On the non-separable baroclinic parallel flow instability problem

Michael E. McIntyre

J. Fluid Mech., **40**, 273–306 (1972)

J. Fluid Mech. (1970), *vol.* 40, *part* 2, *pp.* 273–306
Printed in Great Britain

On the non-separable baroclinic parallel flow instability problem

By MICHAEL E. McINTYRE

Department of Applied Mathematics and Theoretical Physics,
University of Cambridge

(Received 5 February 1968 and in revised form 7 July 1969)

Perturbation series are developed and mathematically justified, using a straightforward perturbation formalism (that is more widely applicable than those given in standard textbooks), for the case of the two-dimensional inviscid Orr–Sommerfeld-like eigenvalue problem describing quasi-geostrophic wave instabilities of parallel flows in rotating stratified fluids.

The results are first used to examine the instability properties of the perturbed Eady problem, in which the zonal velocity profile has the form $u = z + \mu u_1(y, z)$ where, formally, $\mu \ll 1$. The connexion between baroclinic instability theories with and without short wave cutoffs is clarified. In particular, it is established rigorously that there is instability at short wavelengths in all cases for which such instability would be expected from the 'critical layer' argument of Bretherton. (Therefore the apparently conflicting results obtained earlier by Pedlosky are in error.)

For the class of profiles of form $u = z + \mu u_1(y)$, it is then shown from an examination of the $O(\mu)$ eigenfunction correction why, under certain conditions, growing baroclinic waves will always produce a counter-gradient horizontal eddy flux of zonal momentum tending to reinforce the horizontal shear of such profiles. Finally, by computing a sufficient number of the higher corrections, this first-order result is shown to remain true, and its relationship to the actual rate of change of the mean flow is also displayed, for a particular jet-like form of profile with *finite* horizontal shear. The latter detailed results may help to explain at least one interesting feature of the mean flow found in a recent numerical solution for the wave régime in a heated rotating annulus.

1. Introduction

This paper considers some fundamental aspects of the quasi-geostrophic baroclinic instability problem. Apart from its frequent relevance in laboratory situations involving slow motions of an inhomogeneous fluid in a rotating frame of reference, this parallel-flow instability problem yields a theoretical description of processes known to be important in the earth's atmosphere. It has already been studied extensively (see Pedlosky 1964*a*, Fowlis & Hide 1965). Simple baroclinic instability theory accounts qualitatively for the way in which many large-scale weather systems obtain their kinetic energy. For readers not familiar with the type of dynamics involved, a brief description is given in appendix A.

Of particular interest are forms of the relevant two-dimensional eigenvalue problem, (2.1) below, that are *non-separable* because of the presence of horizontal as well as vertical shear in the mean zonal current $\{u(y,z),0,0\}$. It is precisely such problems that can be expected to model the interesting kind of simultaneous potential and kinetic energy transformation known to play a role in the maintenance of the mid-latitude westerlies, the associated horizontal eddy flux of zonal momentum corresponding to a negative Austausch coefficient.

But the mathematical difficulties have for some time remained a serious theoretical obstacle. Valuable progress has been made with 'two-level models', equivalent to use of the crudest possible finite differencing in the vertical (Eliasen 1961; Pedlosky 1964*b*); the important recent work of Stone (1969), using such a model, will be mentioned later. Another approach that has been used, e.g. by Eady (see Green 1970) and by Brown (1969*a*), is to solve for particular numerical cases by the use of purely finite-difference methods of relatively high resolution. These avoid *a priori* assumptions about the vertical structure, but do not easily yield generality or insight.

In this paper we present a perturbation formalism (§ 3) that was developed in order to provide a flexible analytical approach to the non-separable eigenvalue problem (2.1). The method is applied to a discussion of the perturbed Eady problem, in which $u(y,z) = z+\mu u_1(y,z)$, μ being the perturbation parameter.

The idea of perturbing about a simple form of (2.1) is not new, being implicit for instance in some unpublished work of Stern & Magaard (see Magaard 1963), and having also been put forward by Pedlosky (1965). In the latter's investigation, first correction terms for the perturbed Eady problem were obtained for small μ by means of a somewhat elaborate initial-value approach, which brings in the (singular) complete set of unperturbed eigenfunctions in a way reminiscent of classical perturbation theory. By contrast, the present approach is relatively straightforward, and we can obtain the higher corrections as well, yielding results valid for a *finite* range of μ. An example for which such calculations were carried out in detail is given in § 7 below. A more fundamental consequence is that knowledge of the higher corrections enables us to justify our procedure in a mathematically rigorous way (§ 4).

The latter point gains added importance in view of the fact that Pedlosky's (1965) conclusions on the perturbed stability properties turn out to be in error (although that, indeed, becomes clear upon comparison with the numerical results found by Green 1960, for a particular example).

In § 5 we re-examine the perturbed stability properties. The main results have already been predicted by Bretherton (1966*a*, p. 333), using an indirect but powerful argument in which the idea of the 'critical layer' plays a key role. Our analysis can be looked upon as providing a mathematically rigorous expression of, and so a clear justification of, Bretherton's argument.

In § 6 we go on to develop a linear-theory explanation of the previously mentioned negative Austausch coefficient, for profiles of the form $u = z+\mu u_1(y)$ where $u_1(y)$ is an unspecified function. The discussion is based on the $O(\mu)$ terms, and identification of corresponding physical effects. Evidence that higher terms do not change the qualitative picture in cases of interest is then obtained (§ 7) by computing a sufficient number of terms in two 'realistic' examples, for which

$$u = z + \begin{Bmatrix} 0\cdot 4 \\ 0\cdot 5 \end{Bmatrix} \sin^2 \pi y \quad (0 \leqslant (y, z) \leqslant 1).$$

It is worth mentioning that the results for the first example have recently been compared, and show excellent detailed agreement, with corresponding results independently obtained by Brown (personal communication), using his finite-difference procedure (Brown 1969*a*).

2. The eigenvalue problem

Attention will be focused on a problem that is idealized but adequately embodies the fundamental properties under discussion. As will become clear, refinement would be essentially straightforward.

Consider small-amplitude frictionless adiabatic disturbances to a parallel flow $u(y,z)$ of stably-stratified Boussinesq liquid, whose horizontally averaged buoyancy or Brunt–Väisälä frequency is $N(z)$. The flow is in the x-direction and is limited by boundaries at $z = 0$, H and $y = 0$, L, on which the normal velocity must vanish; x, y, z are Cartesian co-ordinates in a frame of reference rotating about the vertical z-axis with angular velocity $\frac{1}{2}f$. A dimensionless combination of importance in the problem is

$$\epsilon(z) \equiv \frac{f^2 L^2}{N^2 H^2},$$

which is formally of order unity, expressing the anticipated importance of both buoyancy and Coriolis forces. (But with our definition of L, numerical values of ϵ are more like π^2 in cases of interest.) Compressibility would introduce no essential modification provided that H is very much less than the density scale height, and the presence of a horizontal rotation component will have negligible effect if $H/L \ll 1$. Also, very crudely, one could regard the rigid upper boundary as the beginning of e.g. an idealized 'stratosphere of infinite static stability'.

The well known eigenvalue problem for the perturbation pressure

$$\operatorname{Re} \varphi(y,z)\, e^{ik(x-ct)}$$

of a quasi-geostrophic normal-mode wave disturbance can be written in dimensionless form, as

$$(u-c)\,[(\epsilon\varphi_z)_z + \varphi_{yy} - k^2\varphi] + q_y(y,z)\,\varphi = 0, \tag{2.1a}$$

$$(u-c)\,\varphi_z - u_z\varphi = 0 \quad \text{on} \quad z = 0, 1, \tag{2.1b}$$

$$\varphi = 0 \quad \text{on} \quad y = 0, 1, \tag{2.1c}$$

where the dimensionless wave-number k is real and the complex amplitude $\varphi(y,z)$ and phase velocity c are sought as eigenfunction and eigenvalue. The function $q_y(y,z)$, a property of the basic flow analogous to the $-u_{yy}$ of the classical Orr–Sommerfeld problem, is defined in appendix A (ii), which briefly sketches the derivation of (2.1). The boundary conditions (2.1*b*) reflect the active role that can be played by horizontal boundaries in the present problem, due to the importance of vertical vortex-tube stretching.

