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On wave-action and its relatives

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Conservable quantities measuring 'wave activity' are discussed. The equation for the most fundamental such quantity, wave-action, is derived in a simple but very general form which does not depend on the approximations of slow amplitude modulation, linearization, or conservative motion. The derivation is *elementary*, in the sense that a variational formulation of the equations of fluid motion is not used. The result depends, however, on a description of the disturbance in terms of particle displacements rather than velocities. A corollary is an elementary but general derivation of the approximate form of the wave-action equation found by Bretherton & Garrett (1968) for slowly-varying, linear waves.

The sense in which the general wave-action equation follows from the classical 'energy-momentum-tensor' formalism is discussed, bringing in the concepts of pseudomomentum and pseudoenergy, which in turn are related to special cases such as Blokhintsev's conservation law in acoustics. Wave-action, pseudomomentum and pseudoenergy are the appropriate conservable measures of wave activity when 'waves' are defined respectively as departures from ensemble-, space- and time-averaged flows.

The relationship between the wave drag on a moving boundary and the fluxes of momentum and pseudomomentum is discussed.

1. Introduction

It is known that the law of conservation of wave-action can be derived (in a very general form not dependent on any approximations such as slow modulations, infinitesimal amplitude, etc.) by essentially the same mathematical procedure as the conservation law for the energy-momentum tensor $T_{\mu\nu}$ of classical theoretical physics. This idea can be traced back at least as far as Sturrock (1962), appears in a slightly different form in Whitham (1970), and has been systematically developed by Hayes (1970) and further clarified by Bretherton (1979). One considers an ensemble of disturbed-flow solutions labelled by a smoothly-varying parameter α, and the mean flow is defined by averaging over α. On replacing certain space and time differentiations occurring in the usual definition of $T_{\mu\nu}$ by differentiations with respect to α, and

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averaging, one immediately obtains a conservable wave property associated with the invariance of the mean flow to changes in the value of α (§ 5 below). By 'wave property' is meant an expression which may be evaluated to a consistent first approximation from linearized theory, and so on at higher orders in wave amplitude.

The conservable wave property thus obtained depends on the way in which 'disturbance' and 'mean flow' are defined. In this note we show that a wave property of remarkable analytical simplicity, to be denoted in what follows by A and called 'the wave-action', results from using the generalized Lagrangian-mean (GLM) description of waves on a mean flow given in the preceding paper (Andrews & McIntyre 1978*b*, hereafter denoted by IV). The fact that a simple yet exact result is obtained is related to the fact that the GLM description enables a suitable disturbance particle-displacement field $\boldsymbol{\xi}(\mathbf{x}, t)$, with zero mean, to be defined exactly. This provides the simplest way of expressing Hamilton's variational principle for the disturbance (§ 5 below and references), and hence of defining a disturbance-associated analogue of $T_{\mu\nu}$ exactly.

The simple analytical form of A facilitates an alternative, *elementary* derivation of its equation direct from the general equations of motion, without referring to any variational formulation; and this is done first, in § 2. Such a derivation is especially convenient when departures from conservative motion, such as the effects of viscosity and heat conduction or radiation, are to be allowed for, as is crucial to some applications. In § 3 we show generally that the flux of A across any undisturbed material boundary vanishes exactly. This is a desirable property when using the wave-action equation in problems involving reflexion of nonlinear waves from a boundary. The simplicity of these basic results is very appealing, and seems to support the view that A is, at least from a theoretical standpoint, the most fundamental measure of 'wave activity' for finite-amplitude disturbances on arbitrary mean flows.

It is important to know how A reduces to more familiar, approximate forms; of particular interest in practice is the useful formula

$$\mathsf{A} \doteq \hat{E}/\hat{\omega} \tag{1.1}$$

derived by Bretherton & Garrett (1968) for conservative, slowly-varying waves of infinitesimal amplitude, where $\hat{\omega}$ is intrinsic frequency and $\hat{E}$ is intrinsic wave-energy density ('intrinsic', that is, to the local wave dynamics, as seen in a frame of reference moving with the local mean flow). To derive (1.1) we follow Sturrock, Whitham and Hayes (*op. cit.*) and identify α with phase shift, as is permissible as an approximation for slowly-varying waves. Essential to our derivation of (1.1) is a 'virial theorem' for the disturbance, obtained by scalarly multiplying the equation of motion by $\boldsymbol{\xi}$. The resulting derivation of Bretherton & Garrett's formula (§ 4 below), although completely elementary and dependent only on the usual approximate definition of $\boldsymbol{\xi}$ appropriate to *linearized* wave theory, does not seem to have been given before. Our derivation shows moreover why a formula like (1.1) cannot generally be expected to hold at finite amplitude, the reason being that the term resembling potential energy in the virial theorem is displacement times restoring force, which equals twice potential energy (as required to obtain Bretherton & Garrett's formula) only in the case of a linear restoring force.

As a corollary of our analysis, it can be remarked that the conservation law of Blokhintsev (1945) for slowly-modulated acoustic waves may also be derived as a special case of the basic equation for A (§ 5.3). This casts fresh light on the physical interpretation of Blokhintsev's conserved wave property, which is usually thought of as an 'energy'. From a general viewpoint, *pseudoenergy* appears to be the more closely related concept. This will emerge from the discussion of the energy-momentum-tensor formalism and its relation to A, given in § 5. (The interesting relation between Blokhintsev's invariant and energy which was established by Cantrell & Hart (1964) is apparently one of those special relations, often encountered in classical wave theories, which depend crucially on the fluid motion being irrotational.)

Also in § 5 we note that in the GLM description the fluxes of momentum and pseudomomentum are closely related (although not identical). This seems to be one of several reasons why momentum and pseudomomentum have sometimes been confused with one another.

2. The general wave-action equation and its corollaries

We use the same notation as in IV and distinguish equation numbers from that paper by the prefix IV. Following Hayes (1970) and Bretherton (1979) we suppose that $\overline{(\;)}$ is an ensemble average and that each field $\varphi(\mathbf{x}, t; \alpha)$ depends differentiably upon the ensemble label α, so that

$$(\bar{\varphi})_{,\alpha} = (\overline{\varphi_{,\alpha}}) = 0, \tag{2.1}$$

where $(\;)_{,\alpha}$ stands for $\partial/\partial\alpha$. The label α may have any dimensionality (Hayes 1970, § 10) but in most applications may be taken to be a single, real variable. It is convenient to leave its range of variation arbitrary for the present.

In IV we showed that a finite-amplitude disturbance particle-displacement field $\boldsymbol{\xi}(\mathbf{x}, t)$ can be defined such that [IV (2.10*b*), (2.7)]

$$\bar{D}^{\mathrm{L}}\boldsymbol{\xi} = \mathbf{u}^{l}, \quad \bar{\boldsymbol{\xi}} = 0, \tag{2.2a, b}$$

where $\bar{D}^{\mathrm{L}}$ is the Lagrangian-mean material derivative $\partial/\partial t + \bar{\mathbf{u}}^{\mathrm{L}}.\nabla$, and $\bar{\mathbf{u}}^{\mathrm{L}}$ and $\mathbf{u}^{l}$ are respectively the mean and disturbance velocities as measured in the GLM description. The relations (2.2) are basic to our development. As in IV we introduce the notation, for any field $\varphi(\mathbf{x}, t)$,

$$\varphi^{\xi}(\mathbf{x}, t) \equiv \varphi\{\mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t), t\}; \tag{2.3}$$

then by definition

$$\bar{\varphi}^{\mathrm{L}} \equiv \overline{\varphi^{\xi}}, \quad \varphi^{l} \equiv \varphi^{\xi} - \bar{\varphi}^{\mathrm{L}}, \quad \overline{\varphi^{l}} = 0, \tag{2.4a, b, c}$$

so that $\mathbf{u}^{l} = \mathbf{u}^{\xi} - \bar{\mathbf{u}}^{\mathrm{L}}$ in (2.2*a*).

