dependence of u near the bottom centre of the cross-section. That is exactly what is seen in figure 7*c*; notice the kinks in the isotachs. It should also be mentioned that the finite amplitude waves found in the numerical solution are indeed very similar in spatial structure to the baroclinic waves of linearized theory (Williams, *op. cit.*).

It would be interesting to extend the calculations to incorporate the Ekman suction due to a lower frictional boundary layer. That would be straightforward (Barcilon 1964), but has not yet been done for this model (although the more sophisticated model studied by Brown 1969*b* incorporates such an effect, among many others). A significant gain in meteorological realism may however require other refinements as well, including non-linear effects (Smagorinsky 1964, p. 3; Thompson 1959, § 1).

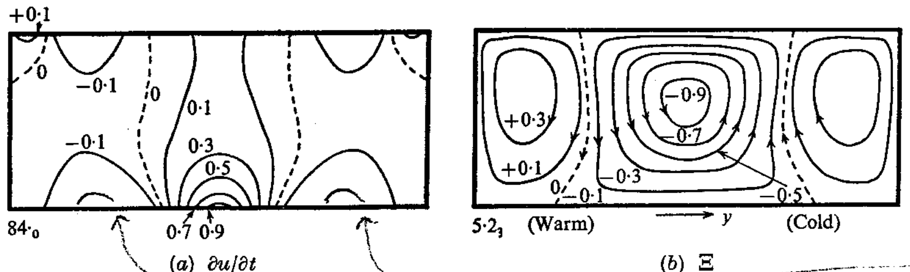

FIGURE 6. Effect of the amplifying wave on the mean flow, for the same case as in figure 5. (*a*) Dimensionless rate of change u_t of the mean zonal momentum. (*b*) Streamfunction Ξ of the associated mean meridional circulation ($w = \Xi_y$, $v = -\Xi_z$). The contour values are to be understood in the same sense as in figure 5. The dimensionalizing scale for Ξ is $Ro\, HU$, $\times$ (amplitude)$^2 \times \exp\{2k\,\mathrm{Im}(c)\,t\}$.

NB missing contours $\partial u/\partial t = -0{\cdot}3$, cf. Fig. 5d.

To relate the above results to the large scale motions of the atmosphere, note that if L is formally identified with 40° of latitude (although $\beta = 0$ in these calculations) and $H = 10\,\mathrm{km}$, and if the wave amplitude is such that the north–south velocity has amplitude $10\,\mathrm{m\,sec^{-1}}$ at the 'tropopause' and $12\,\mathrm{m\,sec^{-1}}$ at the ground, then u_t has respective values of 0·7 and $5{\cdot}6\,\mathrm{m\,sec^{-1}\,day^{-1}}$. This is of the right order of magnitude to be invoked as a partial explanation of the maintenance of the westerly winds (against the frictional retardation that is not included in our model). The zonal wavelength is about $4500 \times 2\pi/k_M \simeq 4500\,\mathrm{km}$, and if the vertical shear of the mean flow is $U/H = 2{\cdot}5\,\mathrm{m\,sec^{-1}\,km^{-1}}$, then the doubling time is $(\ln 2/k\,\mathrm{Im}(c))\,(L/U) = 1{\cdot}5$ day.

8. Concluding remarks

The perturbation method has proved to be a powerful tool, having permitted a significant degree of generalization of our precise mathematical knowledge of the non-separable problem (2.1), (appendix D), the computation of accurate and physically interesting solutions to it (§ 7) and, most important, physical insight into aspects of the processes it describes (§§ 5, 6), through interpretation of the first correction terms.