Since $(u-c)^{-1}q_y$ (see (A 6)) is not in general of the form func (y) + func (z), nor $(u-c)^{-1}u_z$ independent of y, the problem is generally non-separable in the co-ordinates y, z. Certain neutral separable solutions (c real) are possible if u is of the form func $(y) \times$ func (z), and β (see (A 6)) is zero, but are not of great importance in themselves.

As is known from particular solutions, such a system can exhibit more than one type of instability (Phillips 1963, § 3a), but in speaking of the 'baroclinic instability problem' one is thinking of situations in which the vertical shear u_z is the most essential feature of the basic velocity profile. Because of hydrostatic and geostrophic (pressure-Coriolis) balance, the vertical shear is associated with a transverse horizontal density gradient, and thus represents a store of available potential energy (Lorenz 1955). Under suitable conditions some of this mean-flow energy can be released by a growing disturbance in the manner described in appendix A (i), as was first clearly shown by the independent mathematical analyses of Charney (1947), and Eady (1949) (see appendix A (iii)).

We shall take (2.1) as our starting point. Although, in view of the aforementioned controversy, some care will be taken to construct solutions of (2.1) in a mathematically rigorous way, one should remain aware that (2.1) is already the result of several formal approximations. (But it can be noted that our perturbation approach could be used as the basis for a mathematical justification of the latter too, if desired; in that connexion see the footnote to appendix D (ii).)

The assumption of normal-mode form for the solutions needs little discussion here, because the results we shall be interested in concern positive cases of instability. That the existence of instability in the normal-mode sense must imply instability in the solution to the general initial-value problem hardly needs proof, but in any case the kind of analysis needed is not essentially different from that given e.g. by Pedlosky (1964c) and Burger (1966).

3. The perturbation formalism

The convergent perturbation series to be obtained below are based simply upon the use of a generalized Green's function (Courant & Hilbert 1953, p. 354). This device seems more natural, and is certainly much more widely applicable, than the standard eigenfunction expansions elaborated upon in textbooks on theoretical physics. In particular, it does not depend upon a complete set of unperturbed eigenfunctions; cf. Courant & Hilbert (1953, p. 343), Morse & Feshbach (1953, p. 1034), Pedlosky (1965, § 1).

The problem to be considered in detail in this paper is that for which

$$u = z + \mu u_1(y, z), \quad q_y = \mu q_{1y}(y, z), \tag{3.1}$$

where μ is the perturbation parameter. That is, we shall be perturbing about an Eady solution, (A 7).

Note from (A 6) that a transverse gradient of the Coriolis parameter, or β-effect, can be included, as long as β can be written in the form $\mu\beta_1$ within the radius of convergence of the perturbation scheme. For simplicity, $\epsilon(z)$ will be assumed to remain constant. But it should be realized that there would be no formal difficulty in writing $\epsilon = \text{const.} + \mu\epsilon_1(z)$, or in perturbing about any other (e.g. a separable neutral) solution, etc., etc.

Because of the branch points in the k-dependence of the Eady solutions at $k = k_N$ (A 8), it turns out that two cases must be considered separately, namely $k \neq k_N$ and $k = k_N$.

(i) *The case* $k \neq k_N$

It seems natural to pose

$$\varphi = \varphi_0 + \mu\varphi_1 + \mu^2\varphi_2 + \dots. \tag{3.2a}$$

Regarding c as the eigenvalue and the other parameters as fixed, one would also expect that

$$c = c_0 + \mu c_1 + \mu^2 c_2 + \dots. \tag{3.2b}$$

On substituting (3.1) and (3.2) into (2.1) and equating like powers of μ, a succession of boundary value problems is obtained, whose details are given in appendix B. Here we abbreviate the lth problem to

$$\begin{cases} L(\varphi_l) = I_l, & (3.3a)\\ D(\varphi_l) = B_l' + \dfrac{c_l\varphi_0}{(z-c_0)^2}, \equiv B_l, \quad \text{on} \quad z = 0, 1, & (3.3b)\\ \varphi_l = 0 \quad \text{on} \quad y = 0, 1, & (3.3c) \end{cases}$$

where

$$L \equiv \frac{\partial^2}{\partial z^2} + \frac{1}{\epsilon}\left(\frac{\partial^2}{\partial y^2} - k^2\right),$$

$$D \equiv \frac{\partial}{\partial z} - \frac{1}{z-c_0}.$$

Of course $I_0 = B_0 = 0$, so that $\{\varphi_0, c_0\}$ is an Eady mode. For $l \geqslant 1$, I_l and B_l' involve $\varphi_0, \dots, \varphi_{l-1}$ and $c_0, \dots, c_{l-1}$ only, as can be verified from (B 2).

Now the homogeneous problem complementary to (3.3), for $l \geqslant 1$, is the same as the zero-order problem (and its adjoint), and has the non-trivial solution φ_0. This means that the inhomogeneous problem (3.3) has a solution only if the inhomogeneity $\{I_l, B_l\}$ satisfies a certain condition of orthogonality to φ_0 (more generally, to the corresponding solution of the adjoint problem). It is that condition, of course, that determines c_l at each stage.

What the orthogonality condition must be can be found by formally multiplying (3.3a) by φ_0, integrating from 0 to 1 with respect to y and z, integrating by parts twice, and then using the boundary conditions and the fact that $L(\varphi_0) = 0$. Thence

$$-\iint \varphi_0 I_l\, dy\, dz + \int dy \left[\varphi_0 B_l\right]_{z=0}^{z=1} = 0.$$

Referring to (3.3b) and using the identity (A 13) we can rewrite this as an explicit formula for c_l:

$$c_l = \frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})}\left\{\iint \varphi_0 I_l\, dy\, dz - \int dy \left[\varphi_0 B_l'\right]_{z=0}^{z=1}\right\}. \tag{3.4a}$$

Here φ_0 is normalized as in (A 7). For $l = 1$ (3.4a) gives the first correction for c, in terms of φ_0 only; it will be discussed in detail in § 5.

Equation (3.4 a) was derived as a necessary condition for (3.3) to have a solution. It is also sufficient; when (3.4 a) is satisfied a solution is

$$\varphi_l = -\iint \mathfrak{G}(y,z;\,\eta,\zeta)\,I_l(\eta,\zeta)\,d\eta\,d\zeta + \int d\eta\left[\mathfrak{G}\,.\,B_l(\eta,\zeta)\right]_{\zeta=0}^{\zeta=1}, \tag{3.4b}$$

where $\mathfrak{G}(y,z;\,\eta,\zeta)$ is a Green's function in the generalized sense (Courant & Hilbert 1953, p. 354). Here $\mathfrak{G}$ is a solution of

$$L(\mathfrak{G}) = A\varphi_0(y,z)\,\varphi_0(\eta,\zeta) - \delta(y-\eta)\,\delta(z-\zeta), \tag{3.5a}$$

$$D(\mathfrak{G}) = 0 \quad \text{on} \quad z = 0,1, \tag{3.5b}$$

$$\mathfrak{G} = 0 \quad \text{on} \quad y = 0,1. \tag{3.5c}$$

L and D are understood to operate on (y,z) and A is a constant defined so that (3.5) is soluble. It can be seen that A is given by

$$A\iint \varphi_0^2\,dy\,dz = 1.$$

As yet there is arbitrariness in $\mathfrak{G}$ and φ_l to the extent that a constant multiple of φ_0 may be added. (This corresponds to multiplying $\varphi = \Sigma\mu^l\varphi_l$ by a constant, $1+O(\mu)$.) It seems natural in the present context to remove this arbitrariness by requiring that φ_l be 'as small as possible', in terms of a norm such as

$$\left\{\iint \varphi_l\varphi_l^*\,dy\,dz\right\}^{\frac{1}{2}}.$$

The asterisk denotes the complex conjugate. That norm is minimized when φ_l satisfies

$$\iint \varphi_l\varphi_0^* = 0.$$

Correspondingly, we shall choose $\mathfrak{G}$ so that, for all η, ζ,

$$\iint \mathfrak{G}(y,z;\,\eta,\zeta)\,\varphi_0^*(y,z)\,dy\,dz = 0. \tag{3.6}$$

$\mathfrak{G}$ is uniquely defined by (3.5) and (3.6); an explicit representation is given in appendix B.