Let the equation of motion for the total flow be [as in IV (3.2)]

$$Du_i/Dt + 2(\mathbf{\Omega} \times \mathbf{u})_i + \Phi_{,i} + \rho^{-1}p_{,i} + X_i = 0. \tag{2.5}$$

For simplicity we restrict the gravitational potential Φ to be a function of $\mathbf{x}$ alone. We assume also that $\bar{\Phi} = \Phi$, thus excluding the possibility of a wave contribution to Φ, whether stationary or time-dependent; the self-gravitating case can be treated, as in IV, but involves considerable extra manipulation.

The wave-action equation is obtained by scalar multiplication of $(2.5)^{\xi}$ by $\boldsymbol{\xi}_{,\alpha}$. After manipulations, given in appendix A, which are quite like those familiar from the usual derivation of the kinetic energy equation from (2.5), the result takes the form

$$\bar{D}^{\mathrm{L}}\mathsf{A} + \tilde{\rho}^{-1}\nabla.\mathbf{B} = \mathscr{F}, \tag{2.6}$$

where $\mathscr{F}$, to be defined shortly, is zero for conservative motion. A, the wave-action per unit mass, is defined by

$$\mathsf{A} \equiv \overline{\boldsymbol{\xi}_{,\alpha} \cdot (\mathbf{u}^l + \boldsymbol{\Omega} \times \boldsymbol{\xi})} \tag{2.7a}$$

and $\mathbf{B}$, the non-advective flux of wave-action, by

$$\mathsf{B}_j \equiv \overline{p^{\xi} \xi_{i,\alpha} K_{ij}}, \tag{2.7b}$$

where K_{ij} is the (i,j)th cofactor of the Jacobian

$$J \equiv \det (\delta_{ij} + \xi_{i,j}).$$

K_{ij} satisfies

$$(\delta_{jk} + \xi_{j,k}) K_{ji} = J\delta_{ik} = (\delta_{kj} + \xi_{k,j}) K_{ij} \tag{2.8a, b}$$

and (IV, appendix A)

$$K_{ij,j} = 0, \tag{2.9}$$

$$K_{ij} = (1 + \xi_{m,m}) \delta_{ij} - \xi_{j,i} + k_{ij}, \tag{2.10}$$

where k_{ij} is the (i,j)th cofactor of $\xi_{i,j}$. The right-hand side of (2.6) is given by

$$\mathscr{F} \equiv -\overline{\xi_{i,\alpha} X_i^l} + \overline{(p^l)_{,\alpha} q}, \tag{2.11}$$

a wave property representing the rate of generation or dissipation of wave-action associated with departures from conservative motion, q in particular being the departure from adiabatic motion defined in IV (3.5). That is,

$$q = -1/F(S^{\xi}, p^{\xi}) + 1/F(\bar{S}^{\mathrm{L}}, p^{\xi}), \tag{2.12}$$

where

$$\rho = F(S,p)$$

is the equation of state for the fluid and S is entropy per unit mass; we recall that for adiabatic motion $S^{\xi} = \bar{S}^{\mathrm{L}}$ [IV (2.23)], so that q is indeed zero in that case. The density $\tilde{\rho}$ appearing in (2.6) is a mean quantity, satisfying

$$\tilde{\rho} \equiv \rho^{\xi} J = \bar{\tilde{\rho}} \tag{2.13a, b}$$

(IV § 4), and also satisfying the mean-flow mass conservation equation

$$\bar{D}^{\mathrm{L}}\tilde{\rho} + \tilde{\rho}\nabla \cdot \bar{\mathbf{u}}^{\mathrm{L}} = 0. \tag{2.14}$$

This last relation enables (2.6) to be written in the alternative form

$$\partial(\tilde{\rho}\mathsf{A})/\partial t + \nabla \cdot \mathbf{B}^{\mathrm{tot}} = \tilde{\rho}\mathscr{F} \tag{2.15}$$

involving the total flux of wave-action

$$\mathbf{B}^{\mathrm{tot}} = \bar{\mathbf{u}}^{\mathrm{L}}\tilde{\rho}\mathsf{A} + \mathbf{B}, \tag{2.16}$$

the first term of which represents advection of wave-action by the mean flow $\bar{\mathbf{u}}^{\mathrm{L}}$.

Since by (2.7) and (2.10), A and its flux are wave properties, and satisfy a conservation relation [(2.6) or (2.15)] whenever $\mathscr{F}$ is zero, A is an appropriate general measure of wave 'activity'. The generality is considerable. No special assumption is needed about the type of wave involved; indeed, no approximations whatever have been made.

In the case of small wave amplitude a, manipulations starting with (2.10) and similar to those which led to IV (8.11) show that

$$\mathsf{B}_j = \overline{p'\xi_{j,\alpha}} + (\overline{p}\overline{\xi_m\,\xi_{j,\alpha}})_{,m} + O(a^3), \tag{2.17}$$

where it should be noticed that p' is the *Eulerian* pressure disturbance, $p-\overline{p}$. Like the term N_{ij} in IV (8.11), the second term of (2.17) is identically non-divergent $[(\overline{p}\overline{\xi_m\,\xi_{j,\alpha}})_{,mj} = 0]$ because (2.1) shows that $\overline{\xi_m\,\xi_{j,\alpha}} = -\overline{\xi_j\,\xi_{m,\alpha}}$. If A and B are required correct to $O(a^2)$ only, there is no need to invoke the GLM description since $\boldsymbol{\xi}$ simply satisfies the usual linearized relations which comprise the leading approximations to (2.2) and (2.13):

$$(\partial/\partial t + \overline{\mathbf{u}}\,.\,\nabla)\boldsymbol{\xi} \doteq \mathbf{u}^l \doteq \mathbf{u}' + \boldsymbol{\xi}\,.\,\nabla\overline{\mathbf{u}}; \tag{2.18a}$$

$$\overline{\boldsymbol{\xi}} = 0, \quad \overline{\rho}\nabla\,.\,\boldsymbol{\xi} \doteq -\rho^l \doteq -\rho' - \boldsymbol{\xi}\,.\,\nabla\overline{\rho} \tag{2.18b,c}$$

[see IV (2.28), IV (A 8)]. As would therefore be expected, it is straightforward to rederive (2.6) correct to $O(a^2)$, again without invoking the GLM description, by multiplying the linearized equation of motion scalarly by $\boldsymbol{\xi}_{,\alpha}$ and using (2.18) and IV (2.28); in that case, however, the equation of motion for the *mean* flow to zeroth order in a must be used as well (McIntyre 1978). The flux term arises immediately in the form given by the first term of (2.17), since the pressure term in the linearized equation involves $\nabla p'$.

Evidently (2.6) and (2.15) have various corollaries when mean quantities are independent of a time or space co-ordinate, assuming that a suitable ergodic principle holds whereby ensemble averaging can be replaced by time or space averaging, as will often be the way in which (2.6) is applied in practice. Suppose for example that mean quantities are independent of x_1. For a given (deterministic) wave solution we may generate the ensemble envisaged in the general theory by simply translating the disturbance pattern through a distance α in the x_1 direction, for each value of α in the range $(-\infty, \infty)$ (Bretherton 1979). Then from (2.7a)

$$\mathsf{A} = \mathsf{p}_1 \equiv -\overline{\boldsymbol{\xi}_{,1}\,.\,(\mathbf{u}^l + \boldsymbol{\Omega}\times\boldsymbol{\xi})} \tag{2.19}$$

($\partial/\partial\alpha$ being replaced by $-\partial/\partial x_1$ and $\overline{(\quad)}$ now being an average with respect to x_1); it will be noticed that p_1 is just the 1-component of the *pseudomomentum* per unit mass defined in IV (3.1).