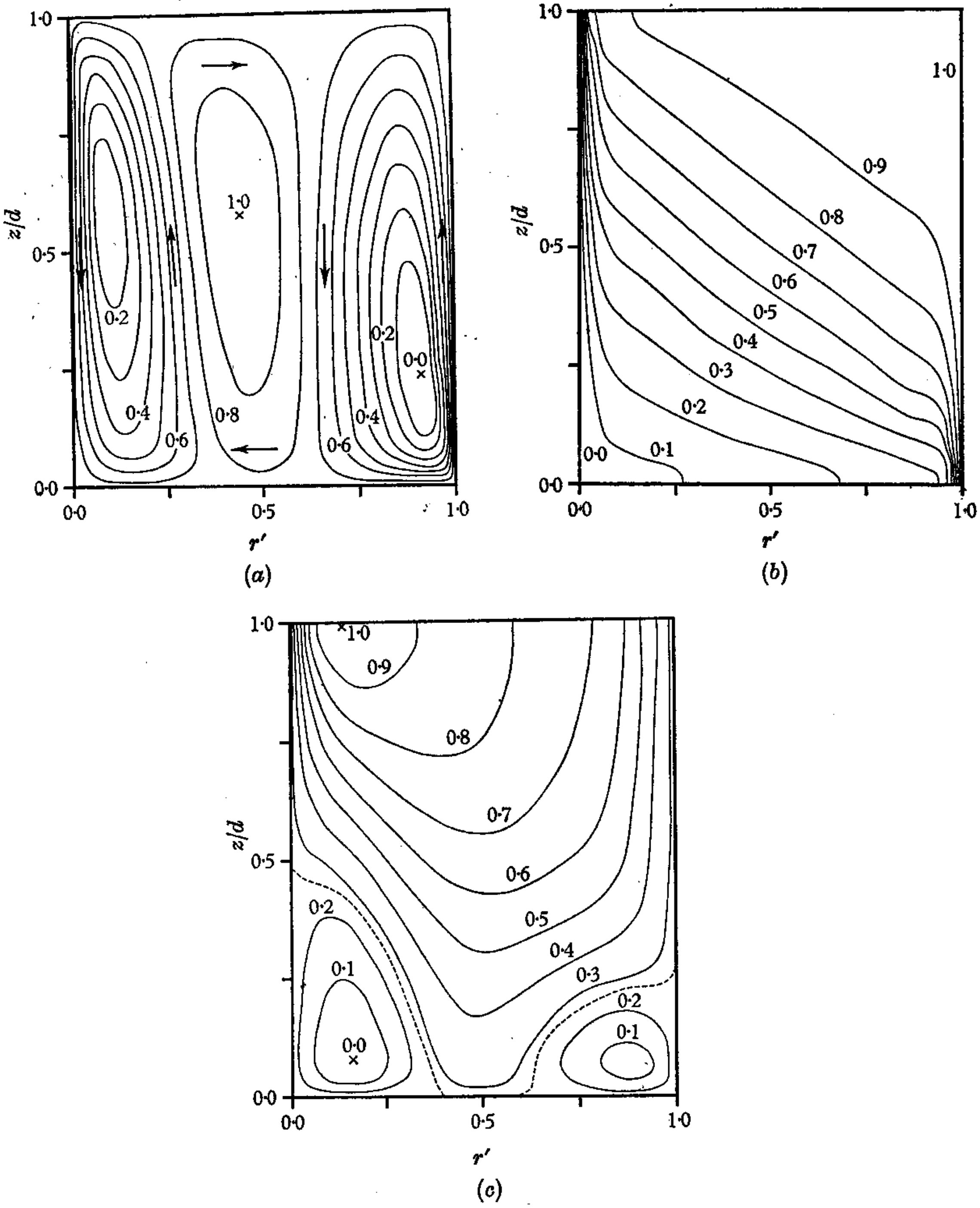

FIGURE 7. Contours of the (steady) zonal averages of the meridional circulation (Stokes stream function), temperature, and zonal velocity in the regular wave regime, with wave-number 5, of convection in a rotating annulus, from a numerical solution obtained by Williams (1969). The inner cylindrical cold wall $r = 2$ cm ($r' = 0$) is on the left of the meridional cross-section and is held at 17·5° C; the outer hot wall $r = 5$ cm ($r' = 1$) on the right, is held at 22·5° C. The contours are evenly spaced, between $-0{\cdot}05347$ and $0{\cdot}01759$ cm^3 sec^{-1} in (*a*), and between $-0{\cdot}1090$ and $+0{\cdot}3027$ cm sec^{-1} in (*c*). The thermally insulated top and bottom boundaries are 3 cm apart; the top is stress-free while all the other boundaries are no-slip. The rate of rotation is 0·8 rad sec^{-1}, and the viscosity, thermal diffusivity and expansion coefficients are $1{\cdot}008 \times 10^{-2}$ cm^2 sec^{-1}, $1{\cdot}420 \times 10^{-3}$ cm^2 sec^{-1}, and $2{\cdot}054 \times 10^{-4}$ °C^{-1}.

The versatility of this kind of method, within its range of validity, is not confined to the present problem; it is formally capable of handling any non-singular perturbation to *any* well-posed conventional, or unconventional, differential-equation eigenvalue problem. In the present problem, for instance, one could incorporate the corrections to (2.1) representing higher approximations to Euler's equations of motion, as has recently been done for the non-geostrophic effects in an independent study by Derome & Dolph (1969). In the latter type of connexion especially, the automatic elimination of physically different modes from consideration (e.g. inertia–gravity waves) is often an advantage, rather than otherwise.

As well as the calculations of § 7, the results of § 5 on perturbed stability properties bear upon the theoretical interpretation of the baroclinic wave motions found in the heated rotating *annulus experiments* (Fowlis & Hide 1965). The lack of a well-defined inviscid short wave cutoff for a wide class of profiles $u(y,z)$ (as in (*b*), figure 3) shows that the concept of an inviscid limit for the so-called upper transition curve is probably not well-defined theoretically. (It should be mentioned that, worse still, recent theoretical considerations relevant to the symmetric régime above the transition curve have indicated that the inviscid limiting behaviour of the basic symmetric flow itself is exceedingly pathological in many cases of interest (see McIntyre 1969*b*).)

Two points are worth making about the non-uniform validity over $0 \leqslant z \leqslant 1$ of the φ representation when $k \geqslant k_N$. One is that it is possible, although cumbersome, to modify the perturbation scheme so as to give a uniformly valid representation, if φ_l and c_l are allowed to depend on μ (McIntyre 1967, p. 202). We then lose the convenience of having true power series in μ. The other is that in any case, the non-uniformity resulting from the present scheme may be quite mild in practice, and the φ representation still useful. For a recent exploitation of that, see McIntyre (1969*c*).

In § 6 we discussed the first-order effect of the horizontal shear of a profile of form $u = z + \mu u_1(y)$ upon a growing Eady wave, and in § 7 we presented numerical calculations to show that the $O(\mu)$ trend persists up to $\mu = 0{\cdot}5$ for the case $u_1 = \sin^2 \pi y \; (0 \leqslant y \leqslant 1)$, and that it does in that case give a qualitative idea of the actual mean flow change. Differential advection by any given reasonably smooth $u_1(y)$ always brings about an $O(\mu)$ eddy stress that transports zonal momentum against the mean flow gradient u_{1y}, *if* the motion is not too closely constrained laterally. (But there is at least one kind of profile, namely parabolic y-dependence, for which the latter proviso is not necessary.)

A qualitatively similar behaviour of the eddy stress is a feature of the recent results of Stone (1969) for a model whose lateral constraints are deliberately made 'slight'; his basic approximation scheme, in contrast to ours, requires u_y to be small at the outset, via an assumption that the y-scale of the basic flow $u(y,z)$ is large (as in Miles 1964). Here 'large' implies comparison with the scale NH/f of the fastest growing waves, which emerges naturally as a y-scale, as well as an x-scale, from Stone's analysis. His velocity profiles are different from ours, the horizontal shear being confined to the upper half of the two-level model used.

Finally, it should not be forgotten that by our particular choice of zero-order solution we have confined our attention to a particular kind of instability. The reader is referred to Brown (1969*a*) for some interesting results not subject to that restriction.

Much of this research was carried out while the writer was a research student at Cambridge under F. P. Bretherton, whose guidance, criticism and encouragement are gratefully acknowledged. The writer also thanks J. W. Elder for the use of certain computer programs developed by him, and the Cambridge University Mathematical Laboratory for the use of their computing facilities. Opportunities to discuss aspects of the work with J. Pedlosky and J. A. Brown, and access to unpublished results of Brown, were greatly appreciated. The work was supported by a Commonwealth Scholarship awarded by the Commonwealth Scholarship Commission in the United Kingdom, and was also supported by National Science Foundation Grant GA-402X at the M.I.T. Meteorology Department.

Appendix A. Theoretical background

(i) *Dynamics of the baroclinic instability*

The following brief description is not novel, nor is it intended to replace discussions such as those given by Bretherton (1966*b*) and, from a somewhat different point of view, by Holmboe (1959). But it quickly makes the instability plausible, and gives a substantially correct feel for the dynamics.