In summary, the solution to the perturbed problem is given formally by

$$\{\varphi, c\} = \left\{\sum_0^\infty \mu^l\varphi_l, \quad \sum_0^\infty \mu^l c_l\right\},$$

where all the terms for $l \geqslant 1$ are defined by (3.4), together with the recursion formulae written out in appendix B.

(ii) *The case* $k = k_N$.

Equation (3.4 a) shows that the series just derived fail if $c_0 = \frac{1}{2}$, i.e. at the critical neutral wave-number $k = k_N$. But expansions in powers of $\lambda \equiv \mu^{\frac{1}{2}}$ turn out to be appropriate:

$$\varphi = \varphi_0^N + \lambda\varphi_1^N + \lambda^2\varphi_2^N + \ldots, \tag{3.7a}$$

$$c = \tfrac{1}{2} + \lambda c_1^N + \lambda^2 c_2^N + \ldots, \tag{3.7b}$$

where φ_l^N and c_l^N are not to be confused with the previous lth coefficients in the μ expansions. Although there are some features of interest, the essential ideas are the same, and the details are relegated to appendix C.

Dependence of c_l on the lower eigenfunction corrections

As is easy to verify, (3.4*a*) depends on $\varphi_0, \varphi_1, \ldots, \varphi_{l-1}$. Although (3.4*a*) is the simplest and most convenient form for computational purposes, it can be noted that as a consequence of the self-adjointness of L, c may be found in terms of $\varphi_0, \ldots, \varphi_{[\frac{1}{2}l]}$ only, where $[\frac{1}{2}l]$ denotes the largest integer $\leqslant \frac{1}{2}l$. That can be shown by first forming the equation

$$\iint \sum_{j=0}^{[\frac{1}{2}l-\frac{1}{2}]} \{\varphi_j(3.3\,a)_{l-j} - \varphi_{l-j}(3.3\,a)_j\}\, dy\, dz,$$

and then integrating by parts, using the boundary conditions. Alternatively and more elegantly (L. Segel, private communication; Morse & Feshbach 1953), the result can be derived from variational considerations. This generalizes a result given by Joseph (1967).

4. Mathematical interpretation and justification of the formulae

I_l is singular at $z = c_0$, and so it is necessary to define the meaning of expressions such as (3.4*a*, *b*) when the unperturbed eigenvalue c_0 is real, as is the case for $k \geqslant k_N$.

That is easy if we assume that u_1 and q_{1y} are analytic functions. The whole process can then be carried out in a domain $\mathscr{D} \equiv \Gamma_z \times \Gamma_y$, where Γ_y is the interval $0 \leqslant y \leqslant 1$, and Γ_z is a contour in the complex z-plane which joins 0 and 1 and avoids $z = c_0$ (see figure 1). Γ_z could depend on y, and must also be chosen so that u_1 has no singularities between Γ_z and the real axis (the shaded region). Once an appropriate Γ_z has been chosen, then (3.4) (and (C 6)) are unambiguous even for real c_0. Clearly $\mathfrak{G}$ can be understood as being defined by (B 3) of appendix B; it is convenient to suppose that both z and ζ lie on Γ_z.

Complex z more useful than y $\because$ interested in cases $u_z \neq 0$

Having chosen an appropriate $\mathscr{D}$ (which need not be complex if c_0 is not real), one can prove, for $|\mu| <$ some finite positive μ_0, that the c expansion is convergent, that the φ expansion is uniformly convergent over $\mathscr{D}$, and that the expansions do in fact represent a solution in $\mathscr{D}$ of the original problem (2.1). The proof is quite straightforward but somewhat tedious. It is given in appendix D for $k \neq k_N$; the proof for $k = k_N$ is similar.

After the perturbed $c = \sum_0^\infty \mu^l c_l$ has been found for given μ within the radius of convergence, it must then be asked whether any point $z_c(y)$ for which $u(y, z) = c$ falls within the shaded region or on the real axis. If so, the eigensolution on $\mathscr{D}$ is generally not the continuation of a solution regular in the physical domain of real y and z. If not, it is. (That the eigenfunction φ, as opposed to the terms in its representation $\Sigma\mu^l\varphi_l$, cannot then be singular in, or on the boundary of, the shaded region, will probably be obvious to the reader. In any case, it could be proved using continuation, upon substituting for c (now known) in (2.1*a*) and e.g. for $(u-c)\,\varphi_z$ (known) in (2.1*b*), thus considering (2.1) as an inhomogeneous boundary value problem for φ in which $(u-c)^{-1}\,q_y$ can be taken as a known regular analytic function of z for each y.) In this latter case, φ being smoothly behaved for real y and z in $0 \leqslant (y,z) \leqslant 1$, it must represent a physically meaningful solution, and we shall speak of an 'admissible' eigensolution.†

If the perturbed c happened to fall exactly on the real axis, further discussion would be needed. However, that possibility seems to be of little interest.

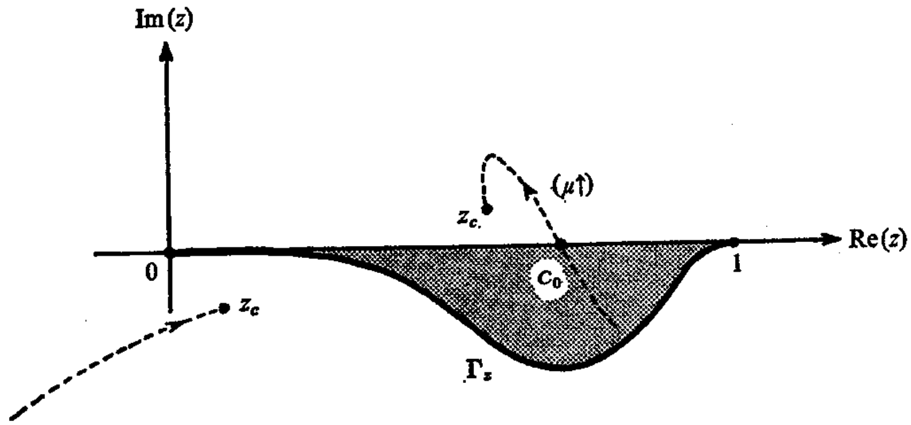

FIGURE 1. An example of an 'admissible' (for $\mu > 0$) configuration in the z plane; $\Gamma_z(y)$ must be chosen so that u_1, as a function of z, has no singularities in the shaded region for any y in Γ_y. Possible paths of the point z_c as μ varies are illustrated by the dotted lines. Note that $z_c(\mu)$ will in general be multivalued, and that that might possibly require extra care in the choice of Γ_z for a finite value of μ.

5. The first-order instability properties of the perturbed Eady problem

As a first, simple application, we discuss the stability of the mean flow $u = z + \mu u_1$, for $\mu \ll 1$ and any sufficiently differentiable function $u_1(y,z)$. It will appear below that the first-order results are qualitatively useful over a fair range of μ values of practical interest, especially for the short waves $k > k_N$, (A 8).‡ For $k \geqslant k_N$ it is also assumed that u_1 is analytic.

† When solutions are found that have Im $(c) \neq 0$ and are admissible by our definition, they must occur in complex conjugate pairs, Im $(c) \gtrless 0$, corresponding to two appropriate choices of Γ_z. One solution is 'damped' (but not dissipative!) and the other amplifying. The latter solution is the physically interesting one, but to be self-consistent one should also admit the former under the definition; we emphasize this point only because confusion about it sometimes seems to occur in the literature. *Any* non-singular normal mode of an approximate (e.g. non-dissipative) problem must of course be expected to represent a physically meaningful approximate solution, in the natural sense that over a finite time interval it approximates a solution (as opposed to a normal mode) of whatever are being regarded as the exact equations, e.g. equations with small diffusion coefficients. See the related discussion by Lin (1961).