As pointed out by Peierls (1976) in another context, pseudomomentum in theoretical physics is the quantity whose conservation is associated with translational invariance of the mean flow, as opposed to translational invariance of the whole physical problem (which latter invariance gives conservation of *momentum*). This will be made more precise in § 5 below, but we can already see why the GLM description is the most natural formulation within which to express the pseudomomentum concept. Conservation of pseudomomentum, as distinct from momentum, is connected with invariance to a translation of the disturbance pattern *while mean particle positions are kept fixed*, as distinct from a displacement of the whole system, particles as well as disturbance pattern (Peierls, *op. cit.*). A general expression of the pseudomomentum concept therefore depends on an equally general expression of the idea of 'fixed mean particle positions'. This idea cannot be directly expressed within a purely field-theoretic or Eulerian description, which does not keep track of where fluid particles are. But it is precisely this idea that is expressed, without approximation, by the

GLM description, when we fix $\mathbf{x}$ but replace $\boldsymbol{\xi}(\mathbf{x}, t)$, in our example, by $\boldsymbol{\xi}(\mathbf{x}-\alpha\hat{\mathbf{x}}, t)$ where $\hat{\mathbf{x}}$ is the unit vector $(1, 0, 0)$.†

3. The flux of wave-action at an undisturbed boundary

We now show that any undisturbed boundary impermeable to the fluid is also impermeable to the flux of wave-action.

By 'undisturbed boundary' we do not of course mean one where $\boldsymbol{\xi}$ or $\boldsymbol{\xi}.\mathbf{n}$ vanishes, neither of which would be appropriate at a boundary which is reflecting finite-amplitude waves. An undisturbed boundary Σ will be defined, rather, as one which maps into itself under the mapping $\mathbf{x} \rightarrow \mathbf{x}+\boldsymbol{\xi}(\mathbf{x}, t)$; that is, the image Σ^{ξ} of Σ is the same surface as Σ. It follows that the shape of Σ is independent of the ensemble label α (expressing in a very general way the idea that the boundary has no 'undulations'). Now as α varies, the tip of the vector $\boldsymbol{\xi}(\mathbf{x}, t)$, whose tail is at a fixed point $\mathbf{x}$, moves along Σ; therefore

$$\boldsymbol{\xi}_{,\alpha}.\mathbf{n}^{\xi} = 0 \quad \text{on} \quad \Sigma, \tag{3.1}$$

where $\mathbf{n}^{\xi}$ is a vector normal to Σ at the point $\mathbf{x}+\boldsymbol{\xi}$. If $\mathbf{n}$ is normal to Σ at $\mathbf{x}$, we have

$$n_i^{\xi} \propto K_{ij} n_j \tag{3.2}$$

by IV (A 12) *et seq*. Thus (3.1) implies that

$$\overline{p^{\xi}\xi_{i,\alpha} K_{ij}} n_j = 0 \quad \text{on} \quad \Sigma, \tag{3.3}$$

which in virtue of (2.7*b*) is the required statement, namely that the non-advective flux of A across Σ vanishes, $\mathbf{B}.\mathbf{n} = 0$.

An important special case is that in which Σ is immobile as well as undisturbed; then by IV § 4.2 we have $\bar{\mathbf{u}}^{\mathrm{L}}.\mathbf{n} = 0$ on Σ so the total flux $\mathbf{B}^{\mathrm{tot}}$ of A given by (2.16) has zero normal component at each point $\mathbf{x}$ of Σ. If on the other hand Σ is moving, so that $\bar{\mathbf{u}}^{\mathrm{L}}.\mathbf{n} \neq 0$, (3.3) and (2.16) imply that $\mathbf{B}^{\mathrm{tot}}.\mathbf{n} = \tilde{\rho}\mathsf{A}\bar{\mathbf{u}}^{\mathrm{L}}.\mathbf{n}$, which is simply an alternative way of stating the fact that the moving, undisturbed boundary is impermeable to wave-action.

Finally, it is noted that $\boldsymbol{\xi}.\mathbf{n} = \boldsymbol{\xi}_{,\alpha}.\mathbf{n} = O(a^2)$ on Σ for small wave amplitude a, as is evident intuitively, or from (3.2) and (2.10). Thus the first term of the linear approximation (2.17) to $\mathbf{B}$ vanishes on Σ, to leading order, as well as the full expression for $\mathbf{B}$ itself.

† Similarly, the conservable wave property derived from A when the mean flow is invariant under rotation (see Bretherton 1979; Andrews & McIntyre 1978*a*, i.e. paper III of the present series) may be called the *angular pseudomomentum*. A co-ordinate-independent expression for it may straightforwardly be written down using the tensor definition of the azimuthal averaging operator given in the footnote to § 2.1 of IV, identifying α with the angle λ appearing there in the rotation tensor and noting that $\varphi_{,\alpha}|_{\alpha=0} = \epsilon_{kmn} z_n x_m \varphi_{,k}$ for any scalar field φ, and

$$\varphi_{i,\alpha}|_{\alpha=0} = \epsilon_{kmn} z_n x_m \varphi_{i,k} - \epsilon_{ipn} z_n \varphi_p$$

for any vector field φ_i, where the rotation axis is taken through the origin and parallel to the unit vector $\mathbf{z}$. Use of these relations for p^l_{α} and $\boldsymbol{\xi}_{,\alpha}$ in (2.6) immediately gives the exact equation for angular pseudomomentum. (The $O(a^2)$ approximation to this, analogous to (2.6) after substitution of (2.17), is given explicitly in Bretherton (1979) and also in our paper III [see III (A 19)]. In those references polar co-ordinates were used, a device which simplifies the expression for angular pseudomomentum because $\partial/\partial\alpha$ becomes just partial – not covariant – differentiation with respect to the azimuthal angle $-\lambda$, and vector fields may be averaged by naïvely averaging their components.)

4. The virial theorem, and Bretherton & Garrett's equation

If $(2.2a)$ is substituted into the identity

$$\xi_i \bar{D}^L u_i^l = \bar{D}^L(\xi_i u_i^l) - u_i^l \bar{D}^L \xi_i,$$

there results

$$\xi_i \bar{D}^L u_i^l = \tfrac{1}{2}(\bar{D}^L)^2 (\xi_i \xi_i) - u_i^l u_i^l. \tag{4.1}$$

Therefore scalarly multiplying $(2.5)^\xi$ by $\boldsymbol{\xi}$ and averaging, using the fact that

$$(D\mathbf{u}/Dt)^\xi = \bar{D}^L(\mathbf{u}^\xi),$$

and noting $(2.2b)$, $(2.4b, c)$, (2.13) and $(A\,7)$, gives

$$\overline{|\mathbf{u}^l|^2} + 2\overline{\mathbf{u}^l.\boldsymbol{\Omega}\times\boldsymbol{\xi}} - \tilde{\rho}^{-1}\overline{\xi_i K_{ij}(p^\xi)_{,j}} - \overline{\xi_i(\Phi_{,i})^l} = \tfrac{1}{2}(\bar{D}^L)^2\,\overline{|\boldsymbol{\xi}|^2} + \overline{\boldsymbol{\xi}.\mathbf{X}^l}. \tag{4.2}$$

Following Eckart (1963, and references) we call this a 'virial theorem' for the disturbance, by analogy with the corresponding result in classical particle dynamics.

