex conjugate, then multiply by  $\frac{1}{2}\rho_{00}$  to get

$$\frac{d}{dt} \left( \frac{E_1}{\omega_1} \right) = (\text{real}) \text{ quantity symmetric in } j = 1, 2, 3$$

therefore

$$\frac{d}{dt} \left( \frac{E_1}{\omega_1} \right) = \frac{d}{dt} \left( \frac{E_2}{\omega_2} \right) = \frac{d}{dt} \left( \frac{E_3}{\omega_3} \right) \quad (3.39)$$

which implies that  $(E_2/\omega_2) - (E_1/\omega_1)$  and the two other similar expressions are each constants of the motion. With the help of (3.39) it can be shown (Bretherton (1964) *J. Fluid Mech.* **20**, 457) that (3.37) soluble analytically in terms of Jacobi elliptic functions; see also Simmons (1969) *Proc. Roy. Soc. A* **309**, 551. It follows that for *isolated triads* the amplitudes fluctuate periodically.

Note incidentally that the sum  $(E_1/\omega_1) + (E_2/\omega_2) + (E_3/\omega_3)$  is not a constant of the motion, because the wavefield no longer involves a single phase-shift  $\alpha$  as on p. 68. The result (3.39) has a natural counterpart in quantum theory in terms of numbers of particles created or destroyed in particle–particle interactions; see Peierls (1979) *Surprises in Theoretical Physics*, eq. (5.2.4), also Peierls (1991) *More surprises in Theoretical Physics*. However, we still have conservation of wave-energy and wave pseudomomentum (the background being translationally invariant). From (3.39) and (3.35) (which, remember, was already used in arriving at (3.39)) we have

$$\frac{d}{dt} (E_1 + E_2 + E_3) = 0 \quad (3.40 \text{ a})$$

$$\frac{d}{dt} \left( \frac{E_1 \mathbf{k}_1}{\omega_1} + \frac{E_2 \mathbf{k}_2}{\omega_2} + \frac{E_3 \mathbf{k}_3}{\omega_3} \right) = 0 \quad (3.40 \text{ b})$$

This is a good check on the algebra: the conservation of pseudomomentum and wave-energy (more generally *pseudoenergy*) under nonlinear interactions is a known consequence of the fact that the background is both translationally invariant and steady (Andrews & McIntyre (1978), On wave-action and its relatives, *J. Fluid Mech.* **89**, 647–664).



## 4.

### A closer look at the effects of non-uniform $\bar{u}(z)$ and $N^2(z)$

#### §4.1 Various forms of the linearized equations

Again, we think in two dimensions but write equations in form such that generalization to three dimensions easy by rotating the horizontal axes. If no dissipation or external forcing, (3.17) is

—1.39—

$$\mathcal{D}_t u' + \bar{u}_z w' = -p'_x \quad (4.1 \text{ a})$$

$$\mathcal{D}_t w' - \sigma' = -p'_z \quad (4.1 \text{ b})$$

$$\mathcal{D}_t \sigma' + N^2 w' = 0 \quad (4.1 \text{ c})$$

$$u'_x + w'_z = 0 \quad (4.1 \text{ d})$$

where  $\mathcal{D}_t = \partial/\partial t + \bar{u}(z)\partial/\partial x$  as before. [Note  $\mathcal{D}_t$  commutes with  $\partial_x$  but not  $\partial_z$ ] Note also Lagrangian disturbance from; it is simplest in nondissipative case since, with definitions (3.6) ff,

$$\mathcal{D}_t^2 \xi = \mathcal{D}_t(u^l) = \mathcal{D}_t(u' + \zeta \bar{u}_z) = \mathcal{D}_t u' + w' \bar{u}_z \quad (\text{from (3.21)}) \quad (4.2)$$

and  $\sigma' = -N^2 \zeta$  (from (3.22))

so

$$\mathcal{D}_t^2 \xi = -p'_x \quad (4.3 \text{ a})$$

$$\mathcal{D}_t^2 \zeta + N^2 \zeta = -p'_z \quad (4.3 \text{ b})$$

$$\xi_x + \zeta_z = 0, \quad \text{from (3.20)}. \quad (4.3 \text{ c})$$

This is actually the best starting point for getting (3.24) and its generalizations). Note setting  $\xi = 0$  recovers case of oscillation of vertical column with frequency  $N$  (p. 18). [\*This system is *self-adjoint*; see section §4.2 on page 79.\*]

Analogue of (2.3): Take  $\partial(4.1 \text{ a})/\partial z - \partial(4.1 \text{ b})/\partial x$ :

$$\mathcal{D}_t(u'_z - w'_x) + \bar{u}_{zz} w' + \sigma'_x = 0 \quad (\text{using (4.1 d)}) \quad (4.4)$$

(vorticity again). Then  $\mathcal{D}_t$  (4.4)  $-\frac{\partial}{\partial x}$  (4.1 c):

$$\mathcal{D}_t^2(u'_z - w'_x) + \bar{u}_{zz} \mathcal{D}_t w' - N^2 w'_x = 0.$$

Finally,  $-\partial/\partial x$  of this, and use (4.1 d) again:

$$\boxed{\mathcal{D}_t^2 \nabla^2 w' - \bar{u}_{zz} \mathcal{D}_t w'_x + N^2 w'_{xx} = 0} \quad (4.5)$$

For disturbances of form

$$w' = \hat{w}(z) e^{ik(x-ct)} \quad (4.6)$$

this becomes

$$\boxed{\hat{w}_{zz} + \left\{ \frac{N^2(z)}{(\bar{u} - c)^2} - \frac{\bar{u}_{zz}}{(\bar{u} - c)} - k^2 \right\} \hat{w} = 0} \quad (4.7)$$

the celebrated *Taylor–Goldstein equation*. (Case  $c = 0$  also called *Scorer’s Equation*.)

Now notice the condition for validity of ‘slow variation’ in  $z$ ; we used a local plane-wave solution satisfying (2.3) instead of (4.5) or (4.7), i.e. we neglected the  $\bar{u}_{zz}$  term. For slow variation to be valid we therefore need

$$\frac{N^2 H^2}{U^2} \gg 1 \quad (4.8)$$

where  $H$  is a height scale for the  $z$ -dependence of the mean flow and  $U$  ditto for  $\bar{u}$  and  $(\bar{u} - c)$ . This is consistent with the condition  $m \gg H^{-1}$  (vertical wavelength  $\ll H$ ) since  $m$  is generally of order  $N/U$ , from the dispersion relation, if we assume  $U \simeq (\bar{u} - c)$ . The condition (4.8) is sometimes referred to as that of ‘large Richardson number’, though the usual definition of Richardson number  $Ri$  is not quite the same; it is

$$Ri = \frac{N^2(z)}{\bar{u}_z(z)^2}, \quad (4.9)$$

which is generally a function of  $z$ .  $Ri$  and  $N^2 H^2/U^2$  are generally of the same order of magnitude (in simple cases characterized by the one height scale  $H$ ).

Analogue of (4.5) starting from (4.3):

Take  $\partial(4.3 a)/\partial z - \partial(4.3 b)/\partial x$ :

$$\mathcal{D}_t^2(\xi_z - \zeta_x) + 2\bar{u}_z \mathcal{D}_t \xi_x - N^2 \zeta_x = 0 \quad (4.10)$$

Take  $-\partial/\partial x$  of this, and use (4.3 c):

$$\boxed{\mathcal{D}_t^2 \nabla^2 \zeta + 2 \bar{u}_z \mathcal{D}_t \zeta_{zx} + N^2 \zeta_{xx} = 0} \quad (4.11)$$

(*Exercise:* Check now that (4.5) gives  $\mathcal{D}_t$  (4.11) when we substitute  $w' = \mathcal{D}_t \zeta$ . Note that

$$\begin{aligned} w_z &= \mathcal{D}_t \zeta_z + \bar{u}_z \zeta_x \\ \text{so } w_{zz} &= \mathcal{D}_t \zeta_{zz} + 2 \bar{u}_z \zeta_{zx} + \bar{u}_{zz} \zeta_x \end{aligned}$$

and notice that last term cancels the  $\bar{u}_{zz}$  term in (4.5).)

**These equations contain several new and basic phenomena**, plural:

- trapping (waveguide formation, by shear  $\bar{u}_{zz}$  **or**  $N^2(z)$  structure)
- shear instability
- "absorption" (irreversible degradation of disturbance, often involving wave breaking).

—I.41—

One can now proceed in the usual way to study various initial-value and boundary-value problems for these linearized equations — evolution of an arbitrary initial disturbance in an unbounded shear flow, disturbance due to flow over a boundary undulating in some given way, etc. It is often convenient to exploit homogeneity in  $x$  and to use Fourier transformation, with basic solutions  $\propto e^{ikx}$ . There is a vast literature on this, arising from attempts to understand observed phenomena like lee-wave trains behind mountains, and 'billow clouds' (a manifestation of shear instability) — more generally the circumstances leading to turbulence in oceans and atmospheres, e.g. CAT (clear-air turbulence), which can present a hazard to aviation.

[Skip to §4.4.]

—I.39.a—

## §4.2 Digression: self-adjointness of (4.3) and some of its consequences

This is just as easy to do in three dimensions, for general  $\bar{\mathbf{u}}(\mathbf{x})$  (we are still, however, confining ourselves to linearized theory, and  $\bar{\mathbf{u}}$  may still be taken to be a mean or a basic, undisturbed flow (in the linearized theory under discussion it doesn't matter which), and its time dependence neglected). In

**Non-examinable**  
(and needs revision)

our Boussinesq linearized model,  $\nabla \cdot \bar{\mathbf{u}} = 0$  and  $\nabla \cdot \boldsymbol{\xi} = 0$ . We replace (4.3a,b) by (with  $\mathcal{D}_t$  now  $\partial/\partial t + \bar{\mathbf{u}} \cdot \nabla$ ):

$$\mathcal{L}(\boldsymbol{\xi}) \equiv \mathcal{D}_t^2 \boldsymbol{\xi} + \frac{1}{\rho_{00}} (\boldsymbol{\xi} \cdot \nabla) \nabla \bar{p} = -\nabla p' \quad (4.12)$$

i.e. the linear operator  $\mathcal{L}$  has cartesian components  $\mathcal{L}_{ij} = \delta_{ij} \mathcal{D}_t^2 + \frac{1}{\rho_{00}} \frac{\partial^2 \bar{p}}{\partial x_i \partial x_j}$ . (It takes a little nontrivial algebra to get (4.3a,b') from the three-dimensional Eulerian linearized equation  $\mathcal{D}_t \bar{u}' + \bar{u}' \cdot \nabla \bar{\mathbf{u}} = -\frac{1}{\rho_{00}} \nabla p' + \mathbf{n} \sigma'$  (set  $p_1 = \bar{p} + p'$  and  $\sigma_1 = \bar{\sigma} + \sigma'$  on page 17). You have to assume that  $\bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = -\frac{1}{\rho_{00}} \nabla \bar{p} + \bar{\sigma} \mathbf{n}$ , i.e. basic flow is a dynamically possible steady flow (unforced). It is easy, however, to check that (4.3ab') does reduce to (4.3a,b) for the unidirectional (and hydrostatic) basic flow assumed there.) (If you want to prove the full three-dimensional form, it's useful to define  $\mathbf{u}^l = \mathbf{u}' + \boldsymbol{\xi} \cdot \nabla \bar{\mathbf{u}} = \mathcal{D}_t \boldsymbol{\xi}$ .)

The self-adjointness of  $\mathcal{L}$  says that, for any pair of fields  $\boldsymbol{\xi}^{(1)}(\mathbf{x}, t), \boldsymbol{\xi}^{(2)}(\mathbf{x}, t)$ ,

$$\boldsymbol{\xi}^{(1)} \cdot \mathcal{L}(\boldsymbol{\xi}^{(2)}) = \mathcal{S}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}) + (4\text{D divergence}) \quad (4.13)$$

where  $\mathcal{S}(\cdot, \cdot)$  is a *symmetric* scalar-valued differential operator on pairs of fields:

$$\begin{aligned} \mathcal{S}(\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}) &= \mathcal{S}(\boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(1)}) \\ &= -(\mathcal{D}_t \boldsymbol{\xi}^{(1)}) \cdot (\mathcal{D}_t \boldsymbol{\xi}^{(2)}) + \frac{1}{\rho_{00}} \xi_i^{(1)} \xi_j^{(2)} \frac{\partial^2 \bar{p}}{\partial x_i \partial x_j} \end{aligned} \quad (4.14)$$

(This is rather easy to prove once you have (4.3ab'); you have to use  $\nabla \cdot \bar{\mathbf{u}} = 0$  and  $\nabla \cdot \boldsymbol{\xi} = 0$ ; note incidentally that  $\nabla \cdot \bar{u}^l$  is *not* zero.) The four-dimensional divergence in (4.13) is

$$\frac{\partial}{\partial t} (\boldsymbol{\xi}^{(1)} \cdot \mathcal{D}_t \boldsymbol{\xi}^{(2)}) + \nabla \cdot (\bar{\mathbf{u}} \boldsymbol{\xi}^{(1)} \cdot \mathcal{D}_t \boldsymbol{\xi}^{(2)}). \quad (4.15)$$

(4.13) holds also for  $\boldsymbol{\xi}^{(2)} \cdot \mathcal{L}(\boldsymbol{\xi}^{(1)})$  provided (1) and (2) are swapped in (4.15), trivially.

### Special cases of interest:

1.  $\boldsymbol{\xi}^{(1)} = \boldsymbol{\xi}^{(2)} = \boldsymbol{\xi}$  such that (4.3ab') holds, and *average* over a wavelength or period of a plane wave on a uniform basic flow:  $\overline{\mathcal{S}(\boldsymbol{\xi}, \boldsymbol{\xi})} = 0$  ('equipartition' of wave-energy; cf. top p. 89 and mid. p. 70).

2.  $\xi^{(1)} = \frac{\partial \xi}{\partial x}$ ,  $\xi^{(2)} = \xi$  satisfying (4.3ab') and average in  $x$  for  $x$ -independent basic state;  $\bar{S} = 0$  then (because  $\mathcal{D}_t$  and  $\frac{\partial}{\partial x}$  then commute, and  $\overline{\left(\frac{\partial}{\partial x}(\cdot)\right)} = 0$  on RHS (4.14)), and (4.15)  $+\nabla \cdot (\overline{\xi p'})$  gives conservation of pseudomomentum (quasimomentum).

You might well ask: why don't we somehow include the  $-\nabla$  in (4.3ab') and the  $\nabla \cdot$  in ((4.3c)) in 'the operator'  $\mathcal{L}$  which exhibits self-adjointness? The answer is that, strictly speaking, the operator involved is not really  $(\cdot)$  but rather  $\Pi(\mathcal{L}(\cdot))$ , where  $\Pi(\mathbf{v})$  is a projection operator from a function space of arbitrary vector fields  $\mathbf{v}(\mathbf{x})$  to a function space of nondivergent vector fields:

$$\Pi(\mathbf{v}) = \mathbf{v}^{(\text{nd})} \quad \text{where} \quad \mathbf{v} = \mathbf{v}^{(\text{nd})} + \mathbf{v}^{(\text{irr})},$$

the nondivergent and irrotational parts of  $\mathbf{v}$ , with the former satisfying  $\mathbf{v}^{(\text{nd})} \cdot \boldsymbol{\nu} = 0$  on a rigid boundary with normal  $\boldsymbol{\nu}$ . The decomposition is unique (assuming all fields evanesce fast enough at large distances, as appropriate); this is because  $\mathbf{v}^{(\text{irr})}$  is then given by  $\nabla\phi$  where

$$\begin{aligned} \nabla^2\phi &= \nabla \cdot \mathbf{v} \\ \partial\phi/\partial\nu &= -\mathbf{v} \cdot \boldsymbol{\nu} \quad \text{on a rigid boundary.} \end{aligned}$$

We may then work within the space [actually a Lie algebra] of nondivergent vector fields  $\xi$  satisfying  $\xi \cdot \boldsymbol{\nu} = 0$  on boundary, and regard the complete equations of motion as

$$\Pi(\mathcal{L}(\xi)) = 0;$$

note  $\Pi$  annihilates  $-\nabla p'$ , and recall that  $\nabla \cdot \mathbf{u}^l \neq 0$ , i.e.  $\nabla \cdot (\mathcal{D}_t \xi) \neq 0$ . The reason why this doesn't affect the previous calculation is that irrotational vector fields are orthogonal to nondivergent ones, in the sense

$$\iiint \mathbf{v}^{(\text{irr})} \cdot \mathbf{w}^{(\text{nd})} dV = 0,$$

for any irrotational  $\mathbf{v}^{(\text{irr})}$  and any nondivergent  $\mathbf{w}^{(\text{nd})}$ , such that  $\mathbf{w}^{(\text{nd})} \cdot \boldsymbol{\nu} = 0$  on the boundary.

## §4.3 Two basically different types of solution:

**Type I:** disturbances of form

$$\text{func}(z) e^{ik(x-ct)}, \quad k \text{ real, } c \text{ constant,} \quad (4.16)$$

(=  $\omega/k$  in earlier notation) as in (4.6). We may distinguish

{ Type Ia:  $c$  real: ‘internal gravity waves’    Type Ib:  $k \operatorname{Im}(c) > 0$ : (exponentially grow

**Type II:**

disturbances which

$$\sim \operatorname{func}(t) \times \operatorname{func}(z) e^{ik[x - \bar{u}(z)t]} \text{ as } t \uparrow \infty \quad (4.17)$$

or more generally

$$\sim \operatorname{func}(t) \times \operatorname{func}(z) \cdot \operatorname{func}[x - \bar{u}(z)t] \text{ as } t \uparrow \infty \quad (4.18)$$

[N.B. Type II is not, of course, a classical ‘normal mode’ (and the problem, small disturbances on shear, is not a classical small-oscillations problem).] Good general discussions of Type I and Type II can be found respectively in Banks, Drazin & Zaturka (1976) *J. Fluid Mech.* **75**, 149 and Brown & Stewartson (1980) *J. Fluid Mech.* **100**, 811. See also both editions of Phillips, O. M., *Dynamics of the Upper Ocean*, C.U.P.; Hartman (1975) *J. Fluid Mech.* **71**, 89; Marcus & Press (1977) *J. Fluid Mech.* **79**, 525 and references (Kelvin (1887) *Phil. Mag.* **24**(5), 188!); Yamagata (1976) *J. Oceanog. Soc. Japan* **32**, 162; Shepherd (1985) *J. Atmos. Sci.* **42**, 1868.

**Remark:** All these solutions are unaffected by introducing a mean flow  $\bar{v}(z)$  in the  $y$  direction (i.e. they all apply unchanged when mean flow changed from  $(\bar{u}(z), 0, 0)$  to  $(\bar{u}(z), \bar{v}(z), 0)$ , since  $\partial/\partial y$  and therefore  $\bar{v} \partial/\partial y$  are zero (‘Squire’s Theorem’). [There is a trivial additional equation  $\mathcal{D}_t v^l = \mathcal{D}_t v^l + \bar{v}_z w^l = 0$ , decoupled from the other equations.]

## §4.4 Type Ia solutions

For Type Ia solutions, which include steady internal-gravity-wave patterns in a slowly-varying mean flow under (4.8) etc., the main phenomenon we have not yet discussed is total internal reflection and the resulting

**Trapping** (of type Ia disturbances). Take  $c = 0$  in (4.7), without loss of generality:

$$\hat{w}_{zz} + m^2(z) \hat{w} = 0 \quad (4.19)$$

where

$$m^2(z) = \frac{N^2(z)}{\bar{u}^2(z)} - \frac{\bar{u}_{zz}}{\bar{u}} - k^2 = l^2(z) - k^2 \quad \text{say.}$$

The function  $l^2(z)$  is sometimes called ‘Scorer’s parameter’ in the lee-wave literature (even though it’s a function and not a parameter). We have

$$\operatorname{sgn}(\hat{w}_{zz}/\hat{w}) = -\operatorname{sgn} m^2;$$



that is,  $\hat{w}(z)$  has ‘oscillatory behaviour’ (concave towards) or ‘exponential behaviour’ (concave away from,  $z$ -axis) according as  $m^2$  is +re or -re.

—I.42—

Clearly if we have (e.g.) a boundary at  $z = 0$  on which  $w' = 0$ , and  $l^2$  decreasing with height (either because  $N^2$  decreases or  $\bar{u}$  increases, or both), there is the possibility of solutions like

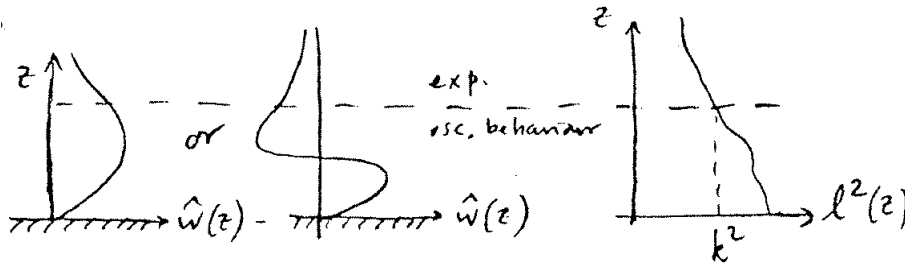
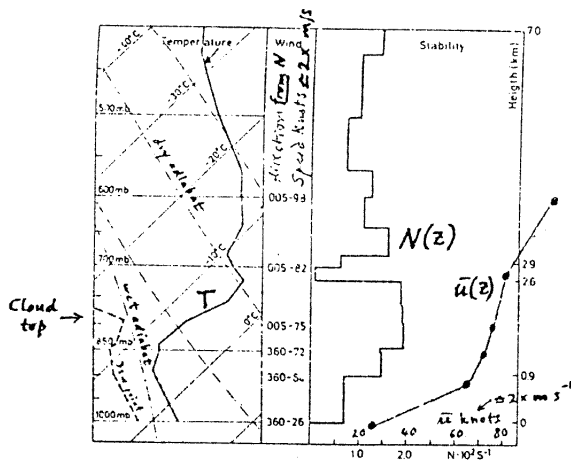


Figure 4.1: [This is classical ‘total internal reflection’: solution locally like an Airy function here.] [Cf. potential well for Schrödinger’s equation.] [as is easy to confirm by solving (4.19) numerically, if you feel the need].



For example the radiosonde at Jan Mayen Island ( $71^\circ\text{N}$ ,  $8\frac{1}{2}^\circ\text{W}$ ) gave the sounding shown at left, at the time of the ship-wake pattern I showed in the satellite picture (the information — and satellite picture — is taken from Gjevik & Marthinsen (1978) *Q. J. Roy. Meteorol. Soc.* **104**, 947). Here  $\bar{u}$  was almost exactly from the north, and increased with height while  $N^2$  decreased, according to their estimate at left, above altitudes of about 3 km. [To estimate  $N^2$

—N42—

one needs to make some allowance for compressibility; the chart at far left is a standard meteorologist’s tool to help do this roughly ‘by eye’.] So it appears that  $l^2$  did decrease with height, from its typical values in the first few km. Gjevik & Marthinsen give a rough calculation giving values of  $k$  agreeing quite well with values  $\sim 2\pi/9$  km measured from the satellite photo (provided we use the component of mean wind perpendicular to the local wavecreases — Squire again). This is an example of an internal-gravity-wave

*waveguide*. Clearly the ocean thermocline can act similarly as a waveguide, and (on a larger scale) the whole stratosphere (see bottom left, of fig. 1.1 on page 1.1), although to deal with the latter case properly we would need to take compressibility fully into account.

One can now go on and apply a lot of standard bits of mathematics to (4.19) — e.g. if we set sensible boundary conditions such as  $\hat{w}(0) = 0$ ,  $\hat{w} \rightarrow 0$  as  $z \rightarrow \infty$ , and if  $\bar{u} > 0$  everywhere so  $l^2$  nonsingular, then regarding  $k^2$  as an eigenvalue we have a Sturm–Liouville system with *real* eigenfunctions  $\hat{w}(z)$

— note this  $\Rightarrow$  no phase change with height, *unlike* our ~~example~~ example of (untrapped) waves. Eigenvalues  $k^2$  real, infinite decreasing sequence, but most (or possibly all) *negative*;<sup>1</sup> there is ‘room’ for only a finite number of modes in a layer of finite depth (e.g. think of the trivial case  $l^2 = \text{const.} > 0$ ,

$\hat{w}(0) = \hat{w}(H) = 0$  ~~etc.~~ etc:  $\hat{w} \propto \sin m z$ ,  $m^2 = l^2 - k^2$ ,  $n$  modes with  $k^2 > 0$ , real  $x$ -waveno, where  $n$  is largest integer such that  $n^2 < l^2 H^2 / \pi^2$ ; when  $\bar{u}$  and  $N$  are const. this is the square of  $N H / \bar{u} \pi$  as before).

Another nice result is that the expression for the eigenvalue (Rayleigh quotient)

$$k^2 = I(\hat{w}) \equiv \frac{\int (l^2 \hat{w}^2 - \hat{w}_z^2) dz}{\int \hat{w}^2 dz}, \quad (4.20)$$

obtained by multiplying (4.19) by  $\hat{w}$  and integrating, assuming  $\hat{w} = 0$  at limits of integration when integrating  $\hat{w} \hat{w}_{zz}$  by parts, is *stationary* for small variations  $\delta \hat{w}(z)$  in  $\hat{w}$  which vanish at the boundaries (limits of integration):

$$\delta I(\hat{w}) = O(\delta \hat{w}^2) \quad (4.21)$$

Proof is simple standard exercise in the calculus of variations (replace  $\hat{w}$  by  $\hat{w} + \delta \hat{w}$  in RHS (4.20), neglect square of  $\delta \hat{w}$  and its  $z$ -derivative  $\delta \hat{w}_z$ , integrate by parts to get rid of  $\delta \hat{w}_z$ , using  $\delta \hat{w} = 0$  at boundaries, and use ‘numerator of (4.20) =  $k^2 \times$  denominator’ as well as (4.19); this gives (4.21)). The stationarity property (4.21) implies that (4.20) is a *good* way of calculating  $k^2$  if one has a rough approximation to the mode structure  $\hat{w}(z)$  (the relevant extension of ‘Rayleigh’s principle’). It *also* means that we can get an analogous expression for the *group velocity* in the  $x$  direction

$$c_g = \frac{\partial}{\partial k} (c k) \Big|_{c=0} = k \frac{\partial c}{\partial k} \Big|_{c=0} \quad (4.22)$$

<sup>1</sup>If *all* are negative there are no waveguide modes, and we say that the flow is ‘super-critical’ (to all modes). This always happens if  $l^2$  is small enough or (usually) if  $\bar{u}$  fast enough.

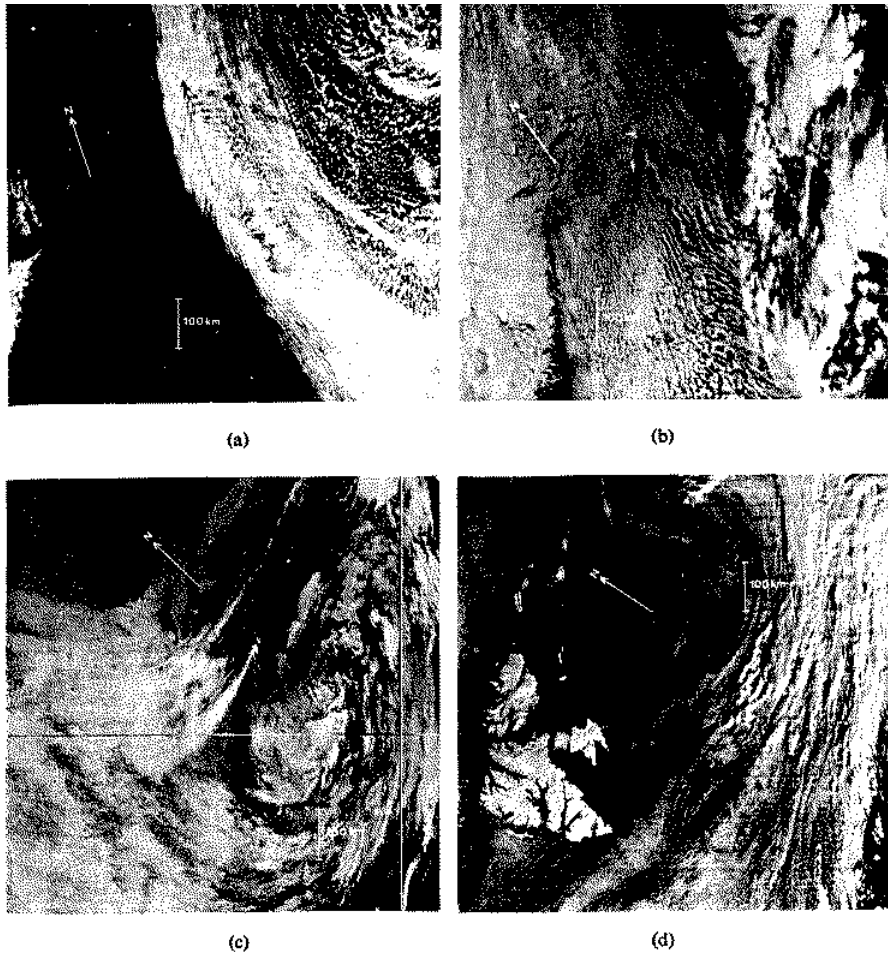


Figure 4.2: Three-dimensional lee-wave pattern. Section of a VHRR photograph taken by NOAA 5. The scale indicates the flight direction of the satellite. The scale perpendicular to the flight direction is diminished by a factor 0.72. (a) Jan Mayen 1 Sept. 1976, 1117 GMT (visible band). On the left, Scoresby Sound, east Greenland. (b) Jan Mayen 29 Dec. 1976, 1122 GMT (infrared band). (c) Jane Mayen 8 Oct. 1976, 1314 GMT (infrared band). (d) Spitzbergen 19 Sept. 1976, 1137 GMT (visible band).

since (4.21) means that in (4.20) we can make small changes  $dc, dk$  [ $l^2$  becomes

$$\frac{N^2}{(\bar{u} - dc)^2} - \frac{\bar{u}_{zz}}{\bar{u} - dc} = l^2|_{dc=0} + \left( \frac{2N^2}{\bar{u}^3} - \frac{\bar{u}_{zz}}{\bar{u}^2} \right) dc + O(dc^2) \quad ],$$

and *neglect* the implied changes in  $\hat{w}$ , so that

$$2k dk = dc \frac{\int \left( \frac{2N^2}{\bar{u}^3} - \frac{\bar{u}_{zz}}{\bar{u}^2} \right) \hat{w}^2 dz}{\int \hat{w}^2 dz} = dc \frac{\int \left( \frac{N^2}{\bar{u}^3} + \frac{l^2}{\bar{u}} \right) \hat{w}^2 dz}{\int \hat{w}^2 dz}$$

whence

$$c_g = \frac{2k^2 \int \hat{w}^2 dz}{\int \left\{ \frac{N^2(z)}{\bar{u}^2(z)} + l^2(z) \right\} \frac{\hat{w}^2(z)}{\bar{u}(z)} dz} \quad (4.23)$$

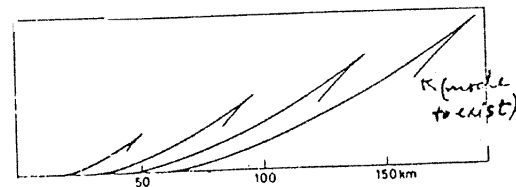
(You should check that this does indeed give  $c_g$  in the trivial case  $\hat{w} = \sin mz$ ; but remember that this  $c_g$  is *relative to our present frame of reference*, i.e. is  $\bar{u} + \hat{c}_g$  where  $\hat{c}_g$  is the *intrinsic* group velocity given by (2.7). Remember moreover that if  $\bar{u} > 0$  we have  $c = 0$  so intrinsic phase and group velocities are in *minus x* direction.) The expression (4.23) explains at once why observed mountain waves are *lee* waves, occurring downstream of their source (like ship waves); evidently, when  $l^2 > 0$  and  $\bar{u}$  is one-signed,

$$\text{sgn } c_g = \text{sgn } \bar{u}. \quad (4.24)$$

Moreover it is clear that (since it can be shown — Sturm–Liouville again — that  $\hat{w}(z)$  is well-behaved as  $k^2 \downarrow 0$ )<sup>2</sup>

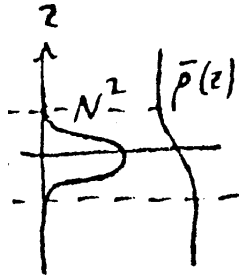
$$c_g \downarrow 0 \quad \text{as} \quad k^2 \downarrow 0 \quad (\hat{c}_g \rightarrow c) \quad (4.25)$$

so that we can begin to see that the dispersion properties in the horizontal are likely to resemble those of surface gravity waves in an ocean of finite depth, and it is not surprising that the satellite pictures of lee waves from a single mountain show patterns qualitatively like ship waves. E.g. this is the calculated pattern for the case shown earlier (source reference, QJRMS **104**, 947). (**Exercise:** Note also that if  $l^2 > l_{\min}^2 > 0$  above surface layer (more realistic) then as  $k \downarrow 0$ , trapped modes cease to exist at some  $k \leq l_{\min}$ .)



<sup>2</sup>But we have to be a little careful in the unbounded (e.g. semi-infinite atmosphere) case. E.g. if  $l^2 \xrightarrow{z \rightarrow \infty} 0$ , vertical range of  $\hat{w}$  extends upwards like  $k^{-1}$ ; then  $c_g \downarrow 0$  like  $k$  not  $k^2$ ; this has implications for the theory of ‘solitary waves’ and related large-amplitude ‘solitary disturbances’, e.g. the ‘Morning Glory’ of North Australia.

### §4.5 A special case of trapped waves: the thin thermocline



A special case of trapped waves, of special theoretical interest, is that of the ‘*thin thermocline*’: a layer  $\mathcal{L}$  of depth  $\ll k^{-1}$  outside which  $N^2$  is zero. Take

$$\begin{cases} \bar{u} = \text{const.} > 0 \\ c = 0 \end{cases}$$

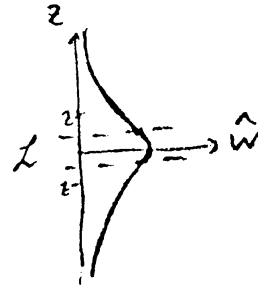
Then (4.7) becomes

$$\hat{w}_{zz} + \underbrace{\left( \frac{N^2(z)}{\bar{u}^2} - k^2 \right)}_{= -k^2 \text{ outside } \mathcal{L}} \hat{w} = 0 \tag{4.26}$$

The solution  $\propto e^{-|kz|}$  outside  $\mathcal{L}$ , and approximately satisfies

$$\hat{w}_{zz} + \frac{N^2(z)}{\bar{u}^2} \hat{w} = 0 \tag{4.27}$$

within  $\mathcal{L}$  (a simple case of ‘matched asymptotics’ in the *limit of thin  $\mathcal{L}$*  (fundamental to notion of ‘layer’ mode) or large  $k^{-1}$ ). Moreover  $\hat{w}$  in (4.27) can be taken as a constant,  $\hat{w}_0$ , to leading order, so the jump in  $\hat{w}_z$  across  $\mathcal{L}$  is



$$[\hat{w}_z] = -\hat{w}_0 \int \frac{N^2(z)}{\bar{u}^2} dz \tag{4.28}$$

To match outer solution, which has  $\hat{w}_z = \pm k \hat{w}_0$  just outside  $\mathcal{L}$ , we have

$$2|k| = \int_{\mathcal{L}} \frac{N^2(z)}{\bar{u}^2} dz; \tag{4.29}$$

but  $\bar{u}$  is constant and  $N^2 = -g \rho_{00}^{-1} \frac{d\bar{\rho}}{dz}$  in notation of p. 17 so  $2|k| = \frac{1}{\bar{u}^2} \left( \frac{-g \Delta \rho}{\rho_{00}} \right)$ , therefore

$$(\text{intrinsic phase speed})^2 = \bar{u}^2 = \frac{1}{2} g' |k|^{-1} \tag{4.30}$$

where

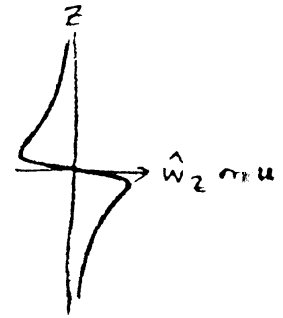
$$g' = -\frac{\Delta \bar{\rho}}{\rho_{00}} g \quad (> 0) \tag{4.31}$$

the ‘reduced gravity’ as it is often called. (The phase speed is  $\frac{1}{\sqrt{2}} \cdot \sqrt{\frac{g'}{g}}$  times the intrinsic phase speed of surface gravity waves in a semi-infinite ocean: reason for  $\sqrt{2}$  is that we have *two* semi-infinite layers and therefore twice the effective mass for a given displacement of the thermocline  $\mathcal{L}$ .)

The same result can of course be got by assuming a ‘two-layer model’ at the outset and applying continuity of (absolute) pressure at the interface;<sup>3</sup> the present derivation illustrates that such ‘layer models’ are often *justifiable as the limit of a continuous system*. This is just as well, since we shall see that *it is only the continuous system that can be stable* for any finite range of amplitudes. The reason is that continuity (4.1 d) implies

$$u' = i k^{-1} \hat{w}_z e^{i k x} \quad (4.32)$$

which develops a jump discontinuity (of strength (4.28)) as  $\mathcal{L}$  shrinks to an interface. So in this limit the model predicts infinite shear at the interface. (Note that the associated *vorticity* is due entirely to  $\sigma'_x$  in (4.4) [ $n \times \nabla \sigma$  in (2.1)].)



If we idealize the local motion as itself approximating to a horizontal flow  $\bar{u}(z)$ , we see that it is indeed unstable, and get our first example of a

## §4.6 Type Ib disturbance:

(solution of (4.7) with  $c_i = \text{Im } c > 0$ , representing an exponentially growing instability). Note first a consequence of pseudomomentum conservation [(3.24) with RHS = 0] which holds for general  $\bar{u}(z)$  and  $N(z)$ . For type Ib disturbances

(real parts understood)

$$\begin{aligned} \xi &= \hat{\xi} e^{i\theta} \\ u^l &= \mathcal{D}_t \xi = i k (\bar{u} - c) \hat{\xi} e^{i\theta}, \\ w^l &= i k (\bar{u} - c) \hat{\zeta} e^{i\theta} \quad (\text{where real parts } \textit{must} \text{ be taken!}), \\ \xi_x &= i k \hat{\xi} e^{i\theta}, \\ \zeta_x &= i k \hat{\zeta} e^{i\theta}, \end{aligned} \quad (4.33)$$

so that pseudomomentum  $\wp$  per unit mass is

<sup>3</sup>plus kinematics: particles at interface assumed to stay at interface

$$\begin{aligned}\varphi &= -\overline{\xi_x u^l} - \overline{\zeta_x w^l} \\ &= \underline{k^2 |\xi|^2} \{c_r - \bar{u}(z)\} \quad \text{where } c_r = \text{Re}(c)\end{aligned}\quad (4.34)$$

[Use  $[\overline{\text{Re}\{a e^{i\theta}\}}]^2 = \frac{1}{2}|a|^2$ , and  $\overline{\text{Re}\{ \}} \overline{\text{Im}\{ \}} = 0$ ,  $|\xi|^2 = \xi^2 + \zeta^2 = \frac{1}{2}|\hat{\xi}|^2 + \frac{1}{2}|\hat{\zeta}|^2$ , where  $\overline{(\quad)}$  denotes  $x$ -average.]

We see at once that  $\text{sgn } \varphi = \text{sgn}(c_r - \bar{u}(z))$ . Since pseudomomentum is conserved, the only way to get a growing, free (i.e. not driven externally, e.g. from boundary) disturbance,

$$\overline{|\xi|^2} \propto e^{2k c_i t} \quad (c_i = \text{Im } c > 0), \quad (4.35)$$

is for  $\varphi$  to take *both signs* (in different parts of the flow). Explicitly, integrate (3.24) with respect to  $z$  (with RHS = 0) and assume  $\zeta = 0$  at boundaries: (*no wave sources*)

$$\frac{\partial}{\partial t} \int \varphi dz \quad (4.36)$$

and so from (4.34) and (4.35):

$$2k c_i \int k^2 \overline{|\xi|^2} \{c_r - \bar{u}(z)\} dz = 0 \quad (4.37)$$

which proves that *if* an exponentially growing disturbance is possible, it must comply with

$$\boxed{\min \bar{u}(z) < c_r < \max \bar{u}(z)}. \quad (4.38)$$

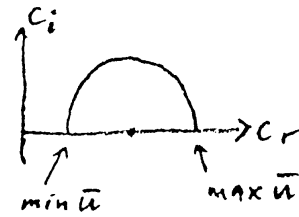
[We note in passing a beautiful extension of this result (Howard's semicircle theorem): If  $c_i > 0$ ,

$$c = c_r + i c_i \quad \text{lies in the semicircle} \longrightarrow \quad (4.39)$$

This is a consequence of (3.24) and a generalization of wave-energy 'equipartition', a '**virial theorem**' got by multiplying (4.3a,b) scalarly by  $\xi$  and averaging (cf.  $\partial \xi / \partial x$  to get  $\varphi$  conservation). This was first shown by Eckart (1963) *Phys. Fluids* **6**, 1042. The proof (in this Lagrangian form) extends trivially to the non-Boussinesq, compressible case.] -I.46-

In our case,  $\overline{\xi(4.3a)} + \overline{\xi(4.3b)}$  gives [since e.g.  $\xi \mathcal{D}_t^2 \xi = \mathcal{D}_t(\xi \mathcal{D}_t \xi) - (\mathcal{D}_t \xi)^2 = \mathcal{D}_t(\frac{1}{2} \mathcal{D}_t(\xi^2)) - (u^l)^2$ , and similarly  $\zeta \mathcal{D}_t^2 \zeta$ ]

$$N^2 \overline{\zeta^2} - \overline{|u^l|^2} = -\overline{(\zeta p')_z} - \frac{1}{2} \mathcal{D}_t^2 \overline{|\xi|^2}. \quad (4.40)$$



Notice this *proves* ‘*wave-energy equipartition*’ for plane waves on a uniform basic state, for which RHS is zero. =  $\partial^2/\partial t^2$  since indep. etc.

Now assuming as always that  $N^2 \geq 0$ , we have

$$|\mathbf{u}^l|^2 - |\zeta p'_z - \frac{1}{2} \frac{\partial^2}{\partial t^2} |\boldsymbol{\xi}|^2 = N^2 \bar{\zeta}^2 \geq 0$$

Integrate over  $z$  and use

$$|\mathbf{u}^l|^2 = k^2 |\bar{u} - c|^2 |\boldsymbol{\xi}|^2, \quad \frac{\partial^2}{\partial c^2} |\boldsymbol{\xi}|^2 = 4 k^2 c_i^2 |\boldsymbol{\xi}|^2,$$

and fact that we have no boundary sources ( $\wp$ -flux vanishes at limits of integration):

$$\int (k^2 \underbrace{|\bar{u} - c|^2}_{=\bar{u}^2 - 2\bar{u}c_r + c_r^2 + c_i^2} - 2k^2 c_i^2) |\boldsymbol{\xi}|^2 dz \geq 0.$$

[ $c_i = \text{Im } c$ ,  $c_r = \text{Re } c$ , as before.] Note  $\int \mathbf{u} |\boldsymbol{\xi}|^2 = \int c_r |\boldsymbol{\xi}|^2$  by (4.36). Therefore

$$k^2 \int |\boldsymbol{\xi}|^2 \{\bar{u}^2 - |c|^2\} dz \geq 0$$

i.e.

$$|c|^2 \leq \frac{\int k^2 |\boldsymbol{\xi}|^2 \{\bar{u}(z)\}^2 dz}{\int k^2 |\boldsymbol{\xi}|^2 dz} \quad (4.41)$$

therefore, *a fortiori*

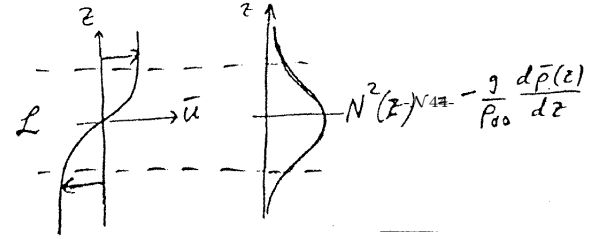
$$\boxed{|c|^2 \leq (\max |\bar{u}(z)|)^2} \quad (4.42)$$

But this is true in all  $x$ -moving reference frames!