‡ It is sometimes said that the basic quasi-geostrophic approximation (appendix A(ii)) becomes invalid at short wavelengths. However, it can be shown that that is not the case; the physical reason is that the height scale of the wave (see appendix A(i)), and hence the mean flow vertical velocity difference seen by the wave, diminishes as the wavelength for large enough k.

When $k \neq k_N$, (3.2*b*) is the appropriate expansion. Referring to (3.4*a*), we have

$$c = c_0 + \frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})} K\mu + O(\mu^2), \tag{5.1}$$

where, as can be verified from (B 2),

$$K = \iint \frac{-\epsilon^{-1} q_{1y} \varphi_0^2}{z - c_0} dy\, dz - \int dy \left[\frac{-u_1 + (z-c_0) u_{1z}}{(z-c_0)^2} \varphi_0^2 \right]_{z=0}^{z=1}. \tag{5.2}$$

From (A 6), q_{1y} is related to u_1 by

$$q_{1y} = \beta_1 - \epsilon u_{1zz} - u_{1yy}, \tag{5.3}$$

with the obvious definition of the constant β_1. Only the zero-order eigenfunction φ_0, given by (A 7), is involved in this first correction to c_0.

When $k = k_N$, (3.7*b*) is appropriate. More explicitly, it may be shown (appendix C) that for $k = k_N$

$$c = \tfrac{1}{2} \pm \left\{ \frac{\alpha_N^2 - 1}{8\alpha_N^4} K_N \mu \right\}^{\frac{1}{2}} + O(\mu), \tag{5.4}$$

$\alpha_N = 1{\cdot}2$, p 297

where K_N is defined by (5.2), with $\varphi_0 = \varphi_0^N$ and $c_0 = \frac{1}{2}$, as in (A 10).†

When c_0 is real these formulae must be interpreted in accordance with the discussion of § 4. The most interesting thing about them is that, in the short-wave neutral régime $k \geqslant k_N$ of the zero-order problem, Im (c) is non-zero in general, even though c_0 and φ_0 are real. For $k > k_N$ the imaginary contribution to c comes entirely from the half-residue at $z = c_0$ of the first integrand in (5.2). Taking the Γ_z shown in figure 1, we have (3.4a), (A.13)

$$\operatorname{Im}(c) = -\pi \frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})} \int_0^1 \mu \epsilon^{-1} \{q_{1y} \varphi_0^2\}_{z=c_0} dy + O(\mu^2), \quad \text{when} \quad k > k_N. \tag{5.5}$$

Note that, to first order, Im (c) depends on q_{1y} at $z = c_0$ only. *Whenever* (5.5) *is positive, we have an admissible amplifying mode* (as well as its 'damped' conjugate, by the conjugate choice of Γ_z). *If* (5.5) *is negative, there is no admissible normal mode to which* φ_0 *is a first approximation.*

The half residue can be given an illuminating physical interpretation as a critical-layer quasi-potential-vorticity flux, following the discussion given by Bretherton (1966*a*). His argument shows clearly why instability is to be expected whenever $\int \mu q_{1y} \varphi_0^2\, dy$ has the appropriate sign at the unperturbed critical level $z = c_0$.

The factor multiplying the integral in (5.5) is positive for the lower wave $c_0 < \frac{1}{2}$, and negative for the upper wave $c_0 > \frac{1}{2}$, as can most easily be seen from (A 13). Therefore, the lower wave is destabilized by a positive weighted-average

† The formula (5.1) could have been obtained very simply, although not rigorously, by the Tollmien argument (Lin 1955, p. 122). One then has to *assume* the existence of neighbouring eigensolutions. (Conversely, the present type of analysis justifies the Tollmien argument). It is possible to derive (5.4) in a similarly simple way if one is prepared to assume also, writing $c - \frac{1}{2} \equiv \Delta(k, \mu)$, that $(\partial\Delta^2/\partial\mu)_k = -(\partial\Delta^2/\partial k)_\mu\, (\partial k/\partial\mu)_\Delta$ at the singular point $\{k = k_N, \mu = 0\}$. (The latter relation is thus true, but its truth does not seem obvious *a priori*.)

quasi-potential vorticity gradient $\int \mu\epsilon^{-1} q_{1y}|_{z=c_0} \sin^2 m\pi y\, dy$, such as might, especially for $m = 1$, be associated with a horizontally-jet-like profile. The upper wave is destabilized by a negative gradient.

With regard to the interpretation of (5.4), it is evident that in virtue of the two sign possibilities in (5.4) for *each* Γ_z, there is always, in general, exactly one conjugate pair of admissible solutions at $k = k_N$. An exception occurs when $K_N \mu$ is real positive (which cannot be so unless $\int q_{1y} \sin^2 m\pi y\, dy = 0$ at $z = \frac{1}{2}$).

These results and the related arguments of Bretherton (1966*a*) greatly clarify the connexion between baroclinic instability theories with and without 'short wave cutoffs'; see also Bretherton (1966*b*). (This connexion, or seeming lack of it, had puzzled many investigators in the past.) They also show that the conclusion stated by Pedlosky (1965, abstract), that 'only the vertically antisymmetric and horizontally symmetric component of the velocity deviation affects the stability of the flow' (to $O(\mu)$), is incorrect. As can be seen from our discussion, the vertically (and horizontally) symmetric component of q_{1y}, and hence of u_1 in general, is also involved, at short wavelengths.†

Examples

It is of interest first of all to apply (5.1) and (5.4) to the simple case $u_1 = 0$, $\mu q_{1y} = \beta = \text{const.}$, for comparison with the results of Green (1960; see also Garcia & Norscini 1969). A comparison is presented in figure 2 for $\mu\epsilon^{-1} q_{1y}\, (= \epsilon^{-1}\beta) = 1$. The y-dependence has been suppressed by replacing $\sin^2 m\pi y$ by $\frac{1}{2}$, and setting $m = 0$ elsewhere, so as to correspond to Green's y-independent formulation. The sign of q_{1y} is positive throughout the flow. Accordingly, the lower wave is destabilized and the upper wave disappears. The first-order formulae are quite accurate for this finite perturbation, except at the longer wavelengths and, for Re(c), at $k = k_N$. As can be shown from symmetry, the first-order change in growth rate is zero for $k < k_N$.

It is not surprising that perturbing about the Eady solution does not yield the long-wave phenomena discovered by Green, since the β-effect is dominant in these very long waves, and cannot be regarded as a perturbation. Indeed, it may be verified in general that (5.1) is not uniformly valid near $\kappa = 0$, the first correction behaving like κ^{-2}. The 'critical' κ (cf. Garcia & Norscini 1969) is,

† Nor, it should be added, does the statement appear to be true for the long-wave end of the spectrum considered in Pedlosky's (6.6). It certainly seems inappropriate, in principle, because of the fact that the perturbation method is not uniformly valid in the limit of small total wave-number κ, as will be remarked upon shortly. The corresponding results of Green (1960) and Garcia & Norscini (1969) seem, in a practical sense, a sufficient counter-example. (Pedlosky's analysis does include the β-effect, via a trivial modification, and hence should relate to Green's problem for small β.)

It should perhaps be pointed out that the argument given in § 5 of Pedlosky's paper appears to be in error (F. P. Bretherton, private communication). It does not seem to be a straightforward matter to produce a corrected version. For instance, suppose that (c_N, c_N^*) is the conjugate pair of admissible perturbed eigenvalues that is found in general at $k = k_N$, as was indicated above. Then (cf. (3.5), etc., of the paper in question) the quantities $-k^2 c_N c_N^*$, $-ik(c_N + c_N^*)$, are *not* regular analytic functions of μ at $\mu = 0$ and $k = k_N$, except in certain special cases. That can be seen immediately from (5.4); see also appendix C.

however, qualitatively indicated by the condition $\mathrm{Re}\,(c_0+\mu c_1) = 0$, even though the $O(\mu)$ change in $\mathrm{Im}\,(c)$ is zero. Note that if the y-dependence is reintroduced, κ cannot approach zero; the limit $\kappa \to 0$ implies infinite zonal *and* meridional length scales, and so is not of very great interest in practice. (See footnote §6.)