Now for periodic, plane, conservative waves of infinitesimal amplitude a, on a uniform, steady basic flow given by

$$\bar{\mathbf{u}}^L = \bar{\mathbf{u}} + O(a^2) = \text{constant} + O(a^2), \tag{4.3}$$

we may write

$$\boldsymbol{\xi} \propto \exp i(\mathbf{k}.\mathbf{x} - \omega t - \alpha), \tag{4.4}$$

where the real part is understood and where $\mathbf{k}$ and ω are constants. By letting the phase α vary over the range $(0, 2\pi)$, following Sturrock (1962), Hayes (1970) and Whitham (1970), we may generate an ensemble of wave solutions to which the result (2.6) may be applied. In this simple case we have, correct to $O(a)$,

$$\boldsymbol{\xi}_{,\alpha} = \omega^{-1}\boldsymbol{\xi}_{,t} = -k_i^{-1}\boldsymbol{\xi}_{,i} = \hat{\omega}^{-1}\bar{D}^L\boldsymbol{\xi} \tag{4.5a, b, c}$$

(no summation over i), where $\hat{\omega}$ is the intrinsic wave frequency, defined as

$$\hat{\omega} \equiv \omega - \mathbf{k}.\bar{\mathbf{u}}. \tag{4.6}$$

Still working correct to $O(a)$, we see from (4.5) and $(2.18a)$ that, since $\nabla\bar{\mathbf{u}} = O(a^2)$,

$$\mathbf{u}' = \mathbf{u}^l = \bar{D}^L\boldsymbol{\xi} = \hat{\omega}\boldsymbol{\xi}_{,\alpha}. \tag{4.7}$$

So the leading approximation to $(2.7a)$ is

$$\mathsf{A} = \hat{\omega}^{-1}\overline{\mathbf{u}'.(\mathbf{u}' + \boldsymbol{\Omega}\times\boldsymbol{\xi})} \tag{4.8}$$

and the leading approximation to the first term of (2.17), $\hat{\mathsf{B}}_j$ say, is

$$\hat{\mathsf{B}}_j = \hat{\omega}^{-1}\overline{p'u_j'}. \tag{4.9}$$

These formulae will usually remain true as leading approximations for slowly-varying, *almost*-plane waves on slightly unsteady, slightly non-uniform mean flows (the main exception to this statement being Rossby waves, as noted below). In the same approximation, the virial theorem (4.2) reduces to

$$\overline{|\mathbf{u}'|^2} + 2\overline{\mathbf{u}'.\boldsymbol{\Omega}\times\boldsymbol{\xi}} \doteq \tilde{\rho}^{-1}\overline{\xi_i K_{ij}(p^\xi)_{,j}}, \tag{4.10}$$

noting that $(\Phi_{,i})^l = \xi_j \Phi_{,ij} + O(a^2)$, since $\Phi' = 0$, and that $\Phi_{,ij}$ is negligible for a slowly-varying gravitational acceleration. So (4.8) may be rewritten

$$\mathsf{A} \doteq \hat{\omega}^{-1}\{\tfrac{1}{2}\overline{|\mathbf{u}'|^2} + \tfrac{1}{2}\tilde{\rho}^{-1}\overline{\xi_i K_{ij}(p^{\xi})_{,j}}\}. \tag{4.11}$$

Now it may be shown (appendix B) that, in the circumstances assumed, the second term within braces equals the 'acoustic' (compressibility) energy plus the available potential energy of the wave motion, per unit mass (so (4.10) generalizes the classical equipartition-of-energy theorem). Thus we recognize the expression within braces as $\tilde{\rho}^{-1}$ times the density of wave-energy $\hat{E}$, correct to $O(a^2)$, as defined by Bretherton & Garrett (1968). This is the result (1.1).

Provided that $\hat{\mathbf{B}}$ can to leading order be consistently evaluated as for a plane wave in a homogeneous medium, we have

$$\hat{\mathbf{B}} \doteq \hat{\mathbf{c}}_g \tilde{\rho} \mathsf{A} \tag{4.12}$$

where $\hat{\mathbf{c}}_g$ is the intrinsic group velocity $\nabla_{\mathbf{k}} \hat{\omega}$. An elementary proof (based essentially on a generalization of Stokes' classical argument) is given by Hayes (1977).† Therefore the generalized wave-action equation (2.6) reduces, in the conservative case $\mathscr{F} = 0$, to

$$\bar{D}^L(\hat{E}/\tilde{\rho}\hat{\omega}) + \tilde{\rho}^{-1}\nabla . (\hat{\mathbf{c}}_g \hat{E}/\hat{\omega}) = 0. \tag{4.13}$$

In virtue of (2.14) (in which we may here consistently approximate $\tilde{\rho}$ by $\bar{\rho}$ and $\bar{\mathbf{u}}^L$ by $\bar{\mathbf{u}}$) we may rewrite (4.13) as

$$\frac{\partial}{\partial t}\left(\frac{\hat{E}}{\hat{\omega}}\right) + \nabla . \left(\mathbf{c}_g \frac{\hat{E}}{\hat{\omega}}\right) = 0, \tag{4.14}$$

where the absolute group velocity

$$\mathbf{c}_g = \hat{\mathbf{c}}_g + \bar{\mathbf{u}}. \tag{4.15}$$

Equation (4.14) is equivalent to Bretherton & Garrett's approximate form of the wave-action conservation law, and it justifies our use of the term 'wave-action' for the *exactly* conservable wave property A. Again we emphasize that the derivation of (4.14) is general, even though no variational formulation has been invoked. The generality stems from the description of the disturbance in terms of particle displacements.

It can easily be shown that essentially the same derivation applies to cases like classical 'water waves' involving waveguide structure. The only additional consideration is that the divergence term then arising in the virial theorem (the first term on the right of (B 2), which is negligible for almost-plane waves) integrates to zero when $\hat{E}$ is defined in the usual way by integrating the three-dimensional wave-energy density across the waveguide. We assume that either $\boldsymbol{\xi} . \mathbf{n} = 0$ or $p' = 0$ at the waveguide boundary.

It is noteworthy that the restriction to $O(a^2)$ is essential to the relation (1.1) between A and wave-energy in Bretherton & Garrett's sense. The pressure term in the virial

† The circumstances assumed in Hayes' proof and in (4.11) hold for most types of wave, the main exception being Rossby waves on a beta-plane, for which homogeneity of the medium and thus Hayes' argument are vitiated by the effectively strong spatial variation of the Coriolis parameter (Longuet-Higgins 1964). However it may be shown, following Longuet-Higgins (*op. cit.*), that even for Rossby waves we have $\nabla . \hat{\mathbf{B}} = \nabla . (\hat{\mathbf{c}}_g \tilde{\rho} \mathsf{A})$. An explanation for this at a deeper level is given by Bretherton & Garrett [1968, equations (3.12), (3.13)].

theorem does not generally represent potential energy when $\boldsymbol{\xi}$ is finite; this is clear by analogy with the fact that potential energy for a particle on a nonlinear spring is not half displacement times restoring force. [The main exception is the case of almost-plane waves in an incompressible, uniformly-stratified fluid, for which the dynamics is approximately linear even at amplitudes of order unity because $\mathbf{u}.\nabla$ is small (e.g. Drazin 1969; Grimshaw 1975).]