It is true in particular for a frame of reference moving in the  $x$  direction with velocity  $\frac{1}{2}(\min \bar{u} + \max \bar{u})$ , so that  $\max |\bar{u}| = \max \bar{u} = -\min \bar{u}$ , which minimizes RHS (4.42) and gives the strongest result implied by bit, viz. (4.39).



We now give the promised *example* of a type Ib disturbance. We are interested (see tope p. 88) in  $\bar{u}(z), N^2(z)$  looking like this: (ubiquitous in Nature — for reasons already hinted).<sup>4</sup> Again we consider a thin layer  $\mathcal{L}$  (compared to  $k^{-1}$ ). But this time it is safer (easier at least) to use (4.11) rather than (4.5), (4.11), *since it turns out that  $w$  is not continuous across  $\mathcal{L}$ , although  $\zeta$  is.* For type Ib disturbance, (4.11) becomes



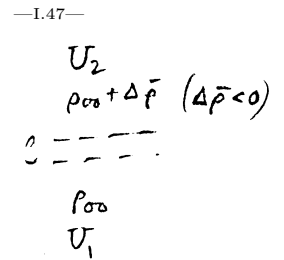
$$\boxed{(\bar{u} - c)^2 \hat{\zeta}_{zz} + 2 \bar{u}_z (\bar{u} - c) \hat{\zeta}_z + \{N^2 - k^2 (\bar{u} - c)^2\} \hat{\zeta} = 0} \quad (4.43)$$

or

$$\boxed{\{(\bar{u} - c)^2 \hat{\zeta}_z + \{N^2(z) - k^2 (\bar{u} - c)^2\} \hat{\zeta} = 0} \quad (4.44)$$

(exhibiting self-adjointness of the differential operator).

Consider, then, a (gravest-mode) disturbance in which whole layer  $\mathcal{L}$  is displaced,  $\zeta = \hat{\zeta}(z)c^{ik(x-ct)}$  where  $\hat{\zeta}$  continuous across layer  $\mathcal{L}$  in limit of thin layer. Then (4.44) can be integrated across  $\mathcal{L}$  to give in place of (4.28)



$$\begin{aligned} [(\bar{u} - c)^2 \hat{\zeta}_z] &= -\hat{\zeta}_0 \int_{\mathcal{L}} N^2(z) dz \\ &N^2 \gg k^2 (\bar{u} - c)^2 \text{ in limit } (\mathcal{L} \text{ shrinks holding } \Delta \bar{\rho} \text{ const.}) \\ &= -g' \hat{\zeta}_0 \quad (4.45) \\ &\left( g' \text{ is same 'reduced gravity' } - \frac{\Delta \bar{\rho}}{\rho_{00}} g \right) \end{aligned}$$

Rest of calculation same as before: if  $N^2 = 0$  and  $\bar{u} = \text{const.}$  outside  $\mathcal{L}$  ( $U_1$  below  $\mathcal{L}$  and  $U_2$  above  $\mathcal{L}$ ) then  $\hat{\zeta} \propto e^{-|kz|}$  (because  $\hat{\zeta}_{zz} - k^2 \hat{\zeta} = 0$  from (4.43) (irrotational))<sup>5</sup> outside  $\mathcal{L}$  so  $\hat{\zeta}_z = \pm k \hat{\zeta}_0$  just above/below  $\mathcal{L}$ , and (4.45) gives

$$\begin{aligned} &-(U_1 - c)^2 |k| - (U_2 - c)^2 |k| = -g' \\ &\times -\frac{1}{2} |k|^{-1}: \\ &c^2 - (U_1 + U_2)c + \frac{1}{2} \{(U_1^2 + U_2^2) - g' |k|^{-1}\} = 0 \\ &\boxed{c = \frac{1}{2}(U_1 + U_2) \pm \left\{ \frac{1}{2} g' |k|^{-1} - \frac{1}{4} (U_1 - U_2)^2 \right\}^{1/2}} \quad (4.46) \end{aligned}$$

<sup>4</sup>e.g. Woods, J. D. (1968) *J. Fluid Mech.* **32**, 791 — direct observation (within the Mediterranean oceanic thermocline) of almost precisely the situation of p. 86.

<sup>5</sup>[*Exercise:* Note extension to  $N \rightarrow \text{const.}$  ( $\neq 0$ ) outside  $\mathcal{L}$  (trivial).] (almost) (note  $m$  depends on  $c$ , complex)

This demonstrates the possibility of Type IIb, unstable disturbances,<sup>6</sup> since we can have  $\text{Im } c > 0$  when

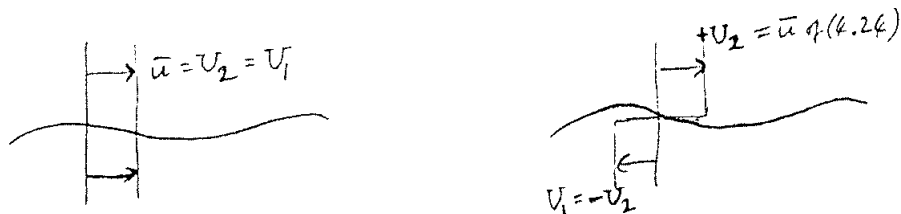
$$(U_1 - U_2)^2 > 2g' |k|^{-1}. \quad (4.47)$$

It should be noted that when  $U_1 = U_2 = \bar{u}$  and  $c = 0$  we recover (4.30).

(not  $p'$ )

Again, we can get (4.46) by assuming a ‘two-layer’ model *ab initio*; here continuity of absolute pressure across the interface yields (4.45). [*Lamb’s Hydrodynamics* does it this way (Lamb (1932) *Hydrodynamics* §232, following Helmholtz.)]

We get another check by noticing that the *marginally stable* disturbance [with ‘=’ in (4.47) instead of ‘>’] has the *same* wavenumber  $k$  as the stationary, trapped gravity wave (4.30) with  $\bar{u} = \frac{1}{2}(U_1 - U_2)$ . E.g. think of case  $U_2 = -U_1 = \bar{u}$  of (4.30)



(Both are dynamically possible and inviscid *steady flows*!)

The two solutions have the same interface shape and the same *absolute pressures*  $P$  on either side of the interface (e.g. apply Bernoulli — equation (0.9) is convenient since already in terms of absolute pressure  $P$  — and neglect  $u'^2$  in  $\frac{1}{2}(\bar{u} + u')^2$  since this is linear theory). So if one is a correct solution, in the thin-layer limit, then so is the other. (Note plausibility that increasing  $|U_2 - U_1|$  distabilizes, via excess aerodynamic ‘lift’!)

<sup>6</sup>Meteorologists often call this mode of instability the ‘Kelvin–Helmholtz instability’.

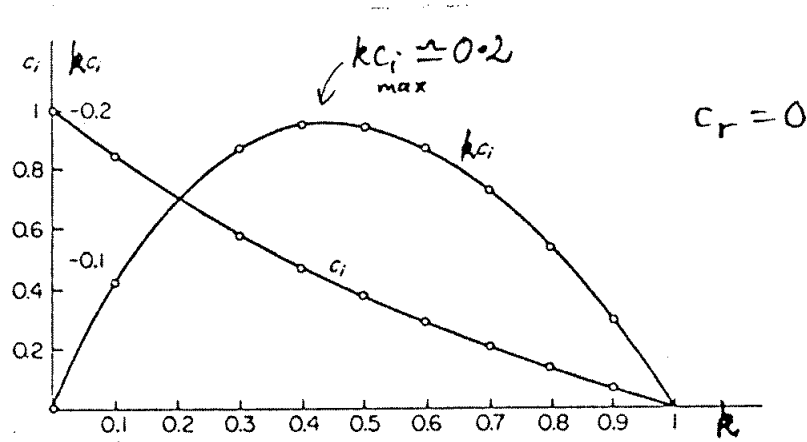
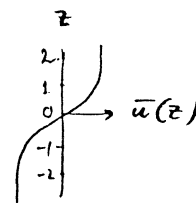


Figure 4.3: Eigenvalues and amplifications for a shear layer:  $U = \tanh y$ . From Betchov & Criminale, Fig. 5.4 (p. 33), Academic Press.

It will be noticed that (4.47) predicts instability for *any*  $g'$ , by taking  $|k|$  large enough, and (4.46) then gives  $c_i = \text{Im } c \sim \frac{1}{2}|U_1 - U_2|$  as  $|k| \rightarrow \infty$  [showing incidentally that Howard's semicircle theorem (4.39) gives a *sharp* bound in at least one case], and so the growth rate  $|k|c_i \rightarrow \infty$ . This of course is where we have to remember that the layer  $\mathcal{L}$  is *not* infinitely thin in real life; we expect our approximations to break down as soon as  $|k|$  exceeds  $H^{-1}$ ,  $H$  being a scale for the thickness of  $\mathcal{L}$ , and the growth rate to reach a *maximum* for some such  $|k|$ . Numerical solution of (4.7) or (4.43) confirms this. E.g. in the case  $N^2 = 0, \gg' = 0, \bar{u}(z) = \tanh z$ , see fig. 4.3.

—1.48—



Note  $c_i \rightarrow 1$  as  $|k| \rightarrow 0$ , in agreement with (4.46) for  $g' = 0, U_{1,2} = \pm 1$ . Much the same result is found when  $N^2 > 0$  provided  $N^2$  is not too large; (4.47) suggests that 'too large' means  $g'$  of order  $U^2|k|$  where  $U$  is a scale for the variation of  $\bar{u}(z)$  with height. Since we expect the most unstable  $|k|$  to be of order  $H^{-1}$ , this suggests that stratification can stabilize the flow altogether, when (in order-of-magnitude terms)

$$g' \gtrsim U^2/H \quad \text{or} \quad g'/H (\approx N^2) \gtrsim U^2/H^2$$

i.e.

$$N^2 H^2 / U^2 \gtrsim 1. \tag{4.48}$$

Notice that this is consistent with the criterion (low Froude number) for buoyancy forces to dominate fluid accelerations §3.2; cf. also (4.8). The same

conclusion can be reached from an order-of-magnitude comparison between the  $N^2$  and  $\bar{u}_{zz}$  or  $\bar{u}_z$  terms in (4.7) or (4.43), or in the wave-energy equation obtained from  $u'(4.1a) + w'(4.1b) + \sigma'(4.1c)/N^2$  [note  $E$  is *no longer conserved* in shear flows, unlike  $\hat{p}$ ].

We can, however, prove a theorem making (4.48) precise, the celebrated Miles–Howard theorem.

**Miles–Howard Theorem:** For (linearized) type Ib disturbances to exist, it is necessary that

$$\boxed{\text{Ri}(z) = \frac{N^2(z)}{\{\bar{u}_z(z)\}^2} < \frac{1}{4} \quad \text{for some } z.} \quad (4.49)$$

*Proof.* Put  $Z(z) = \{\bar{u}(z) - c\}^{1/2} \hat{\zeta}(z)$  in (4.44): Result can be written in the self-adjoint form

$$\{(\bar{u} - c) Z_z\}_z - \left[ \frac{1}{2} \bar{u}_{zz} + k^2 (\bar{u} - c) + \frac{\frac{1}{4} \bar{u}_z^2 - N^2}{\bar{u} - c} \right] Z = 0$$

Multiply by  $Z^*(z)$  (complex conjugate) and integrate, assuming all boundary terms vanish. On integrating some terms by parts, we get

$$\int \left[ (\bar{u} - c) \{|Z_z|^2 + k^2 |Z|^2\} + \frac{1}{2} \bar{u}_{zz} |Z|^2 + (N^2 - \frac{1}{4} \bar{u}_z^2) (\bar{u} - c^*) \left| \frac{Z}{\bar{u} - c} \right|^2 \right] dz \quad (4.50)$$

Take imaginary part:

$$-c_i \left[ \int \left\{ |Z_z|^2 + k^2 |Z|^2 \right\} + (N^2 - \frac{1}{4} \bar{u}_z^2) \left| \frac{Z}{\bar{u} - c} \right|^2 \right] dz = 0 \quad (4.51)$$

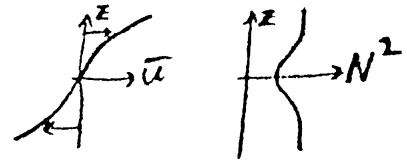
If  $c_i > 0$  but  $N^2 - \frac{1}{4} \bar{u}_z^2 > 0$  everywhere, this gives a contradiction, which proves the theorem. [Howard’s proof].

Recent progress towards a finite-amplitude counterpart of this theorem is reported in Abarbanel *et al.* (1986) *Phil. Trans. Roy. Soc.* **318**, 349–409. Mathematical concepts used are simply explained in McIntyre & Shepherd, *J. Fluid Mech.* It is sometimes said that (4.49) is a consequence of ‘energy considerations’, but this is *not* true except in the rough order-of-magnitude sense leading to the order-of-magnitude statement (4.48). The functional appearing in (4.51) is not ‘energy’ in any of the usual physical senses of the

term;  $Z$  involves  $(\bar{u} - c)^{1/2}$ , corresponding (apart from a factor  $(ik)^{1/2}$ ) to the operator  $(D_t)^{1/2} = (\partial/\partial t + \bar{u}(z) \partial/\partial x)^{1/2}$ .

The Miles–Howard theorem (4.49) shows that stratification is stabilizing, according to linear theory, for small disturbances, when there is *enough* stratification. It should not however be thought that stratification is always ‘stabilizing’.

Cases are known (the first discovered by G. I. Taylor; see also Thorpe (1969) *J. Fluid Mech.* **36**, 679) where a stable unstratified flow is rendered unstable by adding some stratification ( $N^2 > 0$ ) but not enough to violate (4.49). (The cases in question have a *minimum* in  $N^2(z)$  and  $\bar{u}_z(z)$ , and don’t seem to arise very often in practice.



—I.49.a—

—I.49.b—

—I.50—

Throughout the discussion of type Ib (unstable) disturbances, we have been taking it for granted that these disturbances are always ‘trapped’ in the sense that they die off as  $|z| \rightarrow \infty$  (where relevant) — fast enough, for instance, for integrals like (4.51) to converge. We expect that this *will* be the case for an exponentially growing disturbance in an unbounded fluid, when the seat of the instability is localized in some layer of finite depth like the  $\mathcal{L}$  of our example, since information (in non-acoustic modes of fluid motion with finite  $k$ ) is not expected to travel infinitely fast, so the disturbance at large  $|z|$  will only ‘know’ about what was happening at, say,  $z = 0$ , a *long time ago* when the amplitude was exponentially smaller than at present. This suggests that type Ib disturbances will generally *die off exponentially* as  $|z| \rightarrow \infty$  away from the seat of the instability if the latter is indeed localized. [Actually, if  $N^2 = 0$  above or below  $\mathcal{L}$  the motion is irrotational and information does travel infinitely fast — but in that case the structure is  $e^{ikx} e^{-|kz|}$  and so dies off anyway.]

The results (4.38), (4.46) contain a clue as to how to make this idea more precise: the seat of the instability must involve a change of sign of  $\bar{u}(z) - c_r$ . We can in fact *prove* the following

**Theorem** (generalized Charney–Pedlosky theorem). *If  $\{\bar{u}(z - c_r)\} >$  some positive constant  $A$  throughout a semi-infinite region  $z \geq z_0$ , or  $\{\bar{u} - c_r\} < -A$  throughout that region, and if  $\bar{u}$  is bounded so that  $|\bar{u}(z - c)| <$  another positive constant  $B$ , then any type-Ib disturbance bounded as  $z \uparrow \infty$  must have pseudomomentum flux  $F = -\overline{\zeta_x p'}$  satisfying*

$$\boxed{|F| < C e^{-Dz}} \tag{4.52}$$

where  $C$  and  $D$  are positive constants, and  $\boxed{D \propto \text{growth rate } k c_i}$ . (A similar result holds as  $z \downarrow -\infty$  where relevant.)

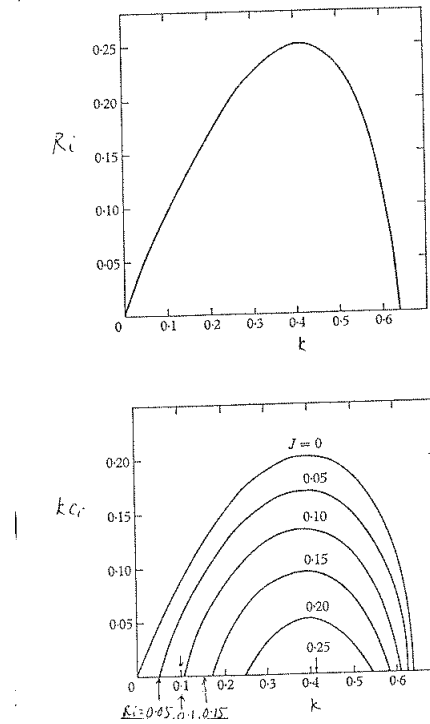


Figure 4.4: Note on a heterogeneous shear flow, from Thin thermocline (long-wave) approximation gives cut-off at  $Ri = 0.05, 0.1, 0.15, \dots$  (not bad!).  $J = Ri = \frac{N^2(z)}{(\bar{u}_z)^2}$  Wavelength =  $\frac{2\pi}{k} \cdot \frac{H}{2}$ , so  $k \simeq 0.4 \Rightarrow$  wavelength  $\simeq 8H$ .

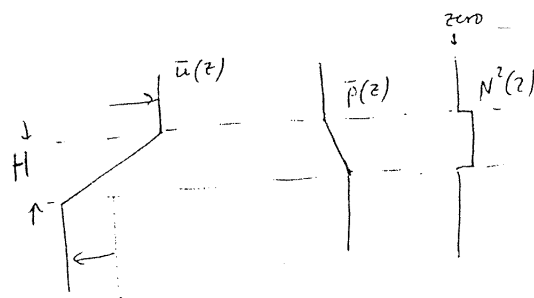
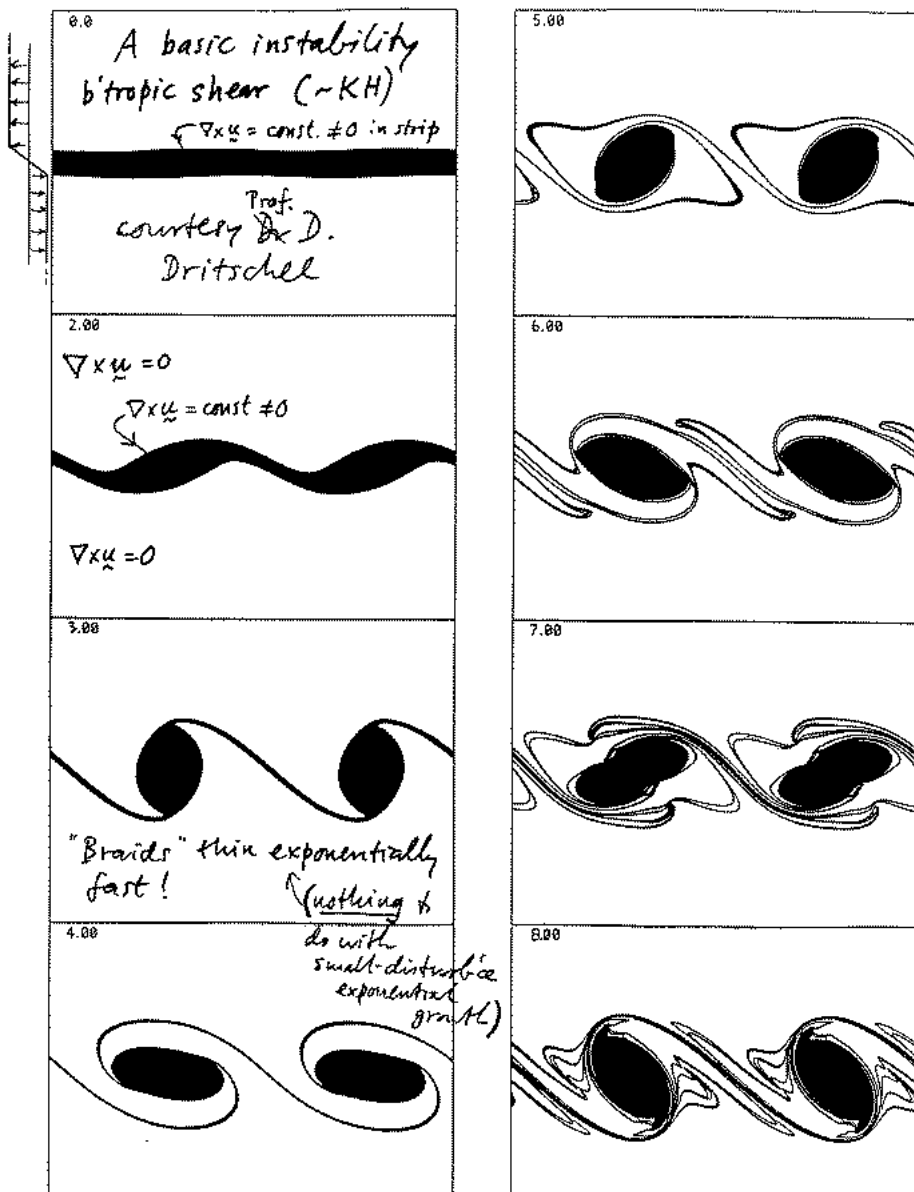


Figure 4.5: Basic profiles (piecewise linear)



Notice corollary that, *according to linear theory*, a *fast-growing* disturbance cannot (*ipso facto*) penetrate far from the seat of the instability. This applies even when a disturbance with the same phase speed  $c_r$  but *constant* amplitude ( $c_i = 0$ ) would *not* be trapped — in which case linear theory would be a bad guide to how far a real disturbance would penetrate from the seat of the instability once growth had stopped.<sup>7</sup> In such a case, I call the ‘trapping’ implied by *apparent trapping* (as opposed to the *real trapping* exemplified on p. 83). The point has often been overlooked in the literature on linear instabilities — so much so that M. A. Weissman and I felt the need to publish a polemic about it in 1978 (McIntyre, M. E., Weissman, M. A., (1978) On radiating instabilities and resonant overreflection, *J. Atmos. Sci.* **35**, 1190). The point proves important in other problems, too, e.g. the behaviour of weather systems which affect medium-range weather forecasting. To prove (4.52), we recall from (4.34) that

$$\tilde{p} = k^2 \overline{|\xi|^2} \{c_r - \bar{u}(z)\}$$

[ $\tilde{p}$  has same sign as intrinsic phase speed  $c_r - \bar{u}$ ] and for type Ib,  $\partial/\partial t$  of such a quadratic quantity ( $\propto e^{2k c_i t}$ ) is  $2k c_i$  times the quantity, so that

$$\frac{\partial \tilde{p}}{\partial t} = 2k^3 c_i \overline{|\xi|^2} (c_r - \bar{u}).$$

—I.51—

Now (in the absence of dissipation or external forcing), (3.24)  $\Rightarrow$  pseudo-momentum is conserved locally:

$$\frac{\partial \tilde{p}}{\partial t} + \frac{\partial F}{\partial z} = 0$$

therefore

$$\frac{\partial F}{\partial z} = +2k^3 c_i \overline{|\xi|^2} (\bar{u} - c_r), > 2k^3 c_i A \overline{|\xi|^2} \quad (4.53)$$

if we take the case  $(\bar{u} - c_r) > A$ . but we can bound  $|F|$  itself by a constant times  $\overline{|\xi|^2}$ , using the equation of motion to eliminate  $p'$ . In the present case this is (4.3a), whence

$$\begin{aligned} |F| &= |-\overline{\zeta_\xi p'}| = |\overline{\zeta p'}| \leq (\overline{\zeta^2})^{1/2} (\overline{p_x'^2})^{1/2} \quad (\text{Schwartz' inequality}) \\ &\leq (\overline{\zeta^2})^{1/2} \{(\overline{D_t^2 \xi})^2\}^{1/2}, \quad \text{using (4.3a),} \\ &\leq (\overline{\zeta^2})^{1/2} k^2 |\bar{u} - c|^2 (\overline{\zeta^2})^{1/2} \\ &< \overline{|\xi|^2} k^2 B^2. \quad (|\xi|^2 = \xi^2 + \zeta^2 \geq \xi^2 \text{ or } \zeta^2.) \end{aligned}$$

<sup>7</sup>which requires nonlinear theory to describe it (the cessation of growth!) theoretically, of course.



Combining this with (4.53) and writing  $D = \frac{2k c_i A}{B^2}$  we get

$$\frac{\partial F}{\partial z} > D |F| \quad (4.54 \text{ a})$$

If we had taken the case  $(\bar{u} - c_r) < -A$  we would have obtained

$$\frac{\partial F}{\partial z} < -D |F| \quad (4.54 \text{ b})$$

Note  $k c_i > 0$  (growing disturbance assumed), so  $D > 0$ .

One or other of (4.54a,b) holds in  $z \geq z_0$ . If (4.54a), we deduce that  $F \leq 0$  ( $z \geq z_0$ ); for if  $F > 0$  for some  $z > z_0$ , (4.54a) would imply that  $|F| \geq (\text{const.}) \times e^{+Dz}$  as  $z \uparrow \infty$  contradicting boundedness of the disturbance. So

$$F \leq 0 \quad \text{when (4.54a) holds.} \quad (4.54 \text{ c})$$

Similarly

$$F \geq 0 \quad \text{when (4.54b) holds,} \quad (4.54 \text{ d})$$

and (4.52) now follows. (For  $z \downarrow -\infty$  we similarly obtain  $|F| < C e^{+Dz}$ .) [The theorem was originally proved by Charney & Pedlosky (1963) *J. Geophys. Res.* **68**, 6441, but using conservation of wave-energy rather than pseudomomentum, so they had to restrict  $\bar{u}$  to be *constant* for  $z \geq z_0$ .]

## Type II or ‘sheared’ disturbances

have not been discussed yet. They are the simplest solutions describing how disturbances to a shear flow are irreversibly ‘absorbed’ or ‘degraded’. It is remarkable how far linear theory for ostensibly non-dissipating disturbances goes towards predicting such processes — which are going on all the time in the real atmosphere and oceans. Type II disturbances *always* arise, e.g., when one solves an initial-value problem for

arbitrary disturbances to a shear flow — hence the expectation that they should occur ubiquitously in Nature. Since they have the structure (oscillatory) func  $(x - \bar{u}(z)t)$  at large  $t$ , they develop finer and finer vertical scales as  $t \uparrow$ , when  $\bar{u}_z \neq 0$ . Their essential properties are thus well represented by the simplest example, which is all that we study here, viz. the

—1.52—

### Case of linear shear

$\bar{u} = \Lambda z$  ( $\Lambda = \text{const.}$ ). Then (4.5) simplifies to

$$D_t^2 \nabla^2 w' + N^2 w'_{xx} = 0 \quad \left( D_t = \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} \right) \quad (4.55)$$

We show that this has type II solutions, exactly ( $\forall t$ ) of the form

$$w' = f(t) e^{ik(x-\Lambda z t)} \quad (4.56)$$

(or *any* func ( $x - \Lambda z t$ )) For then

$$\nabla^2 w' = k^2 f(t) (1 + \Lambda^2 t^2) e^{ik(x-\Lambda z t)}$$

Note that  $D_t$  annihilates the exponential (or *any* func ( $x - \Lambda z t$ )), so in first term of (4.55) only  $-k^2 \left[ \frac{d^2}{dt^2} \{f(t) (1 + \Lambda^2 t^2)\} \right] \cdot e^{ik(\cdot)}$  survives. So (4.56) satisfies (4.55) if, and only if,  $f$  satisfies the *ordinary* differential equation

$$\frac{d^2}{dt^2} \{(1 + \Lambda^2 t^2) f(t)\} + N^2 f(t) = 0. \quad (4.57)$$

(“Only if” is emphasized because there are some erroneous results in the literature asserting, by implication, that there are solutions of the form (4.56) for which  $f(t)$  does not satisfy (4.57); see the Brown & Stewartson paper cited earlier.<sup>8</sup> The correct solution for this case was first given by Eliassen, Høiland and Riis in 1953, also independently in O. M. Phillips’ *first* edition (1966). The problem originally *looked* difficult because the prejudice that one should think in terms of ‘normal modes’ (type I) led to the (simple) solution (4.56) being described as a (complicated) superposition of a continuum of singular type Ia disturbances — the so-called ‘continuous spectrum’ of neutral normal modes, leading to mathematical subtleties and pitfalls.)

(4.57) has two independent solutions (expressible in terms of hypergeometric functions<sup>9</sup> of  $t^2$ ). For large  $t$  the leading term in their asymptotic expansion in inverse powers of  $t$  satisfies

$$\frac{d^2}{dt^2} \{\Lambda^2 t^2 f(t)\} + N^2 f(t) = 0 \quad (4.58)$$

which has solutions  $f(t) \propto t^\mu$  where  $\mu$  is a constant satisfying

$$(\mu + 2)(\mu + 1) + \text{Ri} = 0 \quad (\text{Ri} = N^2/\Lambda^2 = \text{const.}).$$

[note  $t^\mu = e^{t \log \mu}$  on oscillatory function of  $t$  if  $\text{Ri} > \frac{1}{4}$  here]

Therefore

$$\mu = -\frac{3}{2} \pm \sqrt{\frac{1}{4} - \text{Ri}}. \quad (4.59)$$

$\frac{1}{4}$ : (the magic number again!)

The implied amplitude dependence always  $\Rightarrow$  local static instability (heavy fluid over light): this is another situation where the waves *always break*, as we now show.

<sup>8</sup>Brown & Stewartson, *op. cit.* (*J. Fluid Mech.* **100**, 811).

<sup>9</sup>No text?

This becomes obvious when we consider horizontal displacements  $\xi$ . In fact equations (4.3) give a very direct view of what is going on, in terms of ‘inclined-plane’ dynamics, as with the simple plane waves discussed on p. 27; but this time the inclination is changing with time. The planes on which disturbance quantities are constant make an angle  $\cot^{-1}(\Lambda)$  with the horizontal. With this in mind we can rederive the solution directly from equations (4.3). One could use rotating or sheared (oblique) axes, but it is simpler just to take the component instantaneously along the ‘inclined planes’ by adding (4.3b) at  $\Lambda t$  times (4.3b), which eliminates  $p'$  if  $p' \propto e^{ik(x-\Lambda t z)}$ :

$$\Lambda t D_t^2 \xi + D_t^2 \zeta + N^2 \zeta = 0$$

Assuming that  $\xi$  and  $\zeta$  have the same structure:  $\xi = \hat{\xi}(t)e^{ik(x-\Lambda t z)}$  and  $\hat{\zeta}$  similarly, we note again that  $D_t^2 \xi = e^{ik(\cdot)} \hat{\xi}''(t)$  etc, and from (4.3c) that

$$\xi = \Lambda t \zeta. \tag{4.60}$$

Therefore

$$\underbrace{\frac{d^2 \hat{\xi}}{dt^2} + \frac{1}{\Lambda t} \frac{d^2}{dt^2} \left( \frac{\hat{\xi}}{\Lambda t} \right)}_{\text{(true with or without } \hat{s})} + \frac{N^2}{\Lambda^2 t^2} \hat{\xi} = 0. \tag{4.61}$$

The underbraced term will be negligible for large  $t$  ( $\gg \Lambda^{-1}$ ), so large- $t$  asymptotic behaviour governed by

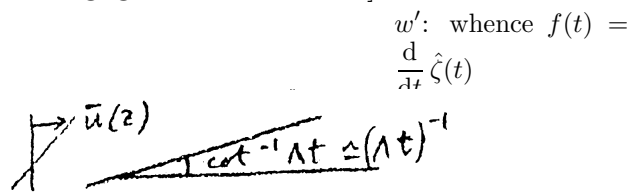
$$\boxed{\frac{d^2 \hat{\xi}}{dt^2} + (N^2/\Lambda^2 t^2) \hat{\xi} = 0} \quad (t \uparrow \infty) \tag{4.62}$$

to which the solutions are (again writing  $N^2/\Lambda^2 = \text{Ri}$ )

$$\hat{\xi}(t) \propto t^{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \text{Ri}}} \quad (t \gg \Lambda^{-1}) \tag{4.63}$$

[You should check via (4.60) and the relation  $w' = D_t \zeta$  that (4.63) is consistent with (4.59), and that middle term (4.61) is negligible for  $t \gg \Lambda^{-1}$ .]

Equation (4.62) is the same as the equation for a particle on an inclined plane, under a restoring force acting vertically and equal to  $N^2$  per unit vertical displacement. The situation is closely comparable to that depicted on p. 27, apart from the fact that the angle of inclination is fixed there, and changing here. In particular, solution still **valid at finite amplitude**.



Now note from (4.63) that  $\xi$  is unbounded as  $t \uparrow \infty$ , showing that light fluid elements must eventually slide underneath heavier ones. motion becomes statically unstable locally (cf. p. 45)<sup>10</sup>



-N46-

Thus, remarkably, our dissipationless, ideal-fluid model has led to a firm prediction that type II disturbances *must, in practice, if not otherwise dissipated, lead to small-scale turbulence and dissipation.* This is quite fundamental to understanding the

-I.54-

irreversible processes that go on all the time in naturally-occurring stratified (shear) flows — and we have already glimpsed in §3.4 some of the reasons why the observed *mean* flows may depend on just such ‘irreversibly transient’ processes. [‘wave transience’ or  $\frac{\partial}{\partial t}$  terms in (3.30) do *not* have time integral zero, integrated over lifetime of disturbance.] The unstable breakdown of type II disturbances is also the basic process underlying the so-called ‘critical-layer absorption’ or ‘singular absorption’ phenomenon. If a packet of internal gravity waves propagates in a shear flow with  $\text{Ri} \gtrsim 1$ , say, it is found that almost all the disturbance gets ‘caught’ near those levels, if such exist, where  $\bar{u}(z)$  falls within the range of horizontal phase speeds  $\omega/k$  possessed by the Fourier components whose superposition comprises the wave packet.<sup>11</sup>

-N47-

## A numerical experiment

-N48-

Here are some pictures from a numerical experiment by D. Fritts *J. Geoph. Res.* **87C**, 7997 (1982) in which equations (4.1) were solved with the  $\bar{u}(z)$  profile shown in fig. 4.6. The waves were generated by prescribing  $w' = \hat{w}_0(t) \cos kx$  at  $z = 0$ , where  $w_0(t)$  is shown in fig. 4.6 also.

<sup>10</sup>This prediction is self-consistent, since our solution is another exact solution of the *nonlinear* equations, for the same reason that the simple plane waves on p. 27 are  $(\mathbf{u}' \cdot \nabla \mathbf{u}' = 0 = \mathbf{u}' \cdot \nabla \sigma')$  for structure func  $(x - \Lambda t z)$ .

<sup>11</sup>**Exercise:** Show that the ‘ray-tracing equations’ for internal gravity waves with dispersion relation

$$\omega = \hat{\omega}(k, m) + k \bar{u}(z), \quad \bar{u} = \Lambda z,$$

where  $\hat{\omega}$  is the intrinsic frequency given by (2.6), predict that a wave packet of frequency  $\omega$  approaching its ‘critical level’  $\omega = k \bar{u}(\hat{\omega} = 0)$  *never escapes* (ray time infinite). This corroborates picture in case of ‘large’ basic-flow  $\text{Ri}$  ( $2 \gg 1$  in practice!). We can now understand why ‘lee waveguides’ (pp. 83–84) depend on  $\bar{u}(z)$  being one-signed!

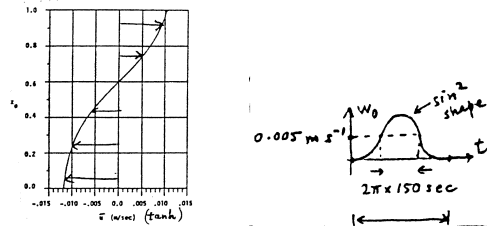




Figure 4.6: **Left figure:**  $\max \bar{u}_z = 0.01 \text{ s}^{-1}$ ,  $N^2 = \text{const.} = 2 \times 10^{-4} \text{ s}^{-2}$ , so  $\min \text{Ri} = 2$  for basic flow.  $z_0 = 3.75$  metres (height of domain),  $k = 0.8 \text{ metres}^{-1}$ . (Numbers are chosen with oceanic thermocline in mind.)

**Right figure:**  $w_0(t)$ . The duration is 3 of the units  $\frac{2\pi}{k \bar{u}(0)} = 2\pi \times 100 \text{ sec}$ .

Some results are shown in fig. 4.7. Various ways of viewing the result: Time increases as we go down this sequence, and is denoted by boxed numbers giving time in units of  $\frac{2\pi}{k \bar{u}(0)} = 2\pi \times 100 \text{ sec}$ . The left column shows isopycnals,  $\rho \propto n^2 z + \sigma$ .

The middle sequence shows where this *linear* calculation predicts fluid is

locally statically unstable  (and also where  $\text{Ri}_{\text{total}}$ , the local Richardson number for basic flow plus disturbance, i.e. for total motion,  $< 1$  and (the  $\dots$  contour)  $< \frac{1}{4}$ ) (the  $- - -$  contour). A corresponding *fully non-linear* numerical simulation shows small-scale instabilities developing in  regions as expected. In a real fluid these would usually lead to turbulence.

At timestep  $\boxed{6}$  the solution is now unmistakably type II, over range of levels  $\bar{u}(z) = \omega/k$ . ? dependence of (Amplitude  $\sim$  *Fourier transform* of  $\hat{w}_0(t)$ )  
Booker, J. R., Bretherton, F. P. (1967), The critical layer for internal gravity waves in a shear flow, *J. Fluid Mech.*, **27**, 513–539.)

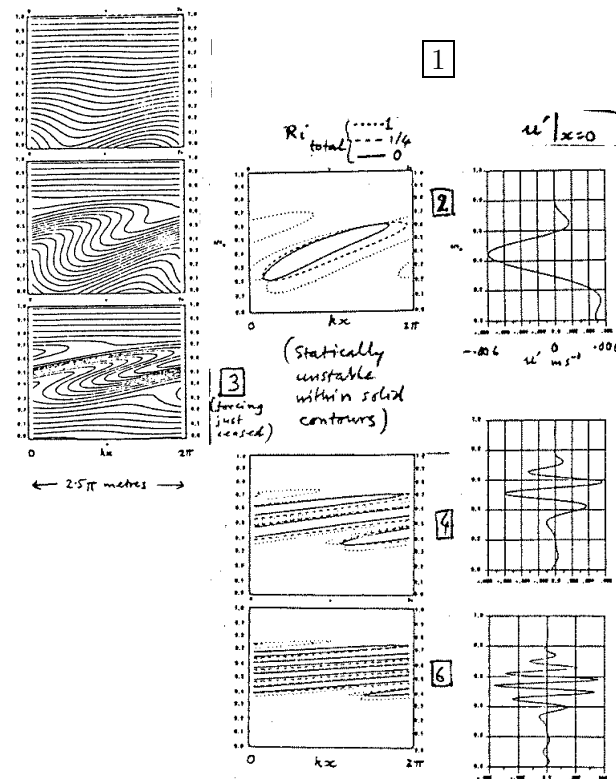


Figure 4.7: Results from D. Fritts, *J. Geoph. Res.* **87C**, 7997 (1982). The left column shows isopycnals,  $\rho \propto n^2 z + \sigma$ . The middle sequence shows where the linear calculation predicts fluid is locally statically unstable. See text for further description.

## 5.

# The simplest model exhibiting a gravitational restoring mechanism — the ‘shallow-water’ equations

—1.55—

These will be especially important when we subsequently add rotational (Coriolis) effects, to get the simplest ‘rotating, stratified’ system. The fluid system is a layer (depth  $h$  say) of homogeneous, incompressible fluid ( $\rho$  strictly constant) with a free upper surface (total pressure  $P = 0$ , or constant, on that surface). Thus all the stable stratification is concentrated in the free surface ( $N^2$  has delta-function form). The basic assumptions are

1. horizontal length scales  $L \gg h$  (whence ‘shallow’);
2. hydrostatic balance holds;
3. viscosity can be neglected: the flow is frictionless; and
4. the horizontal velocity components  $\mathbf{u}_H = (u_H, v_H, 0)$  are functions of time  $t$  and horizontal position  $x, y$  alone, so that  $\frac{\partial u_H}{\partial z} = \frac{\partial v_H}{\partial z} = 0$ .

We also assume for simplicity that the bottom boundary is rigid and flat, though one can easily generalize to gently sloping boundaries.

Writing  $\nabla_H = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)$ , we can show that the momentum and mass-conservation equations, together with the boundary conditions, then imply the following *two-dimensional* momentum and mass equations, all variables (including  $h$ ) being functions of  $(x, y, t)$ :

$$\underbrace{\frac{\partial \mathbf{u}_H}{\partial t} + \mathbf{u}_H \cdot \nabla_H \mathbf{u}_H}_{\text{(call this } \mathcal{D}_H \mathbf{u}_H / \mathcal{D}t)} = -g \nabla_H h \quad \text{(momentum)} \quad (5.1 \text{ a})$$

$$\frac{\partial h}{\partial t} + \nabla_H \cdot (h \mathbf{u}_H) = 0 \quad \text{(mass)} \quad (5.1 \text{ b})$$

Equation (5.1a) is simply the horizontal projection of the  $\mathcal{D}/\mathcal{D}t$  form of the frictionless three-dimensional momentum equation, since assumption (4) reduces the horizontal components of  $\mathbf{u} \cdot \nabla \mathbf{u}$  to  $\mathbf{u}_H \cdot \nabla_H \mathbf{u}_H$ , and assumption (2) implies that, at any fixed height  $z$ , the (total, unmodified) pressure

$P = \text{const.} + \rho g h$ , so that  $-\rho^{-1}\nabla_H P = -g\nabla_H h$ . Equation (5.1b) is the vertically integrated mass-conservation equation under assumption (4).<sup>1</sup> One can derive (5.1b) *either* by considering the rate of flow of mass into a vertical-sided volume whose horizontal projection is an arbitrary (or, alternatively, a small) region of the  $xy$  plane, *or* (perhaps slightly easier) by integrating the incompressible mass-conservation equation  $\nabla \cdot \mathbf{u} = \nabla_H \cdot \mathbf{u}_H + \partial w / \partial z = 0$  with respect to  $z$  from bottom to top of the layer, say from 0 to  $h$ . For we have

$$\int_0^h \nabla_H \cdot \mathbf{u}_H \, dz = h \nabla_H \cdot \mathbf{u}_H, \quad \text{using assumption (4);} \quad (5.2)$$

and

$$\int_0^h (\partial w / \partial z) \, dz = [w]_0^h = w|_{z=h} = \frac{\partial h}{\partial t} + \mathbf{u}_H \cdot \nabla_H h, \quad (5.3)$$

using the boundary conditions that fluid particles remain on the upper and lower boundaries, i.e. that  $w = 0$  on  $z = 0$  and that  $w =$  material rate of change of  $h$  at  $z = h(x, y, t)$ . Hence

$$h \nabla_H \cdot \mathbf{u}_H + \frac{\partial h}{\partial t} + \mathbf{u}_H \cdot \nabla_H h = 0,$$

hence (5.1b).