As was said in § 2, the fluid is stably stratified with buoyancy frequency N, but possesses available potential energy associated with a small slope $(\partial z/\partial y)_\rho$ of the lines of constant density in a meridional or yz plane. To fix ideas, suppose first that $(\partial z/\partial y)_\rho$ is positive and constant.

Imagine an initial disturbance involving a small transverse horizontal velocity v' with a wave-like x-dependence, wavelength $2\pi L_w$ say. If, hypothetically, buoyancy forces represented the only constraint on the disturbance, fluid elements drifting to the left or to the right of the main current would just tend to move along the sloping constant-density surfaces. But other effects are of course present; it is only because of one of them, the Coriolis force, that the undisturbed constant-density surfaces can slope at all. If these other effects were such as to make fluid elements move *more nearly horizontally*, i.e. on paths with positive slope less than $(\partial z/\partial y)_\rho$, then potential energy could clearly be released. The buoyancy force could do work against whatever was causing the fluid particles to move on their shallower trajectories.

Now the instability is possible because sideways-drifting fluid elements can indeed be made to move along such paths, in the simple situation we are considering, by a combination of two things. The first is the presence of rigid boundaries that are either horizontal, or nearly so, with slope less than $(\partial z/\partial y)_\rho$. The second is the resistance to horizontal divergence that arises from a sufficiently strong Coriolis effect f; this 'rotational stiffness' has the effect of making the kinematical constraint due to a boundary felt throughout a substantial depth ($\propto f$) of fluid (although not in the simple Taylor–Proudman sense appropriate to a homogeneous fluid). The penetration height scale is in fact fL_w/N (Walin 1969).

Note that if $(\partial z/\partial y)_\rho$ were allowed to vary with height, a level where $(\partial z/\partial y)_\rho$ is relatively small could play the same role as a rigid boundary (see Green 1960, § 8; McIntyre 1969*c*).

In either case, it is the resulting kinematical-rotational constraint that gives rise to the pressure field against which the buoyancy force is enabled to do work. From the point of view of vorticity, the buoyancy force can be thought of as slowly stretching or compressing the very strong tubes of absolute vertical vorticity of the rotation f, the effect of which is described by the dominant term $f\,\partial w/\partial z$ in the vertical vorticity equation.

The work done by buoyancy, then, appears as kinetic energy of the *horizontal* relative velocities associated with the resulting 'spin-up' relative vorticity. To a first approximation, the Coriolis force does the actual accelerating. (One should note the complete contrast with e.g. Solberg's symmetric baroclinic instability (see McIntyre 1969*a* and references), a type of essentially non-geostrophic sloping convection in which buoyancy *can* contribute directly to the acceleration of a fluid element.)

The horizontal velocities produced by vortex-tube stretching can indeed reinforce the original disturbance, giving exponential growth, provided that there is an appropriate phase change with height. It is found that the surfaces of constant phase of v' must slope forwards–downwards, so that $\partial z/\partial x < 0$, i.e. they slope in the sense 'opposite to that of the velocity profile'. To see this, and to understand among other things the necessary role of differential advection by the vertical shear u_z, a more detailed description is needed (see e.g. Bretherton 1966*b*).

(ii) *The basic formulation*

This is well known (Phillips 1963; Pedlosky 1964*a*) and will be sketched only briefly, for the Boussinesq liquid case. First, the hydrostatic and geostrophic approximations are made. The latter signifies an approximate balance between horizontal pressure and Coriolis forces, the condition for which is formally expressed by the smallness of the Rossby number,

$$Ro = U/fL \ll 1, \tag{A 1}$$

where U is a characteristic horizontal velocity and L a horizontal length (taken for convenience as the channel width, in the present problem). The time scale is assumed $\gtrsim L/U$. In this approximation the departure ψ from the horizontally-averaged hydrostatic pressure becomes a stream function for the dimensionless horizontal velocities, after ψ is made dimensionless by the scale $f\rho_0 UL$, where ρ_0 is an average density for the whole (Boussinesq) fluid. The approximate velocities (scale U) are then

$$u = -\psi_y, \quad v = \psi_x, \tag{A 2}$$

in the x and y directions respectively. The vertical velocities are small of order

$(RoH/L)\,U$, but important because of vortex-tube stretching. We assume $\epsilon \equiv f^2L^2/N^2H^2 \sim 1$, in the formal limit $Ro \to 0$ implied by (A 1).

Under all the above assumptions it can be shown that N^2, and thus ϵ, can be taken as a horizontal and time average and thus as a function of height z only, and that the vorticity equation for inviscid adiabatic motion can be reduced to a single approximate equation involving the vertical component only of the dimensionless absolute vorticity, $Ro^{-1} + \psi_{xx} + \psi_{yy} + O(Ro)$:

$$\left(\frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right)[Ro^{-1} + \psi_{xx} + \psi_{yy} + (\epsilon\psi_z)_z] = 0. \tag{A 3}$$

The ϵ term represents the stretching of vertical vortex tubes. The vertical velocity is related to ψ through the adiabatic equation. To sufficient accuracy,

$$w = -\epsilon\left(\frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right)\psi_z. \tag{A 4}$$

The scale for w is $(RoH/L)\,U$. In virtue of the hydrostatic relation, $-\psi_z$ represents the local density anomaly due to the motion. In (A 3) the quantity in square brackets is related to Ertel's potential vorticity in the manner explained by Charney & Stern (1962, p. 163) and will be called a 'quasi-potential-vorticity'.

The eigenvalue problem for normal-mode disturbances to the mean flow $u(y,z)$ is now obtained by posing

$$\psi = -\int^y u(\eta, z)\,d\eta + \psi'; \quad \psi' = \mathrm{Re}\,\{\varphi(y,z)\,e^{ik(x-ct)}\} \tag{A 5}$$

where, formally, $|\psi'| \ll |\psi|$. The dimensionless wave-number k is considered real, but c and $\varphi(y,z)$ may be complex. Then (A 3) yields the linearized equation (2.1a). The coefficient q_y that appears in (2.1a) is the transverse gradient of mean quasi-potential-vorticity,

$$q_y = \beta - (\epsilon u_z)_z - u_{yy}; \tag{A 6}$$

$\beta = d(Ro^{-1})/dy \doteq$ const. is included to represent the earth's planetary vorticity gradient or north–south variation of f, where relevant. The boundary conditions for (2.1a) are that w and $v = \psi'_x$ vanish on horizontal and vertical boundaries respectively, which yields (2.1b) and (2.1c).