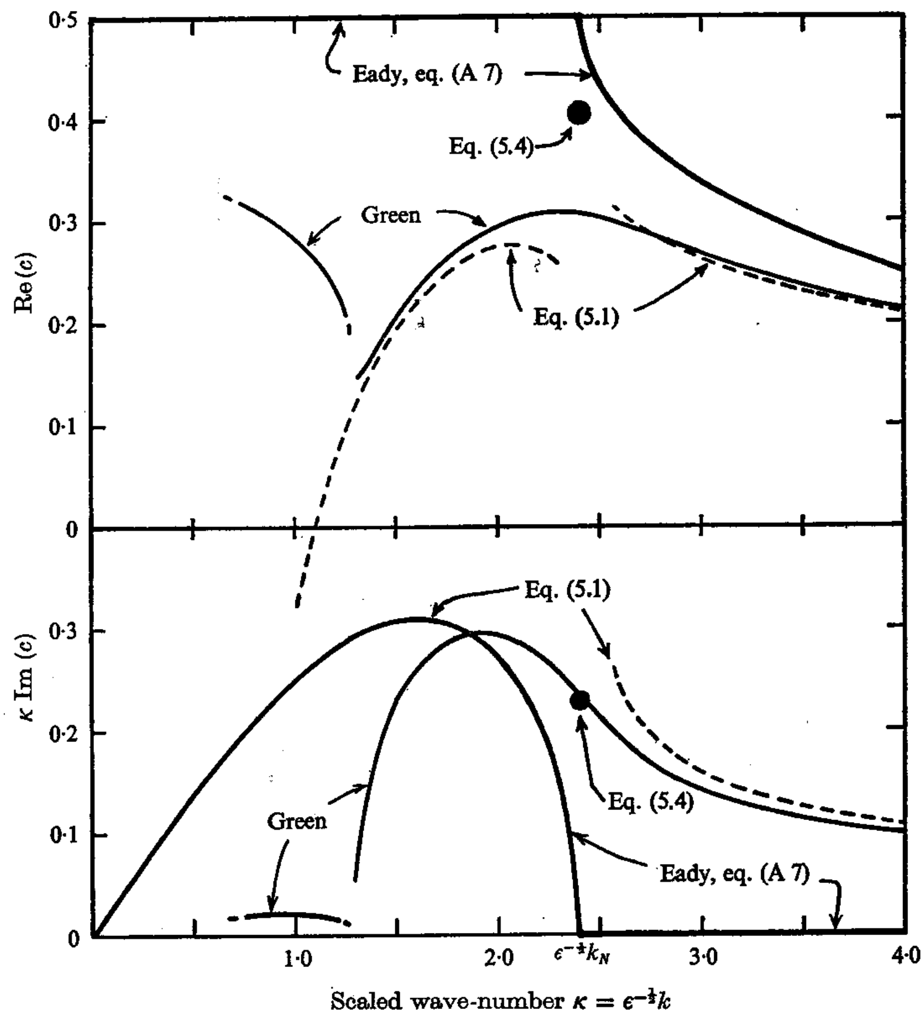

FIGURE 2. Comparison of first correction results from (5.1) and (5.4) with some numerical results of Green (1960), for $u = z$, $\epsilon^{-1}\beta = 1$, $m = 0$ (see text). Upper graphs: $\mathrm{Re}(c)$; lower: scaled growth rate $\kappa\,\mathrm{Im}\,(c) = \epsilon^{-\frac{1}{2}}k\,\mathrm{Im}\,(c)$. Note that the first correction to the growth rate is zero for $k < k_N$, but not for $k = k_N(\bullet)$ or $k > k_N$. For accuracy of comparison, the graphs of Green's results have been re-drawn, using his original data; in the case $\epsilon^{-1}\beta = \frac{1}{2}$ (not shown) the agreement at short wavelengths is even closer, upon correcting an inaccuracy in Green's corresponding published figure (*op. cit.*, p. 242; Green, private communication).

The formulae are illustrated further by the calculations presented in figure 3. There $\beta = 0$, $m = 1$, $\epsilon = 9$, and (*a*) $u = z_s(1-2y_s^2)$ and (*b*) $u = z_s - y_s^2$, where $y_s = (y-\frac{1}{2})$, $z_s = (z-\frac{1}{2})$.

In the first example, (a), it happens that $q_y = 0$ at $z = \frac{1}{2}$. There is a critical neutral mode (which, incidentally, is a separable solution). The corresponding wave-number k'_N in figure 3 was estimated from the formula

$$k_N'^2 = k_N^2 - \frac{4\epsilon K_N}{\alpha_N^2}\mu + O(\mu^2), \tag{5.6}$$

which can be established by the perturbation method (or again, obtained heuristically by the Tollmien argument). The perturbation formulae yield no admissible perturbed normal modes for $k > k_N$, at $O(\mu)$.

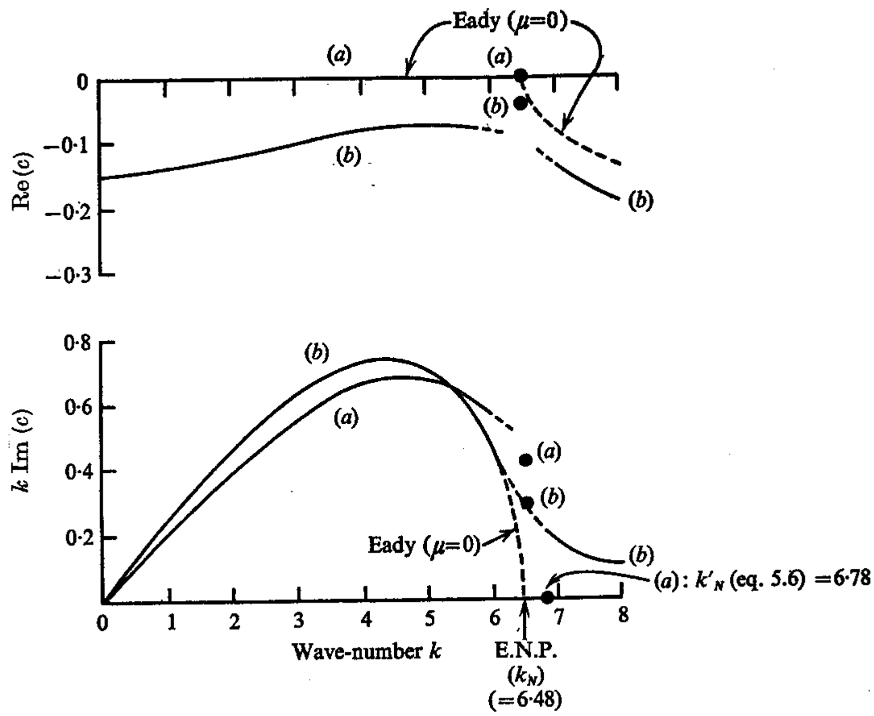

FIGURE 3. Examples of perturbed growth rate and phase velocity curves for the profiles
(a) $u = z_s(1-2y_s^2)$, (b) $u = z_s - y_s^2$, $\quad -\frac{1}{2} \leqslant (y_s, z_s) \leqslant \frac{1}{2}$,
calculated from (5.1), except at the Eady neutral point (E.N.P.), where (5.4) and (5.6) were used (●). Note that $k = 0$ means that $\kappa = \epsilon^{-\frac{1}{2}}\pi$, not $\kappa = 0$. In these calculations $\epsilon/\pi^2 = 0{\cdot}912$, and $m = 1$ (gravest mode).

That probably means, in this example, that there are indeed no eigensolutions for $k > k'_N$, but that unstable modes exist for $k < k'_N$ even though the Eady neutral waves do not serve as first approximation to some of them. This tentative interpretation is confirmed by perturbing about the neutral separable solution (McIntyre 1967).