It remains to show that the model (5.1) is a self-consistent approximation under the assumptions (1)–(4), and that (1)–(4) are themselves mutually consistent. **First**, assumption (4) is consistent with equation (5.1a) and assumption (2), because  $-\nabla_H P$  is  $z$ -dependent because of its proportionality to  $-\nabla_H h$ , so horizontal accelerations are  $z$ -dependent (in the absence of friction). Hence if (4) is true at some initial time it remains true, at least over a bounded time interval. **Second**, the motion described in (5.1) has  $w \neq 0$  in general, as already implied; but since  $\nabla \cdot \mathbf{u} = 0$  in three dimensions we have  $w \lesssim U h/L$  just as on p. 47, so that  $\mathcal{D}w/\mathcal{D}t \lesssim (h/L) \times$  typical magnitude  $A$ , say, of  $\frac{\mathcal{D}_H \mathbf{u}_H}{\mathcal{D}t}$ . Let  $P'$  be the departure of  $P$  from hydrostatic balance; then need to show  $\rho^{-1} \nabla_H P' \ll A$ . Now  $-\rho^{-1} \partial P' / \partial z = \mathcal{D}w/\mathcal{D}t$  by definition of  $P'$ ; therefore

$$|\rho^{-1} \nabla_H P'| = \left| \nabla_H \int_z^h \frac{\mathcal{D}w}{\mathcal{D}t} \right|_{z=\tilde{z}} \, d\tilde{z} \lesssim \frac{1}{L} \cdot h \cdot \frac{h}{L} A = \frac{h^2}{L^2} A \ll A, \quad (5.4)$$

<sup>1</sup>*Exercise:* Show that small disturbances about a uniform state of rest  $h = h_0$ ,  $\mathbf{u}_H = 0$  satisfy the classical wave equation  $h''_{tt} - c_0^2 \nabla_H^2 h' = 0$  where  $c_0 = \sqrt{g h_0}$ . These nondispersive waves are the gravity waves of the system.



by assumption (1), showing its consistency with dynamical behaviour under assumption (2).

It is not necessarily true that the assumed conditions would remain approximately true of a real fluid layer over arbitrarily long times, although the smaller  $h^2/L^2$  is, the longer we might expect validity to persist. We shall see shortly that strong Coriolis effects ( $\gg \mathcal{D}u/\mathcal{D}t$ ) help assumption (4) to remain true, via a phenomenon called ‘rotational stiffness’, or the ‘Taylor–Proudman effect’. Note also [non-examinable]:

- (a)  $\exists$  precise analogy (equations are the same) between (5.1) and two-dimensional gas dynamics, with the correspondence

$$\left. \begin{array}{l} h \text{ in (5.1)} \leftrightarrow \text{density } \rho \text{ in gas dynamics} \\ \frac{1}{2} g h^2 \text{ in (5.1)} \leftrightarrow \text{pressure } p \text{ in gas dynamics} \end{array} \right\} \begin{array}{l} \text{with compressible} \\ \text{eqn of state } p \propto \rho^2 \end{array}$$

(so the ‘gas’ is a fictitious one with specific heat ratio  $\gamma = c_p/c_v = 2$ , sound speed  $\left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} = (g h_0)^{1/2}$ .)

- (b) Can extend theory (at some cost in accuracy) to include turbulent friction, and prescribed pressures and frictional stresses on top surface — giving it direct practical, not only theoretical, importance, e.g. in tidal and storm-surge forecasting in shallow seas like the North Sea.



# Part II

## Rotating flow



Typical atmospheric and oceanic flows (see next page) strongly feel Coriolis effects from the Earth's rotation. Part II of these notes develops the basic theory necessary to understand such flows, and their role in atmospheric and oceanic circulation and transport. The two most basic ideas are those of *Rossby wave propagation* and *potential-vorticity inversion*.

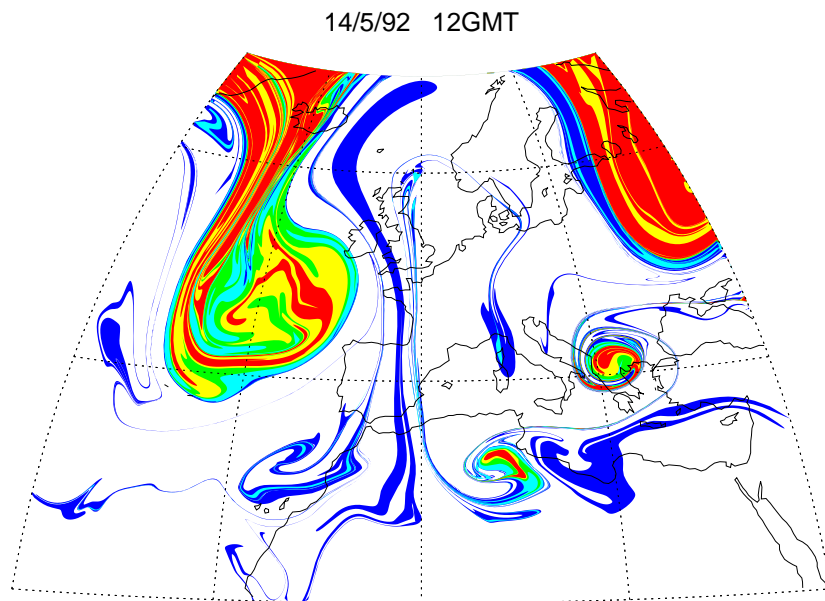


Figure 5.1: Passive tracer on upper-air stratification surface, from Appenzeller, C., Davies, H. C., Norton, W. A. (1996), *J. Geophys. Res.*, **101**, 1435–1456.

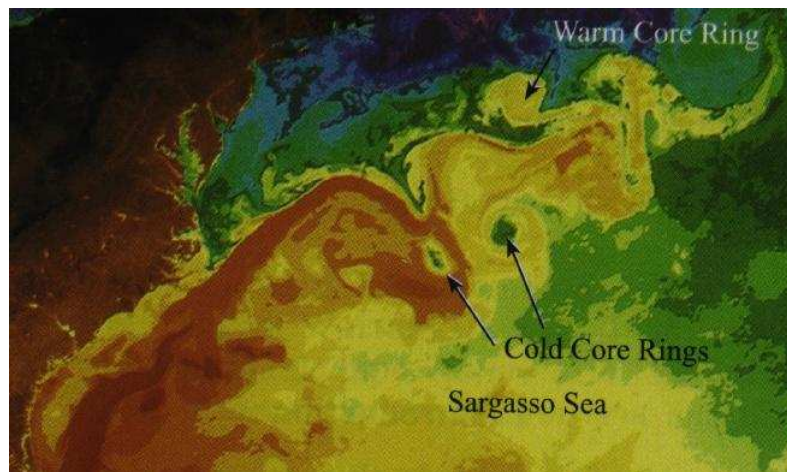


Figure 5.2: Gulf Stream eddies, from C. Garrett, *The Dynamic Ocean*, in Batchelor, G. K., Moffatt, H. K., Worster, M. G. (2000), *Perspectives in Fluid Dynamics*, CUP.

## 6.

### The analogy between 2-D, constant- $N$ stratified flow and 2-D, homogenous rotating flow:

Homogeneous, incompressible flow in a rotating frame of reference (take  $\mathbf{F}$  = viscous force in (0.2 a)):

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \tilde{\chi} + \nu \nabla^2 \mathbf{u},$$

where  $\rho$  is strictly constant and  $\tilde{\chi} = \chi - \frac{1}{2} \varpi^2 |\boldsymbol{\Omega}|^2$ . Substituting  $P = -\rho \tilde{\chi} + p$ , we get

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \quad \text{Also } \nabla \cdot \mathbf{u} = 0. \quad (6.1)$$

(Note: position of rotation axis no longer appears explicitly.)

For *two-dimensional motion* there exists an exact analogy with uniformly-stratified Boussinesq fluid. The analogy is summarized in the following table:



$\mathbf{z}$ $\uparrow \rightarrow \mathbf{x}$ <b>Stratified:</b> $\downarrow \mathbf{g}$ $N$ const.; $\frac{\partial}{\partial y} = 0$ ; $v = 0$ ( $y$ -cpt. of velocity)	$\mathbf{z}$ $\uparrow \rightarrow \mathbf{x}$ <b>Rotating:</b> $\rightarrow \mathbf{\Omega}$ $\frac{\partial}{\partial y} = 0$ ; $v = v(x, z)$ ; $\rho = \text{const.}$
$\frac{\mathcal{D}u}{\mathcal{D}t} = -\frac{1}{\rho_{00}} \frac{\partial p}{\partial x} + \nu \nabla^2 u$ $\frac{\mathcal{D}w}{\mathcal{D}t} + N \left( \frac{-\sigma}{N} \right) = -\frac{1}{\rho_{00}} \frac{\partial p}{\partial z} + \nu \nabla^2 w$ $\frac{\mathcal{D}}{\mathcal{D}t} \left( \frac{-\sigma}{N} \right) - N w = \kappa \nabla^2 \left( \frac{-\sigma}{N} \right)$ $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$	$\frac{\mathcal{D}u}{\mathcal{D}t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u$ $\frac{\mathcal{D}w}{\mathcal{D}t} + 2\Omega v = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w$ $\frac{\mathcal{D}v}{\mathcal{D}t} - 2\Omega w = \nu \nabla^2 v$ $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$

Because  $\partial/\partial y = 0$ , the material derivative  $\mathcal{D}/\mathcal{D}t = \partial_t + u \partial_x + w \partial_z$ . Coriolis terms  $2\Omega v$  and  $-2\Omega w$  appear only in the second and third equations on the right, because  $\mathbf{\Omega}$  points in the  $x$  direction.) The two sets of equations are the same if  $\nu = \kappa$  and if we identify  $v$  with  $-\sigma/N$ ,  $2\Omega$  with  $N$ , and  $\rho$  with  $\rho_{00}$ .

### Notes:

0. All 2-d, constant- $N$ , ‘stratified’ motions are also possible ‘rotating’ motions.
1. The contribution  $\frac{1}{2} \rho v^2$  to the relative kinetic energy for the rotating system is the analogue of the stratified ‘available potential energy’. [\*When  $N = \text{constant}$ , no approximations are involved in equation (0.16), apart from the Boussinesq approximation.\*]
2.  $p$  does *not* mean the same thing in both systems. [So the analogy fails if there is a free surface, or other boundary where the boundary condition involves  $p$ .]
3. Analogue of the Froude number,  $U/NL$  or  $U/NH$ , is the *Rossby number*,  $U/2\Omega L$  (if  $L$  or  $H$  is the length scale in the direction  $\parallel \mathbf{g}$  or  $\perp \mathbf{\Omega}$ ).



# 7.

## Inertia waves or ‘Coriolis waves’

(also called ‘epicyclic waves’ in astrophysics)

—II.2—

The restoring effect to which these waves owe their existence is often called ‘rotational stiffness’, which as we’ll see has some very striking effects.

Viscosity is now set to zero ( $\nu = 0$ ). A simple solution of the linearized equations is obtained by supposing  $\partial/\partial z = 0$  as well as  $\partial/\partial y = 0$ :

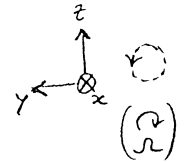
$$\begin{aligned} u_t &= -\rho^{-1} p_x \\ w_t + 2\Omega v &= 0 \\ v_t - 2\Omega w &= 0 \\ u_x &= 0, \end{aligned} \tag{7.1}$$

where, as on previous page,

$$\mathbf{\Omega} = (\Omega, 0, 0,)$$

Solution:  $u = 0, \therefore \partial p/\partial x = 0$ ;  $w = \hat{w}(x) e^{-2i\Omega t}, v = i\hat{w} e^{-2i\Omega t}$ . Simplest possible illustration of how the restoring effect analogous to ‘ $N$ ’ works.

Note, fluid particles move in circles, in the sense opposite to that of  $\mathbf{\Omega}$ , as suggested by the dashed circle in the sketch. ( $v$  is  $\pi/2$  out of phase with  $w$ , because of the factor  $i$ .)



More generally, with  $\hat{w}$  any function of  $\mathbf{k} \cdot \mathbf{x}$  alone (cf. (2.12 d)) and  $\partial/\partial z \neq 0$ , but still with  $\partial/\partial y = 0$ , we can reproduce the two-dimensional counterparts of (2.6) and (2.7) together with their geometrical interpretation:

$$\begin{aligned} w &= \hat{w}(\mathbf{k} \cdot \mathbf{x}) e^{-i\omega t}, & v &= \frac{2i\Omega}{\omega} \hat{w} e^{-i\omega t}, & u &= -\frac{m}{k} \hat{w} e^{-i\omega t} \\ \omega &= \pm \frac{2\Omega k}{(k^2 + m^2)^{1/2}}, & \mathbf{k} &= (k, 0, m) \\ \mathbf{c}_g &= \pm \frac{2\Omega}{(k^2 + m^2)^{3/2}} \{m^2, 0, -k m\} \end{aligned}$$

Notice immediately that these formulae can be written in **vector form** — therefore independent of choice of axes relative to  $\mathbf{\Omega}, \mathbf{k}$  — as follows<sup>1</sup>

<sup>1</sup>Note that plane-wave motion is ‘two-dimensional’, in the sense required by the analogy with stratified motion. The plane of  $\mathbf{\Omega}$  and  $\mathbf{k}$  replaces the  $xz$  plane. There is a velocity component  $\perp$  to that plane, but  $\mathbf{u}$  and  $p$  are independent of the direction  $\perp$  that plane.

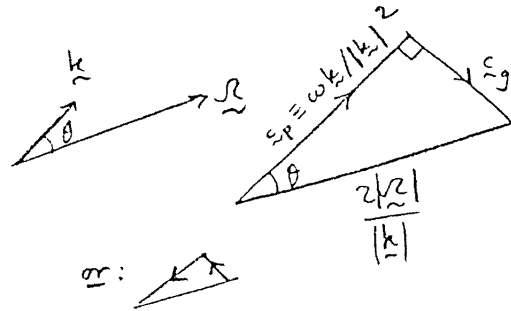
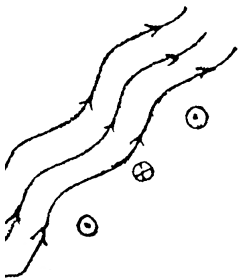


Figure 7.1: Signs in formulae imply that arrows play follow-the-leader, as shown.



$$\mathbf{u} = \hat{\mathbf{u}}(\mathbf{k} \cdot \mathbf{x}) e^{-i\omega t}, \quad \hat{\mathbf{u}} = \mathbf{u}_0 \pm i \mathbf{u}_0 \times \frac{\mathbf{k}}{|\mathbf{k}|} \quad (\mathbf{u}_0 \text{ real, } \perp \mathbf{k})$$

$$\omega = \pm \frac{2\boldsymbol{\Omega} \cdot \mathbf{k}}{|\mathbf{k}|} \quad (= \pm 2|\boldsymbol{\Omega}| \cos \theta)$$

$$\mathbf{c}_g = \pm \frac{\mathbf{k} \times (2\boldsymbol{\Omega} \times \mathbf{k})}{|\mathbf{k}|^3} \quad \left( |\mathbf{c}_g| = \frac{2|\boldsymbol{\Omega}| \sin \theta}{|\mathbf{k}|} \right),$$

making the geometric interpretation more apparent: The wavemotion can be thought of as self-induced motion of tubes of basic vorticity  $2\boldsymbol{\Omega}$ , due to these tubes having been twisted into a spiral as shown: the linearized vorticity equation is  $\partial\boldsymbol{\omega}/\partial t = 2\boldsymbol{\Omega} \cdot \nabla \mathbf{u}$  ( $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$ ); note  $\mathbf{u}$  and  $\therefore \boldsymbol{\omega} \perp \mathbf{k}$  in this motion.

[**But NB:** the behaviour can be very different indeed if the basic vorticity is dominated by a basic shear flow  $\mathbf{U}$ : e.g.  $\boldsymbol{\Omega} = (\Omega, 0, 0)$ ,  $\mathbf{U} = (0, V(z), 0)$ : (7.1) above replaced by

-N50-

$$\begin{aligned} w_t + 2\Omega v &= 0 \\ v_t + (V_z - 2\Omega) w &= 0. \end{aligned}$$

So frequency  $\omega = \{2\Omega(2\Omega - V_z)\}^{1/2}$ . This is an example of what is variously called ‘inertial instability’ or ‘Rayleigh instability’!]

-II.2.a-

# 8.

## ‘Rotational stiffness’

—II.3—

What is the rotating-flow counterpart of equation (3.10), expressing the general tendency of 2D stratified flow to be  $x$ -independent, or as nearly  $x$ -independent as it can be, at low Froude number?

-N51-

Answer: ditto at low *Rossby* number; more precisely, for  $\min(R_U, R_T) \ll 1$  where  $R_U$  and  $R_T$  correspond (partially, see below) to the Froude number and its temporal counterpart defined in (3.3 a),(3.3 b), on p. 46. *But there is now an extra dimension available  $\perp x$ .*

-N52-

Let us now rename the axes (anticipating standard notation to be used below) such that the  $z$ -axis  $\parallel \Omega$ . Then the main points are that

-N53-

4. 3-D rotating problems are in some respects *qualitatively* (not exactly) analogous to 2-D stratified problems. But note new phenomena arising from the extra available dimension! A famous example is the “Taylor column” due to an obstacle obstructing slow motion of fluid in the direction *perpendicular* to  $\Omega$ , between boundaries  $\perp \Omega$ . See Batchelor’s textbook, *Introduction to Fluid Dynamics*, plate 23, figures 7.6.3. (Continuing the numbering from the last section)

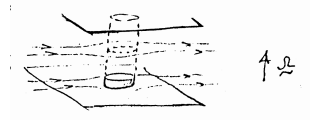

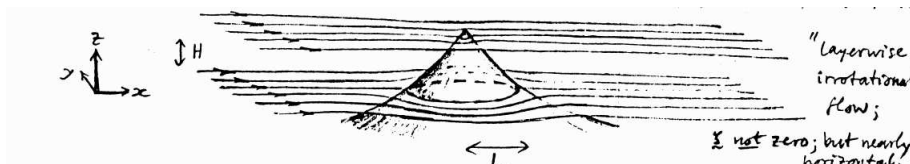


Figure 8.1: Unstratified low-Rossby-number flow past an obstacle.

5. 3-D stratified motion [ e.g.  ] is *not* analogous to any homogeneous rotating flow.



### Formal definition: (strictly) geostrophic flow

Flow satisfying the limiting forms for ‘zero’ Rossby numbers  $\frac{1}{2\Omega T}$ ,  $\frac{U}{2\Omega L}$ , of the inviscid, unforced ( $\mathbf{F} = 0$ ) momentum and mass-continuity equations in a rotating frame. That is, strictly geostrophic flow is  $(\mathbf{u}, p)$  satisfying

$$2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p \quad ; \quad \nabla \cdot \mathbf{u} = 0.$$

The first relation (balancing the Coriolis pressure-gradient terms) expresses what is called **geostrophic balance**.

### Taylor–Proudman theorem

For strictly geostrophic flow we have, taking the  $z$ -axis  $\parallel \boldsymbol{\Omega}$ ,

$$\frac{\partial p}{\partial z} = 0 \quad ; \quad \frac{\partial \mathbf{u}}{\partial z} = 0.$$

-N54-

**Proof:** With  $\boldsymbol{\Omega} \parallel z$ , we have  $\nabla p \perp z$ , so  $p_z = 0$ ;




$\therefore \nabla p = \text{func}(x, y)$  (because of geostrophic balance)

$\therefore (u, v) = \text{func}(x, y) = \frac{(-p_y, p_x)}{2\Omega} \Rightarrow u_z = 0 = v_z; \quad u_x + v_y = 0;$

$\therefore w_z = 0;$  so, in summary,  $\partial \mathbf{u} / \partial z = 0$ ,

using mass conservation/continuity. [Analogue of (3.10), 2D stratified.] [See Batchelor’s textbook, plate 23, figures 7.6.2.]

### Geostrophic contour of a container finite in the $\boldsymbol{\Omega}$ direction

Let the depth of the container measured along the  $\boldsymbol{\Omega}$  direction be  $d$ : a geostrophic contour is a curve on (either) boundary (or projected on to the plane  $\perp \boldsymbol{\Omega}$ ) such that  $d = \text{const}$ . Note different possibilities, e.g. *cylinder* , *sphere* , hemisphere .



Any curve is a  
geostrophic contour.



This closed  
geostrophic contour.



All geostrophic  
contours are open.

(It is easy to guess immediately that *real* flows at *small* Rossby number will be very different in the three cases.)

Notice that the Taylor–Proudman theorem says nothing whatever about evolution in time. The information it gives is like a constraint in classical mechanics, such as saying that a bead must stay on a curved wire of fixed shape, without saying anything about the velocity and acceleration of the bead.

### Converse of Taylor–Proudman theorem

If  $\psi(x, y)$  is the streamfunction of **any two-dimensional solution**, meaning  $\partial/\partial z$  (all fields) = 0, of the (**viscous or inviscid**) incompressible homogeneous-fluid equations in an inertial frame of reference ( $\mathbf{\Omega} = 0$ ), and if  $\psi$  is *single-valued*, then  $\psi$  also provides a solution of the equations in the rotating frame of reference, with constant  $\mathbf{\Omega}$  parallel for the  $z$ -axis.

**Proof:** The only equation containing  $\mathbf{\Omega}$  is the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p + (\text{anything independent of } \mathbf{\Omega}) .$$

With  $(u, v, 0)$  equal to, say,  $(-\psi_y, \psi_x, 0)$ , we have  $2\mathbf{\Omega} \times \mathbf{u} = 2|\mathbf{\Omega}|(-v, u, 0) = -2|\mathbf{\Omega}|(\psi_x, \psi_y, 0)$ . So when  $\mathbf{\Omega}$  is made  $\neq 0$ , we need only add  $2|\mathbf{\Omega}|\psi$  to the  $p$  field — permissible because  $\psi$  is single-valued — keeping the same  $\mathbf{u}(\mathbf{x}, t)$ . Then we still have a solution of the equations.

Note, however, that this says only that the equations are satisfied. In a boundary-value problem with a  $p$ -dependent boundary condition that is satisfied when  $\mathbf{\Omega} = 0$ , the solution would cease to satisfy the boundary condition when  $\mathbf{\Omega}$  is made  $\neq 0$ .



## 9.

### Slow 3-D motions (of a homogeneous fluid — not yet the real atmosphere or ocean)

$\boldsymbol{\Omega}\|z$  — call this ‘vertical’. [For any 2D motion, e.g. plane waves, we have the stratified analogue, but must think of  $\mathbf{g}$ , for that purpose, as ‘horizontal’ and lying in the  $(\boldsymbol{\Omega}, \mathbf{k})$  plane.] —II.4—

Slow motion, in the sense below, is a 3-D generalization of the analogue of **2D** stratified flow for low Froude number,  $U/NH \ll 1$ , and  $(1/NT) \ll 1$ . The extra dimension in which rotational stiffness acts — it acts, remember, in the two dimensions  $\perp \boldsymbol{\Omega}$ , as compared with the one dimension  $\parallel \mathbf{g}$  — gives rise to many more interesting possibilities.

What happens to geostrophic motion when inviscid fluid is confined between boundaries *nearly*  $\perp \boldsymbol{\Omega}$ ? and moves across geostrophic contours? (Or when, for some other reason,  $w_z = -u_x - v_y$  is forced to be slightly nonzero.) It can’t then be *strictly* geostrophic.

Assume

$$\begin{aligned} R_T &\equiv \frac{1}{2\Omega T} \ll 1 \\ R_U &\equiv \frac{U}{2\Omega L} \ll 1 \end{aligned} \tag{9.1}$$

(no longer ‘exactly zero’), where  $T, U, L$  scales for time, horizontal velocity [relative], horizontal  $(x, y)$  length. Let  $H$  be scale for height  $z$ ,  $W$  scale for vertical velocity, entailing that  $w \approx W \approx H w_z$ . Assume  $\frac{H}{L} \lesssim 1$ . Equations (inviscid, incompressible, unstratified) are

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u - 2\Omega v = -p_x \tag{9.2 a}$$

$$\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + 2\Omega u = -p_y \tag{9.2 b}$$

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = -p_z \tag{9.2 c}$$

$$w_z = -u_x - v_y \tag{9.2 d}$$

(Remember  $p = \textit{modified pressure} \div \textit{density}$ .)

From (d),  $w \lesssim U H/L$ ; then (a)–(c) give

$$\boxed{\begin{aligned} u &= -\frac{1}{2\Omega} p_y \\ v &= \frac{1}{2\Omega} p_x \end{aligned}} \quad (9.3)$$

the geostrophic relations now with relative error  $O(R)$  where  $R = \max(R_U, R_T)$ .

$$\text{Again noting } w \lesssim U H/L \text{ we get, from (c), } p_z \lesssim \max\left(\frac{1}{T}, \frac{U}{L}\right) \frac{UH}{L} \approx R \underbrace{2\Omega U L}_{\frac{H}{L^2}}$$

$$H p_z \lesssim \underbrace{\max(R_U, R_T)}_R \max(L p_x, L p_y) \cdot \frac{H^2}{L^2} \quad (9.4)$$

so throughout depths of order  $H$  we can take

$$p = \text{func}(x, y) \left[ 1 + O\left(\underbrace{R_T + R_U}_R \times H^2/L^2\right) \right] \quad (9.5)$$

(Taylor–Proudman theorem with error terms)

—II.4.a—  
—II.5—

If  $p(x, y)$  is given,<sup>1</sup> then the approximate velocities that would be deduced from (9.3) are called the ‘geostrophic velocities’.

This first approximation, (9.5) and (9.3) and  $w \simeq 0$  (meaning  $\ll U H/L$ ) is the Taylor–Proudman theorem again. As before, it tell us *nothing* about the time dependence on the slow time scale  $T$ . To get that, we can write  $u_0, v_0, p_0$  for a solution of (9.5)(9.3), and

$$\begin{aligned} u &= -\frac{1}{2\Omega} p_y + u_1 \\ v &= \frac{1}{2\Omega} p_x + v_1 \end{aligned} \quad (9.6)$$

— and consider the next correction  $u_1, v_1$  explicitly. (The corrections are sometimes called ‘ageostrophic velocities’.) [In a ‘dimensionless formulation’ this is the same thing as considering the second terms in an expansion in powers of one of the small parameters<sup>2</sup> (page 49)  $R_T, R_U$ .] **Alternatively**, we can form the vertical component of the *vorticity equation* by cross-differentiating (9.2 a & b), before making any approximations. Write  $\nabla \times \mathbf{u} = (\xi, \eta, \zeta)$

—N55—

<sup>1</sup>(e.g. by observation of large-scale field)

<sup>2</sup>recall remarks on p. I.21 (footnote). But by committing oneself to a particular nondimensionalization, one can lose sight of some of the possibilities!



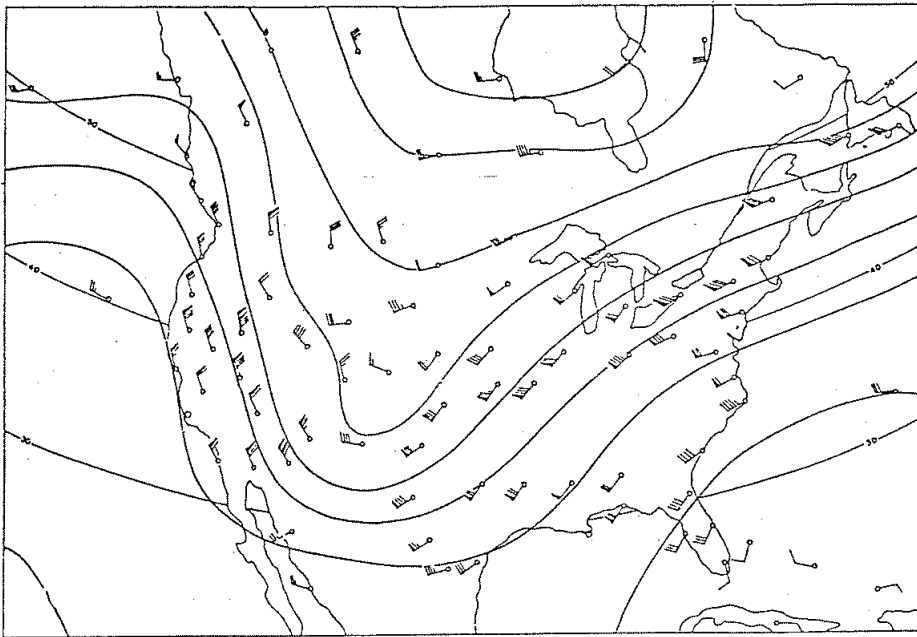


Figure 9.1: Large scale motions observed in the atmosphere: An example of a flow pattern at the 500-mb level ( $\sim 18000$  feet MSL), 0000UT, January 21, 1959. Isolines are height contours at 400-foot intervals. Arrows indicate observed winds at this level (triangular barb = 50 knots, full barb = 10 knots,  $1/2$  barb = 5 knots). The contours are essentially of constant  $p$ , at  $8\frac{1}{2}$  mbar intervals. From Phillips, N. A. (1963), *Reviews of Geophysics* **1**, 123–76

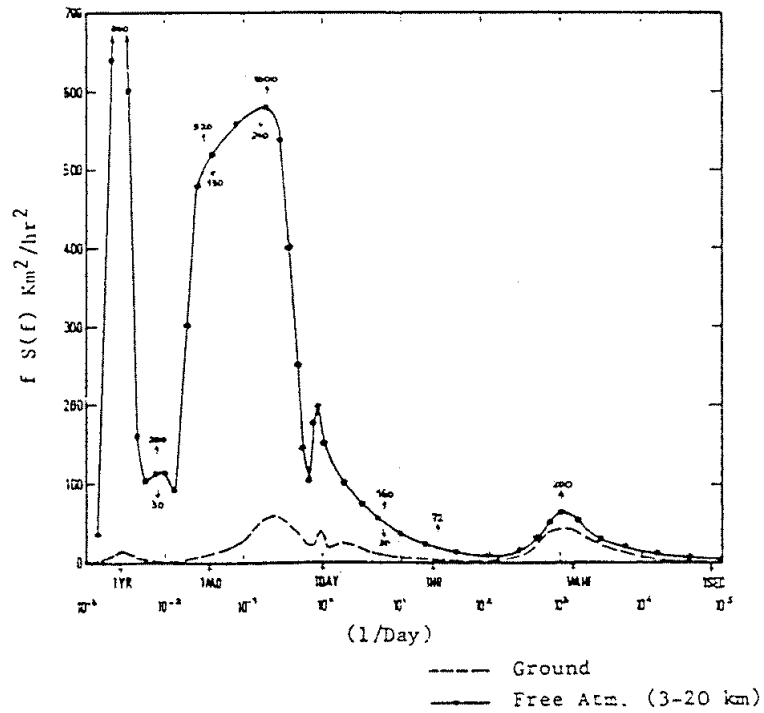


Figure 9.2: Average kinetic energy of E-W wind component in the free atmosphere (solid line) and near the ground (dashed line). Number indicate maximum values of kinetic energy at particular periods (after Vinnichenko (1970) *Tellus* **22**, 158-166).

(not to be confused with the particle displacements defined on page 63), i.e.,  $\xi = w_y - v_z$ ,  $\eta = u_z - w_x$ ,  $\zeta = v_x - u_y$ . Then

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = (2\Omega + \zeta) \frac{\partial w}{\partial z} + \xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y}. \quad (9.7)$$

$2\Omega w_z$  describes stretching of the vortex lines of the (strong, stiff, elastic!) background rotation. We may expect this term to be important. Indeed, order-of-magnitude estimation now shows that the leading approximation to (9.7) is

$$\boxed{\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = 2\Omega w_z} + O\left(\frac{2\Omega W}{H} R_U\right) \quad (9.8)$$

where, furthermore,  $\zeta$  and  $\mathbf{u}$  are now given to sufficient accuracy by (9.3). In particular we may define a stream function

$$\psi \equiv \frac{p}{2\Omega} \quad (9.9)$$

such that in (9.8) we can take  $\mathbf{u} = (u, v, 0)$  with

$$\boxed{u = -\psi_y, v = \psi_x; \zeta = \psi_{xx} + \psi_{yy} \quad [\text{functions of } (x, y) \text{ only}].} \quad (9.10)$$

Finally, for consistency with (9.1), we must have not merely  $w \lesssim UH/L$  but (since  $\zeta \approx U/L$  in (9.8))

$$w \lesssim R \frac{UH}{L} \quad (R \equiv \max(R_U, R_T) \text{ as before}) \quad (9.11)$$

This uses the assumption made below (9.1) that  $w \approx w_z H$ , and shows, in turn, that (9.4) is even more strongly satisfied. Its RHS can be multiplied by a *further* factor  $R = \max(R_U, R_T)$ .]. Furthermore, (9.8) now  $\Rightarrow$

$$\boxed{w \text{ is not only small, but depends linearly on } z.} \quad (9.12)$$

\*Can also show  $u_1$  and  $v_1$  are  $z$ -independent, by repeating the argument that led to (9.5) but with the sharper estimate (9.11) for  $w$  hence  $Dw/Dt$  in (9.2c). Thus (9.5) holds with relative error  $O(R^2)$ , not merely  $O(R)$ , as  $R \downarrow 0$ . The Coriolis terms in (9.2a,b), hence  $(u, v)$ , must also be  $z$ -independent correct to two orders in  $R$ .\*

Eq. (9.8) with (9.10) are the simplest example of what are usually called ‘**quasi-geostrophic equations**’ for rapidly rotating flow. They give us an asymptotically self-consistent model, which succeeds in describing the time-dependence of rapidly-rotating flows close to Taylor–Proudman conditions.



# 10.

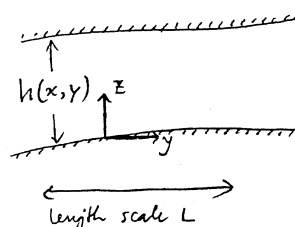
## Two examples: Rossby waves and flow over a gently sloping obstacle

### §10.1 Rossby waves

Consider the slow  $t$ -dependence due to slightly-sloping boundaries:

$$\text{so (9.11) restricts } \frac{\Delta h}{h} \lesssim R \quad (10.1)$$

$$\boxed{w_{\text{top}} - w_{\text{bot.}} = \mathbf{u} \cdot \nabla h .} \quad (10.2)$$



Linearized: (9.8)  $\rightarrow \zeta_t = \frac{2\Omega \nabla h}{h} \cdot \mathbf{u}$ , using (9.12) above. Specialize to case

$$2\Omega h^{-1} \nabla h = (0, -\beta) ; \quad \beta \text{ const..}$$

Using (9.10),

$$\boxed{\frac{\partial}{\partial t} \nabla^2 \psi + \beta \psi_x = 0 .} \quad (10.3)$$

Note that this includes the possibility of *steady flow along geostrophic contours* (which are  $y = \text{const}$ ): a solution of (10.3) is

$$\boxed{\psi = \text{func}(y) \quad (v = 0) .} \quad (10.4)$$

[Clearly, also, a solution of (9.8) *before* linearization.]

If  $v \equiv \psi_x \neq 0$ , then (10.3) shows at once that *the motion must be time-dependent*. The time-dependence is such that there is a restoring effect against upslope or downslope displacements (i.e. in the  $y$  direction). This is shown mathematically by the fact that (10.3) admits wave solutions

$$\psi = \epsilon e^{i(kx + ly - \omega t)} \quad (10.5)$$

( $\epsilon = \text{const}$ , small), provided that

$$\boxed{\omega = \frac{-\beta k}{k^2 + l^2} .} \quad (10.6)$$

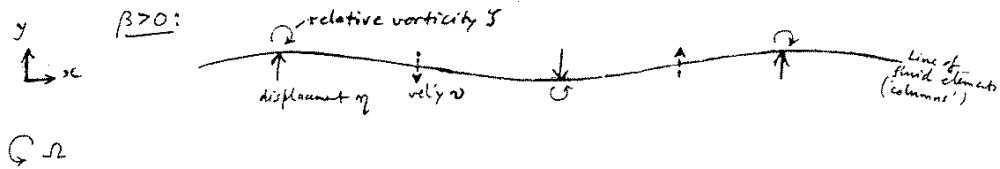
(Note phase speeds *and* frequencies  $\uparrow$  as wavelengths  $\uparrow$ .) In this situation, the validity of our approximations is not uniform in  $y$  even though  $h_y/h$  can be made exactly constant — the horizontal divergence  $u_x + v_y$ , neglected in (9.3), has the same sign along a wavecrest, and an error in  $v$  must accumulate over large  $y$ -distances. But these two-dimensionally plane-wave solutions can be superposed to represent disturbances of effectively *finite*  $y$ -extent, over which the approximations are valid.

(Recall that one has the same problem of cumulative error, and the same resolution of it, with the vertical dependence in Boussinesq plane internal gravity waves.)

—II.7—

These slow waves are called (topographic) **Rossby waves**. **NB! The dispersion relation (10.6) has only one branch!** Only the negative sign is allowed, corresponding to the fact that the differential equation (10.3) has only one  $\partial/\partial t$ . This is utterly unlike the more familiar kind of wave problem, in which time-reversibility leads to an even number of  $\partial/\partial t$  factors, and an even number of branches of the dispersion relation (usually two). Here the physical system *is* nondissipative and therefore time-reversible — but we need to remember that we’re in a rotating frame. The time-reversed solution involves making the frame (e.g. the Earth!) rotate the other way, changing the sign of  $\beta$ .

We can make a picture that helps us to understand the Rossby-wave restoring effect and the one-signedness of the phase speed, as follows. The case  $l = 0$  is simplest:



Suppose that a line of fluid columns is displaced from rest as shown. Those displaced in  $+y$  direction are *compressed* ( $\beta > 0$ :  $\downarrow_y$  ~~compression~~) and therefore have negative  $\zeta$ , as suggested by the  $\ominus$ , and vice versa.

[During the displacement,  $\text{sgn } 2\Omega w_z$  in equation (9.8) =  $-\text{sgn } v$ . Alternatively, if  $\eta$  now stands for displacement in the  $+y$  direction, then  $v = \eta_t$ , and (10.3) is equivalent to  $\zeta_t + \beta \eta_t = 0$ ; integrating from rest gives  $\zeta = -\beta \eta$ .]

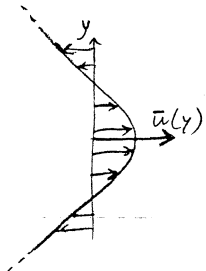
But  $\nabla^2 \psi = \zeta$ , so  $\psi = \nabla^{-2} \zeta$  with suitable boundary conditions (in this case periodicity). This illustrates a central concept (“**invertibility**”), the principle that the vorticity field may be regarded as determining the velocity field. Recall general insight from  $\psi = \nabla^{-2} \zeta$  (soap film picture).

In this case the velocity field is as indicated by the dashed arrows . The velocity is  $\frac{\pi}{2}$  out of phase with displacement, and *lags it in x*. **The whole pattern must therefore be propagating leftwards.**

To summarize:  $\pm y$ -displacement  $\Rightarrow$  vortex compression or stretching  $\Rightarrow$  relative vorticity  $\Rightarrow$  velocity field lagging displacement by  $\pi/2 \Rightarrow$  propagation in  $-x$  direction.

note use of invertibility principle at penultimate step  
←

Note for later reference that a similar propagation mechanism exists in a shear flow  $\bar{u}(y)$  with  $\beta = 0$ , if the *mean vorticity gradient*  $-\bar{u}_{yy}$  is nonzero. E.g. isolated region of positive vorticity gradient ( $\bar{u}_{yy} < 0$ ), see diagram on right. The linearized equation is now  $D_t \nabla^2 \psi - \bar{u}_{yy} \psi_x = 0$ , or  $D_t \zeta - \bar{u}_{yy} D_t \eta = 0$ , where  $D_t = \partial_t + \bar{u} \partial_x$ , so  $\zeta = +\bar{u}_{yy} \eta$  and there is again a tendency for a disturbance pattern like the one above to propagate against the basic flow.



\*This propagation mechanism is also present in equations (4.5), (4.19) etc., and plays a role in shear instabilities (cf. p. 93) and, to some extent, in lee-wave patterns (equation (4.19)).\* We shall see that recognition of this propagation mechanism provides a good way of understanding how a number of kinds of simple shear instabilities work. **In fact the Rossby-wave mechanism is basic to almost everything about the most environmentally important flows, the slow wavelike and eddying (vortical, tracer-transporting) flows in the atmosphere and ocean — e.g. shear instabilities, teleconnections, vortex coherence, ‘blocking’, Atlantic ‘Meddies’,...**

-N57-

—II.8—

Coming back to the simpler, present case of topographic Rossby waves about relative rest, we note also that the changes in  $\zeta$  due to vortex stretching  $2\Omega w_z$  can be summed up in a variant of the **idea of ‘potential vorticity’**. **This idea, too, is basic, to all the flows just mentioned.** Here we define the potential vorticity in the form (first noted by Rossby in 1936):

$$Q = \frac{2\Omega + \zeta(x, y, t)}{h(x, y, t)} . \tag{10.7}$$

then (10.2) and (9.8) [and  $w_z = (w_{\text{top}} - w_{\text{bottom}}/h) \Rightarrow$

$$\frac{DQ}{Dt} = 0 . \tag{10.8}$$

To verify this result, it’s easiest to start with  $\mathcal{D}/\mathcal{D}t$  of (10.7). Thus,  $h$  times

(10.8)  $\Leftrightarrow$

$$\frac{\mathcal{D}(2\Omega + \zeta)}{\mathcal{D}t} = \frac{2\Omega + \zeta}{h} \frac{\mathcal{D}h}{\mathcal{D}t} = \frac{2\Omega + \zeta}{h} \mathbf{u} \cdot \nabla h = (2\Omega + \zeta) \frac{w_{\text{top}} - w_{\text{bottom}}}{h}, \quad (10.9)$$

which because of (9.12) agrees with (9.8) to within the quasi-geostrophic approximations (small  $R$ ), e.g.  $\zeta \ll 2\Omega$ , assumed as before. (NB: *no* linearization here.)

Thus the Rossby restoring effect may be thought of as due to **displacements**  $\boldsymbol{\eta}$  across a gradient of potential vorticity. This makes the analogy with the restoring force in a shear flow, due to a *vorticity* gradient, more immediate.<sup>1</sup>

Actually — did you notice? — the above calculation agrees even better with (9.7)! The result (10.8) does not depend on the quasi-geostrophic approximations, but only on assuming that  $u, v$  independent of  $z$  — which may be true for other reasons, e.g. ‘shallow-water approximation’  $H^2/L^2 \ll 1$ . (Exercise: show that the last two terms of (9.7) are not then separately zero, but add to zero.) (10.8) can be regarded as a special case of Ertel’s theorem (0.5 b), derived by taking  $\alpha$  to be a ‘dye’ distribution (dynamically passive tracer) whose gradient is approximately vertical.

The Rossby-wave dispersion relation (10.6) implies a group velocity

$$\mathbf{c}_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l} \right) = \beta \left\{ \frac{k^2 - l^2}{(k^2 + l^2)^2}, \frac{2kl}{(k^2 + l^2)^2} \right\}. \quad (10.10)$$

Geometrically, a curve  $\omega = \text{const.}$  ( $> 0$ ) in the  $(k, l)$  plane is a circle touching the  $l$ -axis thus:

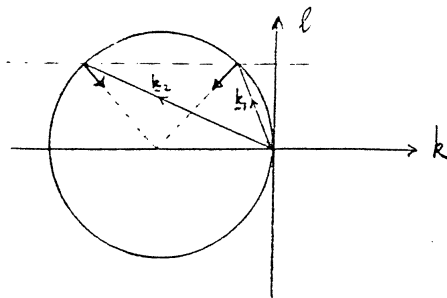


Figure 10.1:

<sup>1</sup>and is the form of the idea that generalizes to *all* types of Rossby waves, including some that depend on *stratification* rather than boundaries — e.g. stratosphere, Sun’s interior.