5. Pseudomomentum and pseudoenergy

5.1. A *and the energy-momentum-tensor formalism*

For background see, for example, Landau & Lifshitz (1975) and Sturrock (1962). An extensive bibliography is given in Jones (1971). Let $L(\varphi, \varphi_{,\mu})$ be a Lagrangian density depending on a set of fields $\varphi(\mathbf{x}, t)$ and their derivatives $\varphi_{,\mu}$, where μ stands for x_i $(i = 1, 2, 3)$ or t. If the fields describe a conservative physical system governed by a variational principle

$$\delta \int L \, d\mathbf{x} \, dt = 0, \tag{5.1}$$

for variations $\delta\varphi$ which vanish outside some finite hypervolume in space-time, then the canonical energy-momentum tensor (Landau & Lifshitz, *op. cit.*, § 32) is the four-dimensional tensor

$$T_{\mu\nu} = \varphi_{,\mu} \, \partial L / \partial \varphi_{,\nu} - L\delta_{\mu\nu}, \tag{5.2}$$

where summation over all the relevant fields is understood. Using the Euler–Lagrange equations for (5.1) and the chain rule for differentiation (Landau & Lifshitz, *loc. cit.*) it is readily shown that $T_{\mu\nu}$ satisfies a conservation relation

$$T_{\mu\nu,\nu} = 0 \tag{5.3}$$

for each μ such that L has no explicit dependence on the corresponding space or time co-ordinate. (The precise physical interpretation of the components of $T_{\mu\nu}$ will depend *inter alia* on the choice of variational principle.)

Now as Bretherton (1979) points out, the generalized Lagrangian-mean description enables us to treat the particle displacements $\xi_i(\mathbf{x}, t)$ formally as fields, and to use Hamilton's principle as the variational principle (5.1), taking

$$L = L(\xi_i, \xi_{i,j}, \xi_{i,t}; \text{mean fields}) \tag{5.4}$$

$$= \tilde{\rho}\{\tfrac{1}{2}(\bar{\mathbf{u}}^{\mathrm{L}} + \bar{D}^{\mathrm{L}}\boldsymbol{\xi})^2 + \boldsymbol{\Omega} \times (\mathbf{x} + \boldsymbol{\xi}) . (\bar{\mathbf{u}}^{\mathrm{L}} + \bar{D}^{\mathrm{L}}\boldsymbol{\xi}) - \Phi(\mathbf{x} + \boldsymbol{\xi}) - e(\tilde{\rho}/J, \bar{S}^{\mathrm{L}})\}. \tag{5.5}$$

Here $e(\rho, S)$ is the internal energy per unit mass expressed as a function of ρ and S; it should be noted that the last term in (5.5) is just $e(\rho^{\xi}, S^{\xi})$; this follows from (2.13*a*), together with the fact that $\bar{S}^{\mathrm{L}} = S^{\xi}$ for conservative motion [IV (2.23)]. The permitted variations of $\xi_i(\mathbf{x}, t)$ are unrestricted except as above, that is, they must vanish outside a finite hypervolume; but variations of the mean fields $\bar{\mathbf{u}}^{\mathrm{L}}$, $\tilde{\rho}$, etc., are subject to the restrictions noted by Newcomb (1962), Bretherton (*op. cit.*) and Dewar (1970), expressing conservation of mass and entropy together with the 'Lin constraint' arising from the underlying pure-Lagrangian description of the *mean* flow (see also Penfield 1966). This is the scheme given by Bretherton (1971), with the important difference that neither the slow-modulation nor the small-amplitude approximation is needed here; see also Bretherton (1979).

We now seek a mathematical analogue of $T_{\mu\nu}$ for the *disturbance problem*, that is to say the problem in which we temporarily regard the mean fields, including $\tilde{\rho}$, as given, and vary only $\xi_i(\mathbf{x},t)$. Obviously (5.1) must still be true under any subclass of allowed variations, including this one. Also, (5.1) is still true (when only $\boldsymbol{\xi}(x,t)$ is varied) if we replace L by $L-L_0$, where L_0 is the 'undisturbed' Lagrangian, given by

$$L_0 = L(0,0,0;\text{ mean fields}) \tag{5.6}$$

in the notation of (5.4). Note that $\overline{L-L_0}$ is an $O(a^2)$ wave property. So if we define

$$\mathsf{T}_{\mu\nu} = \overline{\xi_{m,\mu}\,\partial L/\partial \xi_{m,\nu}} - \overline{(L-L_0)}\,\delta_{\mu\nu}, \tag{5.7}$$

also an $O(a^2)$ wave property, we have as a corollary of (5.3) that

$$\mathsf{T}_{\mu\nu,\nu} = 0 \tag{5.8}$$

for each μ such that the *mean flow* has no explicit dependence on the μth space-time co-ordinate. Since translational or temporal invariance of the mean flow, as opposed to such invariance of the complete physical system, is involved, $\mathsf{T}_{\mu\nu}$ is not physically an energy-momentum tensor, but instead involves the pseudoenergy, the pseudomomentum, and their fluxes (Peierls 1976).

It is easy to check from (5.5), (5.7) and (2.19) that in fact we have

$$-\mathsf{T}_{it} = \tilde{\rho}\mathsf{p}_i \quad (i = 1,2,3). \tag{5.9a}$$

Recall from the remarks following IV (3.1) that the sign was chosen for conformity with past convention, and also with the sign convention inherent in (5.2); it makes the direction of $\mathbf{p}$ agree with the direction in which wave crests are moving, so that the analogue of (1.1) is

$$\mathbf{p} \doteq +\hat{E}\mathbf{k}/\hat{\omega}. \tag{5.10}$$

Similarly we have that the flux of pseudomomentum (with the same sign convention) is

$$-\mathsf{T}_{ij} = \bar{u}_j^{\mathrm{L}}\tilde{\rho}\mathsf{p}_i - \overline{p^{\xi}\xi_{m,i}K_{mj}} + \overline{(L-L_0)}\,\delta_{ij}, \tag{5.9b}$$

where in the last term of (5.5) we use the facts that $\partial J/\partial\xi_{m,j} = K_{mj}$, that $\tilde{\rho}$ and $\bar{S}^{\mathrm{L}}$ are mean fields, which are not being varied, and that $\partial\epsilon(\rho,S)/\partial\rho = p/\rho^2$.

Moreover, if we now make a slight extension of the definition (5.7) of $\mathsf{T}_{\mu\nu}$ to include the case $\mu=\alpha$ [Sturrock 1962, equation (4.7); Hayes 1970, equation (11); Whitham 1970, equation (60)] we obtain

$$\mathsf{T}_{\alpha t} = \tilde{\rho}\mathsf{A}, \quad \mathsf{T}_{\alpha j} = \bar{u}_j^{\mathrm{L}}\tilde{\rho}\mathsf{A} + \mathsf{B}_j = \mathsf{B}_j^{\mathrm{tot}}; \tag{5.11a,b}$$

and A is *always* conserved, i.e.

$$\mathsf{T}_{\alpha\nu,\nu} = 0, \tag{5.12}$$

simply because (2.1) insists that mean quantities are always independent of α, and because extra terms of the form $\mathsf{T}_{\alpha\alpha,\alpha} = -\overline{(L-L_0)}_{,\alpha}$ arising from the extra dimensions of the extended space $(\mathbf{x},t;\alpha)$ are likewise identically zero. We have thus rederived (2.6) in the conservative case $\mathscr{F}=0$, and verified Hayes' (1970) result that the *mathematical* structure leading to wave-action conservation is essentially that of the energy-momentum-tensor formalism.

The temporal analogue of (5.9*a*), namely

$$\mathsf{T}_{tt} = \tilde{\rho}\mathsf{e} - \overline{(L-L_0)}, \tag{5.13a}$$

where [cf. (2.19)]

$$\mathsf{e} = \overline{\boldsymbol{\xi}_{,t}\cdot(\mathbf{u}^l+\boldsymbol{\Omega}\times\boldsymbol{\xi})}, \tag{5.14}$$

may be appropriately called the *pseudoenergy*. Its flux is

$$\mathsf{T}_{tj} = \overline{u}_j^{\mathrm{L}}\tilde{\rho}\mathsf{e} + \overline{p^{\xi}\xi_{m,t}K_{mj}}, \tag{5.13b}$$

and it is conserved when the mean flow is *steady*.