The second example (b) has a non-zero potential vorticity gradient at $z_s = 0$ as well as elsewhere. In this respect it is a less special case. The short wave instability appears in the same way, and for the same reason, as in Green's problem. Again, the correction to the growth rate is zero for $k < k_N$, by symmetry.

$O(\mu)$

6. The tendency of baroclinic waves to generate a counter-gradient momentum flux

Consider a mean flow that is baroclinically unstable when $\mu = 0$ ($\epsilon > \pi^2/4\alpha_N^2$; see (A 8)) with horizontal shear that is independent of height:

Then (5.3) reduces to
$$\left.\begin{aligned} u(y,z) &= z+\mu u_1(y). \\ q_{1y} &= \beta_1 - u_{1yy}. \end{aligned}\right\} \qquad (6.1)$$

This includes the cases $u = z + (0{\cdot}4, 0{\cdot}5)\sin^2 \pi y$ examined in detail in § 7, which bear sufficient qualitative resemblance (although that point should not be pushed too far) to zonal mean profiles both in the atmosphere (Lorenz 1967) and in laboratory analogues (Williams 1969; figure 7 *b*, *c* below) for one to hope for insights that are heuristically useful. For this mathematically simplest way of introducing horizontal shear it will prove easy to see, in quite an elegant degree of generality, its first-order effect on the horizontal wave structure and the associated momentum flux or Reynolds stress component, $-\rho\overline{u'v'}$.

The result that will be obtained below could be simply expressed by saying that, to first order, the horizontal phase of a gravest ($m = 1$) unstable Eady mode is distorted in the horizontal by the differential advection u_{1y} in the 'obvious' sense (see figure 4), and that that effect is guaranteed to predominate, in any given case, provided ϵ is greater than some number ϵ_0 ($> \pi^2/4\alpha_N^2$) formally of order unity. (This last form of proviso is always sufficient, but is not necessary in all cases, in particular when u_1 has the simple form $ay^2 + by$ implying that q_{1y} is constant. Since $\epsilon \propto L^2$ one may think of $\epsilon > \epsilon_0$ as meaning that the wave is not too closely constrained laterally.)

The result and its physical interpretation are not really obvious without the analysis, since the instability mode involves a subtle balance between advection and *propagation* effects. (Recall in that connexion that for a barotropic, or classical inviscid shear instability, the phase lines bend oppositely to the 'obvious way'.) The perturbation method permits an unambiguous discussion of how that balance is altered, under various circumstances, by introducing horizontal shear.

A horizontal structure of the kind illustrated in figure 4 is of interest because the associated Reynolds stress $-\rho\overline{u'v'}$ transports x-momentum *against* the mean gradient u_y, as can be seen immediately from figure 4*c*. Such a process is known to be important in the large-scale atmospheric zonal momentum balance in middle latitudes (Phillips 1963, p. 152).

The trend revealed by the $O(\mu)$ terms will be borne out by the finite μ calculations for $u_1 = \sin^2 \pi y$ presented in § 7.

Before turning to details, we should point out that the stress $-\rho\overline{u'v'}$ is not the only significant mechanism of zonal momentum transfer, in the type of rotationally dominated flow under consideration. This point will be discussed in § 7. It is true, however, that for such flows the *vertically averaged* momentum transfer is described completely by the vertical average of $-\rho\overline{u'v'}$.

The expression (3.4*b*), with $l = 1$, may be written

$$\begin{aligned}\varphi_1 &= \iint \mathfrak{G}\varphi_0(\eta,\zeta)\frac{\epsilon^{-1}q_{1\eta}(\eta,\zeta)}{\zeta-c_0}\,d\eta\,d\zeta \\ &\quad + \int d\eta\left[\mathfrak{G}\varphi_0(\eta,\zeta)\frac{-u_1+c_1+(\zeta-c_0)\,u_{1\zeta}}{(\zeta-c_0)^2}\right]_{\zeta=0}^{\zeta=1}, \\ &= \varphi_{1I}+\varphi_{1B},\end{aligned} \tag{6.2}$$

say. We are perturbing about an unstable wave, with $k < k_N$, and the expressions are uniformly valid in the real, physical domain.

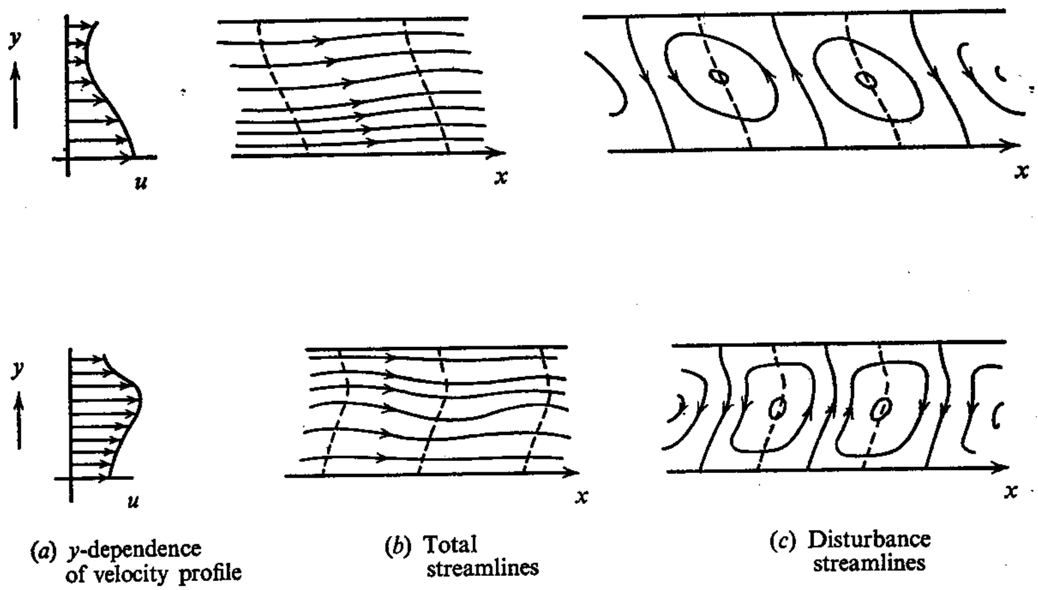

FIGURE 4. Schematic diagrams representing the wave pattern in some given horizontal plane, for two hypothetical mean-flow velocity profiles $u = z+\mu u_1(y)$ whose y-dependences are depicted on the left.

In virtue of (A 2) and (A 5), the lines of constant phase in figure 4 are the same as the lines of constant phase of $\mathrm{Re}\,\{\varphi\, e^{ik(x-ct)}\}$ in a horizontal plane. The shape of the latter is given by the y dependence of $-\mathrm{ph}\,(\varphi)$, $= -\mathrm{ph}\,(\varphi_0+\mu\varphi_1)+O(\mu^2)$, at given z. It is convenient to consider the phase of $(\varphi_0+\mu\varphi_1)$ relative to the phase of φ_0, the latter phase being independent of y. Call this relative phase $\mu\Phi$; then

$$\mu\Phi(y,z) = \mathrm{ph}\,(\varphi_0+\mu\varphi_1)-\mathrm{ph}\,(\varphi_0) = \mu\frac{\mathrm{Im}\,(\varphi_0^*\varphi_1)}{|\varphi_0|^2}+O(\mu^2). \tag{6.3}$$

Consider the gravest mode $m = 1$. Take $u_1 = u_1(y)$, and substitute (6.2) into (6.3). The boundary contribution $-\Phi_B$ to $-\Phi$, arising from the second term in (6.2), may be written, using the form (B 3) for $\mathfrak{G}$ and recalling (A 7), as

$$\begin{aligned}-\Phi_B &= -\frac{\mathrm{Im}\,(\varphi_0^*\varphi_{1B})}{|\varphi_0|^2}+O(\mu) \\ &= \frac{1}{\sin\pi y}\sum_{n=2}^{\infty}\theta_n(z)\,C_n\sin n\pi y+\mathrm{func}\,(z)+O(\mu),\end{aligned} \tag{6.4a}$$

where $$C_n = 2\int_0^1 [u_1(\eta)\sin\pi\eta]\sin n\pi\eta\, d\eta, \tag{6.4b}$$

and $$\theta_n(z) = \mathrm{Im}\left\{\frac{\chi_1^*(z)}{|\chi_1(z)|^2}\left[G_1^n(z;\zeta)\frac{\chi_1(\zeta)}{(\zeta-c_0)^2}\right]_{\zeta=0}^{\zeta=1}\right\}, \tag{6.4c}$$

the second (and irrelevant) contribution to (6.4a) being the $n = 1$ term.