The solid arrows give the direction of  $\mathbf{c}_g$ . Note  $l_1 = l_2$ ;  $\mathbf{k}_1$  and  $\mathbf{k}_2$  could represent incident and reflected waves from a wall  $x = \text{const}$ . Note that the reflected wave has a smaller wavelength  $2\pi/|\mathbf{k}|$ . [This has interesting oceanographic implications, re conditions near western boundaries to ocean basins.]

—II.9—

It will have been noticed that the fluid system we are talking about exhibits a hierarchy of two kinds of balance or equilibrium, and oscillations about them:

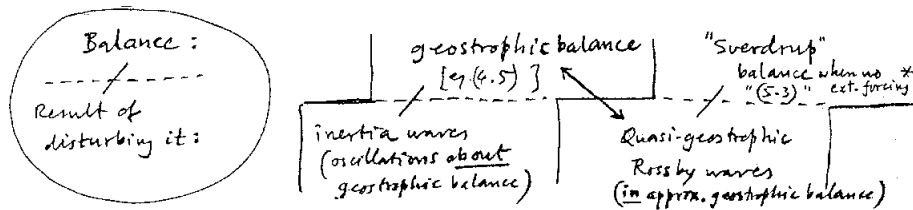


Figure 10.2:

<sup>2 3</sup> [We could have drawn a similar diagram summarizing aspects of the simplest models of a compressible, non-rotating atmosphere; when height scale  $\ll \mathcal{H} \equiv \mathcal{R}T/g$ :

$\mathcal{R}$  = pressure scale height for perfect gas (p. 22)

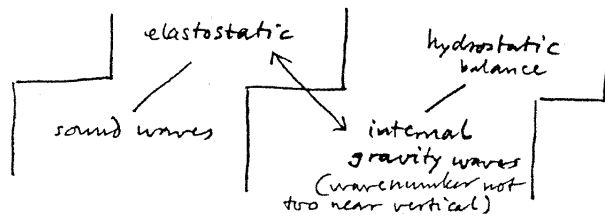


Figure 10.3:

The Boussinesq approximation can be extended to this case, if elastostatic balance holds: Spiegel & Veronis (1960) *Astrophys. J.* **131**, 442; also Gill's book.

Cf. mechanical mass-spring systems with stiff and slack springs, giving modes of oscillation with disparate frequencies. In low-frequency modes, the stiff

<sup>2\*</sup>: I.e. tendency of fluid elements to stay at same  $y$ -value (more generally, tendency of a contour of constant potential vorticity  $q$  to 'resist' deformation from its rest position.

<sup>3†</sup>: I.e. tendency of fluid elements to stay at their undisturbed 'buoyancy level', i.e. tendency of a surface of constant density or potential density to 'resist' deformation from its resting level.

springs are effectively rigid and only the slack ones matter (if the masses are all similar).

Awareness of such hierarchies of balances has played a crucial role in the development of the computer simulations of large-scale atmospheric motions that are used today for numerical weather forecasting, climate studies, and hypothesis-testing ‘numerical experiments’.

In many ways, Rossby waves behave rather like internal gravity waves, qualitatively speaking, apart from the restriction on the direction of phase propagation:

- (1) There are exact solutions of Long’s type (superpositions where all components have the same value of  $\omega/k$ ), applicable to modelling lee Rossby waves due to uniform flow  $\bar{u} > 0$  over topography  $h(x, y)$ , if  $\beta > 0$ , const. (Cf. §3.1.)
- (2) Rossby waves can ‘break’, irreversibly deforming constant- $q$  contours (cf. constant- $\sigma_1$  surfaces for internal gravity waves). For a spectacular example recently observed in the real atmosphere, see the cover of *Nature* vol. **305**, 13–19 October 1983!
- (3) Plane Rossby waves (10.5) (on an unbounded  $xy$  domain with const.  $\beta$ , or ‘ $\beta$ -plane’, as it’s often called) are subject to resonant-triad interactions (cf. §3.5) (Longuet-Higgins & Gill (1967) *Proc. Roy. Soc. A* **A299**, 120), and associated instabilities (Gill, *Geophys. Fluid Dyn.* **6**, 29). Again, the triad member with largest  $|\omega|$  is unstable.
- (4) Rossby waves can induce substantial *mean-flow changes* if they break or otherwise dissipate (cf. §3.3).

—II.10—

## §10.2 Low-Rossby number flow over a gently sloping obstacle or shallow bump (a finite-amplitude solution of the quasi-geostrophic equations)

Assume now that  $h(x, y) = \text{const.} = h_0$  say, except in a finite region  $\mathcal{R}$ . Assuming steady flow, with constant velocity  $(U, 0)$  at large distances,<sup>4</sup> we

---

<sup>4</sup>We may think in terms of undisturbed conditions upstream. Note that this is *unlike* the mountain lee-wave (non-rotating internal gravity wave) problem,  $\exists$  no inertia waves that can be steady and penetrate far upstream ( $\Omega$  in wrong direction for analogy). So

get a very simple solution from (10.8); viz.,

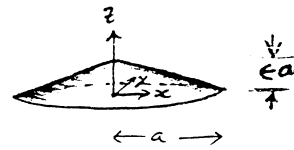
$$\frac{2\Omega + \zeta}{h} = \text{const.} = \frac{2\Omega}{h_0}$$

[More generally, would have  $\frac{2\Omega + \zeta}{h} = \text{func}(\psi)$  for steady flow.] Thus the solution for the velocity field is  $(U, 0)$  plus the (unique) flow that is irrotational and evanescent away from  $\mathcal{R}$  and has vorticity  $\zeta = 2\Omega(h - h_0)/h_0$  within  $\mathcal{R}$ .

Denote the streamfunction of the latter, evanescent flow by  $\psi(x, y)$  say.

A simple example is a cone; see diagram:

In this example,



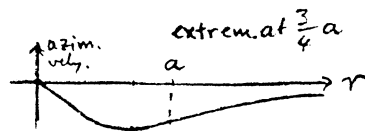
$$\left\{ \begin{array}{l} \mathcal{R} \equiv \{x, y \mid r \equiv (x^2 + y^2)^{1/2} \leq a = \text{const.}\} \\ h_0 - h = \epsilon(a - r) \quad \text{within } \mathcal{R} \quad (\epsilon \approx R_U \ll 1) \end{array} \right.$$

$$\zeta \equiv \psi_{rr} + \frac{1}{r} \psi_r \left\{ \begin{array}{l} = \alpha(r - a) \quad \text{within } \mathcal{R} \\ = 0 \quad \text{outside } \mathcal{R} \end{array} \right. \quad \left( \alpha \equiv \frac{2\Omega\epsilon}{h_0} \right)$$

We expect  $\psi_{rr}$  continuous<sup>5</sup> across  $r = a$  [but if  $h$  had a jump discontinuity, only  $\psi_r$  would be cts.]. The unique solution is

$$\psi = -Uy + \psi' : \left\{ \begin{array}{l} \psi' = \alpha \left( \frac{1}{9} r^3 - \frac{1}{4} a r^2 \right) + \text{const.} \quad \text{within } \mathcal{R} \\ \psi' = -\frac{1}{6} \alpha a^3 \ln r + \text{const.} \quad \text{outside } \mathcal{R} . \end{array} \right.$$

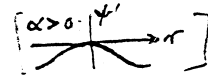
The corresponding velocity field is that of a vortex with azimuthal velocity profile



the assumption of no upstream influence is better justified here than for Long's model of finite-amplitude lee waves.

<sup>5</sup>Again, the correctness of this boundary condition can be confirmed by taking the *limit* in which a smooth shape approaches the cone shape

$$\begin{aligned}\psi'_r &= \text{azimuthal velocity} \\ &= \alpha \left( \frac{1}{3} r^2 - \frac{1}{2} a r \right) \quad (r \leq a) \\ \text{Extremum} &= -\frac{3}{16} a^2 \alpha, \quad \text{at } r = \frac{3}{4} a\end{aligned}$$



In plan view, the streamlines of  $\psi'$  are circles:

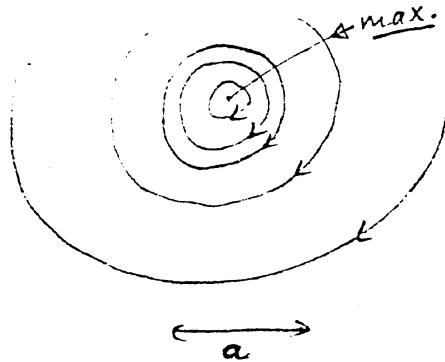


Figure 10.4:

When does the *total flow* have closed streamlines? Evidently, it has them if and only if  $\frac{3}{16} \alpha a^2 > U$ , i.e. if and only if

$$\left( \frac{\epsilon a}{h_0} \right) > \frac{16}{3} \left( \frac{U}{2\Omega a} \right).$$

—II.11—

Figure 10.5 shows some computer-drawn pictures of the streamlines of the total streamfunction  $\psi \equiv -Uy + \psi'$ . There are closed streamlines in second and third cases, conspicuously visible in the third case. The streamfunction is

$$\psi = -Uy + UaA \begin{cases} = \frac{1}{6} \ln \left( \frac{r}{a} \right) & (r \geq a) \\ = \frac{5}{36} + \frac{1}{9} \left( \frac{r}{a} \right)^3 - \frac{1}{4} \left( \frac{r}{a} \right)^2 & (r \leq a) \end{cases}$$

lines  $\psi = 0, \pm \frac{1}{2} Ua, \pm Ua, \dots$  are plotted, see fig. 10.5.

N.B. This kind of solution is strictly relevant to steady flow of slightly-viscous fluid, or of inviscid, initially undisturbed flow *only if*  $\exists$  *no closed streamlines* — as in left-hand picture only.

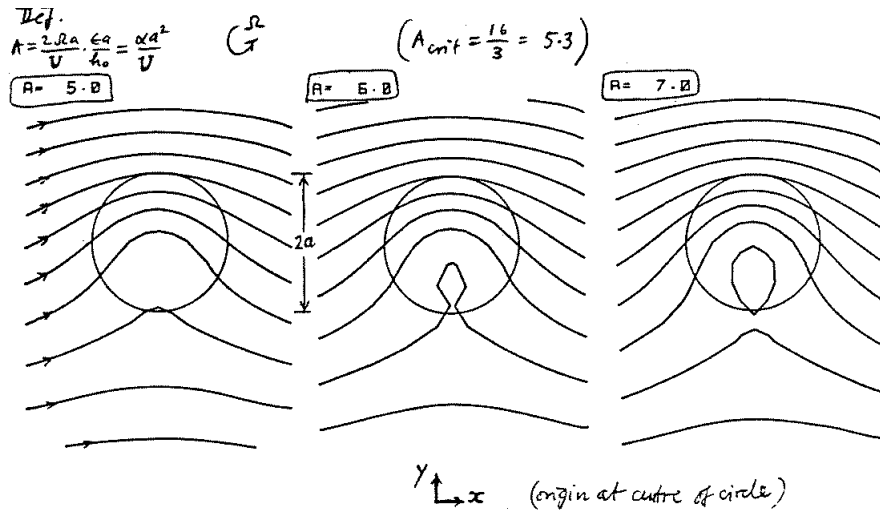


Figure 10.5:

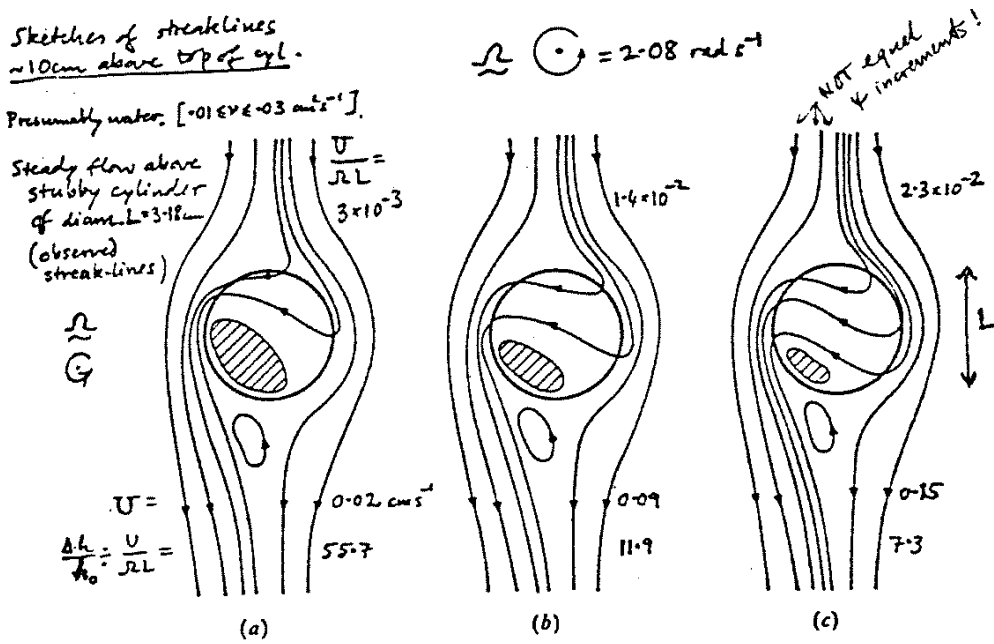
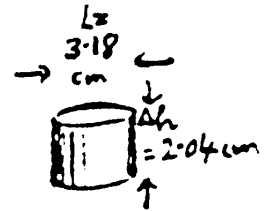


Figure 10.6: Drawing of the relative fluid motion past a cylindrical body moving perpendicular to the rotation axis, [84]. The values of the Rossby number are: (a)  $\epsilon = 3 \times 10^{-3}$ , (b)  $\epsilon = 1.4 \times 10^{-2}$ , (c)  $\epsilon = 2.3 \times 10^{-2}$ .

Figure 10.6 shows experimentally-observed steady flows over a (different-shaped) obstacle (Hide & Ibbetson (1966) *Icarus* **5**, 279); Greenspan, p.174).

We can reasonably define a ‘Taylor column’ to exist when  $\exists$  closed streamlines. Above theory predicts Taylor columns for  $\frac{\Delta h}{h_0} R_U^{-1} >$  some critical value (by reductio ad absurdum — “if no closed streamlines,…” etc. etc.) — but does *not* predict the precise flow pattern.



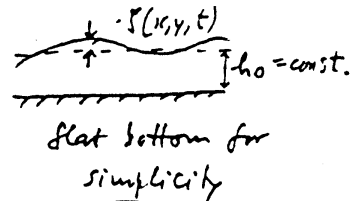
# 11.

## Quasi-geostrophic shallow-water flow

The final sections take two major steps towards the far-reaching generalizations spoken of earlier, from simple two-dimensional vortex dynamics to the most general quasi-horizontal balanced motion, on timescales  $\gg$  acoustic, gravity and ‘inertial’ or Coriolis timescales.

Take the case with  $\left\{ \begin{array}{l} f = 2\Omega = \text{constant} \\ \text{inviscid flow} \end{array} \right\}$ .

We use scale-analytic arguments paralleling those on p. 121. New feature is the free surface, see drawing. Let  $q_a$  be absolute vorticity  $f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ , layer depth  $\underbrace{h_0 + \zeta}_{= h \text{ say}}$  Horizontal velocity compo-



—11.12—

—N58—

nents  $u(x, y, t)$ ,  $v(x, y, t)$ ; define  $\mathbf{u}_H = (u, v, 0)$ ;  $\nabla = \nabla_H = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)$ .

NB new notation since we now need  $\zeta$  for free surface displacement

Equations are

$$\frac{\partial u}{\partial t} + \mathbf{u}_H \cdot \nabla u - f v = -g \frac{\partial h}{\partial x} \quad (x\text{-momentum}) \quad (11.1 \text{ a})$$

$$\frac{\partial v}{\partial t} + \mathbf{u}_H \cdot \nabla v + f u = -g \frac{\partial h}{\partial y} \quad (y\text{-momentum}) \quad (11.1 \text{ b})$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}_H) = 0 \quad (\text{mass conservation}) \quad (11.1 \text{ c})$$

$$\Rightarrow \frac{\partial q_a}{\partial t} + \nabla \cdot (\mathbf{u}_H q_a) = 0 \quad (11.2 \text{ a})$$

$$\text{or } \frac{\mathcal{D}_H q_a}{\mathcal{D}t} + q_a \nabla \cdot \mathbf{u}_H = 0 \quad (\text{vorticity}) \quad (11.2 \text{ b})$$

(where  $\frac{\mathcal{D}_H}{\mathcal{D}t} = \frac{\partial}{\partial t} + \mathbf{u}_H \cdot \nabla_H$ )

From (11.1 c) and (11.2 a):

$$\frac{\mathcal{D}_H}{\mathcal{D}t} \left( \frac{q_a}{h} \right) = 0 \quad (\text{exact potential vorticity, Rossby 1936}) \quad (11.3)$$

If  $R_T \equiv \frac{1}{fT} \ll 1$ ,  $R_U \equiv \frac{U}{fL} \ll 1$  (notation and scaling assumptions as in §9), first approximation to (11.1 a,b) is geostrophic balance (relative error  $O(R)$ ;  $R = \max(R_T, R_U)$ ):

$$u \simeq -\partial\psi/\partial y, \quad v \simeq \partial\psi/\partial x, \quad \psi = g\zeta/f \quad q_a \simeq f + \nabla^2\psi. \quad (11.4)$$

(where  $\nabla^2 = \nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ ). Now if horizontal scale  $L \lesssim \boxed{L_D \equiv c_0/f}$  where  $c_0 = \sqrt{gh_0}$ , the gravity-wave speed for  $f = 0$ , then the fractional free-surface displacement  $\zeta$  is *small*:

$$\begin{aligned} \frac{\zeta}{h_0} &= \frac{f}{gh_0} \psi = \frac{f}{c_0^2} \psi \simeq \frac{f}{c_0^2} LU = \frac{f^2 L^2}{c_0^2} \frac{U}{fL} \\ &= \frac{L^2}{L_D^2} R_U \ll 1. \end{aligned} \quad (11.5)$$

So

$$\begin{aligned} \frac{q_a}{h} &\simeq \frac{f + \nabla^2\psi}{h_0 + \zeta} \simeq \frac{f}{h_0} \left\{ \left(1 + \frac{\nabla^2\psi}{f}\right) \left(1 - \frac{\zeta}{h_0}\right) \right\} \simeq \frac{f}{h_0} \left\{ 1 + \frac{\nabla^2\psi}{f} - \frac{\zeta}{h_0} \right\} \\ &\simeq \frac{1}{h_0} \left\{ f + \nabla^2\psi - \frac{f^2}{gh_0} \psi \right\} = \frac{1}{h_0} Q, \text{ say.} \end{aligned} \quad (11.6)$$

This new  $Q$  is called the ‘**quasi-geostrophic PV**’ to distinguish it from the ‘**exact PV**’,  $q_a/h$ .

Putting the approximations (11.4) and (11.6) into (11.3), material conservation of Rossby’s ‘exact PV’, gives

$$\frac{\mathcal{D}_H Q}{\mathcal{D}t} = 0 \quad \text{with} \quad (11.7 \text{ a})$$

$$\nabla^2\psi - L_D^{-2}\psi = Q - f \quad (11.7 \text{ b})$$

(where again  $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$  and  $\psi$  is function  $(x, y, t)$ ). Note need for *boundary conditions* to invert (11.7 b); this might NOT be quite trivial, e.g. uniform flow at  $\infty$ , as in the example of §10.2. The lengthscale  $L_D \equiv c_0/f$  is called the ‘**Rossby deformation length**’, or ‘**Rossby length**’ for brevity. In the more old-fashioned literature it’s called the ‘Rossby Radius Of Deformation’ (as if things were always axisymmetric).



**Generic form:**

- (1) Evolution equation  $\frac{\mathcal{D}Q}{\mathcal{D}t} = \dots$  (zero if materially conserved, or nonzero material derivative if there is nonadvective (e.g. diffusive) transport) (not linearized)
- (2) Inversion operator

$$\left. \begin{aligned} \psi &= \mathcal{L}^{-1}(Q - Q_0) \\ &= \mathcal{L}^{-1}(\Delta Q) \text{ say} \end{aligned} \right\}; \quad \mathbf{u} = \hat{\mathbf{z}} \times \nabla_H \psi, \quad \hat{\mathbf{z}} \text{ unit vertical vector.}$$

**Examples**

(all in a horizontally unbounded ‘flat earth’ domain, coordinates  $x, y$ )

- (i) *2D vortex dynamics:*

Write  $r^2 = (x - y')^2 + (y - y')^2$

$$\mathcal{L} = \nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\psi(\mathbf{x}, t) = \mathcal{L}^{-1}(\Delta Q) = \frac{1}{2\pi} \iint \Delta Q(x', y') \log r \, dx' dy' = \nabla_H^{-2}(\Delta Q),$$

$$\left\{ \begin{array}{l} Q = q_a = f + \nabla_H^2 \psi \text{ (abs. vortex),} \\ Q_0 = f \text{ (Coriolis parameter)} \end{array} \right.$$

- (ii) Note that *non-rotating, low Froude number dynamics* (pp. 46–49) is the same problem (2D vortex dynamics) on each approximately horizontal isopycnic ( $\sigma_1 = \sigma + \int^z N^2(z') dz' = \text{const.}$ ) surface. This is an immediate corollary of (3.8), with  $Q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$  and  $Q_0 = 0$ .

All logs are base  $e$   
 $t$  dependence not written

- (iii) *Shallow-water quasi-geostrophic (q-g) dynamics* (previous page):

$$\mathcal{L} = \nabla_H^2 - L_D^{-2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - L_D^{-2};$$

$$\psi(\mathbf{x}, t) = \mathcal{L}^{-1}(\Delta Q) = -\frac{1}{2\pi} \iint \Delta Q(x', y') K_0\left(\frac{r}{L_D}\right) dx' dy',$$

$$\begin{aligned} Q &= f + \nabla_H^2 \psi - L_D^{-2} \psi \quad (L_D = \text{const.}) \\ Q_0 &= f \end{aligned}$$

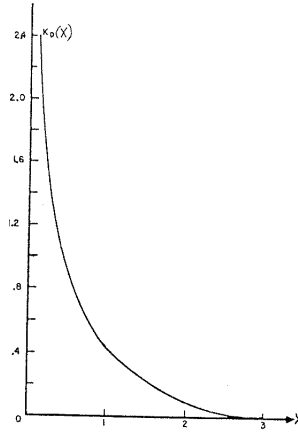


Figure 11.1:

where  $K_0$  is the modified Bessel function

$$\begin{aligned} K_0(X) &\sim -\log X - \log \frac{1}{2} + \gamma + O(X^2 \log X) \quad \text{as } X \rightarrow 0 \\ &\sim \left(\frac{\pi}{2}\right)^{1/2} X^{-1/2} e^{-X} + O(X^{-3/2} e^{-X}) \quad \text{as } X \rightarrow \infty \end{aligned}$$

(also see fig. 11.1)

(iv) *3D stratified quasigeostrophic dynamics* (unbounded  $x y z$  domain) (§17 below)

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z} \frac{f^2}{N^2(z)} \frac{\partial}{\partial z}; \quad Q = f + \mathcal{L} \psi, \quad Q_0 = f$$

In the case  $N^2 = \text{constant}$ :

$$\begin{aligned} \psi(\mathbf{x}, t) = \mathcal{L}^{-1}(\Delta Q) &= -\frac{N}{4\pi f} \iiint \Delta Q(x', y', z') \frac{1}{R} dx' dy' dz' \\ R^2 &= (x - x')^2 + (y - y')^2 + \frac{N^2}{f^2} (z - z')^2 \end{aligned}$$

[If horizontal or nearly-horizontal boundaries are present, then the  $\sigma_1$  distribution on the boundary can be regarded as a (singular) part of the  $Q$  field, as shown below.]

(Non-examinable:) [For further discussion see the review article by Hoskins *et al.* (1985) *Q. J. Roy. Meteorol. Soc.* **111**, 877, **113**, 402, and a paper

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by McIntyre & Norton, *J. Atmos. Sci.* **57**, 1214–1235, which appeared on Millennium May Day, 1 May 2000 and a discussion paper from the millennial Limerick Symposium on my website [www.atm.damtp.cam.ac.uk/people/mem/](http://www.atm.damtp.cam.ac.uk/people/mem/)

In the most accurate models,  $Q$  is replaced by the exact Rossby potential vorticity for shallow-water models, recall (11.3), or by the exact Rossby–Ertel potential vorticity for continuously-stratified models, recall (0.5 a). In such models the velocity field is no longer horizontally nondivergent and the inversion operator no longer linear. The following section briefly elaborates on these points as well as on other aspects of the most accurate models, starting with those in the Millennium May Day paper (figures overleaf).

—II.14—

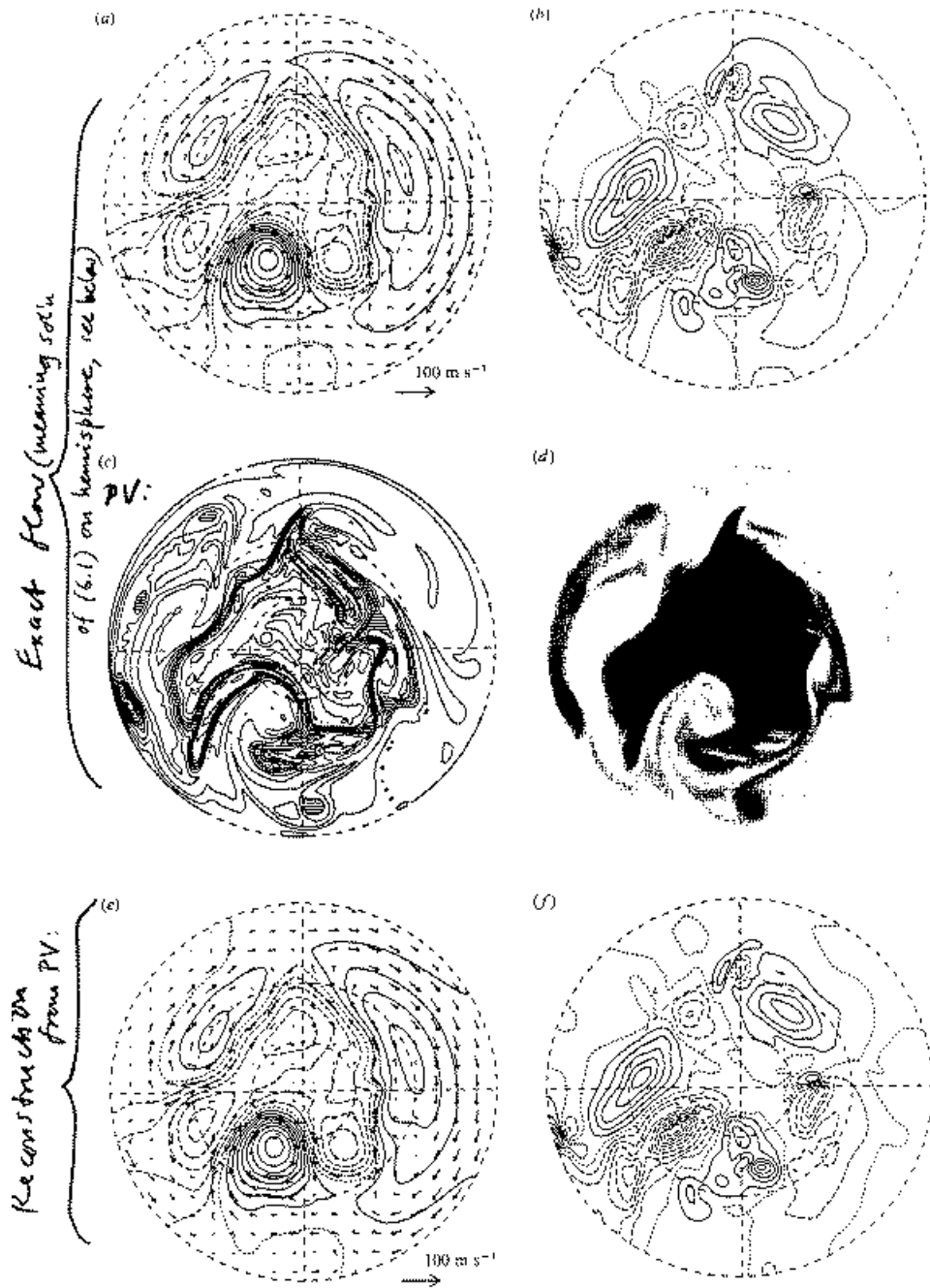


Figure 11.2: Demonstration of balance and invertibility in a shallow-water flow with substantial density variations, from a high-resolution numerical experiment on flow on a hemisphere (McIntyre and Norton, Millennium May Day paper, *J. Atmos. Sci.* **57**, 1214–1235, *Corrigendum* **58**, 949).

# 12.

## A glimpse forward towards highly realistic models

**Notes on fig. 11.2:** Demonstration of balance and potential-vorticity invertibility in a shallow-water flow with substantial  $h$  variations, from a high-resolution numerical experiment on flow on a hemisphere. This was motivated as an atmospheric model but is equally well interpretable as a compressible two-dimensional flow in a hemispherical shell. The system is a shallow water free-surface model (equations (11.1) but on hemisphere,  $f = 2\Omega \sin \phi$  where  $\phi = \text{latitude}$ ) with area-mean depth  $h_0 = 2$  km and corresponding gravity wave speed  $c_0 = 140 \text{ m s}^{-1}$ , or equivalently a fictitious ‘perfect gas’ with ratio of specific heats  $\gamma = 2$  and sound speed  $140 \text{ m s}^{-1}$  at mean density. Solid contours show positive values, long dashed contours negative, and dotted contours zero. The projection is polar stereographic; the radius of the hemisphere is 6371 km. (a): Arrows show the velocity field on the scale indicated; contours show departures of density or layer depth from the area mean value. The contour interval is one twentieth of the mean; in the two-dimensional compressible system it can also be regarded as the anomaly in the square root of the pressure. (b) Divergence field ( $\nabla_H \cdot \mathbf{u}$ ) contoured at intervals of  $0.6 \times 10^{-6} \text{ s}^{-1}$ . (c,d) The quantity  $Q$  defined above is contoured at interval  $1 \times 10^{-8} \text{ m}^{-1}$  in units appropriate  $\text{s}^{-1}$  in units appropriate to the formula  $Q = q_a/h$ . The shading in the contour plot highlights values lying between 4 and 6 of these units. The greyscale representation of the same information in (d) is monotonic from light to dark, from zero at the equator to a maximum value of  $1 \times 10^{-7} \text{ m}^{-1} \text{ s}^{-1}$  near the pole. (e,f) As (a,b), but reconstructed from  $Q$  alone using an accurate nonlinear PV inversion algorithm. —11.15—

**Further notes on fig. 11.2:** Flow is almost inviscid; artificial numerical ‘hyperdiffusion’ affects only the smallest visible scales. What this demonstrates how astonishingly accurate PV inversion can be, at least for the shallow-water equations on a hemisphere (McIntyre & Norton (1990, 2000) *J. Fluid Mech.* **212**, 410; *J. Atmos. Sci.*, May Day 2000, i.e. **57**, 1214).

Not quasi-geostrophic theory; the PV here is the *exactly* conserved quantity  $Q = q_a/h$  — i.e. the ‘ $Q$ ’ of (11.3), *not* (11.7).

Local Froude numbers  $((u^2 + v^2)/gh)^{1/2}$  reach values as high as 0.5, indeed 0.7 in another case looked at; local Rossby numbers are infinite at the

equator where  $f = 0$ . ( $f =$  Coriolis parameter  $-2\Omega \sin \phi$  where  $\phi =$  latitude.)

—II.14.a—

In this realistic, continuously stratified example, what's dynamically relevant is the distribution of Rossby–Ertel PV on each isentropic ( $\theta = \text{const.}$ ) surface where  $\theta$  is potential temperature, a material invariant with a role similar to that of the Boussinesq  $\sigma_1$ .

—II.17—

Re (a), *inertia-gravity waves*: due to both restoring mechanisms: Coriolis and buoyancy; see Part 3 of course

We now note for interest how the PV invertibility principle for realistic continuously stratified, rotating flows can be stated in its most accurate and widely applicable form. The relevant PV is the Rossby–Ertel PV:

$$Q = \rho^{-1}(2\Omega + \nabla \times \mathbf{u}) \cdot \nabla \theta,$$

which as shown before is *exactly* materially conserved ( $DQ/Dt = 0$ ) in 3D flow with no frictional or other non-conservative forces and no diabatic heating ( $D\theta/Dt = 0$ ;  $\theta$  is potential temperature). Then if

- (a) a suitable “balance condition” is imposed, to eliminate gravity and inertia–gravity waves from consideration, and if
- (b) a suitable reference state specified, for instance by specifying the mass under each isentropic surface, as is done in the theory of available potential energy in stratified atmospheres (p. 5), then
- (c) a knowledge of the distribution of  $Q$  on each isentropic surface, and of  $\theta$  at the lower boundary, is sufficient to deduce, diagnostically, all the other dynamical fields such as winds, temperatures, pressures, and the altitudes of the isentropic surfaces,

to some approximation that appears to depend on Rossby and/or Froude numbers being not too large, but which involves far *less* error than would be suggested by simple order-of-magnitude estimates of the kind used, for instance, in p. 137 to construct model (iii). the word ‘diagnostically’ implies the use of information at a single instant only (cf. ‘prognostically’); further discussion in Fermi review<sup>1</sup> and in QJRMS review, reference on next page.

—II.16—

Now look at the cross-sections in the figure on the next page. The real cutoff cyclone is an approximately axisymmetric vortex, and the theoretical one is exactly axisymmetric, and exactly in hydrostatic and cyclostrophic balance, where ‘cyclostrophic’ denotes the exact, steady balance in the radial momentum equation in cylindrical polar coordinates: pressure-gradient force minus Coriolis force balances the relative centripetal acceleration  $|\mathbf{u}|^2/(\text{distance from axis})$ , rather than being considered approximately zero as when geostrophic balance is assumed. These are, respectively, real and realistic examples of the layerwise-2D ‘coherent structures’ that correspond to simple vortices in 2D vortex dynamics, model system (i) of p. 138. Further remarks on p. 148. Such coherent structures have long been recognized

<sup>1</sup>Click on ‘polar cooling thought experiment’ on my home page [www.atm.damtp.cam.ac.uk/people/mem](http://www.atm.damtp.cam.ac.uk/people/mem).

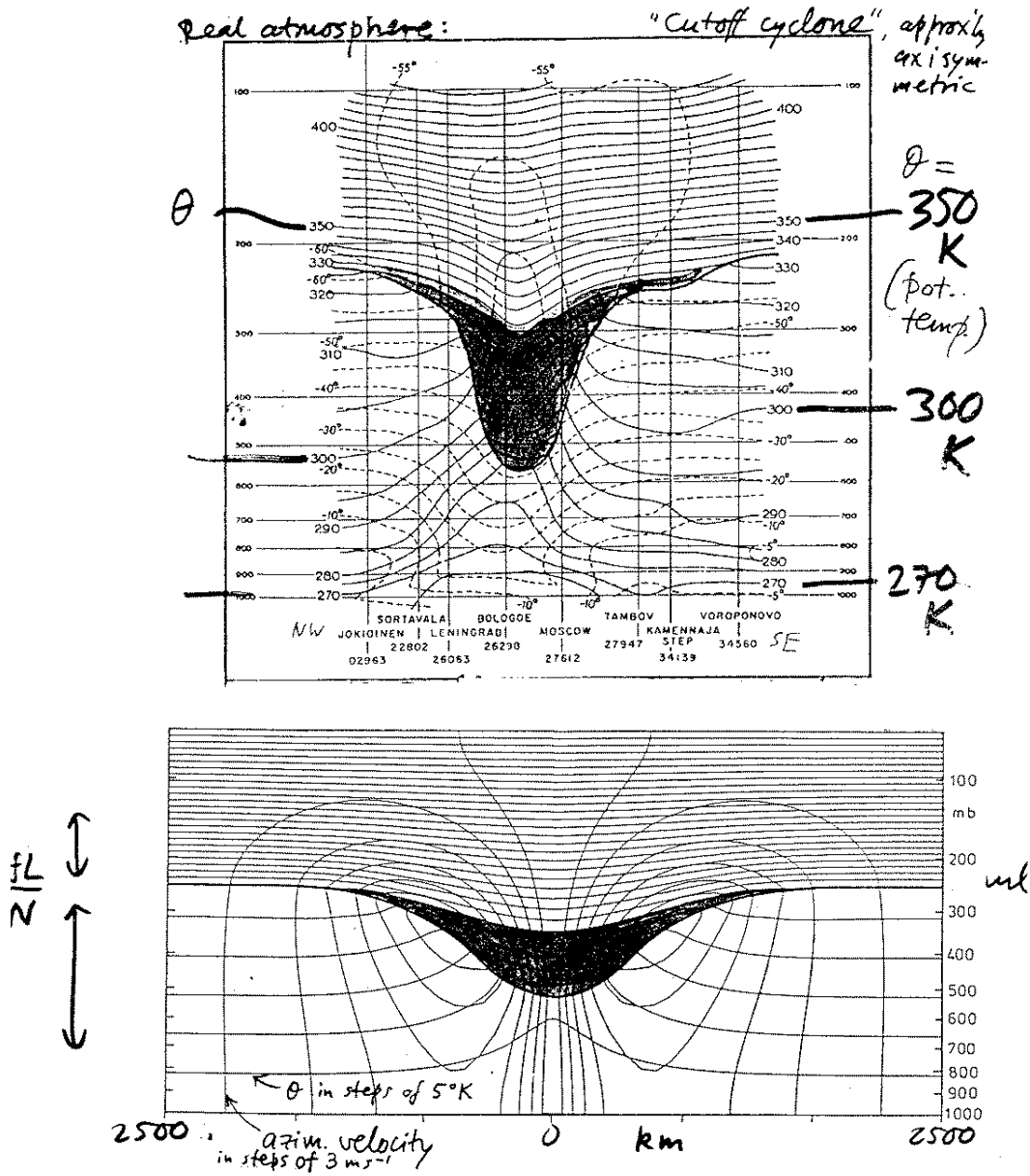


Figure 12.1: Solid quasi-horizontal surfaces are isentropic (constant- $\theta$ ) surfaces. In top picture, - - - - is  $T$  in  $^{\circ}\text{C}$ . In bottom picture, other contours are isotachs for azimuthal velocity. **Whole structure** attributable to a single compact, cyclonic PV anomaly (on  $\theta$ -surfaces intersecting the tropopause). (Peltonen (1963); Palmén & Newton (1969), chapter 10. Calculations by A.J. Thorpe, reproduced in Hoskins, McIntyre & Robertson (1985) *J. Roy. Met. Soc.* **111**, 877–946, and **113**, 502–404.)

by meteorologists interested in the observed behaviour of the atmosphere, a fact that pays tribute to their careful and unprejudiced data analysis, done in the absence of any significant theoretical preconceptions — indeed they got the facts right despite having some wrong theoretical ideas, and found some aspects of the top picture surprising! Only recently (mid-'80s) have such structures been clearly seen as part of a connected theoretical whole; and even today the subtler points of the theoretical framework involved are still being actively researched. The relevant theoretical framework is, however, clear in outline. It is that suggested by the generic form of the model systems (i)–(iv) of §11 and by the example of p.141. In the latter, the details of how the PV inversion operator is defined are complicated (and beyond the scope of this course); the important point is that we still have

(a) an **evolution equation of the simple form**

$$DQ/Dt = 0 \quad (\text{and again only one } t\text{-derivative}) \quad (12.1)$$

(or slight departure from zero from slight nonconservative effects), plus

(b): the **‘PV invertibility’ principle** that the other fields may be constructed at any fixed  $t$  from the PV. These two features are an important key to understanding a vast and complicated-looking theoretical literature on large-scale atmosphere-ocean dynamics and, in particular, to understanding *why* simple models like (i)–(iv) — none of which can claim to be quantitatively accurate in the real atmosphere and ocean — nevertheless have great conceptual importance, far beyond their domains of direct applicability.

Notice **one point glossed over up until now** (though we return to it in the last part of the course when we consider continuous stratification): one needs to specify  $\theta$  at the lower boundary, as well as the isentropic distributions of  $Q$ , in order to get a well-determined inversion. The theoretical example on p. 143 has uniform  $\theta$  on the boundary. More generally,  $\theta$  anomalies on the boundary induce their own vortex structures and are generally important for real weather developments [QJ review]. For frictionless, adiabatic motion, it is easy to see that both the  $\theta$  distribution on a flat lower boundary, and the  $Q$  distribution on each isentropic (constant- $\theta$ ) surface in the interior, are advected *quasi-horizontally* in the sense that  $D_H/Dt$  of these distributions, regarded as functions of time and horizontal position only, vanish. So the problem has the nature, as hinted earlier, of a ‘layerwise-2D vortex dynamics’, in which the vertical coupling is solely through the PV inversion operator.<sup>2</sup> This qualitative character will show up, in simplified form, when we come to derive model (iv) of §11.

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<sup>2</sup>Model (ii) is degenerate in the sense that the vertical coupling vanishes to leading order in Froude number; the first nontrivial, vertically nondegenerate example will be (iv).



# 13.

## 2-D vortex dynamics revisited in the light of the above

We now return to the simplified models, beginning with three aspects of the simplest one of all, model (i) of p. 139, 2D vortex dynamics. Recall that the inversion operation in that case (from vorticity  $Q$  to streamfunction  $\psi$ ) can be thought of in terms of the soap-film analogy:

—II.18—

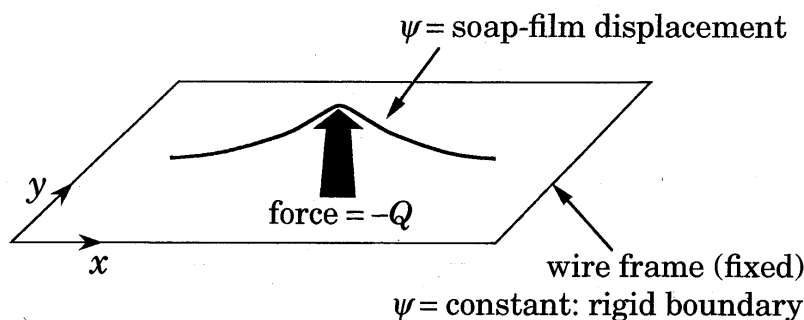


Figure 13.1: Soap-film analogy for vorticity inversion

The three aspects to be emphasized all have their counterpoints in the other cases, including the most general cases glimpsed in §12. That is, the following statements are qualitatively true of all the inversion operators mentioned:

- (1) Local knowledge of  $Q$  does not imply local knowledge of  $\psi$  or  $\mathbf{u}$ ; the inversion is a global process. In particular, as the above sketch illustrates, the inversion depends on specifying suitable boundary conditions to make the inverse Laplacian  $\nabla_H^{-2}$  unambiguous.
- (2) In this system the balance condition, on which invertibility depends, corresponds simply to the absence of sound or external gravity waves. They have been filtered out by the assumption of incompressible, non-divergent motion.
- (3) There is a *scale effect*, whereby small-scale features in the  $Q$  field have a relatively weak effect on the  $\psi$  and  $\mathbf{u}$  fields, while large-scale features

have a relatively strong effect. In particular,  $\psi$  and  $\mathbf{u}$  are to varying degrees insensitive to fine-grain structure in the  $Q$  field. The inverse Laplacian  $\nabla_H^{-2}$  is a smoothing operator, as is the inverse of the modified Helmholtz operator  $\nabla_H^2 - L_D^{-2}$ , and some of the smoothing survives even when followed by the single differentiations in the relations

$$u = -\psi_y, \quad v = \psi_x. \quad (13.1)$$

This scale effect is directly responsible for one of the characteristic peculiarities of Rossby waves, common to the simple case of pp. 127–128 and the more general cases, that intrinsic frequencies increase as wavenumbers decrease. The  $-(k^2 + l^2)^{-1}$  factor in the dispersion relation (10.6) is nothing but  $\nabla_H^{-2}$  in disguise. Again, in examples like p. 97 (which *are* frictionless cases of model (i)) the scale effect is part of the reason why the small-scale features tend to behave like advected passive tracers, a key to understanding ‘2D turbulence’. See also Sheet 4 q. 4.