(iii) *Eady's solution*

When $u = z$, $\beta = 0$, $\epsilon(z) =$ constant, so that $q_y = 0$, it can easily be shown that (2.1) has the following closed form solutions, which were first described by Eady (1949):

$$\left.\begin{aligned} \varphi_{0m}(y,z) &= \sin m\pi y \,.\, \chi_m(z), \\ c_{0m} &= \tfrac{1}{2} \pm \tfrac{1}{2}\alpha_m^{-1}[(\alpha_m - \coth\alpha_m)(\alpha_m - \tanh\alpha_m)]^{\frac{1}{2}}, \\ \text{where} \quad \chi_m(z) &= \kappa_m c_{0m}\cosh\kappa_m z - \sinh\kappa_m z \\ \text{and} \quad \kappa_m &= 2\alpha_m = \epsilon^{-\frac{1}{2}}(k^2 + m^2\pi^2)^{\frac{1}{2}} \quad (m = 1, 2, \ldots). \end{aligned}\right\} \tag{A 7}$$

For each integer m these solutions represent either an amplifying-decaying pair of waves with $c_r = \mathrm{Re}\,(c_{0m}) = \frac{1}{2}$, or a pair of neutral waves ($c = c_r$, $\geqslant$ and $\leqslant \frac{1}{2}$), according as

$$\alpha_m < \quad \text{or} \quad \geqslant \alpha_N, \quad = 1{\cdot}1997 (\alpha_N = \coth \alpha_N). \tag{A 8}$$

Thus there is a short-wave cut-off to the instability, at a critical neutral wave-number $k = k_N = (4\epsilon\alpha_N^2 - m^2\pi^2)^{\frac{1}{2}}$, if k_N is real. For any m such that k_N is real, there is a well-marked maximum growth rate $kc_i\,[c_i \equiv \mathrm{Im}\,(c_{0m})]$ at some $k < k_N$: $k = k_M$, say. The largest of these maxima occurs, if any occur, for $m = 1$. Some numerical values for $m = 1$ are shown in table 1.

ϵ/π^2	ϵ	k_M	κ	c_i	$k_M c_i$
0·9119	9	4·392	1·800	0·1677	0·7366
1·6211	16	6·117	1·719	0·1787	1·0931

TABLE 1

For the long waves in the atmosphere (wavelength 6000 km), these dimensionless maximum growth rates typically correspond to doubling times in the vicinity of 2 days. The structure in an xz plane of an unstable wave is shown in figure 8, from Eady (1949).

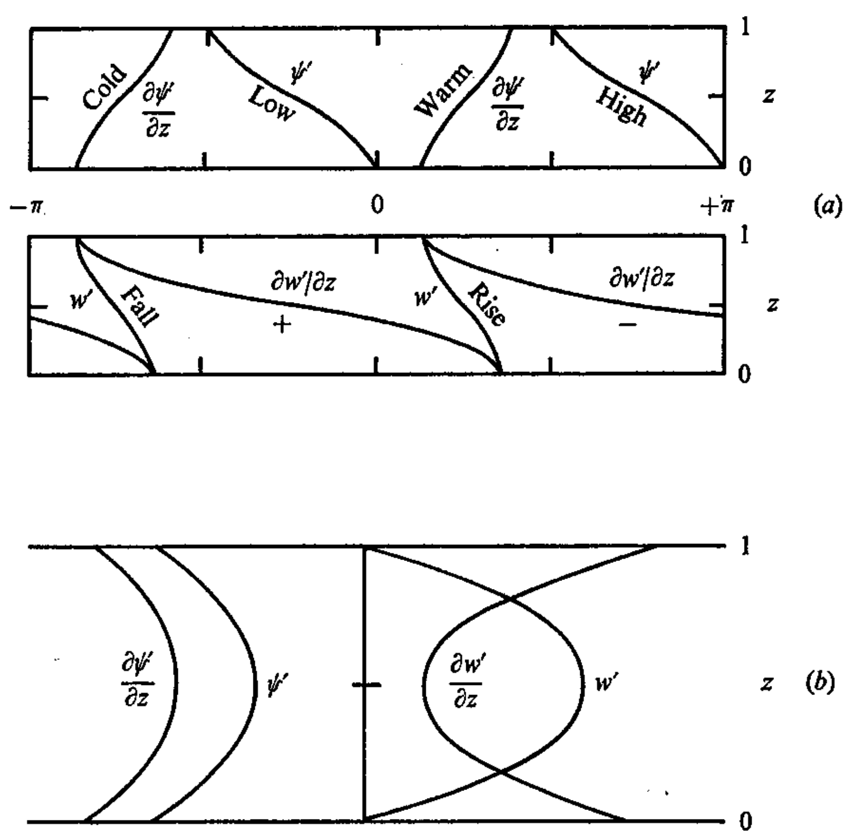

FIGURE 8. (*a*) Negative relative phases and (*b*) amplitudes of an amplifying baroclinic wave on the simple velocity profile $u = z$, from Eady (1949). The negative-phase diagrams give the actual side view of the wave if the x axis points toward the right.

Note that an unstable $\chi_m(z)$ possesses symmetry about $z = \frac{1}{2}$ that is obscured by the otherwise convenient form given. If we define a symmetrical co-ordinate $z_s = z - \frac{1}{2}$, then χ_m is equal to a complex constant times

$$\cosh \kappa_m z_s + (\text{imaginary constant}) \times \sinh \kappa_m z_s. \tag{A 9}$$

At a critical neutral point $k = k_N$ we have $c_{0m} = \frac{1}{2}$, and

$$\varphi_{0m} = \varphi_{0m}^N \equiv \sin m\pi y \,.\, (\alpha_N^2 - 1)^{\frac{1}{2}} \cosh 2\alpha_N z_s. \tag{A 10}$$

The short neutral waves ($k > k_N$) are asymmetrical. For $k \gg k_N$, each is associated with one horizontal boundary exclusively, because of a small penetration height scale fL_w/N, $\ll H$.

Note that if generalized functions are admitted as solutions, there is also, for each k, a continuous spectrum of singular neutral modes with $0 \leqslant c \leqslant 1$ (Pedlosky 1964*c*). At $z = c$, φ_z has a jump discontinuity. These resemble the singular modes discovered by Rayleigh (1895) and used by Orr (1907) to solve the initial value problem for small disturbances to plane Couette flow.