Now if all the θ_n were positive and equal, (6.4a, b) would show at once that $-\Phi_B$, at each height z, would be exactly proportional to $u_1(y)$ (to within an additive function of z).

The θ_n are not equal, but it is possible to show after some manipulation of (6.4c), with the use of (B 3b) and (A 15), that for $k < k_N$ and given ϵ, z,

$$\theta_2 > \theta_3 > \theta_4 > \ldots > 0. \tag{6.5}$$

In words, *the y-dependence of* $-\Phi_B$ *is qualitatively similar to, but more 'smoothed out' than, the y-dependence of* u_1, for any sufficiently simple $u_1(y)$. Note also that $\partial\Phi_B/\partial y = 0$ at $y = 0, 1$, as indeed must be true of $\partial\Phi/\partial y$ because of the boundary conditions. (In establishing (6.5) we use among other things the fact that $p^{-2} - p^{-1}\coth p < 0$, and $d(p^{-2} - p^{-1}\coth p)/dp > 0$ for $p > 0$.)

The remaining contribution to Φ, namely Φ_I, is given by another expression of the form (6.4a) with, say, C_n^I, θ_n^I instead of C_n, θ_n. The C_n^I are given by (6.4b) with $u_{1\eta\eta}$ instead of u_1, and

$$\theta_n^I(z) = \epsilon^{-1}\,\mathrm{Im}\left\{\frac{\chi_1^*(z)}{|\chi_1(z)|^2}\int_0^1 G_1^n(z;\zeta)\frac{\chi_1(\zeta)}{\zeta-c_0}\,d\zeta\right\}. \tag{6.6}$$

Let $u_1(y)$ be given, such that $\Sigma C_n \sin n\pi y\ (\neq 0)$ and $\Sigma C_n^I \sin n\pi y$ are uniformly and absolutely convergent (a very mild restriction), and either let κ_1 be held constant such that $\mathrm{Im}(c_0) \neq 0$ or, alternatively, fix attention on the fastest growing mode. Then we can prove that, uniformly in z, $(\max_y \Phi_I - \min_y \Phi_I)/(\max_y \Phi_B - \min_y \Phi_B) = O(\epsilon^{-1})$ as $\epsilon \to \infty$. This can be interpreted as implying that, *for* $\epsilon >$ *some* ϵ_0 *independent of z, the qualitative result obtained for* Φ_B *also applies to* Φ, as was to be shown. (Here one starts by establishing that, as $\epsilon \to \infty$ under the stipulated conditions, $\theta_n^I = O(1)$ uniformly in $n \geqslant 2$ and z, whereas, for any given $n \geqslant 2$, $\theta_n \geqslant \epsilon \times$ a positive quantity dependent on n but independent of ϵ, z.)†

Note that the u_1 contribution in (6.2) that gives rise to Φ_B does come from a term representing advection, by the mean flow, of the wave pattern; more precisely, of the disturbance 'boundary potential vorticity' (Bretherton 1966a, § 3).

We did not investigate whether or not Φ_I actually does tend to oppose Φ_B (as far as the y-dependence is concerned). A few numerical calculations suggest that, when $u_1 = \sin^2\pi y$, Φ_I does oppose Φ_B, at some but not necessarily all

† We remark that the singular limiting behaviour of Φ can be thought of as reflecting the physical unreality of coherence over width L of an Eady mode (A7) when $L \gg NH/f$, the dominant zonal wave-length. Even the slightest amount of horizontal differential advection $\mu u_1(y)$ will disorganize such a mode if the channel width is too large; conversely, if there *are* modes in the presence of slight horizontal shear, (A7) will no longer represent a first approximation to any of them (cf. Stone 1969). The same remark applies to the singular limit $\kappa \to 0$ mentioned in § 5.

heights z. It is noted here that the part of Φ_I due to an $O(\mu)$ β-effect is y-independent and therefore irrelevant. Also, if u_1 is of the special form ay^2+by, then all of Φ_I is y-independent and so irrelevant.

7. Some finite-μ results, and the effect on the mean flow

A pertinent example of the simple type of profile discussed in § 6 is

$$u = z+\mu\sin^2\pi y. \tag{7.1}$$

We take $\beta = 0$, $\epsilon/\pi^2 = 1{\cdot}62$ ($\epsilon = 16$), and evaluate a number of terms of the series defined by (3.4), etc., for $m = 1$ and $k = k_M$, the zonal wave-number at which the zero-order solution (A 7) has maximum growth rate $k\,\mathrm{Im}\,(c_0)$ for the chosen value of ϵ. From appendix A (iii), $k_M = 6{\cdot}117$, $\kappa = 1{\cdot}719$, $c_0 = \frac{1}{2}+0{\cdot}179i$, $k_M\,\mathrm{Im}\,(c_0) = 1{\cdot}09$.

The first few terms of the c expansion are found to be

$$c = 0{\cdot}5+0{\cdot}179i+0{\cdot}591\mu-0{\cdot}110i\mu^2+0{\cdot}081\mu^3-0{\cdot}193i\mu^4+\ldots. \tag{7.2}$$

Thus the growth rate $k_M\,\mathrm{Im}\,(c)$ is reduced by $O(\mu^2)$, and the phase velocity is increased. (Most of the $O(\mu)$ contribution to the latter, however, is merely a consequence of the increase in the average u as μ increases.) Presumably the wave we are considering is not the dominant wave for the modified profile. But the discussion in § 6 did not depend on the growth rate being maximized with respect to k, and there seems no general reason to believe that any essential features will be lost.

The c and φ expansions were summed for $\mu = 0{\cdot}4$ and $\mu = 0{\cdot}5$, giving, in particular, $c = 0{\cdot}746+0{\cdot}156i$ and $c = 0{\cdot}82_4+0{\cdot}14_0i$ respectively. For comparison, truncation to the terms exhibited in (7.2) gives $c = 0{\cdot}742+0{\cdot}156i$ and $c = 0{\cdot}806+0{\cdot}139i$ respectively, which already come close. Eleven terms were actually calculated, but the last few terms were not, and did not have to be, obtained very accurately. (As one might expect, the higher eigenfunction corrections take on an increasingly complicated spatial structure.) Their main use was as a check on convergence, which appeared safe to an accuracy of 1 or 2 % for the $\mu = 0{\cdot}4$ case,† although less good when $\mu = 0{\cdot}5$. Only the $\mu = 0{\cdot}4$ results will be presented in detail; those for $\mu = 0{\cdot}5$ are very similar.

† The independent finite-difference calculation by Brown mentioned in § 1, for $\mu = 0{\cdot}4$, was done on a 20×40 ($0 \leqslant y \leqslant \frac{1}{2}$, $0 \leqslant z \leqslant 1$) grid and agrees with our $\mu = 0{\cdot}4$ results to better than our roughly estimated accuracy. His c agrees with our value ($0{\cdot}746+0{\cdot}156i$) to three figures. The more stringent test of fitting $|\varphi|_{\max}$ and then comparing detailed results for $-(\overline{u'v'})_y$ gave $[-(\overline{u'v'})_y]_{\max} = 80{\cdot}3$ (cf. our value $81{\cdot}0$, 1 % higher) at bottom centre, and, at top centre, $-(\overline{u'v'})_y = 38{\cdot}9$ (cf. our value $38{\cdot}8$). The contour printout from which the contours in figure 5*d* below were drawn has a resolution of 41×21 points for the half space, and the contours it defines are consistent with Brown's grid point values at each point except for a negligible (0·2 % of max.) inconsistency at the point $80y = 5$, $20z = 18$.