The equations of model (i), namely  $DQ/Dt = 0$  (or  $DQ/Dt = \text{frictional terms}$ ), together with the inversion defined by

$$\psi = \nabla_H^{-2}(\Delta Q) \quad (13.2)$$

and (13.1), summarize with remarkable succinctness the peculiar way in which fluid elements push each other around. The non-localness of the inversion operator and the implied action at a distance (aspect 1) are related of course to aspect 2. The invertibility principle holds *exactly* in this case, because the waves representing departures from balance have been assumed to propagate infinitely fast and to have infinitely stiff restoring mechanisms.

[The succinctness of (13.2) etc. is to be compared with what is involved in thinking directly in terms of Newton’s second law. Newton’s law applied to every fluid parcel is mathematical equivalent to the above, but requires us to think explicitly about subtle aspects of the pressure field. The formulation via (13.2) represents an economy of thinking analogous to, but greater than, the economy that results from treating normal reaction forces as constraints when discussing the dynamics of a roller coaster on a rigid track. This treatment reduces a three-dimensional problem to a one-dimensional problem, making the problem conceptually as well as computationally easier than explicitly using the two normal components of Newton’s second law. In the fluid-dynamical system the pressure field and boundaries also have the nature of constraints: to this extent they play a role analogous to the normal reaction forces in the roller-coaster problem. The power of the viewpoint represented by (13.2) etc. has long been recognized, and made use of,

by aerodynamicists as well as by theoretical meteorologists. Indeed the idea of invertibility, as exemplified by (13.2) and its three-dimensional unstratified generalization (Biot–Savart law), is built into classical low-Mach-number aerodynamical language, in such phrases as “velocity field induced by” a given vorticity field, and in such ideas as the idea that a vorticity anomaly can roll ‘itself’ up into a nearly circular vortex.]

It will have been noticed from (13.2) that diagnosing the  $\psi$  field from the  $Q$  field is almost the same thing, mathematically, as calculating the electrostatic potential induced by a given charge distribution, or the static displacement of a stretched membrane induced by a given pressure distribution on it. Thus strong local anomalies in  $Q$  tend to induce strong circulations around them in the corresponding sense. Such a  $Q$  anomaly, together with its induced velocity field, is exactly what we have in mind when we speak of the coherent structure called a ‘vortex’; and it is this idea that generalizes to the meteorologists’ large-scale ‘cyclones’ and ‘anticyclones’ (and their smaller-scale cousins seen, for instance, in satellite images of vortex streets behind islands), and to the ‘coherent eddies’ observed by oceanographers.

Further examples from model (i) are in the Exercises, and the case of shear instability and its close relation to Rossby-wave propagation (giving details for the linearized, small-amplitude stage in the example of p. 97) will be further discussed in Appendix B<sup>1</sup> together with its relation to the ‘baroclinic instability’ to be discussed in connection with model (iv) below.<sup>2</sup>

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<sup>1</sup>Appendix B is of course non-examinable.

<sup>2</sup>In the section on **Eady’s solution**, also point (viii) near the end of the previous section.



# 14.

## More about the shallow water system and model (iii): Rossby waves

In the shallow-water system (11.1) the PV inversion operators involved are qualitatively not unlike the simple inverse Laplacian, except that the implied action-at-a-distance has a short-range character related to the finiteness of  $c_0$ . This short-range character is part of *why* PV inversion can remain surprisingly accurate as Froude numbers  $|\mathbf{u}|/c_0$  increase (p. 141 again). The short-range character is explicitly brought out in the simplest generalization of model (i), namely the quasi-geostrophic shallow-water equations, model (iii). The modified Bessel function  $K_0$  has values, and gradients, that diminish exponentially rapidly at large distances, in contrast with the relatively long-range logarithmic far field characterizing model (i). [The consequences of this difference have been thoroughly explored in recent PhD work in DAMTP, in the thesis by Waugh, D. W. (1991), in DAMTP library.<sup>1</sup>]

Here we simply note the effect on Rossby-wave propagation, and go on to discuss an example of how the geostrophic balance assumed in model (iii) can be set up starting from an unbalanced initial state — an idealized version of just the kind of initial imbalance that dominated L. F. Richardson’s pioneering attempt at numerical weather prediction. This is often called the ‘Rossby adjustment problem’ (next section).

—11.20—

Consider first Rossby waves in the full shallow-water system (11.1), generalized to the case of a gently sloping bottom boundary. The required modifications to the equations on p. 137 are similar to what was done on p. 125. The only change in (11.1) comes from  $\nabla_H \zeta$  no longer being equal to  $\nabla_H h$ . We retain the meaning of  $h$  as layer depth, so that the vertically integrated mass-conservation equation (11.1 c) still holds. Thus the only change is to replace  $h$  by  $\zeta$  (surface elevation) on the right of (11.1 a,b); note especially that (11.3), exact material conservation of the appropriate PV,  $Q = q_a/h$ , is still true. It is now obvious, e.g., that a state of rest (relative to the rotating reference frame) can have a nonvanishing PV gradient and can thus support Rossby wave propagation. For the qualitative picture on pp. 128–129 still applies:<sup>2</sup> the only essentials are (a) the background PV gradient (let’s say in the  $y$  direction as before), and (b) the PV invertibility principle along with

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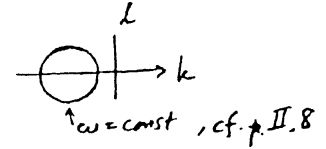
<sup>1</sup>non-examinable

<sup>2</sup>(remember  $\zeta$  there means  $q_a - f = v_x - u_y$  here)

the qualitative insight that PV inversion is like electrostatics, etc., apart from now having a shorter range. This assures us that the signs and phase relations between the velocity and displacement fields are just as on p. 128. The main difference is that the scale effect is greatly modified at large horizontal wavelengths  $2\pi/(k^2 + l^2)^{1/2} \gg 2\pi L_D$ , because of the short-range character of the PV inversion operator.

The simplest analytical model illustrating these features is that based on the corresponding extension of model (iii). All the ingredients of the theory are already present on pp. 125, 137 and we can immediately guess that (10.6) will be replaced by

$$\omega = \frac{-\beta k}{k^2 + l^2 + L_D^{-2}}, \quad (14.1)$$



since the only nontrivial change has been to replace the PV inversion operator  $\nabla_H^{-2}$  by its model-(iii) counterpart

$$\mathcal{L}^{-1} = (\nabla_H^2 - L_D^{-2})^{-1}, \quad (14.2)$$

hence  $k^2 + l^2$  by  $k^2 + L_D^{-2}$  in the denominator. We anticipate that  $\beta$  will be essentially the same measure of the background PV gradient as before. (We had  $\beta = -f h_0^{-1} dh_0/dy$ , where  $h_0(y)$  is the undisturbed layer depth.)

Let us now verify this by writing, more generally than in §11,

$$h = h_0(x, y) + \zeta(x, y, t) = h_{00} - b(x, y) + \zeta(x, y, t), \quad (14.3)$$

say, where  $h_{00}$  is a constant. To get a simplification corresponding to (11.6) we need to assume

$$b/h_{00} \ll 1, \quad (14.4)$$

cf. (11.5); note that (11.4) needs no change since we already introduced  $\zeta$  in place of  $h$  there. The only change in (11.6), (11.7) is to replace  $h_0$  by  $h_{00}$  (just a change of notation) and replace

$$f \quad \text{by} \quad f_{\text{effective}} = f_{\text{eff}} = f \left( 1 + \frac{b(x, y)}{h_{00}} \right). \quad (14.5)$$

The last term comes from the  $b$  contribution to the Taylor expansion of  $\frac{q_a}{h} = \frac{f + \nabla_H^2 \psi}{h_{00} - b + \zeta}$  under condition (14.4). With (11.6) and (11.7) thus modified, the

equations become (with  $Q$  now changed to its model-(iii) meaning, analogous to (11.6), i.e.  $Q$  now defined to be  $\{ \}$  in (11.6) with  $f$  replaced by  $f_{\text{eff}}$ ):

$$\frac{\mathcal{D}_H Q}{\mathcal{D}t} = 0 \quad (14.6 \text{ a})$$

$$\mathcal{L} \psi \equiv (\nabla_H^2 - L_D^{-2}) \psi = \Delta Q, \quad (14.6 \text{ b})$$

where now

$$\Delta Q = Q - f_{\text{eff}}(x, y). \quad (14.6 \text{ c})$$

If we now linearize (14.6 a) about the particular state of rest in which  $Q = Q_0(y) = f_{\text{eff}}(y) = f + \beta y$  with  $\beta$  assumed constant, we get  $\partial Q / \partial t + \beta v = 0$ ,  $\beta = dQ_0 / dy = f h_{00}^{-1} db / dy$  (same as above except  $h_{00}$  replaces  $h_0$ ); hence

$$\frac{\partial}{\partial t} (\mathcal{L} \psi) + \beta \psi_x = 0, \quad (14.7)$$

and (14.1) follows for plane-wave solutions  $\psi \propto \exp(ikx + ily - i\omega t)$ . Note that the case of p. 125 is recovered as the limit

$$g, c_0, L_D, \uparrow \infty \quad (14.8)$$

as one might expect from the intuition that “infinitely strong gravity makes the upper surface rigid”. Note also that (14.4) again prevents uniform validity over an extended range of  $y$ ; cf. bottom of p. 125.<sup>3</sup>

The extended model (iii) system (14.6) allows many other interesting problems to be solved, e.g. the counterpart of the shallow-bump problem of §10.2, p. 132.

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<sup>3</sup>but it can be seen from p. 151 and 128–129 that this is merely a limitation of quasi-geostrophic theory as such, not of the general concepts of ‘quasi-horizontal balanced motion’ and ‘PV invertibility’.





# 15.

## More about the shallow water system and model (iii): inertia–gravity waves and the Rossby adjustment problem

The system (11.1) admits much higher-frequency oscillations ( $\omega \geq f$ , in fact), as can easily be illustrated by returning to the simplest case of a ‘flat’ bottom, meaning  $h_0 = h_{00} = \text{constant}$  ( $b = 0$ ), and seeking the elementary plane-wave solutions of *that* system linearized about relative rest. This gives us our first and simplest example of waves whose restoring mechanism combines the buoyancy mechanism of gravity waves with the Coriolis mechanism of inertia waves. They are generally called ‘inertia–gravity waves’ and, in the present special context of the shallow-water equations (11.1), sometimes also called ‘Poincaré waves’.

—11.22—

The linearized equations are (with  $h = h_{00} + \zeta$ )

$$\frac{\partial u}{\partial t} - f v = -g \frac{\partial \zeta}{\partial x} \quad (15.1 \text{ a})$$

$$\frac{\partial v}{\partial t} + f u = -g \frac{\partial \zeta}{\partial y} \quad (15.1 \text{ b})$$

$$\frac{\partial \zeta}{\partial t} + h_{00} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \quad (15.1 \text{ c})$$

the corresponding linearized vorticity and PV equations being (with  $q_a = f + q$ , say — remember  $q$  is relative vorticity  $v_x - u_y$ , and  $\zeta$  free-surface elevation, both being zero, by assumption, in the undisturbed state of relative rest):

$$\frac{\partial q}{\partial t} + f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (15.2)$$

$$\frac{\partial}{\partial t} \left( \frac{q}{h_{00}} - \frac{f \zeta}{h_{00}^2} \right) = 0. \quad (15.3)$$

The last two equations can be derived equally well by linearizing their exact counterparts (11.2) and (11.3), or directly from (15.1). For instance (15.2) is equivalent to  $\frac{\partial}{\partial x}$  (15.1 b)  $- \frac{\partial}{\partial y}$  (15.1 a), and (15.3) to  $\frac{1}{h_{00}}$  (15.2)  $- \frac{f}{h_{00}^2}$  (15.1 c).

Note that the PV equation (15.3) now contains *no* term of the type  $v \partial Q_0 / \partial y$ , since the basic state of relative rest now has uniform PV, hence no Rossby-wave mechanism: (15.3) tells us that in this case, according to linearized theory, PV anomalies do not propagate but, if introduced initially, just sit around in the same place.

—II.23—

Before writing down the elementary plane-wave solutions to (15.1) we note one further derived equation. Define a new variable

$$\delta = \delta(x, y, t) = \nabla_H \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

the horizontal divergence. Then  $\frac{\partial}{\partial x}$  (15.1 a) +  $\frac{\partial}{\partial y}$  (15.1 b) gives<sup>1</sup>

$$\frac{\partial \delta}{\partial t} - f q = -g \nabla_H^2 \zeta. \quad (15.4)$$

[The nonlinear counterpart of this ‘divergence equation’, obtained from (11.1 a,b), plays an important role in the theory of nonlinear balance and accurate PV inversion used in p. 141.] This is useful since for plane waves

$$q = v_x - u_y = i(kv - lu), \quad \delta = u_x + v_y = i(ku + lv) \quad (15.5 a)$$

and so we can deal with  $q$  and  $\delta$  and work from equations (15.1 c)–(15.4) rather than (15.1 a–c) — by no means essential, but a useful shortcut. Note that, once we have  $q$  and  $\delta$ , (15.5) gives us the velocities

$$u = \frac{ilq - ik\delta}{k^2 + l^2}, \quad v = \frac{-ikq - il\delta}{k^2 + l^2}. \quad (15.6 a)$$

Again for plane waves, we have from (15.1 c), (15.2) and (15.4)

$$-i\omega \zeta + h_{00} \delta = 0 \quad (15.7)$$

$$-i\omega q + f \delta = 0 \quad (15.8)$$

$$-i\omega \delta - f q = g(k^2 + l^2) \zeta \quad (15.9)$$

and from (15.3)

$$-i\omega \left( q - \frac{f}{h_{00}} \zeta \right) = 0. \quad (15.10)$$

We want to include the case of simple gravity waves with  $f = 0^*$ , so let us eliminate  $q$  rather than  $\delta$  from (15.8) and (15.9).

—II.24—

<sup>1</sup>remember the Ex(ercise) on p. 105

Using (15.7) also, to eliminate  $\zeta$  from (15.9), we see that  $(-i\omega)$  (15.9) becomes

$$-\omega^2 \delta + c_0^2(k^2 + l^2) \delta + f^2 \delta = 0 \quad (15.11 \text{ a})$$

where  $c_0 = (g h_{00})^{1/2}$ , the gravity-wave speed again. [Note incidentally that, as on p. 105, we could have postponed introducing plane waves and reduced  $\partial_t$  (15.4) to

$$\frac{\partial^2 \delta}{\partial t^2} - c_0^2 \nabla_H^2 \delta + f^2 \delta = 0 \quad (15.11 \text{ b})$$

using (15.2) and (15.1 c) — another wave equation well known in other branches of physics, the Klein–Gordon equation — e.g. radio waves through a plasma of plasma frequency  $f$ , or (restricting to ID) the waveguides used in microwave technology.] From either version of (15.11) we see that the equations are satisfied provided that

$$\omega^2 = c_0^2(k^2 + l^2) + f^2. \quad (15.12)$$

This is the celebrated dispersion relation for inertia–gravity waves in the simplest shallow-water problem in a rotating frame. What it says physically is that the *restoring forces add* — i.e. ‘frequencies-*squared* add’. Note  $\omega^2 \geq f^2$ : there is now a *low-frequency* cutoff.

What has happened to the zero-frequency solutions with nonzero initial PV? We have swept them under the carpet by eliminating  $q$  and  $\zeta$ . But they are of fundamental interest since<sup>2</sup> they are the counterpart, in this simple linearized problem, of the more general quasi-horizontal balanced flows discussed above. [Historically, it has sometimes been noticed that the dispersion relation, for the most general possible plane-wave solutions of systems like (15.1) and its continuously-stratified analogues, is really a *cubic* — e.g. (15.12) is replaced by

$$\omega\{\omega^2 - c_0^2(k^2 + l^2) - f^2\} = 0 \quad (15.13)$$

because of the three  $\partial_t$ ’s in (15.1) — but then the root  $\omega = 0$  was sometimes dismissed as ‘trivial’, and not thought about any more. This illustrates one way

to make mistakes — in the sense of missing important points — in theoretical research! The trap was to forget that understanding something means being able to view it from all possible angles (which implies taking time to

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<sup>2</sup>\*just like the horizontal, zero-frequency flows needed to complete the solution of the stratified, non-rotating initial-value problem\*

think!)<sup>3</sup> — so that in the present problem, for instance, one becomes interested in the meaning of all the equations, and not just the form of the single equation (15.11 b), elegant though it may be.]

Once we think of looking for zero-frequency solutions, it is easy to see what they are like. Setting  $\omega$  or  $\partial_t$  to zero and writing  $\psi = g\zeta/f$  we see at once from (15.1 a,b) that


$$u = -\psi_y, \quad v = \psi_x, \quad q = \nabla_H^2 \psi \quad (15.14)$$

and that all the other equations, including (15.1 c), are satisfied — for *any* time-independent function  $\psi(x, y)$  including, of course, the plane-wave form  $\psi \propto e^{ikx+ily}$ . We also notice a feeling of *déjà vu* — we have already seen almost the same thing on p. 137, equations (11.4) ff. We have found a special case of the geostrophically balanced situation envisaged in the more generally applicable, but approximate, model (iii) of quasi-horizontal balanced motion. Here, the Rossby-wave restoring mechanism is absent, because of the uniform PV of the basic state ( $\beta = 0$  in (14.1)), so that there is no time-dependence from that mechanism. Also, the linearization giving (15.1) stops the model from becoming time-dependent through advection of any small-amplitude PV anomalies introduced initially: there is no  $\mathbf{u}_H \cdot \nabla_H$  in (15.3), *contrast* (11.7 a). This is why  $\psi$  here is a function of  $x$  and  $y$  only, and not of  $t$ .

It is clear, also, that because we have a **special case of model (iii)** we have a **special case of PV invertibility**, *plus* the additional insight that the ‘balance’ required to make PV inversion possible can be thought of as the **absence of inertia–gravity waves**.

This is another case of an idea being expressed in a form pointing towards the most far-reaching generalizations. Such a concept of balance [together with ideas borrowed from Lighthill’s theory of ‘aerodynamic sound generation’] is one way of seeing the possibility of highly accurate, nonlinear PV inversions such as that illustrated on p. 141 AND, in addition, of seeing the nature of the ultimate limitations on the applicability of the idea. (And all this started from noticing the ‘trivial root  $\omega = 0$ ’!)

The Rossby adjustment problem is an example of an initial-value problem for which both the  $\omega = 0$  and  $\omega \neq 0$  modes are required. The computer demonstration illustrates this in a case with  $y$ -independent geometry:

Note  $v(x, t)|_{t=9.2}$ <sup>4</sup> is middle  curve, already like balanced final state. The computer demonstration actually solves the *nonlinear* initial-value problem, in the  $y$ -independent ‘geometry’, i.e. (11.1) [plus damping

<sup>3</sup>[The physicist John Archibald Wheeler once wrote that “Genius is the ability to make all possible mistakes in the shortest possible time”.]

<sup>4</sup>Note that this  $v$  is roughly proportional to  $\partial h/\partial x$  and hence  $\partial\psi/\partial x$ .

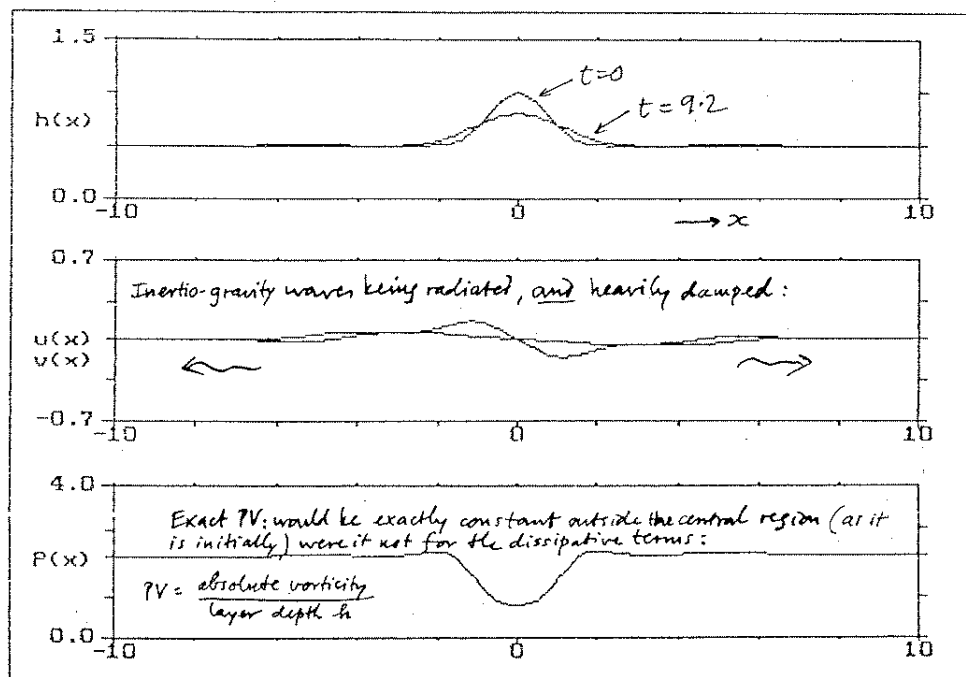
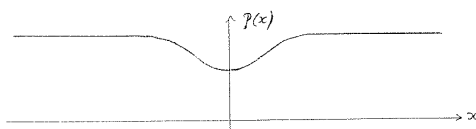


Figure 15.1:  $f = 1$ ,  $\Delta x = 0.1$ ,  $\Delta t = 0.1$ ,  $\alpha_{\max} = 2.5$ , Ratio 1 = 0.25, Ratio 2 = 0.125,  $\nu = 0.2$ . [see GFD summer school demonstration notes]

terms] with  $\partial/\partial y = 0$ , so  $\mathbf{u}_H \cdot \nabla = u \partial/\partial x$  etc. In the case shown,  $u = v = 0$  initially, and  $h (= h_{00} + \zeta)$  initially the function of  $y$  shown by the top graph marked  $t = 0$ . The demonstration allows other cases to be tried, including initial profiles hand-drawn with the mouse; Rossby's original case was  $\zeta$  and  $u$  zero, with  $v \neq 0$  in the central region. Frictional damping terms [see computer demonstration notes] have been added to (11.1) to make the numerical model well behaved in a finite domain. In an infinite domain with no damping, one would see inertia-gravity waves radiating to  $x = \pm\infty$ , leaving behind a balanced steady state, satisfying (15.14) together with the condition

$$\frac{f + \nabla_H^2 \psi}{h_{00} + g^{-1} f \psi} = P(x), \quad (15.15)$$

where  $P(x)$ , the final PV distribution,<sup>5</sup> is a function of  $x$  qualitatively like the initial distribution



and, in the case under consideration,  $\nabla_H^2 = \partial^2/\partial x^2$ . In the linearized version of the (undamped) problem, (15.3) shows that  $P(x)$  is *equal* for all  $t$  to the initial PV distribution,  $= \text{const.} + \left( \frac{q}{h_{00}} - \frac{f \zeta}{h_{00}^2} \right)_{t=0}$  when squares and products of small quantities are neglected,  $u, v, \zeta, q$  being considered small for this purpose. Thus the PV increment is proportional to

$$\Delta Q(x) = q - \frac{f \zeta}{h_{00}} = \nabla_H^2 \psi - L_D^{-2} \psi = \psi_{xx} - L_D^{-2} \psi, \quad [L_D^2 = g h_{00}/f^2] \quad (15.16)$$

—II.28—

which is known from the initial conditions (and is zero outside the central region). We can invert (15.16) to give  $\psi$  (see below), hence  $u, v, \zeta (= f \psi/g)$  and  $h = h_{00} + \zeta$  in the final steady state. The difference between these fields and the initial fields satisfies (15.10) with  $\omega \neq 0$ , and so can be represented as a superposition of inertia-gravity waves. So, in summary, a consideration of the linearized initial-value problem shows two things:

- (1) how the balanced final state is approached from the initial, unbalanced state, and

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<sup>5</sup>PV now being the exact PV of (11.3)

(2) that the balanced state, for which a PV invertibility principle holds, can be defined by saying that inertia-gravity waves are absent.

As already emphasized, it is the latter definition, rather than the alternative one of saying that the balance is geostrophic, that points towards more general, and more generally accurate, ways of defining ‘balance’ and ‘PV inversion’ such as that illustrated on p. 141.

[Note for instance that the top and bottom right panels of p. 141 show  $\delta$  ( $= \nabla_H \cdot \mathbf{u}$ ), making it clear that more general and accurate balance concepts can allow  $\delta \neq 0$ .]

The nonlinear solution of (11.1) for the same initial conditions (as suggested by running the computer demonstration and imagining that the frictional damping is made limitingly small) is qualitatively the same as the linearized solution just described. The main difference, for our purposes, is a spreading of the initial PV distribution by outward advection ( $u \partial/\partial x$ ) as the fluid layer slumps downwards and outwards. The linearized solution neglects this, since  $u \frac{\partial}{\partial x}$  of the PV is a product of small quantities. But since the qualitative nature of  $P(x)$  is still evident — it still looks like the graph below (15.15), just spread out wider — so also is the qualitative nature of  $\psi$ , now deduced by inverting (15.15) rather than (15.16). For, in the case of (15.16), we have

$$\psi = -\frac{1}{2} L_D \int_{-\infty}^{\infty} \Delta Q(x') e^{-|x-x'|/L_D} dx', \quad (15.17)$$

the two-dimensional counterpart of  $\mathcal{L}^{-1}(\Delta Q)$ , middle of p. 138. In the case of (15.15), which can be rewritten

$$\psi_{xx} - g^{-1} f P(x) \psi = h_{00} \Delta P(x), \quad (15.18)$$

where  $\Delta P(x) = P(x) - P(\infty) = P(x) - f/h_{00}$ , we have

$$\psi = \int_{-\infty}^{\infty} h_{00} \Delta P(x') G_P(x, x') dx' \quad (15.19)$$

where  $G_P(x, x')$  is the Green’s function of the operator

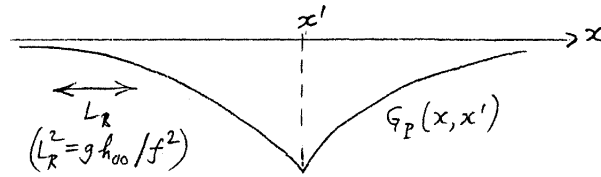
$$\frac{\partial^2}{\partial x^2} - \frac{f P(x)}{g}, = \mathcal{L}_P \quad \text{say,} \quad (15.20)$$

appearing on the left of (15.18). We may define  $G_P$  in the usual way by

$$\mathcal{L}_P G_P = \left( \frac{\partial^2}{\partial x^2} - \frac{f P(x)}{g} \right) G_P(x, x') = \hat{\delta}(x - x'), \quad (15.21)$$

where the  $\hat{\delta}$  here is the Dirac delta function (not  $\nabla_H \cdot \mathbf{u}$ , of course) together with boundary conditions that  $G_P$  evanesces as  $|x| \rightarrow \infty$  with  $x'$  fixed. Note that, to the extent that  $P(x)$  is dominated by the contribution  $f/h_{00}$ ,  $\mathcal{L}_P$  is close to  $\mathcal{L} = \partial^2/\partial x^2 - L_D^2$ , and  $G_P$  close to  $-\frac{1}{2} L_D \exp\{-|x-x'|/L_D\}$ . Qualitative resemblance prevails as long as the initial range of values of  $P$ , which is the same as the final range in the frictionless limit, by (11.3), keeps  $P(x)$  positive and well away from zero. Then (by inspection of (15.21), noting its implications for sign of the second derivative)  $G_P$  still looks qualitatively like  $-\frac{1}{2} L_D \exp\{-|x-x'|/L_D\}$ , i.e. like this sketch,

—II.30—



showing the same exponential evanescence with  $e$ -folding scale  $L_D$ , as  $|x| \rightarrow \infty$ , and the same slope-jump of unity at  $x = x'$ . However, its shape will change slightly as  $x'$  is varied, i.e. it is no longer exactly a function of  $(x - x')$  alone. The relation (15.19) gives a simple illustration of nonlinear PV inversion. If  $P(x)$  is prescribed, (15.19) gives  $\psi$  and hence  $v = \partial\psi/\partial x$  as well as  $h = h_{00} + g^{-1} f \psi$ ; but  $\psi$  is now a *nonlinear* functional of  $P(x)$  in (15.19). [This begins to reveal some of what's involved re p. 141.]



# 16.

## Inertia–gravity waves in a constant- $N$ continuous stratification

Finally, we come back to the case of continuous stratification and the derivation of **model (iv)**. First we note some basic facts about inertia–gravity waves in a constant- $N$  continuous stratification, in the Boussinesq approximation with  $\Omega^2 \varpi/g \ll 1$ .<sup>1</sup> So we simply add a Coriolis term to equation (1.4 a) of p.18, take the effective gravity (gravitational plus centrifugal force per unit mass) to be in the  $-z$  direction, and assume that we can use Cartesian coordinates (‘flat-earth approximation’ — appropriate in the real atmosphere or oceans within sufficiently small regions, say of horizontal extent  $\lesssim$  few hundred km). So the basic dynamical model is taken to be the following, with  $\rho_{00} = 1$  (note that  $\nabla$  is now three-dimensional again):

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla p + \hat{\mathbf{z}} \sigma \quad (16.1 \text{ a})$$

$$\sigma_t + \mathbf{u} \cdot \nabla \sigma + N^2 \hat{\mathbf{z}} \cdot \mathbf{u} = 0 \quad (16.1 \text{ b})$$

$$\nabla \cdot \mathbf{u} = 0 \quad (16.1 \text{ c})$$

$p$  is  $\frac{1}{\rho_{00}}$  (pressure anomaly);

$\hat{\mathbf{z}}$  is unit vector  $\parallel -\mathbf{g}$

—II.31—

with  $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$  ( $\Omega_1, \Omega_2, \Omega_3$  all constant). The elementary plane-wave solutions that satisfy these equations, when  $N^2 = \text{constant}$  also, are called (internal) inertia–gravity waves. Looking for a solution in which  $\mathbf{u}$ ,  $p$  and  $\sigma \propto \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$  we find that it satisfies (16.1) provided that, with  $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ ,

$$\omega^2 = (N \hat{\mathbf{z}} \times \hat{\mathbf{k}})^2 + (2\boldsymbol{\Omega} \cdot \hat{\mathbf{k}})^2. \quad (16.2)$$

cf.(2.6), p.26 and

bottom half p.115

The two restoring forces, buoyancy and Coriolis, add — i.e. frequencies-squared add — in the same way as led to (15.12). Here, however, we have a wavenumber vector  $\mathbf{k}$  of arbitrary orientation hence, as in section §1.1, the possibility of non-hydrostatic motion. One of the best ways of deriving (16.2) — and of seeing how the restoring forces add — is to note that (16.1 c)  $\Rightarrow \mathbf{k} \cdot \mathbf{u} = 0$  and then to consider the projection of (16.1 a) on the plane of the wavefront (see sketch below), as on p.28 for the pure internal-gravity-wave case with  $\boldsymbol{\Omega} = 0$ . Note incidentally that, for the same reason as pointed out on p.29, a single plane wave is a solution of (16.1) without linearization, i.e. without the quadratic terms  $\mathbf{u} \cdot \nabla \mathbf{u}$  and  $\mathbf{u} \cdot \nabla \sigma$  deleted. Other properties,

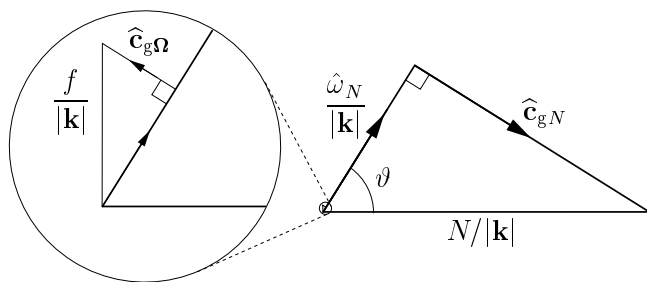
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<sup>1</sup> $\varpi$  = distance from the rotation axis, as on p. 3.

such as group velocity  $\mathbf{c}_g \perp \mathbf{k}$ , also follow as before. Differentiating (16.2), we have

$$\mathbf{c}_g = \frac{\omega_N}{\omega} \mathbf{c}_{gN} + \frac{\omega_\Omega}{\omega} \mathbf{c}_{g\Omega} . \quad (16.3)$$

where  $\mathbf{c}_{gN} = \nabla_{\mathbf{k}} \omega_N$  and  $\mathbf{c}_{g\Omega} = \nabla_{\mathbf{k}} \omega_\Omega$ , each perpendicular to  $\mathbf{k}$ , with directions lying respectively in the  $(\hat{\mathbf{z}}, \mathbf{k})$  and  $(\boldsymbol{\Omega}, \mathbf{k})$  planes, and with respective magnitudes  $|\mathbf{c}_{gN}| = N|\hat{\mathbf{z}} \cdot \mathbf{k}|/|\mathbf{k}|^2$  and  $|\mathbf{c}_{g\Omega}| = |2\boldsymbol{\Omega} \times \mathbf{k}|/|\mathbf{k}|^2$ .



The triangle on the left of figure 9 illustrates this for cases in which either  $\boldsymbol{\Omega}$  is parallel to  $\hat{\mathbf{z}}$ , or the approximation  $N^2 \gg 4\Omega^2$  is valid. In these cases  $\mathbf{c}_{gN}$  and  $\mathbf{c}_{g\Omega}$  have opposite directions, as suggested by the arrows: the two terms in (16.3) oppose each other. In the practically im-

portant case  $N^2 \gg 4\Omega^2$  suggested by the zoom circle at the left of figure 9, we have  $\mathbf{c}_g = \mathbf{c}_{gN} [1 + O(|2\boldsymbol{\Omega}|^2/N|N^2)]$ . The error term is often negligible.

The property  $\mathbf{c}_g \perp \mathbf{k}$  is from dimensional considerations as before,  $\boldsymbol{\Omega}$  and  $N$  both having the dimensions of frequency. (For  $\omega$  is independent of  $|\mathbf{k}|$  and hence of the magnitude of the wavelength; hence  $\mathbf{c}_g$ , which is the gradient of  $\omega$  in wavenumber space, must  $\perp \mathbf{k}$ .) Note also that, because the possibility of nonhydrostatic wavemotion has been brought back in, we now have **a finite range of frequencies at which inertia-gravity wave propagation can take place**, a fact that is evident from the form of (16.2).

The most important case for the real atmosphere and oceans is that in which  $N^2 \gg 4\Omega^2$ , a good approximation throughout most of the atmosphere, and in the ocean thermocline. Then we can approximate (16.2) by replacing  $\boldsymbol{\Omega}$  by its vertical component,  $\frac{1}{2} f \hat{\mathbf{z}}$  say:

$$\begin{aligned} \omega^2 &= N^2(\hat{\mathbf{z}} \times \hat{\mathbf{k}})^2 + f^2(\hat{\mathbf{z}} \cdot \hat{\mathbf{k}})^2 \\ &= N^2 \cos^2 \theta + f^2 \sin^2 \theta \end{aligned} \quad (16.4)$$

where, as on p. 27, the angle  $\theta$  is the angle between the wavenumber  $\mathbf{k}$  and the horizontal. In this case the finite range of frequencies for wave propagation is simply (because  $\sin^2 \theta + \cos^2 \theta = 1$ )

$$f \leq |\omega| \leq N; \quad (16.5)$$

this holds also for arbitrary  $N$  and  $\Omega$  values in the case where<sup>2</sup>  $\boldsymbol{\Omega} \parallel \hat{\mathbf{z}}$ .

<sup>2</sup>The group velocity properties are strange when  $\boldsymbol{\Omega} \parallel \hat{\mathbf{z}}$  and  $2\Omega = N$ ; then  $\mathbf{c}_g = 0$  for all  $\mathbf{k}$ . But this case is seldom important. Also unimportant, except at the equator, is the case  $\boldsymbol{\Omega} \perp \hat{\mathbf{g}}$ , in which there is no low-frequency cutoff:  $0 \leq \omega^2 \leq (N^2 + 4\Omega^2)$ .

The approximation (16.4) suggests the important fact that, in a strongly stratified fluid in the sense that  $N^2 \gg 4\Omega^2$ , **only the vertical component of  $\Omega$  matters dynamically.** (This turns out to be true much more generally.)

Note that in the atmosphere and oceans (and in stellar interiors, to which the same theory is relevant — for our sun,  $2\Omega \simeq 5 \times 10^{-6} \text{ s}^{-1}$ ,  $N \simeq 10^{-3} \text{ s}^{-1}$ , so  $2\Omega/N < 10^{-2}$ ), the effective gravitational potential  $\tilde{\chi} = \chi - \frac{1}{2}\Omega^2 \varpi^2$  is spherically symmetric to good approximation ( $\Omega^2 \varpi/g \ll 1$ ), hence

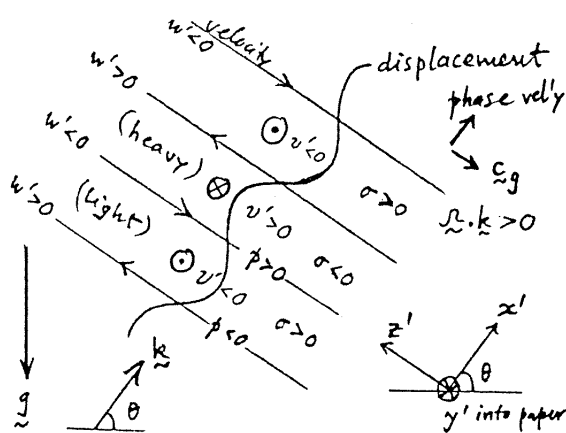
$$f = 2\Omega \sin(\text{latitude}). \tag{16.6}$$

Here is what the fields of motion and buoyancy look like in the case paralleling that on p. 25. Qualitatively speaking the only new feature is the nonzero fluid velocity  $v'$  into the paper ( $y'$  direction) satisfying

LHS is  $+\partial v'/\partial t$

$$-i\omega v' = (2\Omega \cdot \hat{\mathbf{k}}) w',$$

so that  $v'$  is  $\frac{\pi}{2}$  out of phase with  $w'$ , as for pure inertia waves, p. 115. Equation



tion (2.12 a) becomes

$$-i\omega w' = \sigma \cos \theta - (2\Omega \cdot \hat{\mathbf{k}}) v',$$

showing how the restoring effects add, making  $|\omega| > |2\Omega \cdot \hat{\mathbf{k}}|$ , Equation (2.12 b) is unchanged:

$$-i\omega \sigma = -N^2 w' \cos \theta$$

( $v' = \hat{v} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$ ,  $w'$  and  $\sigma$  similarly).

—II.33—

These three equations lead immediately to the dispersion relation (16.2), and to the conclusion that what were circular particle paths for pure inertia waves (p. 115) have now become ellipses, in general. They expand to circles if we make  $\cos \theta = 0$  (with  $\Omega$  not horizontal), i.e.  $\hat{\mathbf{z}} \times \hat{\mathbf{k}} = 0$  in (16.2), i.e. fluid velocity is horizontal and doesn't feel the buoyancy restoring force, same problem as on p. 115. The ellipses shrink to straight lines ( $\because v' = 0$ ) if we make  $2\Omega \cdot \hat{\mathbf{k}} = 0$  (with  $\Omega$  not horizontal); then the Coriolis force is

ineffective to the extent that (16.2) coincides with the pure gravity-wave dispersion relation derived on p. 26. [Note however that the disturbance pressure field is different from before, even in these limiting cases, because a Coriolis term has to be added to Equation (2.12c). This can be shown to tie up with the different group velocity behaviour implied by varying the direction of  $\hat{\mathbf{k}}$  in (16.2), via a consideration of wave-energy or wave-action fluxes.]

Note that, if a problem like that of p. 36 ff with fixed vertical structure  $e^{imz}$  or  $\sin(mz)$  is considered, then (16.4) reduces to (15.12) when  $k^2 + l^2 \ll m^2$  (hydrostatic waves), if we identify  $N^2/m^2$  with  $c_0^2$ .

Inertia-gravity waves conforming to (16.4) are ubiquitous in the atmosphere and oceans, and have often been identified from observations of displacements or velocities that reveal the characteristic elliptical particle paths, or turning of the velocity vector with height. For instance the latter has often been seen in routine meteorological radiosonde soundings; one can tell from these whether the waves have upward or downward group velocity — e.g. picture on previous page is case of downward  $\mathbf{c}_g \cdot \hat{\mathbf{z}}$ , for which a vertical sounding sees  $\mathbf{u}'$  turning like a right-handed screw as height  $\uparrow$  in northern hemisphere;<sup>3</sup> the sonde moves upward through the velocity field much faster (usually) than the vertical phase propagation. One can also deduce  $\omega^2$  from the eccentricity of the ellipse, from a single sounding, if the wave signal is clear enough. Typical velocity amplitudes  $|\hat{\mathbf{u}}|$  in the more conspicuous cases might be a few  $\text{m s}^{-1}$  in the atmosphere, or  $\text{cm s}^{-1}$  to a few tens of  $\text{cm s}^{-1}$  in the oceans. [We do not have a complete understanding of how all the observed waves are generated, although some are generated by flow over topography as in the idealized problem of pp. 29ff. with the Coriolis effects added.]

—II.34—

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<sup>3</sup>i.e. clockwise as seen from *below* — actually it is commoner to see the opposite case of upward  $\mathbf{c}_g \cdot \hat{\mathbf{z}}$ , in the case of the atmosphere.

# 17.

## Quasi-geostrophic motion in a continuously stratified Boussinesq fluid: model (iv)

We now show how the analyses of quasi-geostrophic motion presented in chapters 9–11 carry over to continuously stratified systems. We retain the Boussinesq approximation for simplicity. This leads to **model (iv)** of p. 136, and is another step closer to realistic models of flows like that on p. 123 and like those shown in the first lecture.

We take the same equations but, for the moment, allow  $N^2 = N^2(z)$ :

$$u_t + \mathbf{u} \cdot \nabla u - f v = -p_x \quad (17.1 \text{ a})$$

$$v_t + \mathbf{u} \cdot \nabla v + f u = -p_y \quad (17.1 \text{ b})$$

$$w_t + \mathbf{u} \cdot \nabla w = -p_z + \sigma \quad (17.1 \text{ c})$$

$$\sigma_t + \mathbf{u} \cdot \nabla \sigma + N^2 w = 0 \quad (17.1 \text{ d})$$

$$w_z = -u_x - v_y \quad (17.1 \text{ e})$$

We expect these to apply in two cases:

(1) to systems with  $\boldsymbol{\Omega} = (0, 0, \frac{1}{2} f)$

(1) is easiest case to think of at first

(2) to systems with  $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \frac{1}{2} f)$  and  $4\Omega^2 \ll N^2$  (see below).

In case (2) we allow  $f = f(y)$  hence larger horizontal scales than in section 16, such as the  $10^3$ -km scale of the atmospheric flow shown on p. 123, with  $y$  corresponding to latitude. Write  $R_T = 1/f T$ ,  $R_U = U/f L$  as before, and as before assume

$$R \equiv \max(R_T, R_U) \ll 1 \quad (17.2) \quad \text{---II.35---}$$

so that  $\mathcal{D}/\mathcal{D}t \lesssim R f$ , noting  $w \lesssim U H/L$  (notation as on pp.121ff).