The following identities are useful; we drop the suffix m:

$$\kappa^2 c_0(1-c_0) + 1 = \kappa \coth \kappa, \tag{A 11}$$

$$\{1 - \kappa^2 c_0^2\}\{1 - \kappa^2(1-c_0)^2\} = \kappa^2 \operatorname{cosech}^2 \kappa, \tag{A 12}$$

$$\int_0^1 dy \left[\frac{\{\varphi_0(y,z)\}^2}{(z-c_0)^2}\right]_{z=0}^{z=1} = \frac{\kappa^4(c_0 - \frac{1}{2})}{\kappa^2(1-c_0)^2 - 1}, \tag{A 13}$$

$$\int_0^1 \int_0^1 \{\varphi_0^N(y,z)\}^2 \, dy\, dz = \tfrac{1}{4}\alpha_N^2, \tag{A 14}$$

$$\frac{\chi(1)\,\tilde{\chi}(1-z)}{\kappa(1-c_0)} = \chi(z), \tag{A 15}$$

where $\tilde{\chi}$ denotes the result of replacing c_0 in χ by $(1-c_0)$.

Appendix B

(i) *Definition of B_l, I_l*

It is convenient first to substitute the expansion (3.2*b*) into (2.1*a*, *b*), divide by $(z-c_0)$, and rearrange. (If c_0 is real it is necessary to assume that u_1 is analytic in z and to go into the complex plane, as discussed in § 4.) There results

$$\left.\begin{aligned} L(\varphi) &\equiv \varphi_{zz} + \epsilon^{-1}(\varphi_{yy} - k^2\varphi) \\ &= \frac{-\mu u_1 + \mu c_1 + \mu^2 c_2 + \dots}{z - c_0} L(\varphi) - \frac{\mu\epsilon^{-1} q_{1y}}{z - c_0}\varphi \\ \text{and on } z = 0, 1, \quad D(\varphi) &\equiv \varphi_z - \frac{\varphi}{z - c_0} \\ &= \frac{-\mu u_1 + \mu c_1 + \mu^2 c_2 + \dots}{z - c_0}\varphi_z + \frac{\mu u_{1z}}{z - c_0}\varphi. \end{aligned}\right\} \tag{B 1}$$

We then introduce (3.2a) and require that (B 1) be satisfied separately at each order $\mu^l (l = 0, 1, \ldots)$, giving (3.3). The right-hand sides I_l, B_l are defined recursively by $I_0 = B_0 = 0$ and, for $l \geqslant 1$,

$$(z-c_0) I_l = \sum_{j=1}^{l-1} c_j I_{l-j} - u_1 I_{l-1} - \epsilon^{-1} q_{1y} \varphi_{l-1}, \tag{B 2a}$$

$$(z-c_0) B_l = (z-c_0) B'_l + (z-c_0)^{-1} c_l \varphi_0, \tag{B 2b}$$

where
$$(z-c_0) B'_l = \sum_{j=1}^{l-1} c_j \left\{ B_{l-j} + \frac{\varphi_{l-j}}{z-c_0} \right\} - u_1 \left\{ B_{l-1} + \frac{\varphi_{l-1}}{z-c_0} \right\} + u_{1z} \varphi_{l-1}. \tag{B 2c}$$

$\left(\sum_{j=1}^{0} \text{gives zero, by convention.} \right)$ For convenience, derivatives of φ_l have been eliminated at each stage, by the use of (3.3), as happens to be possible in this problem.

(ii) *The generalized Green's function*

It is a straightforward task to obtain the solution to (3.5) and (3.6) as a sine series in y, whose coefficients are functions of z with discontinuities in their first derivatives at $z = \zeta$. Note that $\mathfrak{G}$ depends on m, both through the operator D, in which the value of c_0 is given by (A 7), and through the presence of φ_0 in (3.5a) and (3.6); thus we write

$$\mathfrak{G} = \mathfrak{G}_m(y, z; \eta, \zeta) = 2 \sum_{n=1}^{\infty} \sin n\pi y \sin n\pi\eta \, G_m^n(z; \zeta). \tag{B 3a}$$

For $n \neq m$, it is found that

$$G_m^n(z;\zeta) = \frac{[\kappa_n c_{0m} \cosh \kappa_n z_< - \sinh \kappa_n z_<][\kappa_n(1-c_{0m}) \cosh \kappa_n (1-z_>) - \sinh \kappa_n (1-z_>)]}{\kappa_n^3 \sinh \kappa_n [c_{0m}(1-c_{0m}) + \kappa_n^{-2} - \kappa_n^{-1} \coth \kappa_n]}, \tag{B 3b}$$

where
$$z_< = \min(z, \zeta), \quad z_> = \max(z, \zeta), \tag{B 3c}$$

with the obvious interpretation if z and ζ both lie on a complex contour Γ_z such as that in figure 1, and where (cf. (A 7))

$$\epsilon\kappa_n^2 = k^2 + n^2\pi^2.$$

This is a one-dimensional Green's function in the usual sense. For $n = m$,

$$\begin{aligned} G_m^m(z;\zeta) = {} & A'\chi_m(z)[\kappa_m c_{0m}^* \zeta \sinh \kappa_m \zeta - (\zeta - c_{0m}) \cosh \kappa_m \zeta] \\ & - (1/\kappa_m) \chi_m(z_<) \cosh \kappa_m z_> + \chi_m(\zeta) \times \text{func}(z), \end{aligned} \tag{B 3d}$$

where
$$A' = \left\{ 2\kappa_m \int_0^1 \chi_m \chi_m^* \, dz \right\}^{-1},$$

$$= \begin{cases} \dfrac{1}{\kappa_m^3 |c_{0m}|^2} = \dfrac{1}{\kappa_m^3 c_{0m}(1-c_{0m})}, & \text{if } c_{0m} \text{ is complex,} \\[2ex] \dfrac{\kappa_m^2 (1-c_{0m})^2 - 1}{\kappa_m^3 [\kappa_m^2 c_{0m}^2 (1-c_{0m})^2 + 3c_{0m}(1-c_{0m}) - 1]}, & \text{if } c_{0m} \text{ is real.} \end{cases}$$

The contribution to G_m^m not explicitly written out may be ignored when $\mathfrak{G}_m$ is used as in (3.4b), that is, with ζ as the dummy variable, in virtue of the solubility condition. (The extra contribution consists of the symmetrizing complement of the first line of (B 3d), plus a further term proportional to $\chi_m(z)\,\chi_m(\zeta)$.)