It should be noted from (5.13*b*) that the $\overline{L-L_0}$ term in (5.13*a*) is not advected by the mean flow. Hence e may be called the 'advected part' of the pseudoenergy; and it is the only part that matters if *all* mean quantities are time-invariant. In the exact theory this stipulation concerning all mean quantities is, however, almost the same as stipulating merely that the mean flow is time-invariant, i.e. steady, since the equations for the mean flow (IV §§ 3, 5, 8) generally forbid it to be exactly steady while mean wave properties such as e are not.

However, when relations correct to $O(a^2)$ only are of interest, it is consistent to neglect $O(a^2)$ mean-flow changes in the invariance argument leading to (5.8), and so conservation of pseudoenergy holds to $O(a^2)$ even when the waves are not steady, provided the $O(1)$ mean flow is steady. In that case the $\overline{L-L_0}$ term in (5.13*a*) can be significant, as can the $\overline{L-L_0}$ term in (5.9*b*) when the invariance in question is spatial rather than temporal.

A useful approximate formula for $\overline{L-L_0}$, showing that it is, however, often negligible to $O(a^2)$ in problems of slowly-varying waves, may be derived with the help of the virial theorem (4.2), after manipulations similar to those in appendix B:

$$\overline{L-L_0} = \tfrac{1}{4}\overline{\rho}\,\overline{(\overline{D}^{\mathrm{L}})^2|\boldsymbol{\xi}|^2} + \tfrac{1}{2}\frac{\partial}{\partial x_j}[\overline{p'\xi_j} + \overline{p}_{,k}\overline{\xi_j\xi_k} + \overline{p}\,\overline{(\xi_j\xi_{k,k}-\xi_{j,k}\xi_k)}] + O(a^3) \tag{5.15}$$

(assuming conservative motion, as elsewhere in this section).

5.2. *Relation between the fluxes of pseudomomentum and momentum*

In IV § 8.1 it was shown that the wave-induced excess momentum flux in the GLM description (the wave property analogous to minus the Reynolds stress in the Eulerian-mean description) is given by

$$-\overline{p^{\xi}\xi_{m,i}K_{mj}} + \overline{p^{\xi}(J-1)}\,\delta_{ij}, \tag{5.16}$$

where J is the Jacobian defined in § 2. The relation between the expression (5.16) and the radiation stress, in cases where the latter concept is definable, is discussed in IV § 8.4. Comparing (5.16) with the expression

$$-\overline{p^{\xi}\xi_{m,i}K_{mj}} + \overline{(L-L_0)}\,\delta_{ij} \tag{5.17}$$

for the non-advective flux of pseudomomentum deducible from (5.9*b*), we note the interesting fact that they are equal for $i \neq j$; that is, the *off-diagonal* parts of (5.16) and (5.17) are equal. An analogous relation exists between the flux of pseudoenergy and the wave-induced excess flux of total energy. These facts lead to alternative derivations of some of the finite-amplitude results on mean-flow evolution given in

IV § 3. For example IV (3.9), i.e. corollary I of theorem I, follows immediately upon comparing the Lagrangian-mean equation of motion IV (8.7a) with the equation for p_i. The latter is (5.8) with $\mu = i$, the $\overline{L - L_0}$ term in (5.9b) being irrelevant since corollary I assumes that all mean quantities are independent of x_i.

The equality of the off-diagonal parts of (5.16) and (5.17) explains moreover why a moving boundary, generating waves in an inviscid fluid, supplies to the fluid a component of pseudomomentum *transverse* to that boundary at a rate just equal to the rate of supply of the corresponding component of momentum. This seems to be one of several reasons why momentum and pseudomomentum have sometimes been mistaken for one another (a misconception traceable back to the time of Rayleigh and Poynting). An analysis of what actually happens in a simple example where internal gravity waves are generated by a moving boundary (McIntyre 1973, § 3) is illuminating in this connexion, since it illustrates the fact that the spatial location of the waves, and therefore of the pseudomomentum may be quite different from the spatial location either of the mean momentum or of the fluid impulse when the latter concept is applicable. The examples given by Gordon (1973), Robinson (1975) and Peierls (1976) illustrate the same point in a different physical context, as do the acoustic examples of Brillouin (1936).

Another misconception sometimes encountered is an expectation that the δ_{ij} term in the acoustic radiation stress (say) should equal minus that in the pseudomomentum flux (5.9b). It is true that by definition the radiation stress differs from the negative of (5.16) in its δ_{ij} term (for reasons which become clear as soon as the complete set of equations governing mean-flow evolution is considered; see IV § 8.4). But the coefficient of the δ_{ij} term in the radiation stress does not, in fact, bear any analytical resemblance to that in the pseudomomentum flux either. This can be seen at once from the differing ways in which they depend on the equation of state of the fluid to leading order. The δ_{ij} term in the acoustic radiation stress is proportional to the variation $\partial c/\partial \rho$ of sound speed with density (see, for example, IV § 8.4), while the corresponding term in the pseudomomentum flux involves $\overline{L - L_0}$, and hence an elastic energy term, proportional to c rather than to $\partial c/\partial \rho$. Moreover $\overline{L - L_0}$ itself is negligible for almost-plane waves, by (5.15), while $\partial c/\partial \rho$ is not. Some further discussion is given in McIntyre (1977, § 5).

5.3. *Relationship with standard $O(a^2)$ results in acoustics*

A well-known result in acoustics is the conservation relation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\bar{\rho}\overline{\mathbf{u}'^2} + \tfrac{1}{2}\bar{c}^2\frac{\overline{\rho'^2}}{\bar{\rho}} + \overline{\rho'\mathbf{u}'}\,.\,\bar{\mathbf{u}}\right) + \nabla\,.\,\overline{\left\{\left(\frac{p'}{\bar{\rho}} + \bar{\mathbf{u}}\,.\,\mathbf{u}'\right)(\bar{\rho}\mathbf{u}' + \rho'\bar{\mathbf{u}})\right\}} = 0, \tag{5.18}$$

where $\bar{c}$ is the speed of sound. Equation (5.18) was shown by Blokhintsev (1945) to hold correct to $O(a^2)$ in the geometric-acoustics (slow modulation) approximation for conservative waves on a steady mean flow. By using information about the structure of plane sound waves, Blokhintsev also showed that the density and flux appearing in (5.18) can be written respectively as

$$\hat{E}\omega/\hat{\omega}, \quad \mathbf{c}_g\hat{E}\omega/\hat{\omega}. \tag{5.19a,b}$$

This can, of course, be viewed as a corollary to Bretherton & Garrett's law (4.14), in virtue of the well-known ray-tracing equation $(\partial/\partial t + \mathbf{c}_g\,.\,\nabla)\,\omega = 0$, which holds for a steady mean state. Alternatively, (5.19) may be derived from (5.8) and (5.13), i.e. direct from conservation of pseudoenergy, using (4.5*a*) instead of (4.5*c*) as in the derivation of Bretherton & Garrett's law. The term $\overline{L-L_0}$ in (5.13*a*) is negligible, by (5.15) and the geometric-acoustics approximation [although interestingly enough it turns out that if $\overline{L-L_0}$ is retained in the calculation there is no need to appeal to the virial theorem (4.2); cf. Hayes 1970, equation (27)]. This identifies Blokhintsev's invariant as the pseudoenergy, to within the geometric-acoustics approximation; and the negligibility of $\overline{L-L_0}$ shows why T_{tt} equals its advected part $\tilde{\rho}\mathsf{e}$ to within the same approximation.