Then we expect geostrophic balance as the first approximation to (17.1 a,b) as before:

$$f u \simeq -p_y, \quad f v = p_x, \quad \text{with relative error } O(R). \quad (17.3)$$

Again we can proceed, as mentioned on p. 122, to consider the next correction in an expansion in powers of  $R$ . But again the vorticity equation offers a

shortcut. Recall (9.7) and its simplification to (9.8); exactly the same thing happens here except that, in case (2) above, we get an extra term in  $df/dy$ . (It seems strange to be working in Cartesian coordinates, i.e. using ‘flat-earth theory’ yet taking account of the variation of  $f$  with latitude; but we shall see shortly that it is a consistent approximation.) **Reverting to the notation  $\zeta$  for the vertical component of vorticity**, we get (9.7) exactly as before except for the extra term in  $df/dy = \beta$  say, since (17.1 a,b) are the same as (9.2 a,b) except for the possible  $y$ -dependence of  $f$ , and we are taking  $-\frac{\partial}{\partial y}$  (17.1 a)  $+\frac{\partial}{\partial x}$  (17.1 b). So under the assumption  $R \ll 1$  we get (with  $\zeta = v_x - u_y \approx U/L$ )

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta + \beta v = f w_z \quad (17.4)$$

with relative error  $O(R)$ , just as when approximating (9.7) by (9.8). **Note that  $\beta$  now means  $df/dy$  and not the topography-related quantity defined above (10.3).** Assume now that

$$\left. \begin{array}{l} \beta \lesssim R f/L; \quad \beta = \text{const with relative error } O(R) \\ \text{and, consistently with this, } f = f_0(1 + O(R)) \\ \text{with } f_0 = \text{constant.} \end{array} \right\} \quad (17.5)$$

Then as in (9.11), we have from estimating LHS (17.4) that

$$w \lesssim R \frac{U H}{L} \quad (17.6)$$

$\therefore$

$$\frac{\mathcal{D}w}{\mathcal{D}t} \lesssim R f \cdot R \frac{U H}{L} \approx R^2 \frac{H^2}{L^2} \cdot \frac{L}{H} \underbrace{|\nabla_H p|}_{(\approx f U \text{ from (17.3)})}$$

Assume also that

$$H^2/L^2 \ll R^{-2} \quad (17.7)$$

(usually satisfied very strongly, and satisfied sufficiently well even in some ‘tall’ laboratory experiments with  $H/L > 1$ ). Then

$$\frac{\mathcal{D}w}{\mathcal{D}t} \ll \frac{L}{H} |\nabla_H p|, \quad (17.8)$$

which means that LHS (17.1 c) (the vertical acceleration) has negligible dynamical effects,  $\therefore$  negligible effects on the horizontal part of the pressure gradient,  $\nabla_H p = (p_x, p_y, 0)$ , which enters (17.1 a,b) and (17.3). Introducing the notation corresponding to that in (9.9) and (11.4), we define

$$\boxed{\psi = f_0^{-1} p, \quad \text{so that } u = -\psi_y, \quad v = \psi_x, \quad \zeta = \nabla_H^2 \psi} \quad (17.9)$$

with relative error  $O(R)$ . Consistent with this level of accuracy we may neglect  $w\partial/\partial z$  in (17.4), because of (17.6). this reduces (17.4) to

$$\frac{\mathcal{D}_g}{\mathcal{D}t} \nabla_H^2 \psi + \beta \psi_x = f_0 w_z \quad (17.10 \text{ a})$$

where, in much the same way as before,

$$\frac{\mathcal{D}_g}{\mathcal{D}t} = \frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}, \quad (17.10 \text{ b})$$

which may aptly be called the ‘geostrophic material derivative’. It differs from  $\mathcal{D}/\mathcal{D}t$  not only in approximating the horizontal velocity by its geostrophic value given by (17.9) or (17.3), **but also in neglecting  $w\partial/\partial z$  altogether**. Note that neglecting  $w\partial/\partial z$  now relies on the smallness of  $w$  as estimated by (17.6) — the factor  $R$  is crucial — since, *unlike* the case of (9.8),  $p, \psi, u, v, \zeta$  may all now depend significantly on  $z$ ; i.e. the Taylor–Proudman theorem fails. This is because of the stratification and associated buoyancy effects: instead of (9.5) we have from (17.1 c) and (17.8) that  $p_z = \sigma$ , *not* zero to the first approximation.<sup>1</sup> This is the essential difference, and the only significant difference, between the present situation and the  $z$ -independent situation summarized by (9.5). In the notation of (17.9),  $p_z = \sigma$  reads

$$\boxed{\sigma = f_0 \psi_z.} \quad (17.11)$$

The departure from Taylor–Proudman conditions can be expressed more explicitly by eliminating  $\psi$  between (17.11) and the geostrophic and hydrostatic relations  $u = -\psi_y, v = \psi_x, \sigma = f_0 \psi_z$  to give

$$\left. \begin{aligned} u_z &= -f_0^{-1} \sigma_y \\ v_z &= f_0^{-1} \sigma_x \end{aligned} \right\}, \quad (17.12)$$

usually called the ‘**thermal wind**’ relations (although ‘**thermal shear relations**’ would be more logical). This is how, for instance, the **cold summer polar mesopause**, implying equatorward temperature and potential-temperature gradients at altitudes  $\sim 80$  or  $90$  km, is connected with wave-induced angular momentum transport and its effect on  $u_z$ , where  $x$  and  $u$  correspond to the eastward direction.

To complete the derivation of model (iv), and its generalization in which horizontal boundaries are important, we substitute the same geostrophic and

<sup>1</sup>There is an implicit assumption here that  $N^2$  is not too small in (17.1 d),  $\gtrsim f^2 L^2/H^2$  in fact, see (17.21) ff.

hydrostatic relations  $u = -\psi_y$ ,  $v = \psi_x$ ,  $\sigma = f_0 \psi_z$  into (17.1 d) to obtain, again using (17.6)

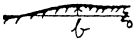
$$w = -\frac{1}{N^2} \frac{\mathcal{D}_g}{\mathcal{D}t} (f_0 \psi_z) = -\frac{\mathcal{D}_g}{\mathcal{D}t} \left( \frac{f_0}{N^2} \psi_z \right) \quad (17.13)$$

$$\begin{aligned} \therefore f_0 w_z = - \left\{ \frac{\mathcal{D}_g}{\mathcal{D}t} \left( \frac{f_0^2}{N^2} \psi_z \right) \right\}_z &= -\frac{\mathcal{D}_g}{\mathcal{D}t} \left\{ \left( \frac{f_0^2}{N^2} \psi_z \right)_z \right\} \\ &\text{(since } (-\psi_{yz} \partial_x + \psi_{xz} \partial_y) \psi_z = 0). \end{aligned} \quad (17.14)$$

Thus (17.10 a) implies that

$$\boxed{\frac{\mathcal{D}_g}{\mathcal{D}t} \left[ \psi_{xx} + \psi_{yy} + \left( \frac{f_0^2}{N^2} \psi_z \right)_z + f \right] = 0}; \quad f = f_0 + \beta y \quad (17.15)$$

$f$ , not  $f_0$  here



-N60-

-II.38-

At an approximately horizontal boundary in the sense of p. 152, say  $z = z_0 + b$  where  $z_0 = \text{constant}$ , we have  $w = \mathcal{D}_g b / \mathcal{D}t$  at the boundary so that, from  $f_0 \times$  (17.13),

$$\boxed{\frac{\mathcal{D}_g}{\mathcal{D}t} \left[ \frac{f_0^2}{N^2} \psi_z + f_0 b \right] = 0 \text{ at boundary.}} \quad (17.16)$$

The extra factor  $f_0$  was inserted to suggest a relationship with (17.15) to be explained below. Note also (by taking the factor  $f_0$  back outside, and the factor  $N^{-2}$  also) that (17.15) can also be written as

$$\frac{\mathcal{D}_g}{\mathcal{D}t} (\sigma + N^2 b) = 0 \text{ at boundary.} \quad (17.17)$$

This is a somewhat disguised way of expressing the obvious consequence of (17.1 d) that the density of a fluid particle moving along the boundary stays the same. \*For a compressible atmosphere the corresponding ‘boundary invariant’ would be the potential temperature  $\theta$ .\* [Note that total density itself is not exactly  $\sigma + N^2 b$ ; rather, it is proportional to a constant plus the total buoyancy acceleration

$$\sigma_1 = \sigma + \int^z N^2(z') dz' \quad (17.18)$$

(recall p. 17). What appears in (17.1 d) as  $w \sigma_z + N^2(z) w$  ( $= w \partial_z \sigma_1$ ) is represented in (17.17) in the approximate form

$$(\sigma_z + N^2) w \simeq N^2 w \simeq N^2 \frac{\mathcal{D}_g b}{\mathcal{D}t} = \frac{\mathcal{D}_g(N^2 b)}{\mathcal{D}t}. \quad (17.19)$$



In fact we shall assume  $b \ll H$ , so the distinction between  $N^2 b$  and  $\int^z N^2(z') dz'$  will disappear anyway.]

Equations (17.15) and (17.16) define **model (iv)<sup>2</sup>** — a remarkably simple and succinct model of what was described in more general terms on p. 144 as a “**layerwise-2D vortex dynamics, in which the vertical coupling is solely through the PV inversion operator**”. The geostrophic material derivative  $\frac{D_g}{Dt}$  **involves only horizontal motion**, by definition — hence ‘layerwise 2D’) — and if we know the distributions of the two material invariants involved, [ ] in (17.15)–(17.16),<sup>3</sup> we can invert to get  $\psi$  and hence, via (17.9), everything else. The inversion problem is of a kind known to be —II.39— robustly well-posed:

$$\left. \begin{aligned} \psi_{xx} + \psi_{yy} + \left( \frac{f_0^2}{N^2} \psi_z \right)_z &= \text{known func.}(x, y, z) \\ \psi_z &= \text{known func. on boundaries } z \simeq \text{const.} \\ \nabla_H \psi &\text{ evanescent at large horizontal distances} \end{aligned} \right\} \quad (17.20)$$

(or  $\psi = \text{const.}$  along an enclosing side boundary). Note that the interior equation is elliptic (Poisson-like)  $\because N^2 > 0$ . In the constant- $N$  unbounded case its 3D Green’s function is  $\{(x - x')^2 + (y - y')^2 + f_0^{-2} N^2 (z - z')^2\}^{-1/2}$ , as mentioned on p. 138; this represents a longer-range interaction than in models (i) and (ii) but a shorter-range interaction than in model (iii). Note that the vertical coupling owes its existence to the Coriolis effects (vertical scale of interaction is proportional to  $f_0$ ). There is a competition between rotational stiffness trying to make  $\partial_z$  small, i.e. increase vertical scales (the Taylor–Proudman conditions expressed by (9.5) ff. being an extreme manifestation of this), and stratification trying to make  $\nabla_H$  small, i.e.  $\partial_x$  and  $\partial_y$  small, increasing horizontal scales (model (ii) of pp. 135 and 46 being the corresponding extreme). In between the two extremes there is a **natural aspect ratio** of vertical and horizontal scales

$$\boxed{\frac{H}{L} \approx \frac{f_0}{N}} \quad (17.21)$$

(Prandtl’s ratio) such that the two effects are comparable, as manifested by comparable magnitudes of  $\psi_{xx} + \psi_{yy}$  and  $(f_0^2 N^{-2} \psi_z)_z$  in (17.20). This state

<sup>2</sup>and its generalizations to vertically bounded domains

<sup>3</sup>The first of these, the PV appropriate to model (iv), is called the ‘quasi-geostrophic PV’ or sometimes ‘pseudo-PV’. Its horizontal gradient can be shown to be  $\propto$  the *isentropic* gradient of Rossby–Ertel PV (gradient on stratification surfaces), with fractional error  $O(R)$ .

of things,<sup>4</sup> with the actual length and height scales conforming to (17.21), is just that which makes  $\sigma$  significant in (17.11) and (17.12) when the typical values of  $\sigma$  are those implied by (17.1 d), assuming that order-of-magnitude equality,  $w \approx RUH/L$ , holds in (17.6), and assuming a scale  $\Sigma$  for the typical variation of  $\sigma$  with  $x, y, t$  (as on p.46):

$$\underbrace{R f_0}_{\frac{\partial}{\partial t} + \mathbf{u}_H \cdot \nabla_H} \Sigma \approx N^2 \underbrace{RUH/L}_w \quad (\text{from (17.1 d)}) \quad (17.22)$$

For if we suppose that (17.21) holds, there follows  $\Sigma \approx NU$  and

$$U/H \approx \Sigma/NH \approx \Sigma/f_0 L, \quad (17.23)$$

consistent with (17.12).

**The following additional points can be made:**

(i) In problems where it is natural to regard the vertical scale  $H$  as given (e.g. thought-experiments like that of p. 36 ff, in which  $H = m^{-1}$  is set by a given forcing, or initial-value problems with the scale set by initial conditions), the length scale

$$L \approx L_D \equiv NH/f_0 \quad (17.24)$$

will inevitably appear. E.g. in a quasi-geostrophic version of the problem of p. 36 ff, with  $\beta = 0$ , one would find the response evanescing like  $\exp(-|x/L_D|)$  rather than propagating to arbitrarily large distances. This structure would be conspicuous if the forcing occupied a region of width  $\ll L_D$ ; then the response in  $\psi$  extends beyond that, exactly as in (15.17). The resemblance is not accidental. E.g. with  $N = \text{const.}$ , the vertical structure  $e^{imz}$  implies a definite hydrostatic internal gravity wavespeed  $N/m$  that can be identified with  $NH$  and also with the  $c_0$  of (15.11 a), (15.12) and (15.17) so  $L_D$  has the same meaning as before,<sup>5</sup>  $L_D = c_0/f_0$ . The  $L_D$  of (17.24) is often called, likewise,

<sup>4</sup>sometimes also called ‘Burger number unity’ conditions since some authors use the term ‘Burger number’ to denote  $N^2 H^2/f_0^2 L^2$ , the square of the dimensionless aspect ratio. Others confusingly call  $f_0 L/NH$  the ‘rotational Froude number’, which has little to do with the mainstream Froude-number concept as on p. 46ff. With “Rossby” ((17.25)) and “Prandtl” we have *four* names associated with *one* concept. (I disclaim responsibility!)

<sup>5</sup>and  $\nabla_H^2 + \left(\frac{f_0^2}{N^2} \psi_z\right)_z = \nabla_H^2 - L_D^2$ , cf. (11.7 b). [For small disturbances about relative rest this can be generalized to arbitrary  $N^2(z)$  via Sturm–Liouville theory, somewhat as on p. 83.]

the *Rossby length* (or more archaically, for historical reasons, the ‘Rossby radius’ or ‘Rossby radius of deformation’) associated with height scale  $H$ .

(ii) A corresponding set of remarks hold in problems where it is natural to regard  $L$  as given; the implied natural height scale

$$H \approx H_R \equiv f_0 L/N \quad (17.25)$$

is now called the ‘Rossby height’ associated with length scale  $L$ . A classic example is the response of the  $\beta = 0$  system, initially at relative rest over a flat lower boundary, to an imposed undulation of that boundary such as  $b = (\epsilon \sin kx)$  times a slowly-growing function of time  $t$ . (Note (17.16) was derived in such a way as not to exclude the possibility of time-dependent  $b(x, y, t)$ . Also, with  $b \approx \epsilon \ll H$  one may apply (17.16) at  $z = z_0$  just as well as at  $z_0 + b$ .) If in (17.16)  $[\ ] = 0$  initially for all  $x, y$  we have

$$\psi_z = -\frac{N^2}{f_0} b(x, y, t) \quad \text{at } z = z_0, \quad \forall t; \quad (17.26)$$

and the quasi-geostrophic PV,  $[\ ]$  in (17.15), is spatially uniform initially so that in the inversion problem (17.20) we have

$$\psi_{xx} + \psi_{yy} + \left( \frac{f_0^2 \psi_z}{N^2} \right)_z = 0, \quad \forall t. \quad (17.27)$$

It is evident that (if we assume  $N = \text{const.}$  again) the response  $\psi \propto \exp(-z/H_R)$ , in the sinusoidal case. [If we made the boundary hump up in an isolated area only, then the far field of  $\psi$  will be like the 3D Green’s function,  $\psi \propto (x^2 + y^2 + f_0^{-2} N^2 z^2)^{-1/2}$ , still conforming to (17.21) even though there is no longer a clear-cut single horizontal scale.] This kind of problem is relevant to understanding the response of the stratosphere to disturbances in the troposphere with length scales  $L \lesssim 10^3$  km for which  $\beta$  and compressibility can to some extent, qualitatively speaking, be ignored.<sup>6</sup> For instance the scale of penetration of the cyclone structures illustrated on p. 140 into the stratosphere is of the order suggested by (17.25). —II.42—

The equation (17.27) and boundary condition (17.26) are also obtained in the stratified counterpart of the problem of pp. 133–135, flow over a shallow hump. Then  $\psi$  represents the departure from uniform flow. As might be expected from (17.25), the flow patterns are qualitatively like those on p. 135,

<sup>6</sup>[It can be shown that to ignore compressibility altogether (i.e. Boussinesq, as here) requires  $H \ll$  density scale height  $H_\rho$  ( $\simeq 7$  or  $8$  km); otherwise the  $\partial_z$  term in (17.15), (17.20) is replaced by  $\rho_0^{-1}(\rho_0 f_0^2 N^{-2} \psi_z)_z$  where  $\rho_0 = \rho_0(z) \propto \exp(-z/H_\rho)$ ; the problem is still qualitatively similar.]

but weakening as  $z$  increases through the Rossby height  $H_{R_a} = f_0 a/N$  based on the horizontal scale  $a$  of the hump. For instance if the flow at small  $z/H_{R_a}$  looks roughly like the top right-hand picture on p.135 then at  $z/H_{R_a}$  somewhat greater (but still  $\lesssim 1$ ) the flow would look more like the top left-hand picture. **In a rather fuzzy sense the stratification is providing an ‘upper lid’,** forcing (negative) vortex stretching to take place,  $f_0 w_z < 0$  in  $(17.4)_{\beta=0}$ , just as in (9.8).

(iii) As already mentioned, if  $b \ll H$  we may apply the general boundary condition (17.16) at  $z = z_0 = \text{const.}$  Note that we do indeed have  $b \ll H$  when the condition corresponding to the inequality in ca. mid p. 134 (the condition for a strong disturbance) is satisfied — i.e. replacing the layer depth  $h_0$  there with a constant of order unity times  $H_{R_a}$  here:

$$\max b = \epsilon a \approx \frac{U}{f_0 a} \cdot \frac{H_{R_a}}{a} a = R_a H_{R_a} \ll H_{R_a} \quad (17.28)$$

if  $R_a$  denotes  $U/f_0 a$ . Applying (17.16) at  $z = z_0$  rather than at  $z = z_0 + b$  greatly simplifies the mathematics of finding explicit solutions. However, in problems with large-scale boundary slopes, analogous to the topographic Rossby-wave problem of p. 127–130, say a slope in the  $y$  direction as there, one has the same problems of nonuniform validity for large  $y \gg L$  as noted near top p. 128 and in the footnote on p. 153. Again this is a limitation of quasi-geostrophic theory, not of the generic picture summarized on p. 139. In the stratified case there are similar problems associated with large-scale slopes of the constant- $\sigma_1$  stratification surfaces. Such slopes (both of  $\sigma_1$ -surfaces and of boundaries) can give rise to Rossby-wave propagation in the same way as in the original problem of pp. 127–130.

(iv) More precisely, what are relevant are gradients (in the  $y$ -direction, say) of one or both of the two material invariants [ ] in (17.15) and (17.16). Let us call these invariants  $Q$  and  $B$  respectively.

Contributions to their  $y$ -gradients arise whenever the slopes of the boundary and the  $\sigma_1$ -surfaces *differ* — just as it was the *difference* between the two boundary slopes that mattered on p. 127-130. In general there are further contributions to the gradients  $Q_y$  and  $B_y$ . For instance if  $\beta = df/dy \neq 0$  in (17.15), then this contributes to  $Q_y$ , as can a nonuniform mean flow  $\bar{u}(y, z)$  in the  $x$  direction — not only through gradients in  $\psi_{yy}$  but also through the  $\partial/\partial z$  term, whose  $y$ -derivative is  $\partial/\partial z$  of  $f_0 \sigma_y/N^2$ , by (17.11). From (17.18),  $\sigma_y/N^2 \simeq \sigma_y/\sigma_{1z} = \sigma_{1y}/\sigma_{1z}$ , **which is minus the  $\sigma_1$ -surface slope.** Similarly,

$$B_y = f_0 \left( \frac{\sigma_y}{N^2} + b_y \right) \Big|_{z=z_0}. \quad (17.29)$$

( ) is minus the **slope difference**, i.e., minus the  $\sigma_1$ -surface slope at  $z - z_0$  plus the boundary slope.

(v) The Rossby waves whose restoring mechanism depends on  $B_y$  alone are called **topographic Rossby waves** if  $b$  is sloping and  $\sigma_1$  are flat, and (for historical reasons) **Eady short waves** if  $\sigma_1$  are sloping and  $b$  is flat,  $b = 0$ . Their dispersion relation for  $\psi \propto e^{i(kx+ly-\omega t)}$  (ignoring  $y$ -nonuniformity) is precisely (10.6)<sup>7</sup> with the following changes. The quantity  $\beta$  in that relation (10.6) (that  $\beta$  being defined above (10.3)) is replaced by  $B_y/H_{RW}$  (which is equal to  $f_0 b_y/H_{RW}$  in the ‘topographic’ case, with the  $h$  above (10.3) identified with  $H_{RW}$  here) **where**

$$H_{RW} = f_0(k^2 + l^2)^{-1/2}/N, \quad (17.30)$$

again illustrating the lid-like effect of the stratification.<sup>8</sup> It is curious that, in the ‘topographic’ case (sloping boundary and flat  $\sigma_1$ -surfaces), the factors  $f_0$  cancel giving

$$\omega = \frac{-b_y N k}{(k^2 + l^2)^{1/2}}. \quad (17.31)$$

**But one should not conclude that rotation is unimportant!** Quite the contrary! [A warning, incidentally, against the common failing of confusing the underlying dynamics itself with too superficial a view of it through this or that particular equation — there is a myth, misrepresenting some of the valid ideas of theoretical physics, that the equations ‘are’ the reality.]

(vi) The Rossby waves whose restoring mechanism depends on  $\beta$ , or on other things that make  $Q_y$  nonzero (more generally, that make isentropic gradients of Rossby–Ertel PV nonzero) can exist **independently of boundaries**. Because of the stable stratification, the  $\sigma_1$ -surfaces can entirely take over the constraining role originally played by the pair of boundaries on p. 127. For instance we can confidently predict the existence of Rossby waves deep within the Sun (recall numerical magnitudes, p. 164;  $N^2 \gg f_0^2$ ) even though no way has yet been found to observe them.

In the case  $\bar{u} \equiv 0$ ,  $Q_y = \beta$ ,  $N = \text{const.}$ , (17.15) linearized (about rest) has plane-wave solutions (which as always can be superposed to give more localized solutions)  $\psi \propto e^{i(kx+ly+mz-\omega t)}$  provided that

$$\omega = \frac{-\beta k}{k^2 + l^2 + \frac{f_0^2}{N^2} m^2}. \quad (17.32)$$

If  $\beta > 0$ , phase velocity always westwards, as before. Also

(this *is* the present  $\beta = df/dy$ )

<sup>7</sup>which reads  $\omega = -\beta k/(k^2 + l^2)$

<sup>8</sup>Remember always that we are talking about timescales  $\gg N^{-1}$ : ‘gravity-wave stiffness’ is effectively large, as well as inertia-wave (Coriolis-wave) stiffness.

$$\mathbf{c}_g = \frac{\beta}{\left(k^2 + l^2 + \frac{f^2}{N^2} m^2\right)^2} \left\{ \left(k^2 - l^2 - \frac{f^2}{N^2} m^2\right), 2kl, 2\frac{f^2}{N^2} km \right\} \quad (17.33)$$

— can have any direction. Note again, stratification is essential.

\*Remark re *trapping*: Variations in  $N^2(z)$  won't do: only basic-flow shear  $\mathbf{u}(z)$ , or rigid boundaries (ocean surface is effectively rigid if  $L$  not too large:  $\frac{f^2 L^2}{g H} \ll 1$ .\*

**Note also the possibility of vertical Rossby propagation**, whose likely importance for the real stratosphere was pointed out in a celebrated paper by Charney & Drazin (1961, *J. Geophys. Res.*) Among other things they asked the question why — in view of the enormous wave-energies involved — does the earth not have a much hotter mesosphere and thermosphere, perhaps even a ‘geocorona’ like the sun’s corona? Today we know why this cannot happen. Rossby waves ‘break’, as mentioned on p. 132, by degenerating into layerwise-2D turbulence<sup>9</sup> long before they reach altitudes where linearized wave theory, generalized to include evanescent mean density  $\rho_0(z) \propto e^{-z/H_\rho}$ , would predict thermally significant wave-energy densities. The video shown in the first lecture, also available at [www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/](http://www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/), though only a shallow-water model, shows rather well what breaking Rossby waves look like when the real wintertime middle stratosphere (say  $\sim 20$ – $40$  km altitude) is observed with sufficient spatial resolution. This has already been achieved for limited timespans (CRISTA project, two space-shuttle missions of several days each, websearch “gyroscopic pump in action”).

It also turns out that the Rossby-wave propagation mechanism is one of the keys to understanding the ‘anti-frictional’ properties — what used to be called the ‘negative viscosity’ — of large-scale atmospheric flow; again see [www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/](http://www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/), also last questions of ex. sheet 3. In brief, the sign of  $\overline{u'v'}$  is easily understandable if the Rossby mechanism is invoked, and the waves are hypothesized to be generated in middle latitudes and to be dissipated by breaking in lower latitudes. The observational evidence, including estimates of real isentropic distributions of PV, strongly supports this hypothesis.

<sup>9</sup>more fundamentally, by irreversibly rearranging Rossby–Ertel PV along isentropic surfaces. The extreme spatial inhomogeneity of the resulting flow (wave-like adjacent to 2D-turbulent flow) is central e.g. to understanding the dynamics and chemistry of the ozone hole; see also figures in my Limerick paper (animation on website; websearch “gyroscopic pump in action”) and web lecture at [www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/](http://www.atm.damtp.cam.ac.uk/people/mem/papers/ECMWF/)

(vii) *Consistency of (17.5) and of cartesian coordinates as a local approximation in mid-latitudes on sphere, when  $\frac{H}{L} \ll 1$  [(2) p. 166].*

Note: *not* same as the usual scaled *spherical* coordinates encountered in the literature. We mean true cartesian coordinates in, e.g., a local tangent plane and perpendicular to it. This represents a self-consistent approximation if  $\frac{L}{a} \ll 1$  ( $a$  = radius of sphere); in the present context we require, for consistency with the quasi-geostrophic approximations:

$$\boxed{\frac{L}{a} \lesssim R.} \quad (17.34)$$

Note that this gives (17.5) above:  $f$  varies by  $O(R)$  over scale  $L$ ; the  $\beta$  term in (17.4), (17.10 a) etc. is no greater than the other terms in order of magnitude, since  $\beta = df/dy \approx 2\Omega/a$ . This is called the ‘ $\beta$ -plane’ approximation.

—II.47—

(viii) If we linearize (17.15) about a  $y$ -independent mean flow  $U(z)$ , and assume a  $y$ -independent disturbance in the sense that

$$\psi = -yU(z) + \psi'(x, z, t), \quad (17.35)$$

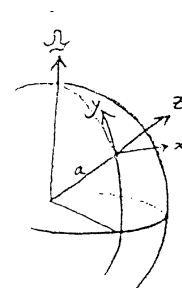
and take  $N$  and  $\beta$  both constant and put  $z = f_0 Z/N$ , then we have

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) [\psi'_{xx} + \psi'_{ZZ}] + Q_y \psi'_x = 0. \quad (17.36)$$

On assuming a solution  $\psi' \propto e^{i\{k(x-ct)\}}$  we get an equation with the same mathematical form as the so-called Rayleigh equation, or Taylor–Goldstein equation (4.7) with  $N = 0$ . As is well known, the Rayleigh equation describes among other things ordinary *shear instabilities* in 2D vortex dynamics, with

$$Q_y = Q_y(Z) = \beta - U_{ZZ}. \quad (17.37)$$

Thus any result for such an instability carries over to the present case — e.g.  $U \propto \tanh Z$ ,  $\beta = 0$  is unstable — but it represents a physically different phenomenon, ‘baroclinic wave instability’ (next section). In practice, boundary conditions from (17.16) usually enter, and may enhance the instability — which is regarded today as the fundamental dynamical instability underlying most cases of real large-scale atmospheric cyclogenesis and oceanic eddy generation. [\* Appendix B shows its close relation to Rossby propagation (interior *or* boundary).\*]



(ix) We return finally to the central concept of part II of this course — the generic structure summarized on p. 138. In **model (iv) with a lower boundary**, the dynamical system is more explicitly representable in the form (neglecting overt dissipation)

(1) Evolution equations

$$\frac{\mathcal{D}_g Q}{\mathcal{D}t} = 0 \quad \text{on each interior level} \quad (17.38 \text{ a})$$

$$\frac{\mathcal{D}_g B}{\mathcal{D}t} = 0 \quad \text{on the boundary;} \quad (17.38 \text{ b})$$

(2) Inversion operator from (17.20)

$$\psi = \mathcal{L}^{-1}\{Q - Q_0, B - B_0\} \quad \text{with (17.9,11)} \quad (17.38 \text{ c})$$

where now

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z} \frac{f_0^2}{N^2} \frac{\partial}{\partial z}.$$

$Q_0$  and  $B_0$  are the  $Q$  and  $B$  fields for fluid at relative rest. As F. P. Bretherton has pointed out (paper in *Quart. J. Roy. Met. Soc.*, 1966) we can actually regard (17.38 a,b) as *one* equation (precisely justifying remark near bottom of p. 140) by introducing Dirac delta functions  $\hat{\delta}(\cdot)$  and replacing

$$Q \quad \text{by} \quad Q + B \hat{\delta}(z - z_0), \quad (17.39 \text{ a})$$

$$Q_0 \quad \text{by} \quad Q_0 + B_0 \hat{\delta}(z - z_0) \quad (17.39 \text{ b})$$

and replacing the middle line of the inversion problem (17.20) by  $\psi_z = 0$ .

The fact that sloping isopycnals are important warns us to be especially careful if top or bottom boundaries *are* present. This is well illustrated by a famous example of baroclinic instability, that discovered by E.T. Eady in the mid-1940's, one of the first two examples ever to be discovered. The other was discovered by J.G. Charney around the same time.



# 18.

## Eady's solution

Continue to assume  $\beta = 0$ ,  $N = \text{const.}$ , and now take  $U = \bar{u} = \Lambda z$ ,  $\Lambda = \text{const.}$  Suppose that  $w' = 0$  on  $z = 0, H$  (**horizontal boundaries**). With the foregoing example (viii) in mind we might expect instability; the boundaries will act a bit like the horizontal stratification surfaces above and below the shear layer in that example.<sup>1</sup> [\*We shall see that Fj\o rtoft's theorem is not contradicted, because the boundary terms will *not* now vanish.\*]

Linearizing (17.38b), we see that the all-important boundary condition is

$$\left( \frac{\partial}{\partial t} + \Lambda z \frac{\partial}{\partial x} \right) \psi'_z - \psi'_x \Lambda = 0 \text{ on } z = 0, H. \quad (18.1)$$

With  $\beta = 0$  and  $N^2 = \text{const.}$ , (17.27) (zero initial disturbance,  $Q_0$  uniform)

$$\Rightarrow \psi'_{zz} + \frac{N^2}{f^2} (\psi'_{xx} + \psi'_{yy}) = 0 \quad (0 < z < H) . \quad (18.2)$$

Seek solution with sinusoidal horizontal structure ( $k, l$  real)

$$\hat{\psi}(z) \sin l y e^{i k(x - ct)} ;$$

$c$  and  $\hat{\psi}(z)$  may be complex-valued, indicating a phase shift with height. To satisfy (18.2), and (18.1) at  $z = 0$ , we may take

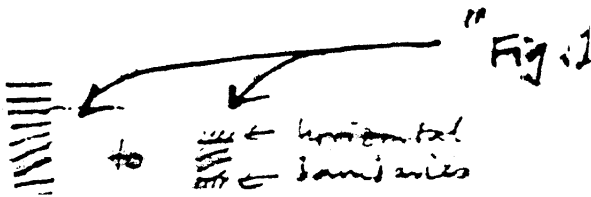
$$\hat{\psi}(z) = \frac{K c}{\Lambda} \cosh K z - \sinh K z \quad (18.3)$$

where  $K^2 = \frac{N^2}{f^2} (k^2 + l^2)$ . (Note that  $K^{-1}$  is a Rossby height for this disturbance.) If (18.1) is to be satisfied at  $z = H$  also, we must have (substituting (18.3) into (18.1))

$$c = \frac{1}{2} \Lambda H \left\{ 1 \pm \gamma^{-1} [(\gamma - \coth \gamma)(\gamma - \tanh \gamma)]^{\frac{1}{2}} \right\} \quad (18.4)$$

(several lines of algebra missed out here), where  $\gamma = \frac{1}{2} K H$ .

<sup>1</sup>In the  $yz$  plane (stratification surfaces for  $\tanh$  profile on left, for  $\Lambda z$  profile on right):



So we have

$$\boxed{\text{instability (Im } c > 0) \text{ for } \gamma < \coth \gamma, \text{ i.e. } \gamma < 1.1997.} \tag{18.5}$$

When  $\gamma \gg 1$  (short waves),

$$c \sim \Lambda H \frac{1}{2\gamma} \quad \text{or} \quad \Lambda H \left(1 - \frac{1}{2\gamma}\right). \tag{18.6 a}$$

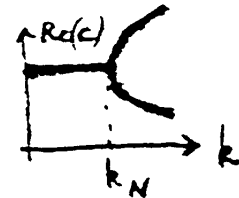
E.g.  $c \sim \frac{\Lambda H}{2\gamma} = \frac{\Lambda}{K}$ ;  $\hat{\psi} \sim e^{-Kz}$  (wave trapped near bottom boundary; doesn't feel top). The other solution is similar but trapped near the top boundary:  $\hat{\psi} \sim e^{-K(H-z)}$ ; be careful to substitute *exact*  $c$  into (18.3) before introducing large- $\gamma$  approximations. As noted in the previous section, these short-wave solutions are essentially the same as *topographic Rossby waves* between boundary and sloping stratification surfaces  $\sigma_1 = \text{constant}$ .

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If  $\min_k \gamma \left( = \frac{NH}{2f} l \right) < 1.1997$  then there is a range  $0 < k < k_N$  say, where the growth rate  $k c_i = k \text{Im } c > 0$ ; its graph looks like the lower of the two heavy curves (ignore the other curves for the moment) in Fig. 18.1 on the next page. The graph of  $\text{Re}(c)$  looks qualitatively like the upper heavy curve (with its large- $k$  asymptotes given by (18.6)). Actually the quantitative results in Fig. 18.1 are for the case  $l = 0$  ( $y$ -independent disturbances), and the abscissa is

$$\frac{NH}{f} k = KH = 2\gamma$$



in our notation. In this case the growth-rate maximum occurs when

$$\gamma = 0.8031$$

in which case

$$-(\gamma - \coth \gamma)(\gamma - \tanh \gamma) = 0.3098,$$

∴

$$\text{maximum growth rate} = \max_k (k c_i) = \frac{f}{NH} \cdot 2\gamma c_i = 0.3098 \frac{f \Lambda}{N}. \tag{18.7}$$

(The  $c$  in fig. 18.1 is dimensionless: (18.4)  $\div \Lambda H$ .) For an idea of numerical magnitudes, take  $\Lambda = 3 \text{ m.s}^{-1} \text{ km}^{-1} = 3 \times 10^{-3} \text{ s}^{-1}$ ,  $f = 10^{-4} \text{ s}^{-1}$ ,  $N = 10^{-2} \text{ s}^{-1}$ , giving growth rates of order  $10^{-5} \text{ s}^{-1}$ , so  $e$ -folding time  $\simeq (10^5 \div 86400) \text{ day} \gtrsim 1 \text{ day}$  (atmosphere, wintertime mid-latitudes). If  $H = 10 \text{ km}$  (ignoring non-Boussinesq effects — not strictly correct since 10 km is

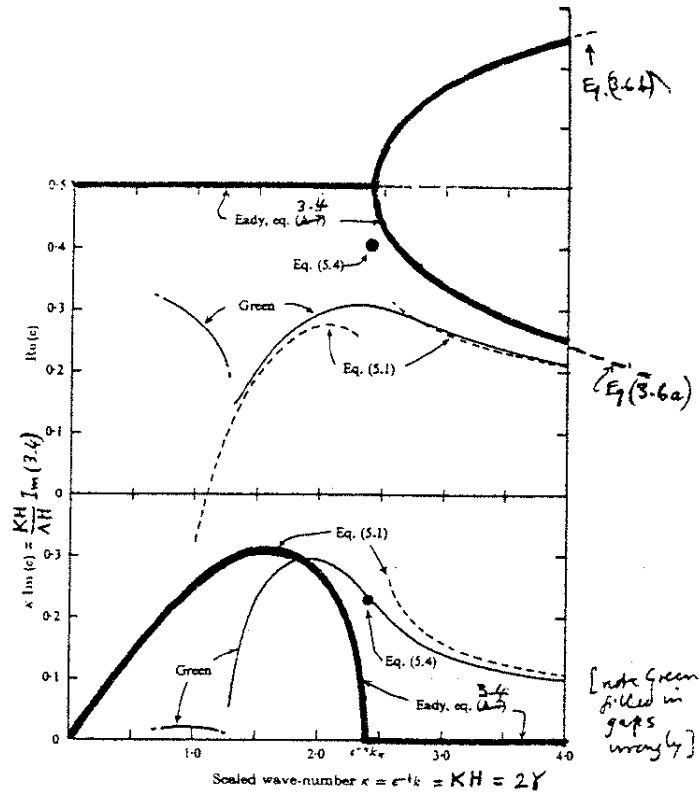
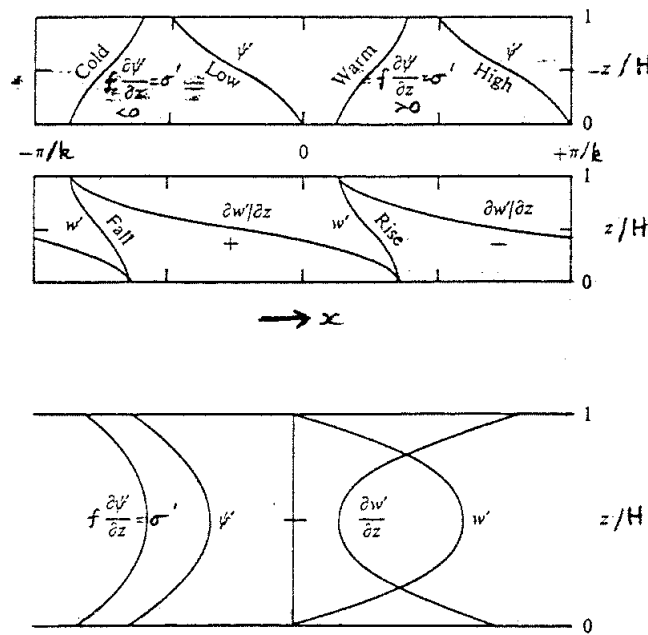


Figure 18.1: Comparison of first correction results from (10.2) and (10.5) with some numerical results of Green (1960), for  $u = z$ ,  $\epsilon^{-1}\beta = 1$ ,  $m = 0$  (see text). Upper graphs:  $\text{Re}(c)$ : lower: scaled growth rate  $\kappa \text{Im}(c) = \epsilon^{-\frac{1}{2}} k \text{Im}(c)$ . Note that the first correction to the growth rate is zero for  $k = k_y$ , but not for  $k = k_y(\bullet)$  or  $k > k_y$ . For accuracy of comparison, the graphs of Green's results have been re-drawn, using his original data; in the case  $\epsilon^{-1}\beta = \frac{1}{2}$  (not shown) the agreement at short wavelengths is even closer, upon correcting an inaccuracy in Green's corresponding published figure *op. cit.*, p. 242; Green, private communication.

comparable to density scale heights in the atmosphere (7–8 km), but can be shown to be qualitatively not very important), then  $k = \frac{f}{NH} \cdot 2\gamma \simeq \frac{10^{-4} \text{ s}^{-1}}{10^{-2} \text{ s}^{-1} 10 \text{ km}} \times 1.6 \simeq (1000 \text{ km})^{-1}$ , or wavelength  $\simeq 6000 \text{ km}$ . For the ocean (away from the Gulf Stream), typical values give  $e$ -folding times of order a month, and  $k^{-1}$  tens of kilometres.

Here are Eady's pictures of the disturbance structure for the fastest-growing,  $l = 0$  case (18.7):



—II.50—

As one might guess from the role of buoyancy in promoting the vortex-stretching that generates the horizontal disturbance motion, the total potential energy of the system is reduced during disturbance growth. This is shown most directly by a calculation by Uryu<sup>2</sup>. Note that this calculation (or any other way of calculating total PE change) must be done correct to  $O$  (amplitude squared). of the *centre-of-mass motions* of material tubes of fluid which are initially of uniform small cross-section and which initially lie parallel to the  $x$ -axis (at  $t = -\infty$ , no disturbance). Such material tubes become wavy as the disturbance grows (and their cross-sections become nonuniform — the tubes become thicker for some values of  $x$  and thinner for others), and their centres of mass move with velocity  $v^{-L}$ , say, in the  $y$ -direction, and  $w^{-L}$  in the  $z$  direction. [ $L$  stands for '(generalized) Lagrangian mean'.] These

<sup>2</sup> Uryu (1979) *J. Met. Soc. Japan* **57**, 1

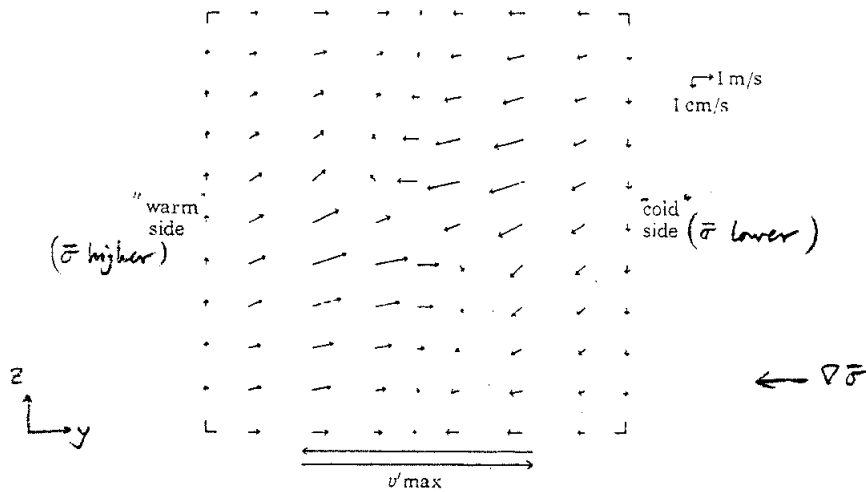


Figure 18.2: Generalized Lagrangian-mean meridional circulation  $v^{-L}, w^{-L}$  associated with a growing Eady wave, from Uryu (1979). The fluid motion is incompressible, but the Lagrangian-mean motion is divergent because of disturbance growth (see text). Parameter values are as follows. Disturbance amplitude, as measured by maximum north–south Eulerian disturbance velocity  $v'$ ,  $11 \text{ m s}^{-1}$  (shown on the same scale as  $v^{-L}$  by the long arrows at bottom): growth rate  $0.7 \text{ day}^{-1}$ ; wavelength  $5000 \text{ km}$ ; width of channel (meridional *half*-wavelength)  $5000 \text{ km}$ ; height of channel  $10 \text{ km}$ ; vertical shear of basic flow  $3 \text{ m s}^{-1} \text{ km}^{-1}$ ; buoyancy (Brunt–Väisälä) frequency  $10^{-2} \text{ s}^{-1}$ ; Coriolis parameter  $10^{-4} \text{ s}^{-1}$ .

centre-of-mass velocities look like this, in the case  $l = \pi/5000 \text{ km}$ : Note that the centres of mass are all *rising* on the ‘warm’ side and *sinking* on the ‘cold’ side — thus *the potential energy of the system is being diminished*. Baroclinic instabilities like these are believed to be one of the main mechanisms whereby potential energy associated with north–south temperature gradients in the atmosphere is converted into the kinetic energy of large-scale weather systems (‘depressions’) in middle latitudes. [‘Depressions’ because at finite amplitude the *minima* of  $\psi'$  (pressure) at the surface  $z = 0$  become more spatially concentrated and hence more prominent on weather maps, than the maxima.]