Appendix C. The expansion at the Eady neutral point $k = k_N$

Posing the expansions (3.7) leads to the sequence, dropping the superscript N,

$$\begin{cases} L(\varphi_l) = I_l, & \text{(C 1}a\text{)}\\ D(\varphi_l) = B_l \quad \text{on} \quad z = 0, 1, & \text{(C 1}b\text{)}\\ \varphi_l = 0 \quad \text{on} \quad y = 0, 1, & \text{(C 1}c\text{)} \end{cases}$$

in which, this time, $I_0 = I_1 = B_0 = 0$, and $z_s B_1 = c_1 \varphi_{0z}$, where $z_s \equiv z - \frac{1}{2}$. The formula (A 10) gives φ_0. The first-order problem is now automatically soluble for φ_1, because of the same symmetry that was associated with the breakdown of the μ expansion. In fact, noting that (C 1a) for $l = 1$ is satisfied by φ_{0z}, and recalling (A 8), we find that

$$\varphi_1 = \frac{c_1 \varphi_{0z}}{\alpha_N^2 - 1}, \quad = \frac{2\alpha_N c_1}{(\alpha_N^2 - 1)^{\frac{1}{2}}} \sin m\pi y \,.\, \sinh 2\alpha_N z_s. \qquad \text{(C 2)}$$

A point of interest that now emerges is that c_1 will not be determined until the problem for φ_2 is considered, and so on.

In the same way as before, recursion formulae give I_l and B_l for $l \geqslant 1$. With the convention $\varphi_{-1} = I_{-1} = I_0 = B_0 = 0$, we can show that for $l \geqslant 1$

$$z_s I_l = \sum_{j=1}^{l-2} c_j I_{l-j} - u_1 I_{l-2} - \epsilon^{-1} q_{1y} \varphi_{l-2}, \qquad \text{(C 3}a\text{)}$$

$$z_s B_l = z_s B_l^{(d)} + c_l \varphi_{0z}. \qquad \text{(C 3}b\text{)}$$

(As before, summations with reversed limits are zero; (C 1a) has been used.) $B_l^{(d)}$ will be defined below, by (C 3c, d). It is zero for $l = 1$, and for $l \geqslant 2$ it will involve c_{l-1} but not c_l, and will turn out to be completely determined at the current (lth) stage. The term $c_l \varphi_{0z}$ will not be determined, since c_l disappears from the solubility condition by symmetry, as before. That condition is

$$-\iint \varphi_0 I_l \, dy \, dz + \int dy \left[\varphi_0 B_l^{(d)} \right]_{z=0}^{z=1} = 0. \qquad \text{(C 4)}$$

But we are free to choose c_{l-1}, since by the same token it could not have been determined previously. It will turn out that (C 4) can always be satisfied by just one such choice (apart from a sign ambiguity when $l = 2$). The solution φ_l which then exists can be split into two parts. One arises from the c_l term in (C 3b), and is just c_l/c_1 times (C 2). The remaining part $\varphi_l^{(d)}$ is independent of c_l, and will thus be determinate at the present stage:

$$\varphi_l^{(d)} = -\iint \mathfrak{G} I_l(\eta, \zeta) \, d\eta \, d\zeta + \int d\eta \left[\mathfrak{G} B_l^{(d)}(\eta, \zeta) \right]_{\zeta=0}^{\zeta=1}. \qquad \text{(C 5)}$$

The whole solution, then, is

$$\varphi_l = \varphi_l^{(d)} + \frac{c_l \varphi_{0z}}{\alpha_N^2 - 1}. \tag{C 6a}$$

This agrees with (C 2), since $\varphi_1^{(d)} = 0$.

We can now complete the recursive definition of B_l by supplying the definition of $B_l^{(d)}$. It will be convenient to define first, for $l \geqslant 2$,

$$z_s B_l^{(c)} = c_1(B_{l-1}^{(d)} + z_s^{-1}\varphi_{l-1}^{(d)}) + \sum_{j=2}^{l-2} c_j(B_{l-j} + z_s^{-1}\varphi_{l-j}) - u_1(B_{l-2} + z_s^{-1}\varphi_{l-2}) + u_{1z}\varphi_{l-2}. \tag{C 3c}$$

Then, using (C 6*a*) above, (C 3*b*), (C 2), and (C 1*b*), we may define $B_l^{(d)}$ and thence B_l by

$$\left.\begin{aligned} B_1^{(d)} &= 0,\\ z_s B_l^{(d)} &= z_s B_l^{(c)} + h_l c_1 c_{l-1}\left(\frac{\alpha_N^2}{\alpha_N^2-1}\right)\frac{\varphi_{0z}}{z_s} \quad (l \geqslant 2),\end{aligned}\right\} \tag{C 3d}$$

where

$$h_l = \begin{cases} 1 & \text{if} \quad l = 2,\\ 2 & \text{if} \quad l \geqslant 3.\end{cases}$$

Finally (C 4) can be re-written, after a little more manipulation, to give c_{l-1} explicitly:

$$c_1 c_{l-1} = \frac{\alpha_N^2 - 1}{8h_l \alpha_N^4}\left\{\iint \varphi_0 I_l \, dy\, dz - \int dy \left[\varphi_0 B_l^{(c)}\right]_0^1\right\} \quad (l \geqslant 2). \tag{C 6b}$$

The expansions (3.7) are now completely defined by (C 6*a*, *b*), the recursive definitions (C 3*a*–*d*), and the definition (A 10) of φ_0. It may be verified that the formulae are all explicit at each stage. (Note that $B_2^{(c)}$ does not depend on c_1.) After the lth stage we know $\varphi_0, \varphi_1, \ldots, \varphi_{l-1}, \varphi_l^{(d)}$, and $c_0, c_1, \ldots, c_{l-1}$ (but not c_l).

Appendix D. Some mathematical details

We give here some of the details, for $k \neq k_N$, of the mathematical justification that is possible using elementary analysis. First, uniform convergence† is established for $|\mu| <$ some sufficiently small $\mu_0 > 0$. Then we prove that the result of substituting the series back into (2.1) is meaningful, and thence that the series do actually represent a solution of (2.1) for appropriate u and q_y of the form (3.1).