Now Cantrell & Hart (1964) showed for the special case of irrotational, homentropic motion (and steady mean flow) that Blokhintsev's conservation relation (5.18) can be derived in that case from conservation of total *energy* together with conservation of total mass and momentum. To this limited extent, then, energy and pseudoenergy are related for irrotational, homentropic motion, at least for slowly-modulated waves. The relationship is presumably a consequence, *inter alia*, of the exact relationship between the densities of pseudomomentum and momentum implied by the result

$$\nabla \times (\bar{\mathbf{u}}^{\mathrm{L}} - \mathbf{p}) = 0 \tag{5.20}$$

established in IV for the case of irrotational motion. Such special results should not distract attention, however, from the fact that in general it is only the *fluxes*, and not the densities, in $\mathsf{T}_{\mu\nu}$ and $T_{\mu\nu}$ which are closely related, as remarked in § 5.2.

Cantrell & Hart's lucid analysis revealed another interesting special result, again for irrotational, homentropic motion only, namely that (5.18) holds without the geometric-acoustics approximation in that case. Indeed it holds with the averaging operators removed; so Cantrell & Hart's result appears to be different in nature from the general results of the present paper, which do depend on averaging, and in particular upon the basic relation $\bar{\boldsymbol{\xi}} = 0$. It should nevertheless be asked whether (5.18) still corresponds to conservation of pseudoenergy, since steady mean flow is one of the assumptions essential to Cantrell & Hart's analysis. The answer appears, however, to be 'no', even if the virial theorem is invoked. We have succeeded in showing that the average of Cantrell & Hart's density, i.e. the first bracket in (5.18), is equal to the pseudoenergy density T_{tt} (including the $\overline{L-L_0}$ contribution) plus an expression of the form $\nabla . \mathbf{F}$, where $\mathbf{F}$ is a lengthy expression representing an $O(a^2)$ wave property. But it appears from our calculations that the difference between the *flux* (5.13*b*) of pseudoenergy and the expression in braces in (5.18) is not reducible, without appealing explicitly to the equation of motion, to the form $\mathbf{F}_{,t}$ plus an identically nondivergent contribution, as would have to be the case in order for (5.18) to remain directly equivalent to pseudoenergy conservation when the geometric-acoustics approximation fails.

6. Concluding remarks

The generality of the basic wave-action equation (2.6) and its corollaries is again emphasized. It may be found useful in areas where the approximations commonly made in wave theory are invalid: for example as an aid to the theoretical study of nonlinear instability of time-dependent mean flows (Davis 1976). Two other examples of possible areas of application that come to mind are planetary waves in the stratosphere, and the scattering theory of acoustic modes in ducts in the presence of a shear flow. Meteorologists studying stratospheric planetary waves are interested in such questions as the heights and latitudes to which the waves propagate or at which they are dissipated; and diagrams representing the flux and density of a conservable wave property such as A (or its relative appearing in the 'generalized Eliassen–Palm relation' (3.12*a*) of Andrews & McIntyre 1978*a*) would help clarify such questions. In acoustics, results for the scattering of duct modes incident on a constriction or some other obstacle are most compactly presented if the modes are normalized in terms of the flux of a suitable conservable wave property, as has been pointed out by Möhring (1977); such a representation of the results also provides a check on their correctness. The wave-action (or more precisely its analogue when ensemble averaging is replaced by time averaging, namely the pseudoenergy) appears to be the only suitable wave property for this purpose when the mean flow is either rotational or heterentropic.

In these applications as well as generally, it will be desirable to have an answer to the following question: to what extent are we entitled to regard A as a uniquely defined entity? Apart from the trivial non-uniqueness stemming from a rescaling of α, the answer depends on whether $\boldsymbol{\xi}$ is uniquely defined. This in turn depends on consideration of a hypothetical initial state of no disturbance, starting from which it is kinematically possible to set up the disturbed motion. (Such a state of no disturbance provides an initial condition, on the basis of which $\boldsymbol{\xi}(\mathbf{x}, t)$ can be computed in principle by integration along mean trajectories; see IV § 2.2.) Now in the case of a stratified fluid, the surfaces of constant density, entropy, potential vorticity and fluid composition must be taken to be undisturbed in the initial state (IV, postulate (**viii**); see also IV (2.23), (4.6) *et seq.*). For *conservative* motion, under at least some circumstances (e.g. those discussed in IV, appendix C), this does provide enough Lagrangian information to ensure a unique instantaneous correspondence between $\boldsymbol{\xi}(\mathbf{x}, t)$ and the Eulerian disturbance fields. The same appears to hold for two-dimensional, homentropic shear flow with a non-vanishing mean vorticity gradient, in which vorticity divided by density carries Lagrangian information. The circumstances assumed in IV, appendix C, are not the most general possible, however, and the point needs further investigation (W. Möhring, personal communication).

If the motion is not conservative, then $\boldsymbol{\xi}$ is not uniquely related to the fields of entropy, potential vorticity, etc., since that relation clearly depends on the history of the diabatic heating pattern associated with the disturbance, and on other departures from conservative motion. The implied non-uniqueness of $\boldsymbol{\xi}$, and therefore of A and its relatives, may in some cases be a necessary price for the great simplification and unification of the theory of nonlinear waves on mean flows which the GLM theory has provided. In other cases there appear to be useful ways of modifying the theory to eliminate the non-uniqueness problem, and these are currently being investigated.

We thank F. P. Bretherton for pointing out the connexion between his and our results and those of Hayes (1970), which in turn led to an appreciation of their intimate connexion with various general concepts in classical theoretical physics. T. Matsuno independently and perceptively suggested that a connexion be sought between the 'generalized Eliassen–Palm relations' derived in our earlier papers, and some generalization of the wave-action concept; and Sir Rudolf Peierls educated us on the closely-related concept of pseudomomentum (and the importance of distinguishing it from momentum, a point appreciated some time ago in solid-state physics). W. Möhring stimulated us to think about possible applications in acoustics, a matter still under investigation. The final version of this paper was written while DGA was supported by National Science Foundation grant NSF-g-76-20070 ATM, and while both authors enjoyed the hospitality of the Advanced Study Program at the National Center for Atmospheric Research.

Appendix A. Derivation of the wave-action equation

Since

$$\overline{\varphi_{,\alpha}} = 0, \tag{A 1}$$

we have

$$\overline{\varphi_{,\alpha}\psi} = -\overline{\varphi\psi_{,\alpha}} \tag{A 2}$$

for any φ and ψ. Also, for any mean field $\bar{\varphi}$, the chain rule gives

$$\{(\bar{\varphi})^\xi\}_{,\alpha} = (\bar{\varphi}_{,i})^\xi\,(x_i+\xi_i)_{,\alpha} = (\bar{\varphi}_{,i})^\xi\xi_{i,\alpha},$$

so that

$$\overline{(\bar{\varphi}_{,i})^\xi\,\xi_{i,\alpha}} = 0. \tag{A 3}$$

We multiply the first term of $(2.5)^\xi$ by $\xi_{i,\alpha}$, average, and use the facts that $\mathbf{u}^\xi = \bar{\mathbf{u}}^{\mathrm{L}} + \mathbf{u}^l$, $\overline{\mathbf{u}^l} = 0$, and $(D\mathbf{u}/Dt)^\xi = \bar{D}^{\mathrm{L}}(\mathbf{u}^\xi)$ [IV (2.14)]. This gives

$$\begin{aligned}\overline{\xi_{i,\alpha}\,\bar{D}^{\mathrm{L}}u_i^\xi} &= \overline{\xi_{i,\alpha}\,\bar{D}^{\mathrm{L}}u_i^l}\\ &= \bar{D}^{\mathrm{L}}\overline{(\xi_{i,\alpha}\,u_i^l)} - \overline{u_i^l\,\bar{D}^{\mathrm{L}}\xi_{i,\alpha}}\\ &= \bar{D}^{\mathrm{L}}\overline{(\xi_{i,\alpha}\,u_i^l)},\end{aligned} \tag{A 4}$$