I should perhaps explain that the other curves on fig. 18.1 are from *perturbation theory* for

$\beta \neq 0$  (but small  $\rightarrow k$  large) (dashed curves)<sup>3</sup> and a numerical calculation for  $\beta \frac{N^2 H}{f^2 \Lambda} = 1$  by Green. Technique: essentially same trick as on line above, equation (4.23) (but must work in complex plane).

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<sup>3</sup>*J. Fluid Mech.* **40**, 273 (1970)

# Part IV

## Appendices





# A.

## Basic equations, Coriolis “forces”, and thermodynamics

(This material, from the Cambridge Summer School in Geophysical and Environmental Fluid Dynamics, is mostly a summary of standard items from fluids textbooks.)

### A.1 Some basic equations and boundary conditions

The Eulerian description is used; so the material derivative  $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$  — the operator for the “rate of change following the fluid”. This gives the rate of change of any field  $f(\mathbf{x}, t)$  not at a fixed point  $\mathbf{x}$ , but at a moving fluid element or “particle”. Its form comes from the chain rule of differential calculus when  $\mathbf{x}$  is made a function of time  $t$ .

**Mass conservation, or “continuity”:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \text{i.e.} \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

( $\rho = \text{density}$ ,  $\rho \mathbf{u} = \text{flux}$ , of mass). For incompressible flow,  $D\rho/Dt = 0$ , implying that

$$\nabla \cdot \mathbf{u} = 0.$$

**Associated boundary condition:** Normal components of velocity must agree, if mass is conserved. That is, we must have

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{u}_b \cdot \mathbf{n} \quad \text{at the boundary (where } \mathbf{u}_b \text{ is prescribed).}$$

Here  $\mathbf{n}$  is a unit vector normal to the boundary. The right-hand side can represent e.g. the normal component of the velocity  $\mathbf{u}_b$  of the boundary material itself, if we have a moving but impermeable boundary, *or*, e.g.,  $\rho^{-1} \times$  mass flux across a fixed but permeable boundary. At a boundary that is both stationary and impermeable,  $\mathbf{u}_b = 0$  and so  $\mathbf{u} \cdot \mathbf{n} = 0$ .

**Newton's second law:** In an inertial frame,

$$\text{Acceleration} = \text{force per unit mass} .$$

In this equation, the right-hand side (RHS) and left-hand side (LHS) can be written in various ways. Defining  $\zeta = \nabla \times \mathbf{u}$ , the vorticity, we can show by vector calculus that

$$\text{LHS} = \frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\partial\mathbf{u}}{\partial t} + \zeta \times \mathbf{u} + \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) \quad (1)$$

and

$$\text{RHS} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \mathbf{F} ,$$

where  $p$  is total pressure, including hydrostatic, and  $\mathbf{g}$  is the gravitational force per unit mass, exactly the gradient of a scalar function. We take  $\mathbf{g} = -\nabla\Phi$ , where  $\Phi$  is called the gravitational potential. The last term  $\mathbf{F}$  stands for all other forces, such as friction, or, especially in large-scale atmospheric models, forces due to unresolved gravity waves.

Wave-induced forces can in a certain sense be *anti*-frictional. The most conspicuous example is the quasi-biennial oscillation of the zonal wind in the equatorial lower stratosphere (QBO). The wave-induced forces drive the stratosphere away from, not toward, solid rotation, as shown in PHH's lectures and in the QBO computer demonstration. Ordinary (molecular-viscous) fluid friction would by itself drive the atmosphere and ocean *toward* solid rotation in the absence of externally applied stresses.

**Viscous force:** For ordinary (molecular-viscous) fluid friction we have

$$\mathbf{F} = \frac{\mu}{\rho} \nabla^2 \mathbf{u} = \nu \nabla^2 \mathbf{u} , \quad \text{say,}$$

in the simplest case of spatially uniform dynamical viscosity  $\mu$ . See e.g. Batchelor's textbook for more general cases. (One might guess  $\mathbf{F} = \rho^{-1} \nabla \cdot (\mu \nabla \mathbf{u})$ , but that's wrong! One must replace the tensor  $\nabla \mathbf{u}$  by its symmetric part. Viscous forces are achiral: they respect mirror-symmetry.)

**Associated boundary conditions:** For viscous fluid motion we need an extra boundary condition on  $\mathbf{u}$ . The commonest cases are of two kinds. **First**, if the boundary is solid, impermeable, and again moving with velocity  $\mathbf{u}_b$ , then

$$\mathbf{u} = \mathbf{u}_b \quad \text{at the boundary;}$$

i.e., the fluid at the boundary must move entirely with the boundary. Tangential as well as normal components must agree. There are a few special cases where this fails, e.g. two-fluid “contact lines” where continuum mechanics itself breaks down.) The agreement of tangential components is called the *no-slip condition* and may still apply when the boundary is solid but permeable. **Second**, in a thought-experiment in which the tangential stress  $\boldsymbol{\tau}$  on the fluid (friction force per unit area) is prescribed at a plane boundary — e.g. at the top of a model ocean strongly constrained by gravity — then  $\boldsymbol{\tau}$  controls the shear at the boundary:

$$\mu \frac{\partial(\mathbf{u} \cdot \mathbf{s})}{\partial n} = \boldsymbol{\tau} \cdot \mathbf{s} \quad \text{at the boundary,}$$

where  $n$  is distance in the  $\mathbf{n}$  direction, outward from the fluid, and  $\mathbf{s}$  is any fixed unit vector normal to  $\mathbf{n}$ , i.e. lying in the boundary. (In more general cases with curved boundaries, we would need to use a more complicated expression for the total stress due to viscosity and pressure. Batchelor’s textbook gives a clear discussion: some knowledge of tensors is required.)

**Boussinesq approximation:** This refers to a set of approximations for flows that feel buoyancy forces, valid in the asymptotic limit  $\Delta\rho/\rho \ll 1$  where  $\Delta\rho$  typifies the range of density variations, with  $g\Delta\rho/\rho$  finite ( $g = |\mathbf{g}| = |\nabla\Phi|$ ). For consistency we need to assume that the motion has height scales  $\ll c_s^2/g$  and that  $|\mathbf{u}| \ll c_s$ , where  $c_s$  is the speed of sound. Mass continuity then reduces to the incompressible case  $\nabla \cdot \mathbf{u} = 0$ , and Newton’s second law simplifies to

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho} \nabla p' + \boldsymbol{\sigma} + \mathbf{F} \quad (2)$$

where the total pressure  $p$  has been replaced by the pressure *anomaly*  $p'$ , in the sense of departure from a background pressure. The background pressure is defined to be in hydrostatic balance with a given background density, both background quantities being functions of  $\Phi$  only. The effects of density anomalies, departures  $\rho'$  from the background density, are represented solely by the upward “buoyancy acceleration”  $\boldsymbol{\sigma} = -\mathbf{g}\rho'/\rho$ . In the pressure-gradient term,  $\rho$  can be taken to be constant. In summary,  $\rho = \text{constant}$  when it measures mass density and inertia but not when it measures buoyancy.

**Equation for buoyancy, and associated boundary conditions:** Generally these involve diffusion of heat or of solutes, with gradients of one thing

influencing fluxes of another. There are great simplifications in the Boussinesq case with constant  $\mathbf{g}$ ; then

$$\mathcal{D}\sigma/\mathcal{D}t = \nabla \cdot (\kappa \nabla \sigma) \quad (\sigma = |\boldsymbol{\sigma}|)$$

may suffice. Here  $\sigma$  represents density anomalies relative to a strictly constant background density. That is,  $\sigma$  describes all of the stratification, and the background none: if the fluid is at rest then the buoyancy frequency (Brunt–Hesselberg–Milch–Schwarzschild–Väisälä frequency) is just  $N^2(z) = \partial\sigma/\partial z$  ( $dz = d\Phi/g$ ). If the buoyancy diffusivity  $\kappa$  is spatially uniform then, even more simply,  $D\sigma/Dt = \kappa \nabla^2 \sigma$ . We usually specify a boundary condition on the buoyancy flux  $\kappa \partial\sigma/\partial n$ , or on  $\sigma$  itself, or on some linear combination of  $\sigma$  and  $\partial\sigma/\partial n$ . It is still consistent to take  $\nabla \cdot \mathbf{u} = 0$  approximately.

**Rotating reference frames:** To the RHS of Newton’s second law we must add

$$- 2\boldsymbol{\Omega} \times \mathbf{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

(see A.2, p. 193 below, for a derivation). Here  $\boldsymbol{\Omega}$  is the angular velocity of the reference frame, assumed constant,  $\mathbf{r}$  is position relative to any point on the rotation axis, and  $\mathbf{u}$  is now velocity relative to the rotating frame. The first and second terms are respectively the Coriolis and centrifugal forces per unit mass — “fictitious forces” felt e.g. by an observer sitting in a rotating room or in a spinning aircraft. It is convenient, and conventional, to recognize that not only  $\mathbf{g}$  but also  $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$  is the gradient of a potential and that there exists, therefore, an “effective gravitational potential”

$$\Phi_{\text{eff}} = \Phi - \frac{1}{2} |\boldsymbol{\Omega}|^2 r_{\perp}^2$$

such that  $\mathbf{g} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\nabla \Phi_{\text{eff}}$ , where  $\Phi$  is again the gravitational potential in the ordinary sense, and  $r_{\perp}$  is the shortest, i.e. perpendicular, distance to the rotation axis.

On the rotating Earth, the level surfaces  $\Phi_{\text{eff}} = \text{const.}$  are only slightly different from the surfaces  $\Phi = \text{const.}$  The  $\Phi_{\text{eff}}$  and  $\Phi$  surfaces tangent to each other at the north pole are only about 11 km apart at the equator. *Brain-teaser:* why does this differ from the equatorial bulge of the actual figure of the Earth, about 21 km? (This is a good exercise in an important transferable skill, that of spotting unconscious assumptions.)

So in the rotating frame Newton’s second law can be written

$$\frac{\mathcal{D}\mathbf{u}}{\mathcal{D}t} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi_{\text{eff}} + \mathbf{F}. \quad (3)$$

It is traditional to display the Coriolis force per unit mass on the LHS, even though in the rotating frame it has the role of force rather than acceleration. Whether terms are written on the RHS or LHS, with appropriate sign changes, is of course entirely a matter of convention. Mathematical equations may look like, but are of course different from, computer code! (There are some myths in the research literature that might come from forgetting this. One example is the misleading, though persistent, idea that the stratospheric Brewer–Dobson circulation is driven by solar heating. The heating term is usually written on the RHS, unhelpfully suggesting that it be thought of as a known forcing, even though, in reality, it is more like Newtonian relaxation toward a radiative-equilibrium temperature.)

The Boussinesq approximation can again be introduced, with  $-\nabla\Phi_{\text{eff}}$  replacing  $\mathbf{g}$ .

**Vorticity equations in a rotating frame:** Same as in a inertial frame except for just one thing: replace the relative vorticity  $\zeta = \nabla \times \mathbf{u}$  by the absolute vorticity  $\zeta^{\text{a}} = 2\boldsymbol{\Omega} + \zeta$ .

Incompressible but not Boussinesq:

$$D\zeta/Dt = \zeta^{\text{a}} \cdot \nabla \mathbf{u} - \frac{1}{\rho^2} \nabla p \times \nabla \rho + \nabla \times \mathbf{F}$$

Boussinesq:

$$D\zeta/Dt = \zeta^{\text{a}} \cdot \nabla \mathbf{u} + \nabla \times \boldsymbol{\sigma} + \nabla \times \mathbf{F}$$

(with  $\boldsymbol{\Omega}$  constant, so that  $D\zeta^{\text{a}}/Dt = D\zeta/Dt$ ). If  $g_{\text{eff}} = |\nabla\Phi_{\text{eff}}|$  can be taken as constant on each level surface  $\Phi_{\text{eff}} = \text{const.}$ , then  $\nabla \times \boldsymbol{\sigma}$  simplifies to  $-\hat{\mathbf{z}} \times \nabla\sigma$  where  $\hat{\mathbf{z}}$  is a vertical unit vector, i.e. parallel to  $\nabla\Phi_{\text{eff}}$ . These equations can be derived by taking the curl of Newton's second law, in the per-unit-mass form (3) or its Boussinesq counterpart.

In the Boussinesq case,  $\rho = \text{constant}$  and so  $\mu/\rho = \nu$  is spatially uniform whenever  $\mu$  is. Then when  $\mathbf{F}$  is viscous,

$$\nabla \times \mathbf{F} = \nu \nabla^2 \zeta .$$

So vorticity behaves diffusively in this case — though not in all other cases, as the example of a viscous jet in inviscid surroundings reminds us. (There, all the vorticity ends up on the interface: diffusive behaviour is countermanded by terms involving  $\nabla\mu \neq 0$ .)

**Ertel’s potential-vorticity theorem:** This is a cornerstone of today’s understanding of atmosphere–ocean dynamics and the dynamics of stratified, rotating flow in other naturally occurring bodies of fluid e.g. the Sun’s interior (<http://www.atm.damtp.cam.ac.uk/people/mem/>). The theorem says that if (a) there exists a thermodynamical variable  $\theta$  that is a function of pressure  $p$  and density  $\rho$  alone *and* is materially conserved,  $D\theta/Dt = 0$ , and (b)  $\mathbf{F} = 0$  or is the gradient of a scalar, then

$$DQ/Dt = 0 \quad \text{where} \quad Q = \rho^{-1} \boldsymbol{\zeta}^a \cdot \nabla \theta, \quad \text{the “Rossby–Ertel potential vorticity”}.$$

For adiabatic motion of a simple fluid (thermodynamic state definable by  $p$  and  $\rho$  alone),  $\theta$  can be taken as specific entropy  $S$  (A.3, p. 194 below) or any function of  $S$ . Adiabatic motion means that fluid elements neither receive nor give up heat, implying that  $S$  is materially conserved. In the atmospheric sciences it is conventional to take  $\theta$  to be the potential temperature, which is a function of  $S$ : for a perfect gas,  $\theta \propto \exp(S/c_p)$ .

**In the Boussinesq approximation:** As above but with  $\rho = \text{constant}$  (implying that the factor  $\rho^{-1}$  can be omitted from the definition of  $Q$ ), and with  $\theta$  replaced by  $\rho'$ , or by  $\sigma = -g_{\text{eff}}\rho'/\rho$  if  $g_{\text{eff}} = |\nabla\Phi_{\text{eff}}|$  can be taken as constant on each level surface  $\Phi_{\text{eff}} = \text{const}$ .

**Bernoulli’s (streamline) theorem** for steady, frictionless, adiabatic motion: First, steady and adiabatic  $\Rightarrow \partial/\partial t = 0$  and  $D\theta/Dt = 0$ . Therefore  $\mathbf{u} \cdot \nabla \theta = 0$ , i.e.,  $\theta = \text{constant}$  along any streamline. Second, the right-hand-most expression in eq. (1) on page 1 allows us to replace  $D\mathbf{u}/Dt + 2\boldsymbol{\Omega} \times \mathbf{u}$  by  $\boldsymbol{\zeta}^a \times \mathbf{u} + \nabla(\frac{1}{2}|\mathbf{u}|^2)$ . With  $\mathbf{F} = 0$  (or  $\mathbf{F}$  workless,  $\mathbf{u} \cdot \mathbf{F} = 0$ ) we can take  $\mathbf{u} \cdot (3)$ , noting that  $\mathbf{u} \cdot \boldsymbol{\zeta}^a \times \mathbf{u} = 0$ , to give simply

$$\mathbf{u} \cdot \nabla \left( \frac{1}{2} |\mathbf{u}|^2 + \Phi_{\text{eff}} \right) = - \frac{1}{\rho} \mathbf{u} \cdot \nabla p, \quad \Rightarrow \quad \mathbf{u} \cdot \nabla \left( \frac{1}{2} |\mathbf{u}|^2 + \Phi_{\text{eff}} + H \right) = 0.$$

The last step introduces the specific enthalpy  $H$  (A.3, p. 194 below) and uses the constancy of  $\theta$ , hence  $S$ , on the streamline. In the notation of A.3,  $dH = V dp = \rho^{-1} dp$  on the streamline. So the Bernoulli quantity  $\frac{1}{2} |\mathbf{u}|^2 + \Phi_{\text{eff}} + H$  is constant on a streamline, even for stratified, rotating flow, in the circumstances assumed.

**Boussinesq Bernoulli quantity:** can be taken as  $\frac{1}{2} |\mathbf{u}|^2 + \rho' \Phi_{\text{eff}} / \rho + p' / \rho$  with  $\rho = \text{constant}$ . If  $\nabla \Phi_{\text{eff}} = \text{constant}$  then  $\rho' \Phi_{\text{eff}} / \rho$  may be replaced by  $-\sigma z$  where  $z$  is vertical distance,  $z = \Phi_{\text{eff}} / |\nabla \Phi_{\text{eff}}|$ . So then  $\frac{1}{2} |\mathbf{u}|^2 - \sigma z + p' / \rho$  is constant on a streamline.

## A.2 Coriolis and centrifugal “forces”

This part of the problem is just the same for continuum mechanics as for particle dynamics. The best tactic is to view everything from an *inertial* (sidereal) frame of reference.

Let  $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ , be an orthogonal triad of unit vectors that rotate rigidly with constant angular velocity  $\boldsymbol{\Omega}$ . (The triad can be right handed too, but that’s not essential.) Thus  $\hat{\mathbf{x}}$  is time dependent, with time derivative

$$\dot{\hat{\mathbf{x}}} = \boldsymbol{\Omega} \times \hat{\mathbf{x}}; \quad \text{similarly} \quad \dot{\hat{\mathbf{y}}} = \boldsymbol{\Omega} \times \hat{\mathbf{y}}, \quad \dot{\hat{\mathbf{z}}} = \boldsymbol{\Omega} \times \hat{\mathbf{z}}. \quad (\text{S1})$$

Consider the position  $\mathbf{X}(t)$  of a single particle (viewed, as always, in the inertial frame). Let  $\mathbf{X} \cdot \hat{\mathbf{x}} = X(t)$ ,  $\mathbf{X} \cdot \hat{\mathbf{y}} = Y(t)$ ,  $\mathbf{X} \cdot \hat{\mathbf{z}} = Z(t)$ ; thus (by orthogonality)

$$\mathbf{X}(t) = X(t) \hat{\mathbf{x}}(t) + Y(t) \hat{\mathbf{y}}(t) + Z(t) \hat{\mathbf{z}}(t). \quad (\text{S2})$$

Take the first time derivative, using (S1) and the properties of vector multiplication:

$$\dot{\mathbf{X}}(t) = \dot{X}(t) \hat{\mathbf{x}}(t) + \dot{Y}(t) \hat{\mathbf{y}}(t) + \dot{Z}(t) \hat{\mathbf{z}}(t) + \boldsymbol{\Omega} \times \mathbf{X}(t) \quad (\text{S3})$$

$$= \dot{\mathbf{X}}_{\text{rel}}(t) + \boldsymbol{\Omega} \times \mathbf{X}(t), \quad \text{say.} \quad (\text{S4})$$

(Notice now that  $\dot{\mathbf{X}}_{\text{rel}} = \dot{X} \hat{\mathbf{x}} + \dot{Y} \hat{\mathbf{y}} + \dot{Z} \hat{\mathbf{z}}$  is, by definition, the rate of change that  $\mathbf{X}$  would appear to have if it were viewed from a reference frame rotating with angular velocity  $\boldsymbol{\Omega}$ . That is,  $\dot{\mathbf{X}}_{\text{rel}}$  is the particle’s velocity relative to that rotating frame.) Remembering that  $\boldsymbol{\Omega} = \text{constant}$ , we can differentiate (S3) to get the second time derivative of  $\mathbf{X}(t)$ , the (absolute) acceleration:

$$\ddot{\mathbf{X}}(t) = \ddot{X} \hat{\mathbf{x}} + \ddot{Y} \hat{\mathbf{y}} + \ddot{Z} \hat{\mathbf{z}} + \boldsymbol{\Omega} \times \dot{\mathbf{X}}_{\text{rel}} + \boldsymbol{\Omega} \times \dot{\mathbf{X}},$$

using (S1) again. The sum of the first three terms,  $= \ddot{\mathbf{X}}_{\text{rel}}$ , say, give, by definition, the particle’s acceleration relative to the rotating frame. Using (S4) in the last term, we have

$$\ddot{\mathbf{X}}(t) = \ddot{\mathbf{X}}_{\text{rel}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{X}}_{\text{rel}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{X}), \quad (\text{S5})$$

the standard result showing that  $\ddot{\mathbf{X}}$  can be equated to  $\ddot{\mathbf{X}}_{\text{rel}}$  plus, respectively, a Coriolis and a centripetal acceleration. Centripetal = inward (from Latin *centrum*, centre, + *petere*, to seek): note that  $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{X}) = -|\boldsymbol{\Omega}|^2 \mathbf{X} + \boldsymbol{\Omega} \cdot \mathbf{X} \boldsymbol{\Omega} = -|\boldsymbol{\Omega}|^2 \boldsymbol{\varpi}$  where the vector  $\boldsymbol{\varpi}$  measures normal distance from the rotation axis.

If we were now to go into the rotating frame, we could regard  $\ddot{\mathbf{X}}_{\text{rel}}$  as the acceleration entering Newton’s laws provided that we also regard the forces acting as including a Coriolis force  $-2\boldsymbol{\Omega} \times \dot{\mathbf{X}}_{\text{rel}}$  and a *centrifugal* (outward) force  $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{X})$  per unit mass.

### A.3 Thermodynamic relations

There is a mnemonic trick, not widely known, for remembering the thermodynamic relations for a simple compressible fluid. They can all be read off from the single array

$$V \quad A \quad T \quad (A.1)$$

$$E \quad \times \quad G \quad (A.2)$$

$$S \quad H \quad p \quad (A.3)$$

which itself can be remembered aurally: say “*vatg, veshp*” to yourself, with a New York accent if it helps. The notation is conventional. On the corners, read  $T$  as temperature,  $p$  as total pressure (including hydrostatic background),  $S$  as specific entropy ( $= c_p \ln \theta + \text{const.}$  in the case of a perfect gas with  $c_p$  the specific heat at constant  $p$ ),  $V$  as  $1/\rho$ , the specific volume (volume of a unit mass of fluid), and, in between, read  $A$  as Helmholtz free energy,  $G$  as Gibbs free energy,  $H$  as enthalpy, and  $E$  as internal energy, all ‘specific’ in the per-unit-mass sense. The diagram contains all the standard thermodynamic relations and Legendre transformations, of which the most important for fluid dynamics are

$$dE = -p dV + T dS, \quad dH = V dp + T dS \quad (H = E + Vp) .$$

The first of these, scanned left to right, corresponds to the pattern  $\begin{matrix} \cdot & \cdot \\ \vdots & \cdot \\ \cdot & \cdot \end{matrix}$  with  $E$  at mid-left representing  $dE$ , etc., and with the diagonal arrows giving the sign: thus  $p dV$  has a minus (against the arrow) and  $T dS$  has a plus (with the arrow). The four Maxwell relations  $(\partial T/\partial V)_S = -(\partial p/\partial S)_V$ ,  $(\partial V/\partial S)_p = (\partial T/\partial p)_S$ , etc., can be read off in a similar way.

The speed of sound is  $c_s = \sqrt{(\partial p/\partial \rho)_S} = V \sqrt{-(\partial p/\partial V)_S}$ .



# B.

## Rossby-wave propagation and shear instability

This material is from the Cambridge Summer School in Geophysical and Environmental Fluid Dynamics; updated versions with recent literature references are at [www.atm.damtp.cam.ac.uk/people/mem/gefd-supplem-material.html](http://www.atm.damtp.cam.ac.uk/people/mem/gefd-supplem-material.html)

### 1. Introduction

This Appendix explores Rayleigh's inviscid shear instability problem and its relation to the Rossby-wave propagation mechanism, or 'Rossby quasi-elasticity'. The instability problem is the simplest of those solved by Rayleigh in his pioneering work last century on the undular instability of jets and shear layers. It provides us with a robust paradigm for a very basic fluid-dynamical process. It is robust in the sense that it gives essentially the same result as the  $U \propto \tanh(y/b)$  case and practically all the other inviscid shear-layer profiles that you can explore for yourself in computer demonstration 10. These include almost *any* moderately smooth shear-layer profile drawn with the mouse, with  $y$ -scale  $b$  somewhat less than the computational domain size  $L$ .

The essential qualitative result, well documented in many places in a vast research literature, is that almost any shear layer sandwiched between constant-velocity regions is unstable to small sideways undular displacements, with a fastest exponential growth rate equal to a modest fraction, often a fifth or so, of the typical shear. The fastest-growing instability has a radian wavelength of the same order as the shear layer thickness  $2b$ , where radian wavelength means full wavelength  $/2\pi$ .

As already remarked in the lectures, the same qualitative result applies also to the 'KH instability' (Kelvin-Helmholtz, sometimes called Taylor-Goldstein, instability) of a stratified shear layer  $U(z), N^2(z)$  at sufficiently small Richardson number  $Ri = (N/U_z)^2$ . This instability is sometimes visible in the sky as groups of long-crested 'billow clouds' having lifetimes of order ten minutes. The instability commonly occurs when larger-scale disturbances tilt a strongly stratified layer and produce sufficient vertical shear  $U_z$  in the layer (via the horizontal gradient of the buoyancy acceleration, in the

vorticity equation, equivalently the term  $\propto \nabla\rho \times \nabla p$ , cf. Dr Linden's lectures) to bring the local value of  $Ri$  well below 0.25. You can use the computer's Movie Viewer (load KH.IMG). to see the evolution to finite amplitude of a typical KH instability, produced in the laboratory by tilting a thin stratified layer in a long tank (S.A. Thorpe 1973, *J. Fluid Mech.*, **61**, 731)<sup>1</sup>

It can be shown that, for laboratory-scale shear layers in which viscosity might be directly significant, the fastest instabilities are not much affected until Reynolds numbers  $Ub/\nu$  are down to very modest values, of the order of 10 or less — another striking indication of the robustness of the shear instability mechanism in these simplest cases.

Rayleigh's problem is the case of small-amplitude disturbances to an exactly inviscid layer with exactly constant shear, the 'vorticity strip', see fig. §4.6 on p. 97. In this particular case the problem can be explicitly solved in detail, with no more than exponential functions and a modicum of patience — apart from just one tricky technicality, the derivation of equation (6) below. Section 4 gives the full analysis, and shows how it illustrates the fundamental relation between shear instability and the Rossby-wave propagation mechanism. An equivalent visual-verbal description is given in section 5, following the review by Hoskins et al (1985, *Q. J. Roy. Meteorol. Soc.*, **111**, 877–946 and **113**, 402–404). To prepare the way for sections 4 and 5, the appropriate case of simple Rossby waves is analyzed first (sections 2 and 3). The qualitative understanding thus gained shows *why* the instability mechanism is robust, particularly as regards its finite-amplitude consequences illustrated in fig. §4.6 on p. 97.

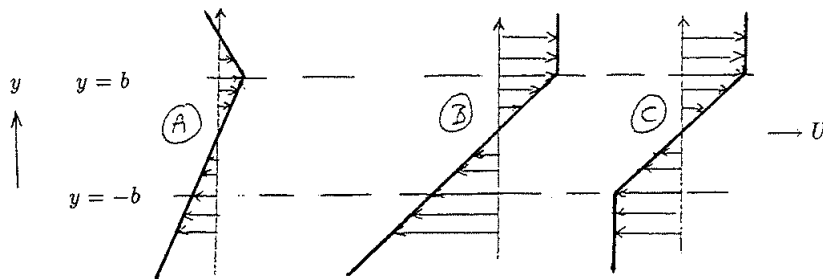
Moreover, that understanding can be extended immediately to the fastest

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<sup>1</sup>You might like to think about how it is that minimum  $Ri = (N/U_z)^2$  tends to occur in the most strongly stratified layers, i.e., where  $N^2$  is *largest*. This is true both in the tilted-tank experiment and in most naturally-occurring situations. The tilting envisaged is one in which some larger-scale disturbance tilts a relatively thin but horizontally extensive layer of relatively strong stratification. Such a layer, by definition, has a strong maximum in  $N$  considered as a function of  $z$ . The key point is that if the layer tilts approximately as a plane, making a small time-dependent angle  $\alpha(t)$  with the horizontal, then the horizontal gradient  $\nabla_H\sigma$  of the buoyancy acceleration  $\sigma$  has approximate magnitude  $\alpha N^2(z)$ , a strong function of  $z$ . This gives rise to horizontal vorticity, appearing mainly as vertical shear  $U_z$  (because of the large ratio of horizontal to vertical scales in this situation), and having the  $z$ -dependence of  $N^2$ , not  $N$ . Specifically,  $U_z \simeq \gamma(t)N^2(z)$  where, if Coriolis forces are negligible, as in the tilted-tank experiment,  $\gamma(t)$  is simply the time integral of  $\alpha(t)$ . Then  $Ri = (N/U_z)^2 = (\gamma N)^{-2}$ . So, when  $\gamma(t)$  increases,  $Ri$  becomes smallest soonest at a maximum, not a minimum, of  $N(z)$ . The same formula  $Ri = (\gamma N)^{-2}$ , hence the same conclusion, can be shown to hold far more generally with suitably modified  $\gamma(t)$ . For instance, in the opposite-extreme case of geostrophic balance the formula  $Ri = (\gamma N)^{-2}$  still holds but with  $\gamma(t) = \alpha(t)/f$  where  $f$  is the Coriolis parameter.

— and likewise robust — three-dimensional ‘baroclinic instabilities’ on horizontal temperature gradients which are usually thought of as accounting for the existence of the mid-latitude cyclones and anticyclones that are conspicuous features of atmospheric weather patterns. Mid-latitude cyclones and anticyclones have horizontal length scales  $L \lesssim 10^3$  km; their oceanic counterparts (scales  $L \lesssim 10^2$  km) place severe requirements on numerical resolution for eddy-resolving ocean circulation models. The extension to three-dimensional baroclinic problems is obtained simply by *replacing vorticity with potential vorticity* and replacing ‘vorticity inversion’ (the inverse Laplacian operator) with ‘potential vorticity inversion’. Rayleigh’s instability itself has direct relevance to some atmospheric weather developments, and ocean-current instabilities, associated with horizontal shear, and in this context is often referred to as a ‘barotropic shear instability’ or a ‘Rayleigh-Kuo instability’. The description of the instability mechanism in section 5 is written so as to apply, suitably interpreted, both to the barotropic and to the baroclinic cases. On first reading, however, it can be viewed simply as a summary of what happens in Rayleigh’s problem, underpinned by the detailed justification available, for those interested, in sections 3 and 4.<sup>2</sup>

Rayleigh’s problem, then, is to find the inviscid, exponentially-growing small disturbances, if any, to the unidirectional basic or background velocity profile  $(u, v) = \{U(y), 0\}$  shown as C in the following diagram:



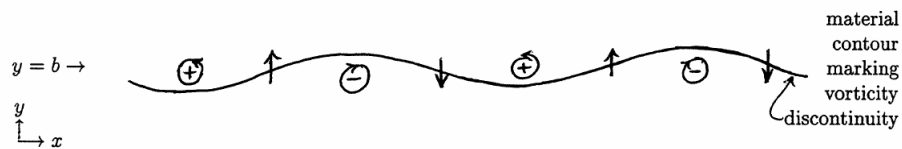
The shear  $U_y = dU/dy$  is piecewise constant in each profile shown. The domain is unbounded. The planetary vorticity gradient  $\beta = df/dy$  is taken to be zero since we are interested, at first, in a paradigm that is equally relevant to large-scale and small-scale flow.

<sup>2</sup>There is a vast and highly technical literature on the linearized theory of shear instabilities that are more complicated, slower-growing, and less robust as regards their finite-amplitude consequences — hence less likely to be practically important, albeit sometimes mathematically intriguing. Some of these more complicated instabilities can be understood in terms of a phenomenon called ‘over-reflection’, as first, I believe, clearly illustrated by A. E. Gill (1965, *Phys. Fluids.*, **8**, 1428–1430), who analyzed an instability arising from the over-reflection of sound waves between two vortex sheets.

## 2. Rossby-wave propagation on a concentrated vorticity gradient

First consider profiles A and B, which are *stable*. They provide another illustration of the Rossby wave propagation mechanism or ‘Rossby-wave quasi-elasticity’, which is basic to most problems in atmosphere–ocean dynamics and, for instance, has an important role in the approximate chemical isolation of ‘Meddies’ (Atlantic Mediterranean Eddies) from their surroundings, and similarly the chemical isolation of the stratospheric polar vortex and ozone hole. (Epigrammatically, “Strong vortices have strongly Rossby-elastic edges.”)

Here the vorticity gradient to which the wave propagation owes its existence is concentrated on a single material contour, namely that material contour whose undisturbed position is  $y = b$ . (The distinction between these Rossby waves and those on a constant, or smoothly varying, basic vorticity or potential-vorticity gradient may be compared to the distinction between surface gravity waves and internal gravity waves. The propagation of surface gravity waves, or ordinary ‘water waves’, can be described as owing its existence to a density or buoyancy gradient concentrated at the water surface, as compared with internal gravity waves on a continuous buoyancy gradient.) The following sketch reminds us of the basic Rossby-wave mechanism:



The encircled signs indicate the sense of the vorticity anomalies  $q'$  caused by displacing the contour; note that the basic-state vorticity is more positive, or less negative, on the positive- $y$  or ‘northern’ side of the contour). The straight arrows indicate the sense of the induced disturbance *velocity* field, i.e. the velocity field resulting from inversion of  $q'$ . The phase of the velocity pattern is important. The velocity pattern is a quarter wavelength out of phase with the material *displacements*, marked by the undular shape of the material contour itself. What follows from this is a matter of simple kinematics. If one makes a movie of the situation in one’s mind’s eye, as viewed from a frame of reference moving with the basic flow  $U$  at  $y = b$ , one can see that the undulations must be propagating relative to the basic flow. (The notion of vorticity inversion allows one to say, epigrammatically, that the undulations are *caused* to propagate *by* the disturbance vorticity anomalies

+ - ... ) The propagation is toward the left in this case; generally it is in whichever direction has the most positive, or least negative, basic-flow vorticity or potential vorticity on the right. This is sometimes called ‘pseudo-westward’ or, more aptly, ‘quasi-westward’.

The all-important quarter-wavelength phase shift is easily understandable from the properties of vorticity inversion, for instance as visualized by the electrostatic and soap-film analogies described in the lectures. For instance a soap film being pushed and pulled in an  $x$ -periodic pattern will show a corresponding pattern of hills and valleys; this tells us that the streamfunction anomalies  $\psi'$  are  $180^\circ$  out of phase with the vorticity anomalies, and hence *in* phase with the material contour displacements. You can also verify this picture from the computer demonstrations.

The next section verifies the correctness of the foregoing picture in an independent way, by using the traditional linearized mathematical theory for small displacements. This prepares the way for a similar mathematical treatment of Rayleigh’s problem. If you are happy to take the theory on trust on first reading, you can skip to section 5 at this point.

### 3. Mathematical verification of the Rossby propagation mechanism

We regard all the basic velocity profiles as limiting cases of smooth profiles with continuous derivatives. This is one way of being sure to get the correct jump conditions across the discontinuities in  $U_y$  — the only tricky point, equation (6) below, in an otherwise straightforward analysis. For smooth profiles  $U(y)$ , with viscosity neglected, the linearized disturbance equation can be written

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) q' - U_{yy} \frac{\partial \psi'}{\partial x} = 0 , \quad (1)$$

where  $-U_{yy} = -d^2U/dy^2$ , the basic or background vorticity gradient giving rise to the Rossby-wave mechanism, and  $q'$  and  $\psi'$  are respectively the disturbance contributions to the vorticity and streamfunction, with the convention

$$(u', v') = (-\partial\psi'/\partial y , \partial\psi'/\partial x) . \quad (2)$$

for the disturbance velocity. The relation between  $\psi'$  and  $q'$  appears in this notation as

$$\nabla^2\psi' = \frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} = q' \quad (3a)$$

$$\left( \psi' \text{ periodic in } x; \quad \psi' \text{ and } \psi'_y \rightarrow 0 \text{ as } |y| \rightarrow \infty \right) , \quad (3b)$$

$\psi'_y$  being shorthand for  $\partial\psi'/\partial y$  as usual. We may summarize the content of (3) more succinctly as

$$\psi' = \nabla^{-2}q' \quad , \quad (4)$$

making explicit the idea of vorticity inversion with boundary conditions of evanescence in  $y$  and periodicity in  $x$  understood here.

In the limit of piecewise constant shear  $dU/dy$ , we have  $U_{yy} = 0$  for  $y \neq \pm b$ . Hence (1) implies that

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) q' = 0 \quad (y \neq \pm b) \quad . \quad (5)$$

By a careful consideration of the limit near  $y = \pm b$  it can also be shown<sup>3</sup> that (1) implies

$$\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left[ \frac{\partial\psi'}{\partial y} \right]_{-}^{+} - \left[ \frac{dU}{dy} \right]_{-}^{+} \frac{\partial\psi'}{\partial x} = 0 \quad \text{at } y = \pm b \quad , \quad (6)$$

where the square brackets denote jumps or differences across  $y = \pm b$ . That is,  $[F(y)]_{-}^{+}$  at  $y = b$  means  $F(y+) - F(y-)$  where  $F(y+) = \lim_{y \downarrow b} F(y)$  and  $F(y-) = \lim_{y \uparrow b} F(y)$ , for any function  $F(y)$  such that the limits exist. The condition (6) says, in a rather inscrutable way, that the initial vorticity distribution moves with the undulating material contour. The inscrutability arises from using the *linearized Eulerian* description — for reasons of mathematical convenience — to describe something that appears simple only in the *exact Lagrangian* description.<sup>4</sup>

<sup>3</sup>One way of deriving (6) from (1) is to use the concept of Dirac delta or ‘point-charge’ functions. When we take the limit in (1),  $U_{yy}$  will tend to a delta function of strength  $[U_y]_{-}^{+}$  centred on  $y = b$ . In the case of profile C one must add a similar contribution centred on  $y = -b$ . Equation (1) can be satisfied in the limit only if the  $\psi'_{yy}$  contribution to  $\nabla^2\psi'$  [see (3a)] likewise tends to a delta function, with  $\psi'$  continuous and  $\psi'_y$  piecewise continuous.

<sup>4</sup>More precisely, the inscrutability of (6) is connected with the noninterchangeability of the two limits involved in deriving it, the first being the limit of small disturbance amplitude, already taken in (1) through the omission of terms like  $q'_y\psi'_x$ , and the second being the limit of infinitely steep vorticity gradients at the material contour! One way to make sense of (6), independently of its derivation from (1), is to recognize that although (for reasons of mathematical convenience) (6) refers to values exactly at  $y = +b$ , for instance, it actually represents physical conditions *at the displaced position*,  $y = b + \eta$ , say, of the material contour. The total velocity field on each side of the contour has been, in effect, extrapolated back to  $y = b$  using one-term Taylor expansions, again neglecting products of small quantities like  $\eta u'_y$  and *ignoring the fact that  $y = b$  may be on the wrong*

Both for profile A and for profile B we have solutions of the form

$$\psi' = \hat{\psi}(y)e^{ik(x-ct)} \quad (7)$$

where

$$\hat{\psi}(y) \propto e^{-|k(y-b)|} \quad , \quad (8)$$

making  $q'$  vanish for  $y \neq b$  and hence satisfying (5). Then (6) gives

$$ik(U - c)(-2|k|) = [U_y]_{-}^{+} ik \quad \text{at } y = b \quad ,$$

so that the intrinsic phase speed is

$$c - U(b) = -\frac{1}{2}G|k|^{-1} \quad (G > 0) \quad (9)$$

where  $G = -[U_y]_{-}^{+}$  at  $y = b$  ( $G > 0$  for profiles A and B). Thus a disturbance of  $x$ -wavelength  $2\pi/k$  centred on  $y = b$  propagates to the *left* with phase speed (9), relative to  $U(b)$ , as anticipated in the lectures. (This agreement between equations and pictures is a good check that we have the sign right in (9). Note also that the intrinsic *frequency*  $-\frac{1}{2}G \operatorname{sgn} k$  is independent of  $|k|$  — inevitable on dimensional grounds, as with internal gravity waves, since profiles A and B have no length scale, and the only relevant property of the basic flow,  $G = -[U_y]_{-}^{+}$ , is a constant having the dimensions of frequency. It follows incidentally that the intrinsic group velocity is zero.)

[Exercise: Verify that, in this case, Rossby waves that have less ‘room’ to propagate will propagate more slowly, in the sense of having smaller intrinsic phase speeds. Take for instance the case in which rigid boundaries are introduced at  $y = b \pm a$  for some positive constant  $a$ ; it is easy to show that this always replaces the  $\frac{1}{2}|k|^{-1}$  in (9) by a smaller quantity.]

#### 4. Mathematical analysis of the instability mechanism

What happens if we add another region of concentrated vorticity gradient, with the opposite sign, as in profile C? The Rayleigh–Kuo and Fjørtoft

---

*side of the contour.* Now if the simple vorticity discontinuity of profile A or B moves with the undulating material contour  $y = b + \eta$ , then the velocity jump  $\Delta_{\eta}u$  across the contour  $y = b + \eta$  must vanish,  $\Delta_{\eta}u = 0$ , for otherwise a sheet of infinite vorticity would have appeared from nowhere (see also section 7 below). The small-amplitude approximation to  $\Delta_{\eta}u$ , expressed in terms of the fields extrapolated back to  $y = b$ , is  $[u' + \eta U_y]_{-}^{+}$ ; so we must have  $[u' + \eta U_y]_{-}^{+} = 0$ . Therefore we must also have  $(\partial/\partial t + U\partial/\partial x)[u' + \eta U_y]_{-}^{+} = 0$ . This is (6), because  $\eta$  is a continuous function of  $y$  (and a differentiable function of  $x$ ) such that  $(\partial/\partial t + U\partial/\partial x)\eta = v' = \partial\psi'/\partial x$ .

theorems now suggest that instability is possible:  $G$  changes sign between regions, evading the Rayleigh stability condition, and  $U$  also changes sign, in the sense required to evade the Fjørtoft stability condition. This still does not guarantee instability; see section 6 below. But we now show directly, following Rayleigh, that profile C does, in fact, have unstable modes provided  $|kb|$  is not too large. (One can see from the decaying exponential structure in (8) that instability will certainly not be found for  $|kb| \gg 1$ . For if  $|kb| \gg 1$ , then (8) and a similar solution  $\hat{\psi} \propto e^{-|k(y+b)|}$  will apply with exponentially small error; the neighbourhoods of  $y = \pm b$  are too far away from each other ( $b \gg |k|^{-1}$ ) to interact significantly, and will behave independently.)