We thereby prove, incidentally, the existence of solutions to (2.1), under much more general conditions than those permitting explicit separable solutions.

When c_0 is not real ($k < k_N$ in the present problem), nothing is assumed about $u_1(y, z)$ except sufficient differentiability, implying boundedness in the closed domain of the problem. When c_0 is real ($k \geqslant k_N$), u_1 is further assumed analytic in z, and the domain of the eigenvalue problem and of the various integrals understood to be some suitable complex $\mathscr{D}$ (§ 4).

† The author did not see the possibility of a straightforward proof of convergence until after the completion of much of the work reported in this paper, when he came across the essential idea in the book by Titchmarsh (1958, p. 226). Titchmarsh also states a perturbation formula that amounts to a generalized Green's representation ((19.5.5), p. 224), although he does not indicate either its conceptually simple nature, or its practical importance for non-standard types of eigenvalue problem.

(i) *Uniform convergence in $\mathscr{D}$ for sufficiently small $|\mu|$*

The proof is straightforward, although to obtain sharp estimates for the radius of convergence of, say, the c expansion, and to avoid obscuring the fact that the latter cannot, obviously, depend upon details of the choice of $\mathscr{D}$ if u_1 is analytic, one would have to use methods deeper than the direct one used here. Therefore no attempt is made to estimate particular radii of convergence numerically, or to construct refined inequalities.

The essence of the proof is to start by considering a series like $\Sigma a_l \mu^l$, where the a_l are defined recursively by specifying $a_1 > 0$ and then defining

$$a_l = M \sum_{j=1}^{l-1} a_j a_{l-j} + N a_{l-1} \quad (l > 1), \tag{D 1}$$

M and N being positive constants. The radius of convergence of $\Sigma \alpha_l \mu^l$ is at least $1/(4Ma_1 + 2N)$ as will now be shown.

First, consider the function

$$g(\mu) = \frac{1}{2M}\left[1 - \left\{1 - \frac{\mu}{\rho}\right\}^{\frac{1}{2}}\right], \tag{D 2}$$

which has the expansion

$$g = \sum_{l=1}^{\infty} d_l \mu^l,$$

say, whose radius of convergence is evidently ρ. Now from (D 2),

$$g^2 = -\frac{\mu}{4M^2\rho} + \frac{g}{M},$$

$$= -\frac{\mu}{4M^2\rho} + \frac{1}{M}\sum_{l=1}^{\infty} d_l \mu^l \quad (\mu < \rho). \tag{D 3}$$

But for $\mu < \rho$ the coefficients of the above power series expansion of g^2 may also be obtained by multiplying the (absolutely convergent) series for g by itself. Comparison with (D 3) then gives the relations $d_1 = (4M\rho)^{-1}$ and

$$d_l = M \sum_{j=1}^{l-1} d_j d_{l-j} \quad (l \geqslant 2). \tag{D 4}$$

This shows that the power series whose coefficients are *defined* by (D 4), with d_1 specified, has radius of convergence $\rho = (4Md_1)^{-1}$. (The function g was of course arrived at in the first place by considering the formal product $(\Sigma d_l \mu^l)^2$.)

If now we choose $d_1 = (a_1 + N/2M)\,(>0)$, then (D 1) and (D 4) imply

$$(a_1 \leqslant d_1), \quad a_2 \leqslant d_2; \quad a_3 \leqslant d_3, \ldots.$$

Therefore the radius of convergence of $\Sigma a_l \mu^l$, where the positive coefficients $a_2, a_3, \ldots$ are related to a_1 by (D 1), is at least

$$\frac{1}{4Ma_1 + 2N},$$

as was asserted. (Clearly this is already a crude estimate.) To apply these ideas to prove convergence of $\Sigma\mu^l\varphi_l$, $\Sigma\mu^l c_l$, we construct first the series $\Sigma a_l\mu^l$, $\Sigma b_l\mu^l$, defined by a scheme slightly more general than (D 1), of the form

$$\left.\begin{aligned} a_l &= M_a^{(1)}\Sigma a_j a_{l-j} + M_a^{(2)}\Sigma a_j b_{l-j} + M_a^{(3)}\Sigma b_j b_{l-j} + N_a^{(1)} a_{l-1} + N_a^{(2)} b_{l-1},\\ b_l &= M_b^{(1)}\Sigma a_j a_{l-j} + M_b^{(2)}\Sigma a_j b_{l-j} + M_b^{(3)}\Sigma b_j b_{l-j} + N_b^{(1)} a_{l-1} + N_b^{(2)} b_{l-1},\end{aligned}\right\} \qquad \text{(D 5)}$$

where $a_1 > 0$, $b_1 > 0$ have been specified, and the summations are taken from $j = 1$ to $j = l-1$. It is then easy to verify that the radii of convergence of the series $\Sigma a_l\mu^l$, $\Sigma b_l\mu^l$ are each at least

$$\left.\begin{aligned} &\frac{1}{4M\max(a_1,b_1)+2N},\\ \text{where}\qquad M &= \sum_{i=1}^{3}\max(M_a^{(i)}, M_b^{(i)}),\\ N &= \sum_{i=1}^{2}\max(N_a^{(i)}, N_b^{(i)}).\end{aligned}\right\} \qquad \text{(D 6)}$$

We are now ready to consider the formulae (3.4), (B 2), giving φ_l and c_l. A hat over a symbol, as in $\hat{I}_l$, will indicate an upper bound, taken over all (y,z) in $\mathscr{D}$ where relevant: for example, $|I_l| \leqq \hat{I}_l$ (= const.). In estimating the Green's representations (3.4*b*) we note that although $\mathfrak{G}(y,z;\eta,\zeta)$ has a logarithmic infinity at $(\eta,\zeta) = (y,z)$, its integral with respect to η or y is finite, and likewise the integral of $|\mathfrak{G}|$. Indeed, there are finite *constants* G_I, G_B such that for all (y,z) in $\mathscr{D}$

$$\iint |\mathfrak{G}|\,d\eta\,d\zeta \leqslant G_I, \quad \int d\eta \sum_{\zeta=0}^{1}|\mathfrak{G}| \leqslant G_B. \qquad \text{(D 7)}$$