since the second term in the penultimate line vanishes by virtue of (A 1) and the relation $u_i^l = \bar{D}^{\mathrm{L}}\xi_i$. The Coriolis term gives

$$\begin{aligned}2\Omega_j\,\epsilon_{ijk}\,\overline{\xi_{i,\alpha}\,u_k^l} &= 2\Omega_j\,\epsilon_{ijk}\,\overline{\xi_{i,\alpha}\,\bar{D}^{\mathrm{L}}\xi_k}\\ &= -2\Omega_j\,\epsilon_{ijk}\,\overline{\xi_{k,\alpha}\,\bar{D}^{\mathrm{L}}\xi_i} \quad \text{since} \quad \epsilon_{ijk} = -\epsilon_{kji}\\ &= 2\Omega_j\,\epsilon_{ijk}\,\overline{\xi_k\,\bar{D}^{\mathrm{L}}\xi_{i,\alpha}} \quad \text{by} \quad (\text{A } 2)\\ &= \Omega_j\,\epsilon_{ijk}\,\bar{D}^{\mathrm{L}}\overline{(\xi_{i,\alpha}\,\xi_k)}\end{aligned} \tag{A 5}$$

by comparison of the first and penultimate lines. The term $\Phi_{,i}$ $(=\bar{\Phi}_{,i})$ gives zero, by (A 3).

To deal with the pressure term, first note that, again by the chain rule,

$$(\varphi^\xi)_{,j} = (\varphi_{,k})^\xi\,(\delta_{kj}+\xi_{k,j}). \tag{A 6}$$

On multiplying by K_{ij}/J and using (2.8*b*), we obtain the inverse relation

$$(\varphi_{,i})^\xi = (\varphi^\xi)_{,j}\,K_{ij}/J. \tag{A 7}$$

Therefore, noting (2.13),

$$\overline{\xi_{i,\alpha}(\rho^{-1}p_{,i})^\xi} = \tilde{\rho}^{-1}\overline{(p^\xi)_{,j}\,\xi_{i,\alpha}\,K_{ij}}.$$

In virtue of (2.9) this may be rewritten as

$$\tilde{\rho}^{-1}\overline{(p^\xi\xi_{i,\alpha}\,K_{ij})}_{,j} - \tilde{\rho}^{-1}\overline{p^\xi\,J_{,\alpha}}$$

since $J_{,\alpha} = K_{ij}\,\xi_{i,j\alpha}$ by the rule for differentiating determinants. The second term is equal to $+\overline{(p^\xi)_{,\alpha}/\rho^\xi}$ by (A 2) and (2.13a). Therefore the result of scalarly multiplying $(2.5)^\xi$ by $\boldsymbol{\xi}_{,\alpha}$ and averaging, introducing the notation of (2.7), is

$$\bar{D}^{\mathrm{L}}\mathsf{A}+\tilde{\rho}^{-1}\mathsf{B}_{j,j}+\overline{(p^\xi)_{,\alpha}/\rho^\xi}=-\overline{\xi_{i,\alpha}X_i^\xi}. \tag{A 8}$$

Now if q is defined as in (2.12) we have, noting that $(\bar{p}^{\mathrm{L}})_{,\alpha}=0$,

$$-\overline{(p^l)_{,\alpha}\,q}=\overline{(p^\xi)_{,\alpha}\left\{\frac{1}{F(S^\xi,p^\xi)}-\frac{1}{F(\bar{S}^{\mathrm{L}},p^\xi)}\right\}},$$

$$=\overline{(p^\xi)_{,\alpha}/\rho^\xi} \tag{A 9}$$

because

$$\overline{(p^\xi)_{,\alpha}/\{F(\bar{S}^{\mathrm{L}},p^\xi)\}}=0.$$

(This is the case of (A 1) obtained by setting φ equal to the indefinite integral of $1/F(\bar{S}^{\mathrm{L}},p^\xi)$ with respect to p^ξ, holding $\bar{S}^{\mathrm{L}}$ constant.)

Putting (A 8) and (A 9) together and noting that $\overline{\xi_{i,\alpha}X_i^\xi}=\overline{\xi_{i,\alpha}X_i^l}$, we get the exact wave-action equation (2.6), noting the definition (2.11).

Appendix B. The pressure term in the virial theorem

We have

$$\frac{1}{\tilde{\rho}}\xi_i K_{ij}(p^\xi)_{,j}=\frac{1}{\tilde{\rho}}\xi_i K_{ij}\{(\bar{p}^{\mathrm{L}})_{,j}+(p^l)_{,j}\}$$

$$=\frac{1}{\bar{\rho}}\xi_i\{\delta_{ij}(1+\xi_{m,m})-\xi_{j,i}+k_{ij}\}\{\bar{p}_{,j}+(p'+\xi_k\bar{p}_{,k})_{,j}\}, \tag{B 1}$$

correct to $O(a^2)$, by (2.4b), (2.10), IV (2.25), IV (2.27), and IV (2.28). After a little manipulation this yields

$$\frac{1}{\tilde{\rho}}\overline{\xi_i K_{ij}(p^\xi)_{,j}}=\frac{1}{\bar{\rho}}\overline{(\xi_j p')}_{,j}+\frac{1}{\bar{\rho}^2\bar{c}^2}\overline{p'^2}+\frac{1}{\bar{\rho}}\left(\bar{p}_{,jk}-\frac{1}{\bar{\rho}\bar{c}^2}\bar{p}_{,j}\bar{p}_{,k}\right)\overline{\xi_j\,\xi_k}$$
$$-\overline{q(\xi_k\bar{p}_{,k}-p')}+O(a^3), \tag{B 2}$$

where the speed of sound $\bar{c}$ is defined by $\bar{c}^{-2}=\{\partial F(\bar{S},\bar{p})/\partial\bar{p}\}_{\bar{S}}$. We have used the fact that, from (2.18c) and (2.12),

$$\xi_{m,m}=-\rho^l/\bar{\rho}=-\frac{1}{\bar{\rho}\bar{c}^2}p^l-\bar{\rho}q$$

$$=-\frac{1}{\bar{\rho}\bar{c}^2}(p'+\xi_j\bar{p}_{,j})-\bar{\rho}q, \tag{B 3}$$

correct to $O(a)$. On the right of (B 2) the first term is negligible for almost-plane waves (or integrates out across a waveguide), the second is twice the acoustic energy, and the third reduces to $2\bar{\rho}^{-1}$ times Lorenz' (1955) formula for the available potential energy when we assume a small disturbance about a hydrostatic mean state under a uniform gravitational field, with $\bar{p}_{,3}\doteq-g\bar{\rho}$ ($g=|\nabla\Phi|$, x_3 taken parallel to $\nabla\Phi$), $\xi_3\doteq-S'/\bar{S}_{,3}$ and $\bar{p}_{,33}\doteq-g\bar{\rho}_{,3}$. For then

$$\frac{1}{\bar{\rho}}\left(\bar{p}_{,jk}-\frac{1}{\bar{\rho}\bar{c}^2}\bar{p}_{,j}\bar{p}_{,k}\right)\overline{\xi_j\,\xi_k}\doteq\frac{1}{\bar{\rho}}\left\{-g\bar{\rho}_{,3}+\frac{g}{\bar{c}^2}\bar{p}_{,3}\right\}\overline{\xi_3^2}, \tag{B 4}$$

and the expression in braces is just $-g\bar{S}_{,3}\,\partial F(\bar{S},\bar{p})/\partial\bar{S}$, as is shown by differentiation of the equation of state $\rho = F(S,p)$ with respect to x_3; this is the appropriate measure of static stability to give Lorenz' formula. The last term in (B 2) involves q and therefore vanishes for conservative motion.

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