For general  $|kb|$ , we have (taking  $k > 0$  to save having to write  $|k|$  all the time):

$$\hat{\psi} = \begin{cases} A \sinh(2kb).e^{-k(y-b)} & (y > b) \\ A \sinh k(y+b) + B \sinh k(b-y) & (-b < y < b) \\ B \sinh(2kb).e^{k(y+b)} & (y < -b) \end{cases} \quad ((10))$$

for some pair of constant coefficients  $A$  and  $B$ . The form of (10) has again been chosen to make  $q'$  vanish for  $y \neq b$  and hence to satisfy (5), and also to make  $\hat{\psi}$  continuous at  $y = \pm b$ . The ratio  $A/B$ , and the constant  $c$ , are still available to satisfy (6) at each interface. Writing  $U_y = \Lambda$  (positive constant) for  $|y| < b$ , dividing (6) by  $ik$ , and writing sh for  $\sinh 2kb$  and ch for  $\cosh 2kb$ , we have

$$\begin{cases} \text{at } y = b: (\Lambda b - c)[\hat{\psi}_y]_{\pm}^+ + \Lambda \hat{\psi} = 0; \text{ also } \hat{\psi} = A \text{ sh}, [\hat{\psi}_y]_{\pm}^+ = -Ak \text{ sh} - (Ak \text{ ch} - Bk); \\ \text{at } y = -b: (-\Lambda b - c)[\hat{\psi}_y]_{\pm}^+ - \Lambda \hat{\psi} = 0; \text{ also } \hat{\psi} = B \text{ sh}, [\hat{\psi}_y]_{\pm}^+ = (Ak - Bk \text{ ch}) - Bk \text{ sh}. \end{cases} \quad (\text{B.1})$$

The first line gives

$$-Ak \text{ sh} - Ak \text{ ch} + Bk + \frac{\Lambda}{\Lambda b - c} A \text{ sh} = 0 \quad . \quad (11a)$$

The second gives

$$Ak - Bk \text{ ch} - Bk \text{ sh} + \frac{\Lambda}{\Lambda b + c} B \text{ sh} = 0 \quad (11b)$$

A nontrivial solution for  $A : B$  exists if and only if the determinant vanishes; write  $\text{sh} + \text{ch} = \exp = \exp 2kb$  (since  $\text{ch} = \frac{1}{2}\{\exp + (1/\exp)\}$  and  $\text{sh} = \frac{1}{2}\{\exp - (1/\exp)\}$ ):

$$\left( -k \exp + \frac{\Lambda \text{ sh}}{\Lambda b - c} \right) \left( -k \exp + \frac{\Lambda \text{ sh}}{\Lambda b + c} \right) - k^2 = 0 \quad . \quad (12)$$



This will give  $c$  (for real, prescribed  $k$ ); it also gives a quick check that we have done our sums correctly so far, since in the large- $k$  limit both  $k \exp$  and  $\text{sh}$  are overwhelmingly greater than  $k^2$ , so that in (12) we have  $(\ )(\ ) = 0$  to an excellent approximation, so that one or other factor must vanish, again to an excellent approximation. The vanishing of the first factor gives  $\Lambda b - c \simeq \frac{1}{2}\Lambda k^{-1}$ , equivalent to (9) (isolated Rossby wave on interface  $y = +b$ ). The second factor similarly gives the wave on  $y = -b$ .

Multiplying-out the product  $(\ )(\ )$  in (12) and noting that  $(\exp)^2 - 1 = 2 \text{ sh. exp}$ , we have

$$2k^2 \text{ sh. exp} - k\Lambda \text{ sh. exp} \cdot \left( \frac{1}{\Lambda b - c} + \frac{1}{\Lambda b + c} \right) + \frac{\Lambda^2 \text{ sh}^2}{\Lambda^2 b^2 - c^2} = 0 .$$

The quantity in parentheses is equal to  $2\Lambda b/(\Lambda^2 b^2 - c^2)$ . Therefore

$$\Lambda^2 b^2 - c^2 = \frac{2kb\Lambda^2 \text{ sh. exp} - \Lambda^2 \text{ sh}^2}{2k^2 \text{ sh. exp}} \quad \left( = \frac{\Lambda^2 b^2}{kb} - \frac{\Lambda^2 b^2 \text{ sh}}{2k^2 b^2 \text{ exp}} \right)$$

or

$$c^2 = \Lambda^2 b^2 \left( 1 - \frac{1}{kb} + \frac{\text{sh}}{2k^2 b^2 \text{ exp}} \right) . \quad (13)$$

If we make  $c$  dimensionless with respect to the total velocity difference  $\Delta U = 2b\Lambda$ , and  $k$  with respect to the shear layer width  $2b$ , say  $C = c/2b\Lambda$ ,  $K = 2kb$ , then (13) becomes

$$C^2 = \left( \frac{c}{2b\Lambda} \right)^2 = \frac{1}{4} - \frac{1}{2}K^{-1} + \frac{\sinh(K)}{2K^2 \exp(K)} = \frac{1}{4K^2} [(K-1)^2 - \exp(-2K)] . \quad (14)$$

Note that  $C^2 = -\frac{1}{4} + \frac{1}{3}K + O(K^2)$  as  $K \rightarrow 0$  (by Taylor-expanding  $\exp(-2K) = 1 - 2K + 2K^2 - \frac{4}{3}K^3 + O(K^4)$ ); note that the first two orders cancel), so that

$$C = \pm \frac{1}{2}i(1 - \frac{2}{3}K + O(K^2)) \quad \text{as } K \rightarrow 0 . \quad (15)$$

This demonstrates the existence of instability for some range of  $K$ : there exists a mode with  $\text{Im } c > 0$ , at least for sufficiently small  $K$ . At this point we get another check that the algebra is correct; the limiting values  $C \simeq \pm \frac{1}{2}i$ , or  $c \simeq \pm \frac{1}{2}\Delta U i$ , agree with those implied by the theory of waves on a single vortex sheet (e.g. Batchelor's textbook, eq. (7.1.20)). The fact that small  $K$  means radian wavelength  $k^{-1} \gg 2b$  suggests that the disturbance should see the whole shear layer as being thin; i.e. as a vortex sheet. Similarly, any other  $U(y)$  profile that goes monotonically between two constant values should give the same long-wave behaviour in an infinite domain. For instance the tanh

profile also checks out in this respect ( $c \sim \frac{1}{2}i \times$  total change in  $U$ ). You could try some mouse-drawn profiles as well, with small-ish but finite  $k$ , but it will be necessary to make the domain size  $L$  somewhat larger than  $k^{-1}$ .

Next we note the *phase relations* implied by (10), (11) and (14) — crucial to a full understanding of what is going on! From (11a, b) respectively we get

$$\frac{B}{A} = \left( \exp -\frac{\Lambda \operatorname{sh}}{k(\Lambda b - c)} \right) = \left( \exp -\frac{\Lambda \operatorname{sh}}{k(\Lambda b + c)} \right)^{-1} . \quad (16)$$

We are interested only in cases where  $k$  and  $K$  are real and  $c$  is pure imaginary, i.e. (14) is negative-valued. The two expressions in large parentheses are then complex conjugates of each other, since they differ only in the sign of  $\pm c$ . It follows that

$$\left| \frac{B}{A} \right| = 1 , \quad (17)$$

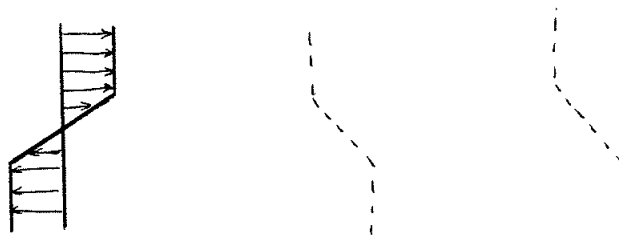
the simplest result consistent with the symmetry of the problem. The relative phases of  $\psi'$  and therefore of  $v' = \psi'_x$  are (when  $c = ic_i$ , pure imaginary):

$$\arg \left( \frac{B}{A} \right) = \arcsin \operatorname{Im} \left( \frac{B}{A} \right) = \arcsin \operatorname{Im} \left( -\frac{\Lambda \operatorname{sh}}{k(\Lambda b - ic_i)} \right) = \arcsin \left( \frac{-\Lambda c_i \operatorname{sh}}{k(\Lambda^2 b^2 + c_i^2)} \right) , \quad (18a)$$

or in dimensionless form, dividing numerator and denominator by  $2b\Lambda^2$ ,

$$\arg \left( \frac{B}{A} \right) = \arcsin \left( \frac{-C_i \sinh(K)}{K(\frac{1}{4} + C_i^2)} \right) , \quad (18b)$$

This gives the phase angle, or fraction-of-a-wavelength times  $2\pi$ , by which the pattern in  $v'$  at  $y = b$  leads that at  $y = -b$ . It is *negative* for the growing mode,  $C_i > 0$ , so in our picture, with  $x$  pointing to the right, the  $v'$  pattern at  $y = b$  is shifted to the *left* of that at  $y = -b$ . The constant-phase lines ‘tilt oppositely to the shear’:



The phase shift (18b) across the shear tends<sup>5</sup> to  $\arcsin(-1) = -\pi/2$ , corresponding to a quarter of a wavelength, as  $K \downarrow 0$  and  $C_i \uparrow \frac{1}{2}$ . Since the

<sup>5</sup>Note that  $C_i$  is real, and  $\rightarrow \frac{1}{2}$  as  $K \rightarrow 0$ , and that  $\sinh(K)/K \rightarrow 1$  as  $K \rightarrow 0$ .

complex displacement amplitude  $\hat{\eta} = \hat{\psi}/(U - c)$ , from  $(\partial/\partial t + U\partial/\partial x)\eta = v'$  and  $v' = \psi'_x$ , and since the  $q'$  pattern at each interface  $\propto \mp\eta$  at  $y = \pm b$ , the phase shift in the  $q'$  pattern is given by

$$\begin{aligned} \arg\left(\frac{B/(-\Lambda b - c)}{-A/(\Lambda b - c)}\right) &= \arg\left(\frac{B(\Lambda b - c)}{A(\Lambda b + c)}\right) = \arg\left(\frac{B}{A}\right) + \arg\left(\frac{1 - 2C}{1 + 2C}\right) \\ &= \arg\left(\frac{B}{A}\right) + 2\arctan(-2C_i) \quad \text{if } C = iC_i, \text{ pure imaginary} \end{aligned} \quad (19)$$

Thus the  $q'$  pattern has a phase shift in the same sense, but *bigger*. On the next page are some numerical values showing how the quantities of interest vary as function of dimensionless wavenumber  $K$ . From left to right: dimensionless wavenumber  $K$ , imaginary part  $C_i$  of dimensionless phase speed (real part being zero), dimensionless growth rate  $KC_i$ , phase shift for  $v'$  or  $\psi'$ , phase shift for  $q'$ , the last two being expressed as fractions of a wavelength.

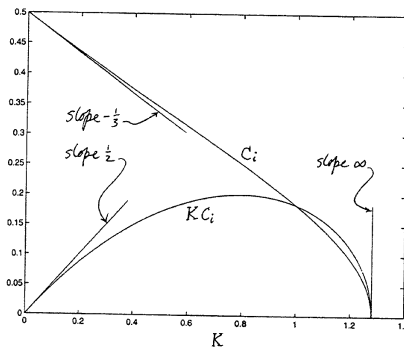
Note from (15) (18b), and (19) that the phase shift for the  $q'$  pattern  $\rightarrow -\pi$ , corresponding to half a wavelength, as  $K \downarrow 0$ . This again is consistent with expectation (and with well known results) for effectively thin shear layers, or vortex sheets. It says that, in the long-wave limit, the displacements  $\eta$  ( $\propto \mp q'$ ) are almost exactly in phase across the shear layer; that is, the layer does undulate almost as a single entity.

Some numerical results for Rayleigh's shear-instability problem: (Dimensionless growth rate  $K C_i = k c/\Lambda$  peaks at  $K = 0.797$ , with max. value  $K C_i = 0.20119$ .)

K	$C_i$	$K C_i$	shifts	
			$\psi'$	$\psi''$
0.100	0.467	0.047	-0.244	-0.484
0.200	0.435	0.087	-0.238	-0.466
0.300	0.404	0.121	-0.231	-0.447
0.400	0.374	0.149	-0.222	-0.426
0.500	0.343	0.172	-0.213	-0.404
0.600	0.313	0.188	-0.202	-0.380
0.700	0.283	0.198	-0.189	-0.353
0.800	0.251	0.201	-0.175	-0.323
0.900	0.219	0.197	-0.158	-0.290
1.000	0.184	0.184	-0.138	-0.250
1.100	0.144	0.159	-0.112	-0.201
1.200	0.094	0.113	-0.075	-0.134

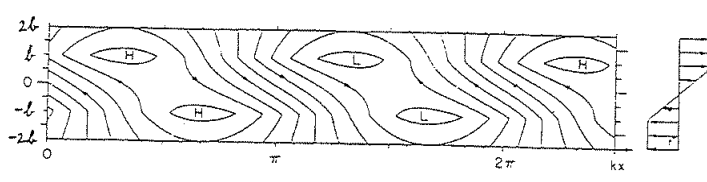
```

2INPUT"FROM "K0
3INPUT"TO "K1
4INPUT"STEP "S
8E2=601020307:REM print format
10FOR K=K0 TO K1 STEP S
20GOSUB 102
30PRINTK,C,K*C,FRACV,FRACQ
50NEXT
99END
102 EX=EXP(K)
103 EM=1/EX
105 SH=.5*(EX-EM)
115 K2=K*K
140 DEN=2*K2*EX
150 C2=.25-(.5/K)+(SH/DEN) (14)
155 C=SQR(ABS(C2))
160 ARGV=ASN(-C*SH/(K*(.25-C2))) (18)
170 FRACV=ARGV/(2*PI)
180 ARGQ=ARGV+2*ATN(-2*C) C+c_i^2 (19)
190 FRACQ=ARGQ/(2*PI)
200RETURN
    
```



$C^2$  changes sign just before  
 K reaches 1.278: 50PRINTK,C  
 1.2783  
 1.2784  $C_i$  { 4.23329659E-3  
 1.2785 { 2.65127772E-3  
 1.2786 { 1.96502458E-3  
 1.2787 { 3.84073147E-3  
 5.06361944E-3

The following picture, taken from Gill's book, gives  $\psi' = \hat{\psi}(y)e^{ik(x-ct)}$  for the fastest growing mode, as a function of  $x$  and  $y$  when the arbitrary constant  $A$  is taken such that  $AB$  is pure imaginary:



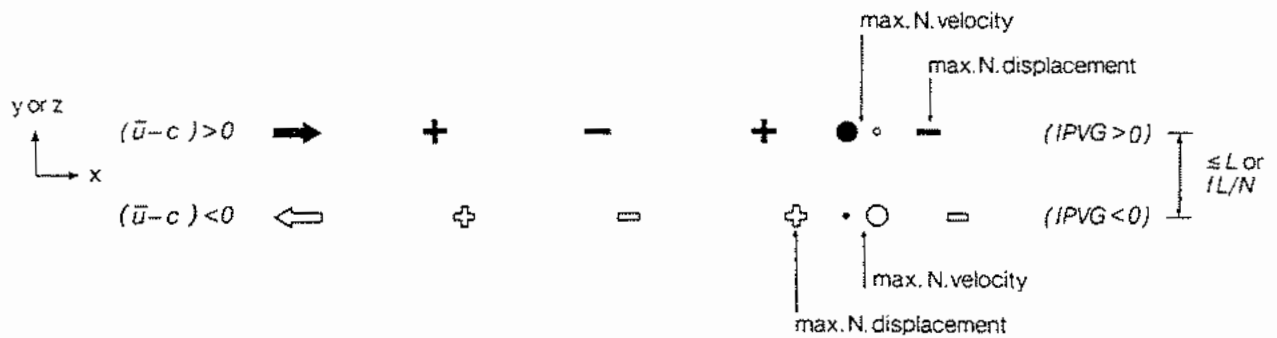
The next section gives a verbal description that serves to summarize the key points about the picture that has emerged. It also tries to make the qualitative robustness of the whole picture more evident, including the pattern of phase shifts and how they are related to the exponential growth with time. As mentioned earlier it can be understood as applying to more than one type of large and small-scale shear instability occurring in the atmosphere and oceans; but on first reading it can be understood simply as a summary of the picture just derived for Rayleigh's problem.

## 5. The essentials of the instability mechanism

This follows section 6*d* of the review article by Hoskins *et al.* cited in section 1. Those of you with a particular interest in large-scale atmospheric and oceanic eddies might be interested not only in the wider meaning of what is to follow, but also in other parts of the review such as the description of what happens, in certain cases, when large amplitudes are attained, and the relationship to cyclogenesis in the real atmosphere (as hinted at in my last lecture). In this connection you should note one point about terminology. Phrases like 'IPV maps', 'IPV distributions', 'IPV anomalies', etc., are used in the review article as a handy abbreviation to signify isentropic or isopycnic maps, distributions, anomalies, etc, of PV, where 'PV' means the quantity  $Q = \rho^{-1} \zeta_{\text{abs}} \cdot \nabla \theta$  defined in the lectures i.e., the Rossby–Ertel potential vorticity. (The full name follows the historical precedents dating from a paper by Rossby published in 1936.) As will be explained in the lectures, it is isentropic or isopycnic *distributions* of PV — and, for Rossby waves and shear instabilities, isentropic or isopycnic *gradients*, and *anomalies*, of PV — that are dynamically significant. They play the role of vorticity gradients and anomalies in two-dimensional vortex dynamics. In the diagram below, taken from the review article, 'IPVG' means (northward) isentropic gradient of PV (and 'N' or 'northward' corresponds to  $+y$  above). Since the review was published it has become apparent, however, that phrases like 'IPV gradient' can be too easily misread as signifying a gradient of something called 'IPV'. Therefore in these notes I shall use phrases like 'PV gradients', leaving tacit the important fact that, in the case of layerwise-two-dimensional stratified flow, this must be understood to mean *isentropic* or *isopycnic* gradients.

On first reading, as suggested, references to baroclinic phenomena can be ignored, and, as appropriate for the case of the strictly two-dimensional flow that is our immediate concern, 'PV gradient' can be read as meaning vorticity gradient (absolute vorticity gradient if in a rotating frame), 'PV anomaly' as meaning  $q'$ , and so on.

As already suggested, the simplest instabilities — by which we mean those with the simplest spatial structures — are also, in many cases, those with the fastest growth rates. These simplest instabilities, including that arising in Rayleigh's problem, are all characterized by a pattern of PV anomalies ( $q'$  anomalies) of the general sort shown schematically by the plus and minus signs in the following diagram:



The pattern can be thought of as a pair of Rossby waves propagating side by side, or one above the other, depending on whether a barotropic (Rayleigh-like) or a baroclinic instability is in question.

Viewed in a reference frame moving with the zonal phase speed  $c$  of the disturbance, each Rossby wave propagates against, and is held stationary by, the local basic flow. From the nature of the Rossby propagation mechanism (recall diagram on page 198), this is dynamically possible if the sign of the basic PV gradient is positively correlated with that of the relative zonal flow ( $U - c$ ), i.e. both signs positive, as in the top half of the diagram, or both signs negative, as in the bottom half. This is evidently the simplest configuration consistent with the Rayleigh–Kuo and Fjørtoft necessary conditions for instability. It will be noticed that if the basic zonal flow  $U$  has a continuous profile then a ‘steering level’ or ‘critical line’ will be present, where by definition  $U - c = 0$ . We shall assume that the basic PV gradient is small or negligible in some region containing the critical line (as in Rayleigh’s problem); the more general case is discussed in the review article. Moreover, for expository purposes we shall restrict attention at first to patterns whose spatial scale is such that, if the induced velocity field associated with each Rossby wave in the diagram did not affect the other, then their phase propagation would be somewhat too slow to hold them stationary against the basic zonal flow.

The essence of the instability mechanism is that the induced velocity fields do, however, overlap significantly. That is why the width  $2b$  in Rayleigh’s

problem, if instability is to occur at a given wavelength  $2\pi/k$ , has to be of order  $k^{-1}$  or less. Similarly, in order to get a baroclinic instability of horizontal scale  $L$ , say, and simple spatial structure, the vertical separation between the two rows of PV anomalies has to be of the order of one Rossby height  $fL/N$  or less, as illustrated by the Eady baroclinic instability problem. The overlapping of the induced velocity fields has the following consequences, under the assumed conditions:

(i) Inasmuch as the PV anomaly patterns are less than a quarter wavelength out of phase with each other, the case shown in the diagram, each half helps the other to propagate against the basic zonal flow. That is, the contributions to the northward velocity induced by each PV pattern partially reinforce each other, making the phase of each pattern propagate upstream faster than it would in isolation. This is how the patterns hold themselves stationary against the basic flow, under the assumed conditions.

(ii) Because of this interdependence between the two counterpropagating Rossby waves, their relative phase tends to lock on to a configuration like that shown. For if the PV patterns were each to shift slightly downstream, i.e. the upper pattern towards the right and the lower towards the left, so as to be more nearly in phase, then each half would help the other to propagate still more strongly, moving the patterns back upstream towards their original relative positions. Conversely, if the patterns were shifted upstream, so as to be more out of phase, then propagation would be weakened, and advection by the basic zonal flow would tend to restore the original phase relation.

(iii) Just as in the diagram on page 3, the northward velocity induced by the upper PV pattern alone is a quarter wavelength out of phase with that pattern. The large black dot in the diagram marks the position of the northward velocity maximum induced by the upper PV pattern alone, for the right-hand-most wave period. This is *less than a quarter wavelength* out of phase with the bottom PV pattern, and therefore with the bottom displacement pattern, as indicated by the position of the small black dot directly below. If we add the velocities induced by the bottom PV pattern (open dots) to get the total velocity field, we see at once that the total velocity is also less than a quarter wavelength out of phase with the displacement pattern. This is true on the top level as well as on the bottom level.

(iv) It follows that the total northward velocity field in each half of the disturbance can be regarded as a sum of sinusoidal contributions in phase with, and a quarter-wavelength out of phase with, the northward displacement field. Moreover, the in-phase contribution has the same sign as the displacement. A velocity in phase with the corresponding displacement implies, by simple kinematics, that both must be growing.

The instability mechanism just described can be summarized in one sentence, by saying that

‘The induced velocity field of each Rossby wave  
keeps the other in step, and makes the other grow.’

These two effects of the induced velocity field are associated respectively with its in-quadrature and in-phase contributions. The pure, exponentially-growing normal mode of linear instability theory describes a situation in which the two PV anomaly patterns have locked on to each other and settled down to a common phase speed  $c$ , such that the rates of growth which each induces in the other are precisely equal, allowing the shape of the pattern as a whole to become precisely fixed, and the growth of all disturbance quantities precisely exponential.

Cases in which the spatial scale is sufficiently large that each wave in isolation would propagate *faster* than the basic zonal flow can be understood in the essentially same way. The main changes needed are in statement (i) of the foregoing, where ‘help’ is replaced by ‘hinder’, ‘faster’ by ‘slower’, and so on. Whereas in the ‘helping’ case the phase shift between the two PV patterns is less than 0.25 of a wavelength, as shown in the diagram, in the ‘hindering’ case the phase shift lies between 0.25 and 0.5 of a wavelength. The relative phase tends to lock on just as before, and the summarizing statement (69) remains true.

In fact this latter case is usually the one which exhibits the largest growth rates, as would generally be expected from the fact that a larger phase shift between the two PV anomaly patterns enables the total induced velocity to be more nearly in phase with the displacement, tending to give a larger growth rate. This is exemplified both by the Rayleigh and by the Eady problem. It can also be checked, as already done for the Rayleigh problem above, that the phase shifts in the northward velocity and geopotential height anomaly patterns are indeed substantially less than those in the corresponding PV anomaly patterns (respectively 0.18 and 0.25 of a wavelength at maximum growth rate, in the two examples), as suggested by the diagram. This can be looked upon as another consequence of the smoothing property of the inversion operator.



## 6. Suppression of shear instabilities by boundary constraints

*Arnol'd's second stability theorem* (discovered in the 1960s, but not widely known until the 1980s) proves that there are cases where neither the Rayleigh–Kuo theorem nor the Fjørtoft theorem rules out instability, yet where the flow is stable (indeed, stable in a certain finite-amplitude sense). These are cases with side boundaries so close to the shear layer that the Rossby-wave propagation mechanism does not have room to operate sufficiently strongly to hold a phase-locked configuration. (This is again a manifestation of the *scale effect* in the vorticity inversion operator. It shows up also in the simple plane-wave dispersion relation  $c - U = -\beta/(k^2 + \ell^2)$ ; when  $y$ -wavenumbers  $\ell$  become large, as would be necessary to fit the waves into a narrow channel, intrinsic phase speeds  $c - U$  become small. See also the Exercise at the end of section 3.)

A relevant case is where  $\beta = 0$  and  $U(y)$  is of the form  $\sin(ay)$ , in which case the critical channel width  $2L = \pi/a$ . So for instance if  $|a| = 0.25\pi$  — note that you can type `sin(.25*pi*y)` when inputting the  $U$  profile — then Arnol'd's second theorem implies that making the half channel width  $L$  less than its default value 2 stabilizes the flow (e.g. McIntyre and Shepherd 1987, *J. Fluid Mech.* **181**, pp. 542, 543). (The result is well known to specialists in instability theory, albeit missed by at least one of the standard monographs on hydrodynamic instability theory!) When  $L$  just exceeds 2, only the longest wavelengths (smallest  $k$  values) are unstable, and only weakly. Try for instance  $k$  values between 0.01 and 0.1, and  $L = 2.01, 2.02, 2.05$ . You may need a relatively fine grid value, say 38. This is quite a delicate check on the correctness of the computer program!

## 7. The continuous spectrum of singular neutral modes

You may be wondering about the origin of the large number of neutral modes that are always found in the computer demonstration (modes with real  $c$  and therefore neither growing nor decaying). A few of these may be ordinary Rossby waves, especially for the larger values of  $\beta$ ; but the majority are likely to correspond, within numerical discretization error, to what is referred to in the literature as the ‘continuous spectrum’ of singular neutral modes. Being singular, these cannot be properly represented by a general-purpose numerical method; but their presence in the continuous problem is likely to be the main reason for the appearance of many neutral modes in the discretized problem.

Quite unlike the instability in which we are interested (which begins as an undulation of the pre-existing vorticity distribution), the continuous spec-

trum modes, and their superpositions including what are called ‘sheared disturbances’, correspond to *artificially changing the initial vorticity distribution* — more precisely, artificially changing the vorticities of fluid elements by small amounts that have an oscillatory  $x$ -dependence — and then letting the system evolve freely. Such (weak) vorticity distributions tend to be sheared over, and thus tend to develop increasingly fine scales in the  $y$  direction, as one might expect of a quantity advected by a total velocity field that is close to pure shear. A consideration of such disturbances is necessary for a full mathematical understanding of the instability problem with arbitrary initial conditions, but is not of primary interest here.

However, it is easy to say simply but precisely what the continuous-spectrum neutral modes are, which may be useful since, despite the clear explanation in Rayleigh’s *Theory of Sound*, p. 391, the subsequent literature contains a certain amount of confusion over what is fundamentally a simple technical point. A singular neutral mode of the continuous spectrum, for given wavenumber  $k$ , is a disturbance with a non-zero velocity jump  $\Delta_\eta u$  on a single material contour  $y = y_0 + \eta$ , with  $\Delta_\eta u$  varying like  $\sin\{kx - kU(y_0)t + \text{constant}\}$  along the contour. In other words, it corresponds to a frozen, sinusoidally varying *sheet of vorticity* inserted as an initial disturbance on exactly the one material contour. Here  $\eta$  is the displacement in the  $y$  direction as before. (In order to be a normal mode, i.e. to have constant spatial shape as time goes on, the whole disturbance generally has to involve undulations of the remaining material contours and hence, in general, a smoothly-varying distribution of disturbance vorticity at any other  $y \neq y_0$ .) Such modes are said to belong to a ‘continuous spectrum’ because they have frequencies  $kU(y_0)$  that vary continuously as  $y_0$  varies. Of course it takes at least two such modes, with vorticity sheets located at two values  $y_1 \neq y_0$  of  $y$  and advected at different speeds  $U(y_1) \neq U(y_0)$ , to begin to describe the ‘shearing-over’ effect; more usually, one has a continuous superposition expressed by an integral.

## 8. Other basic instabilities, especially a 3D one recently discovered

It is arguable that the inviscid shear instability described and analyzed in §§1–6 above is representative of one of the most basic, quintessentially fluid-dynamical classes of instabilities, underlying much high-Reynolds-number fluid-dynamical behaviour both large and small scale, stratified and unstratified, layerwise-two-dimensional (moderate to large Richardson number) and fully three-dimensional (small to zero Richardson number). That is why I have concentrated on it. There are of course many other kinds of fluid

instabilities, some of an obvious kind, such as the convective or Rayleigh-Taylor instability associated with negative stratification ( $N^2 < 0$ ), and others less so, such as the ‘elliptic instability’ discovered by Pierrehumbert and Bayly<sup>6</sup> which, although its significance is still being assessed, seems likely to be another robust paradigm and very basic to an understanding of many fully three-dimensional flows, such as small-scale turbulent mixing (but not layerwise-two-dimensional flows).

A discussion of the elliptic and related instabilities is beyond the scope of the present lectures, although it might be brought in on future occasions, especially if we can develop some suitable computer demonstrations. The physical mechanism is entirely different from the above — it appears to be more closely akin to the Mathieu parametric instability of a pendulum whose point of support is oscillated, and to the ‘resonant triad’ wave–wave interactions that have been extensively studied in connection with oceanic surface and internal gravity waves. In the meantime the interested reader may consult the review by Bayly et al, 1988, *Ann. Rev. Fluid Mech.*, **20**, especially pages 381–384.

## 9. Suppression of shear instabilities by a large-scale strain field

An important paper analyzing this effect is

Dritschel, D. G., Haynes, P. H., Jukes, M. N., Shepherd, T. G., 1991, The stability of a two-dimensional vorticity filament under uniform strain, *J. Fluid Mech.*, **230**, 647–665.

Such a suppression of shear instabilities is essential to understand the existence of thin filaments of vorticity that appear in simulations such as that shown in fig. §4.6 on p. 97. (Meticulous checks were done in these cases to make sure that any small-scale shear instability would be resolved numerically if it occurred.)

This implies an important qualification to the earlier remarks about robustness. Shear instability is robust to finiteness of disturbance amplitude, but not to large-scale strain fields that are stretching the filaments. Stabilization by such stretching can occur for strain rates only a modest fraction, often a sixth or so, of the vorticity contrast in the shear flow. This fact, and its generalization to baroclinic cases, is often critical to the ‘mesoscale developments’, or lack thereof, that in turn can be critical to weather forecasting.

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<sup>6</sup>A Soviet colleague, V. A. Vladimirov, has pointed out that the instability is essentially the same as that discovered by Tsai and Widnall (*JFM* **73**, no 4, 1976) and by Gledzer et al (*Izv. Akad. Nauk SSSR, FAO*, **11**, no 10, 1975). See also Gledzer, E. B., Ponomarev, V. M., 1992, *J. Fluid Mech.* **240**, 1–30.



# C.

## Some basic dimensionless parameters and scales

The table summarizes (using standard notation as far as it exists) some of the basic quantities encountered in the lectures, whose order of magnitude is usually the first consideration when assessing a fluid-dynamical situation.

In fluid dynamics, even more than in other branches of physics, quantities like these have more than one meaning and more than one mode of use. For instance, the nominal length and velocity scales  $L$ ,  $U$  may or may not be true scales in the sense that we can estimate the order of magnitude of  $\partial/\partial x$  as  $L^{-1}$ , of  $\nabla\mathbf{u}$  as  $U/L$ , of  $\nu\nabla^2\mathbf{u}$  as  $\nu U/L^2$ , and so on. In a thin boundary layer with thickness  $\ell$  and downstream lengthscale  $L$ , for instance, we might have  $\mathbf{u}\cdot\nabla\mathbf{u}\sim U^2/L$  and  $\nu\nabla^2\mathbf{u}\sim\nu U/\ell^2$ . Then it is  $U\ell^2/\nu L$  that needs to be of order unity, not the ordinary Reynolds number <sup>1</sup>  $UL/\nu$ , if viscous forces are to balance typical accelerations. Again, it is often relevant to consider  $L$  and  $U$  to be *external* parameters (such as pipe radius and volume flux/pipe area in the Reynolds experiment), especially when interested in ‘scaling up’ or ‘scaling down’ in the sense of identifying the class of problems that reduce to the same problem when suitably nondimensionalized (e.g. half the pipe radius and twice the volume flux requires four times the viscosity, in order to get the same pipe-flow problem). In ordinary low-Mach-number flows, the timescale quite often  $\sim L/U$ , and this is often tacitly assumed when estimating typical material rates of change  $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla \sim U/L$ .

---

<sup>1</sup>not Reynold’s. After Osborne Reynolds who in a famous experiment showed its relevance to whether pipe flow is laminar or turbulent.

Dimensionless parameter	symbol and formula	Interpretation <sup>2</sup>
<b>Mach number</b>	$M = \frac{U}{c_{\text{sound waves}}}$	$\frac{\text{typical advective acceleration}}{\text{wave-induced particle accel.}}$
External <b>Froude number</b>	$Fr_e = \frac{U}{c_{\text{ext. gravity waves}}} = \frac{U}{(gH)^{1/2}}$	”
Internal <b>Froude number</b>	$Fr_i = \frac{U}{c_{\text{int. gravity waves}}} = \frac{U}{NH}$	”

Small values of  $M, Fr_e, Fr_i$  imply the possibility of ‘balanced’ or ‘adjusted’ flows that do not self-excite the waves in question (cf. mass on stiff spring moved gently), e.g. nearly-incompressible flow when  $M$  is small enough, or layerwise-2D stratified flow (e.g. ‘Los Angeles smog’) when  $Fr_i$  is small enough. (Beware: governing equations are ‘stiff’.)  $N$  = buoyancy frequency of stable stratification:  $N^2 = g \partial \ln(\text{potential density})/\partial z$ .

<b>Reynolds number</b>	$R = Re = UL/\nu$	$\frac{\text{typical advective acceleration}}{\text{typical viscous force/mass}}$
Boundary-layer thickness (for flow past obstacle)	$\left(\frac{\nu L}{U}\right)^{1/2} = Re^{-1/2}L$	= diffusion length for time $L/U$
<b>Kolmogorov microscale</b>	$\ell_K = (\nu^3/\epsilon)^{1/4}$	

<sup>2</sup>when  $L, U$  etc are true scales in a flow with simple structure

(Nominal length scale at which viscous dissipation becomes important in three-dimensional turbulence that is dissipating energy at rate  $\epsilon$  per unit mass;  $\epsilon$  has dimensions  $\text{length}^2 \text{time}^{-3}$ .)

Associated velocity and time scales  $U_K \sim (\nu\epsilon)^{1/4}$ ,  $t_K \sim (\nu/\epsilon)^{1/2}$

[*Consistency checks:*  $U_K \ell_K / \nu \sim 1$ , and  $t_K \sim \ell_K^2 / \nu$  (viscous diffusion time)]

Dimensionless parameter	symbol and formula	Interpretation <sup>3</sup>
(gradient) <b>Richardson number</b>	$Ri = N^2/(U_z)^2$ (= $Fr_i^{-2}$ if $U_z = U/H$ )	As for $Fr_i^{-2}$ ; $U_z =$ vertical shear. (Note that $U_z H$ is, quite often, the relevant velocity scale.)
<b>Péclet number</b>	$Pe = UL/\kappa$ ( $\kappa =$ heat diffusivity)	typ. advective rate of change of temp. typ. diffusive rate of change of temp.
<b>Prandtl number</b>	$\nu/\kappa$	$\frac{\text{momentum diffusivity}}{\text{heat diffusivity}}$
<b>Schmidt number</b>	$\nu/\kappa_s$	$\frac{\text{momentum diffusivity}}{\text{solute diffusivity}}$
<b>Rossby number</b> ( <b>Kibel' number</b> in Soviet literature)	$Ro = U/\Omega L \sim U/2\Omega L$ $\sim U/c_{\text{inertia waves}}$	$\frac{\text{typical relative advective accel.}}{\text{typical Coriolis accel.}}$ ( $Ro \ll 1 \Rightarrow$ rotationally stiff)
<b>Ekman number</b>	$E = \nu/\Omega L^2$	$\frac{\text{typical viscous force/mass}}{\text{typical Coriolis accel.}}$
Ekman-layer thickness	(For relevance to spindown time $\Omega^{-1}E^{-1/2}$ , $L$ needs to be a scale in the $\Omega$ direction) $(\nu/\Omega)^{1/2}$	Diffusion length for

<sup>3</sup>when  $L, U$  etc are true scales in a flow with simple structure



scale

time  $\Omega^{-1}$

**Prandtl's ratio of scales**  $H/L \sim f/N$   
 $f = 2\Omega \sin(\text{latitude})$

(Natural vertical-to-horizontal aspect ratio for stratified, rotating flow at low  $Fr_i$  and  $Ro$ )

Associated quantities:

**Rossby length**  $L \sim NH/f$  (also 'Rossby radius'; no standard symbol)

**Rossby height**  $H \sim fL/N$  (no standard symbol)

**Burger number**  $Bu = N^2 H^2 / f^2 L^2$  (sometimes defined the other way up)

( $H$  and  $L$  are vertical and horizontal length scales;  $H \ll L$  in atmosphere and ocean)

**Rayleigh number**  $Ra = \frac{g' H^3}{\nu k}$  ( $g' = g \frac{\Delta \rho}{\rho}$ )

$Ra$  is the product  $RePe$  for vigorous thermal convection (assuming that the velocity scale  $U$  is such that vertical advective acceleration  $U^2/H \sim$  buoyancy acceleration  $\sim g'$ ).

**Nusselt number**  $\frac{\text{total vertical heat or buoyancy flux in a thermally convecting layer}}{\text{conductive heat or buoyancy flux if convection suppressed}}$

**Flux Richardson number**  $Ri_f = \frac{\text{vertical eddy buoyancy flux}}{U_z \times \text{eddy momentum flux}}$

$Ri_f$  arises from the turbulent energy equation for stratified shear flows. It compares the rate at which eddies do work against gravity (in reducing the stable stratification) with the rate at which they acquire energy from the mean shear  $U_z$ .



# D.

## Some useful numbers

(The numbers are mainly from Gill's book.)

Solar 'constant'  $S = (1.36_8 \pm 0.00_1) \times 10^3 \text{ W m}^{-2}$ , where ' $\pm$ ' indicates the order of magnitude of the 11-year variability associated with the solar activity (Schwabe) cycle. This has been known only for a decade or so (*Nature* **332**, p.811, 1988). Not known is whether larger fluctuations, more obviously of climatological significance, say  $\sim \pm 0.5\%$  or more, occurred during past decades; nor is it known to what extent the climate system can nonlinearly amplify solar fluctuations.

Molecular mass of dry air,  $m_a = 28.966$ , of water,  $m_w = 18.016$ , of ozone,  $m_{o_3} = 48.01$

Universal gas constant,  $R_* = 8.3145 \text{ J mole}^{-1} \text{ K}^{-1}$  (*Nature* **331**, p.477; *Amer. Inst. Phys.* 1993); 1 mole is  $10^{-3} \text{ kg}$  times the molecular mass

Gas constant for dry air,  $R = R_*/m_a = 287.04 \text{ J kg}^{-1} \text{ K}^{-1}$ ;  $c_p = 1004$ ,  $c_v = 717$ , same units

Pressure scale height in hydrostatic atmosphere ( $\simeq$  density scale height) =  $RT/g = 7 \text{ km}$  when  $T = 239\text{K}$ ,  $8 \text{ km}$  when  $T = 273\text{K}$ ,  $9 \text{ km}$  when  $T = 307\text{K}$ ; recall  $\text{J kg}^{-1} \text{ K}^{-1} = \text{m}^2 \text{s}^{-2} \text{ K}^{-1}$

Gas constant for water vapor,  $R_w = R_*/m_w = 461.50 \text{ J kg}^{-1} \text{ K}^{-1}$

Molecular weight ratio  $m_a/m_w = R_w/R_a = 1.6078 = 0.62197^{-1}$ ;  $\gamma \equiv c_p/c_v = 1.40$ ;  $\kappa \equiv R/c_p = \frac{2}{7}$ ;

Stefan-Boltzmann constant  $\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$

Gravitational force per unit mass  $g$  (in  $\text{ms}^{-2}$ ) as a function of latitude  $\varphi$  and height  $z$  (in m)

$$g = (9.78032 + 0.005172 \sin^2 \varphi - 0.00006 \sin^2 2\varphi)(1 + z/a)^{-2}$$

Mean surface value,  $\bar{g} = \int_0^{\pi/2} g \cos \varphi d\varphi = 9.7976$

Radius of sphere having the same volume as the earth,  $a = 6371 \text{ km}$  (equatorial radius =  $6378 \text{ km}$ , polar radius =  $6357 \text{ km}$ )

Rotation rate of earth,  $\Omega = 7.292 \times 10^{-5} \text{ s}^{-1}$ ; Coriolis parameter  $f = 2\Omega \sin \phi = 1 \times 10^{-4} \text{ s}^{-1}$  at latitude  $\phi = 43.29^\circ \text{N}$

Mass of earth  $5.977 \times 10^{24} \text{ kg}$

Mass of atmosphere =  $5.3 \times 10^{18} \text{ kg}$

Mass of ocean  $1.4 \times 10^{21} \text{ kg}$

Mass of water in sediments and rocks =  $2 \times 10^{20} \text{ kg}$

Mass of ice on earth =  $2.2 \times 10^{19} \text{ kg}$

Mass of water in lakes and rivers =  $5 \times 10^{17} \text{ kg}$

Mass of water vapor in atmosphere =  $1.3 \times 10^{16} \text{ kg} \simeq \frac{1}{4}\%$  of mass of atmosphere

Mass of ozone in atmosphere =  $3 \times 10^{12} \text{ kg} \simeq 0.00006\%$  of mass of atmosphere  $\simeq 3 \text{ mm}$  layer at surface (Ozone replacement rate, assuming 3 year circulation timescale,  $\simeq 3$  million tonnes per day)

Area of earth =  $5.10 \times 10^{14} \text{ m}^2$ ; Area of ocean =  $3.61 \times 10^{14} \text{ m}^2$ ; Area of land =  $1.49 \times 10^{14} \text{ m}^2$

Area of ice sheets and glaciers =  $1.62 \times 10^{13} \text{ m}^2$

Area of sea ice =  $1.75 \times 10^{13} \text{ m}^2$  in March and  $2.84 \times 10^{13} \text{ m}^2$  in September

Area of Antarctica  $1.41 \times 10^{13} \text{ m}^2$ ; Area of USA  $0.93 \times 10^{13} \text{ m}^2$

Dry adiabatic lapse rate  $\Gamma = g/c_p = 9.76 \text{ K km}^{-1}$

Moist adiabatic lapse rate  $\Gamma_s$  depends on temperature  $T$  and pressure  $p$ :  
e.g. at

30°C	(303K)	and 1000 mbar ( $10^5 \text{ Pa}$ ),	$\Gamma_s = 3.5 \text{ K km}^{-1}$
25	298	”	3.8
20	293	”	4.2
0	273	”	5.2
-30	243	”	8.4

[Gill (p 607) gives an empirical formula for  $\Gamma_s(T, p)$ ]

Typical values of buoyancy (Brunt–Väisälä) frequency  $N$ : most of stratosphere,  $\sim 2 \times 10^{-2} \text{ s}^{-1}$  (period 5 min); most of troposphere,  $\sim 1 \times 10^{-2} \text{ s}^{-1}$  (period 10 min); main ocean thermocline,  $\lesssim 0.2 \times 10^{-2} \text{ s}^{-1}$  (period  $\gtrsim 1$  hour)

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Atmos.–Ocean	Atmosphere–Ocean
JAS	Journal of the Atmospheric Sciences
JATP	Journal of Atmospheric and Solar–Terrestrial Physics
JFM	Journal of Fluid Mechanics
JGR	Journal of Geophysical Research
J. Met. Soc. Jap.	Journal of the Meteorological Society of Japan
JPO	Journal of Physical Oceanography
J. Oceanog. Soc. Jap.	Journal of the Oceanographical Society of Japan
Phil. Mag.	Philosophical Magazine
Phys. Fl.	The Physics of Fluids
Proc. Roy. Soc.	Proceedings of the Royal Society (of London)
Pure & Appl. Geophys. (PAGEOPH)	Pure and Applied Geophysics
QJRMS	Quarterly Journal of the Royal Meteorological Society

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