Then from (3.4*b*)

$$|\varphi_l| \leqslant G_I\hat{I}_l + G_B\hat{B}_l \equiv \hat{\varphi}_l, \quad \text{say, appropriately.} \qquad \text{(D 8)}$$

The other bounds $\hat{c}_l$, $\hat{I}_l$, $\hat{B}_l$, B'_l, are defined in a similar way, using the straightforward estimates that can be written down from (B 2) and (3.4). Writing $Z \equiv (z-c_0)^{-1}$ and using (D 8) to eliminate reference to $\hat{\varphi}_l$, we have the following relations, which define the bounds recursively:

$$|c_l| \leqslant \hat{c}_l \equiv E\hat{\varphi}_0(\hat{I}_l + 2\hat{B}'_l), \qquad \text{(D 9)}$$

$$|I_l| \leqslant \hat{I}_l \equiv \hat{Z}\left[\sum_{j=1}^{l-1}\hat{c}_j\hat{I}_{l-j} + (\hat{u}_1 + \epsilon^{-1}\hat{q}_{1y}G_I)\,\hat{I}_{l-1} + \epsilon^{-1}\hat{q}_{1y}G_B\hat{B}_{l-1}\right], \qquad \text{(D 10)}$$

$$\begin{aligned} |B'_l| \leqslant \hat{B}'_l \equiv {} & \hat{Z}^2 G_I\sum_{j=1}^{l-1}\hat{c}_j\hat{I}_{l-j} + \hat{Z}(\hat{Z}G_B + 1)\sum_{j=1}^{l-1}\hat{c}_j\hat{B}_{l-j}\\ & + \hat{Z}G_I(\hat{Z}\hat{u}_1 + \hat{u}_{1z})\,\hat{I}_{l-1} + \hat{Z}\{\hat{u}_1 + G_B(\hat{Z}\hat{u}_1 + \hat{u}_{1z})\}\,\hat{B}_{l-1}, \end{aligned} \qquad \text{(D 11)}$$

$$|B_l| \leqslant \hat{B}_l \equiv \hat{Z}^2\hat{\varphi}_0^2 E\hat{I}_l + (1 + 2\hat{Z}^2\hat{\varphi}_0^2 E)\,\hat{B}'_l, \qquad \text{(D 12)}$$

where
$$E \equiv \left|\frac{\kappa^2(1-c_0)^2 - 1}{\kappa^4(c_0 - \frac{1}{2})}\right|.$$

In (D 12), reference to $\hat{c}_l$ has been eliminated by means of (D 9).

All we need do now is to note that the pair of recursion relations (D 10) and (D 11) are of the form (D 5), after elimination of $\hat{c}_j$ and $\hat{B}_j$ using (D 9) and (D 12).

(If $\hat{I}_l$ is identified with a_l and $\hat{B}'_l$ with b_l, then $M^{(3)}_a$ is zero but the remaining M's and N's are not.) This shows that $\hat{I}_l$ and $\hat{B}'_l$ and, in virtue of (D 8), (D 9), and (D 12), $\hat{\varphi}_l$ and $\hat{c}_l$, are the (constant) coefficients of majorant series with finite (and constant) radii of convergence. Thus uniform convergence is proved, for $|\mu| <$ some finite μ_0.

(ii) *Proof that the series solve the perturbed problem*

The analysis just given can be extended to prove that $\{\Sigma\mu^l\varphi_l, \Sigma\mu^l c_l\}$, for any $|\mu| <$ some finite μ_0, does solve the eigenvalue problem associated with $u = z+\mu u_1$, $q_y = \mu q_{1y}$. (Note the corollary that existence of a solution is then proved.)

The formalism already ensures that the series satisfy the equation and the boundary conditions term by term. It is sufficient, then, to show that all the infinite series that arise on back substitution are absolutely and uniformly convergent over $\mathscr{D}$, since they are then immediately meaningful in the context of the boundary-value problem.† For the z boundary condition this follows from the term-by-term balance, since there is only one series involved, $\Sigma\mu^l\varphi_{lz}$, whose convergence has not been investigated. But in connexion with the differential equation we must show independently that one of $\Sigma|\mu^l\varphi_{lyy}|$, $\Sigma|\mu^l\varphi_{lzz}|$ is uniformly convergent. It would be straightforward, if tedious, to do this by extending the foregoing proof to include bounds on $|\varphi_{ly}|$ and $|\varphi_{lyy}|$ say, as well as on $|\varphi_l|$. A little more care is needed in estimating the Green's representations for the derivatives; bounds on the first and second y-derivatives of u_1 and q_{1y} will now be involved.

Alternatively, suppose that $u_1(y,z)$ is an analytic function of z whose singularities are bounded away from Γ_z, uniformly over $\mathscr{D}$. Then, since also (y, c_0) can be supposed bounded away from $\mathscr{D}$, the formulae show by induction that $\varphi_l(y,z)$ is analytic in z and that its z-singularities are also bounded away from Γ_z, uniformly in $\mathscr{D}$, and in l also. Thus for any z' on Γ_z, φ_l has an expansion in powers of $(z-z')$ whose radius of convergence $\geqslant$ some number that is non-zero and independent of l as well as of y and z'. Also, by a trivial extension of the previous analysis, $\Sigma\mu^l\varphi_l(z)$ has μ-radius of convergence greater than some constant μ_0, for any z within some neighbourhood of z'. Therefore $\Sigma\mu^l\varphi_l$ can be further expanded as a double series in powers of μ and $(z-z')$, absolutely convergent for $|\mu| < \mu_0$ and small but finite $|z-z'|$. Term-by-term differentiation with respect to z, holding μ at any value within $|\mu| < \mu_0$, then gives the second (or any) derivative with respect to z near $z = z_1$, as another absolutely convergent double power series. Since this can be rewritten as $\Sigma\mu^l\varphi_{lzz}$, the latter must also be absolutely convergent for $|\mu| < \mu_0$, when $z = z'$ in particular. The convergence is uniform over $\mathscr{D}$ as required, since μ_0 can be taken independent of y and z'.

† If the perturbation method were being used to account for higher approximations to the equations of motion, justification would not be quite so straightforward. In the independent analysis of non-geostrophic effects by Derome & Dolph (1969), for instance, the boundary conditions force non-uniformity of convergence at corners such as $y = z = 0$. Although the series are not then immediately meaningful globally, one would still expect pointwise convergence to a solution of the full problem. Indeed, under suitable assumptions, this would follow from considerations of analytic continuation in μ.

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