It should be pointed out that Brown's calculation shows one thing that ours cannot, namely that the wave under consideration is in fact the fastest growing quasi-geostrophic instability at the given value of k. (Brown's method amounts to integrating the linearized Fourier-transformed initial value problem.)

For $\mu = 0{\cdot}4$ the dimensionless growth rate is $k_M \operatorname{Im}(c) = 0{\cdot}95_5$, and figures 5*a* and 5*b* show the contours of modulus and negative relative phase of $\varphi(y, z)$. The dotted line is the mean-flow isotach for which $u = \operatorname{Re}(c)$, i.e. the critical 'level'.

Figure 5*b* is of particular interest. The surface represented by the contours can be thought of as a constant-phase surface in physical space, the x-axis being directed out of the paper, for disturbance pressure or streamfunction ψ' or, equally well, transverse velocity $v' = \psi_x$. The characteristic forwards–downwards slope is evident (cf. figure 8), indicating that the mode is still basically a baroclinic instability, as one would expect. But for a pure Eady wave the phase contours would exhibit no other feature, being horizontal straight lines.

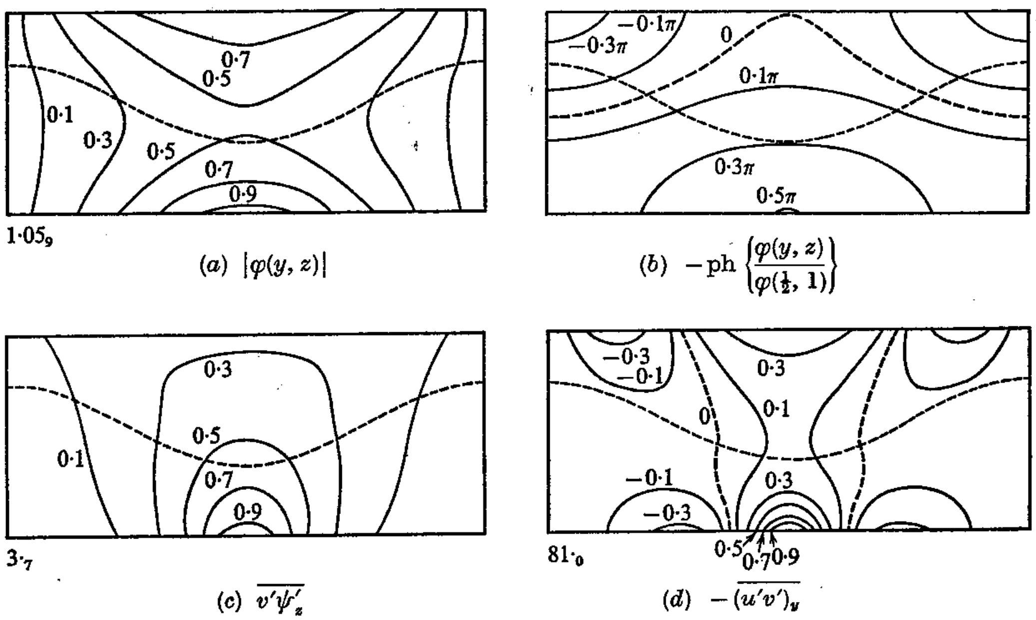

FIGURE 5. Distributions in the meridional or yz plane of quantities associated with an amplifying wave on the mean flow $u = z + 0{\cdot}4\sin^2\pi y$; $\beta = 0$, $\epsilon/\pi^2 = 1{\cdot}62$, $m = 1$, $k = k_M = 6{\cdot}12$. In (*a*), (*c*), and (*d*), the contour values are given as fractions of the maximum (dimensionless) value, shown on the left. The latter corresponds in each case to normalization as in (A 7) of the zeroth approximation φ_0. The dimensionalizing scales may be deduced from appendix A (ii). In (*d*), the eddy contribution to $\partial u/\partial t$, and (*c*), the transverse horizontal eddy flux of heat (i.e. buoyancy), the values are understood to be multiplied by the square of the actual (small) amplitude, times a factor $\exp\{2k \operatorname{Im}(c)t\}$. The thin dotted line in each diagram is the locus of points (y, z) such that $u(y, z) = \operatorname{Re}(c)$.

The actual structure in the horizontal is of the same general nature as that given by the first correction term (§ 6). The resulting Reynolds stress component is indicated by figure 5*d*, which plots the convergence $-(\overline{u'v'})_y$ or associated contribution to the zonal momentum tendency u_t. It is positive where the mean flow is already strongest.

Figure 5*c* shows the dimensionless transverse horizontal eddy heat flux $\overline{v'\psi'_z}$. It is in the positive y-direction, i.e. down the mean gradient, reflecting the fact that the growing wave, being a baroclinic instability, is drawing on mean flow potential energy.

The sharpness and prominence of the maximum at $z = 0$, $y = \frac{1}{2}$ in figure 5d is a higher order (finite μ) effect; in that respect the results go beyond what could have been expected from § 6. The recently published results of Brown (1969a), for a compressible atmosphere on a β-plane and a more complicated u profile in which the horizontal shear increases with height, show a behaviour that is similar in at least some respects. (Note the z-dependence of the y-average of Brown's $C(K_z, K_e) \equiv -u(\overline{u'v'})_y$ shown in his figure 5, in conjunction with u as given by his figure 1 and equation (3.1).)

The zonal momentum tendency

Although calculations of $-(\overline{u'v'})_y$ are suggestive by themselves, they do not actually give the second-order (in wave amplitude) rate of change of u, even though $-(\overline{u'w'})_z$ *is* negligible within the quasi-geostrophic approximation. To find u_t one must also take into account, explicitly or implicitly, the Coriolis force due to the slow mean meridional circulation that arises as a response to the strong dynamical requirement that mean-flow geostrophic and hydrostatic balance continue to hold (Eliassen 1952).

Calculations of u_t and of the stream function Ξ for the mean meridional circulation have been carried out. Their gross features are very much as would have been expected from the pioneering results of Phillips (1954) and Eliasen (1961) for the cruder two-level model. The mathematical framework involved is much the same as in those papers; details are given elsewhere (McIntyre 1967).

For comparison with the eddy contribution $-(\overline{u'v'})_y$ to it, u_t for the case $u = z + 0{\cdot}4 \sin^2 \pi y$ is given in figure 6a; figure 6b shows the associated Ξ (see caption for details); note that the main part of the latter is thermally indirect.† Again, the results for $\mu = 0{\cdot}5$ are qualitatively similar.

The distribution of u_t (figure 6a) is recognizably similar to that of the eddy contribution (figure 5d). It is still positive around $y = \frac{1}{2}$, but considerably reduced at the top, $y = \frac{1}{2}$, $z = 1$, because of the Coriolis force associated with Ξ. The curiously sharp maximum at $y = \frac{1}{2}$, $z = 0$, is present also in the distribution of u_t.

This feature assumes considerable interest when we look at the mean zonal velocity field found in a recent numerical solution for the wave regime in a heated rotating annulus (Williams 1969). This is reproduced in figure 7c (see caption for details). If our result of figure 6 is a representative one, it shows in particular that baroclinic waves would initially tend to bring about a sharp peak in the horizontal

† This corresponds to a small positive rate of transformation of zonal mean kinetic into mean available potential energy, $C(K_z, A_z)$ (in the notation of Brown 1969a) $= +0{\cdot}4_3$ per unit zonal distance, in dimensionless units. Calling the disturbance energies K_e and A_e, the other customarily-defined energetic quantities have the values $C(A_z, A_e) = 14{\cdot}_4$, $C(A_e, K_e) = 6{\cdot}3_6$, $C(K_e, K_z) = 1{\cdot}0_3$; $\partial A_e/\partial t = 8{\cdot}_0$, $\partial K_e/\partial t = 5{\cdot}3_3$, $\partial A_z/\partial t = -14{\cdot}_0$, and $\partial K_z/\partial t = +0{\cdot}6_0$, so that the waves are bringing about a net *increase* in zonal mean kinetic energy, although at a rather small rate in this example. One might expect a greater rate at larger horizontal shear. Similar energy transformations are known to take place in the westerly wind systems of the earth's atmosphere.