On spontaneous imbalance and ocean turbulence: generalizations of the Paparella–Young epsilon theorem

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Abstract Recent progress in understanding the balance–imbalance problem is highlighted, with emphasis on spontaneous-imbalance phenomena associated with the exponentially fast "wave capture" of inertia–gravity waves. These phenomena are excluded from shallow-water models and are outside the scope of the classical Lighthill theory. Also discussed is progress on a different topic, an effort to extend the Paparella–Young *epsilon theorem* to realistic ocean models. The theorem constrains turbulent dissipation rates ε in horizontal-convection thought-experiments, in which mechanically-driven stirring is switched off. The theorem bears on the so-called "ocean heat engine" and "ocean desert" controversies. The original theorem (2002) applied only to very idealized ocean models. Several restrictions on the original proof can now be lifted including the restriction to a linear, thermal-only equation of state. The theorem can now be proved for fairly realistic equations of state that include thermobaric effects, and nonlinearity in both temperature and salinity. The restriction to Boussinesq flow can also be lifted. The increased realism comes at some cost in terms of weakening the constraint on ε. The constraint is further weakened if one allows for the finite depth of penetration of solar radiation. This is collaborative work with Francesco Paparella and William Young.

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1 Introduction

Following the organizers' aims, I had originally undertaken to talk only about the first topic in this paper, spontaneous imbalance and accurate PV (potential vorticity) inversion. Recent progress in that field has been remarkable, throwing a clear light on where the Lighthill paradigm is relevant and where it is not, as well as finding the first accurate and completely self-consistent PV inversion operators. However,

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most of what I had to say about this is already in print elsewhere [12; 13]. Here I give only a quick summary, in Section 2, then move on to the second topic.

The second topic concerns the oceanic MOC, the meridional overturning circulation. It is sometimes called the thermohaline "conveyor belt" despite having nothing like the inextensibility of a real mechanical conveyor belt. The aspect crucial to questions such as the desertification, or not, of the upper ocean under global warming is the rising branch relative to the stratification surfaces in the main thermocline and elsewhere. It is upward material transport across stratification surfaces that is germane to questions about the supply of nutrients to the upper ocean [21], and hence about desertification or not.

It has sometimes been thought that buoyancy forcing alone can "drive the conveyor belt", a scenario much discussed under the heading "horizontal convection". The idea seems to be that the cooling of the sea surface in high latitudes is the main control, not only driving the deep-convective plumes and gravity currents of which the downward branch is composed but also supplying sufficient stirring, hence diapycnal mixing, to sustain the upward branch against the stable stratification N^2. Clearly the plumes and gravity currents must cause a certain amount of stirring and mixing.

Such a picture might tempt one to suppose that the "conveyor belt" circulation is something that can be shut off entirely by reducing the buoyancy forcing – by either warming or freshening the high-latitude sea surface, or both. A contrary view is that the *upward* branch depends, rather, on mechanical stirring by winds, tides, and possibly biota [2; 15; 25; 28]. In that case the upward branch would hardly be affected by shutting off the few tens of sverdrups [5] of flow in the downward branch. The sole effect would be to make the stratification surfaces drift downward, very gradually, without much affecting the upward material transport across them.[1]

The theorem of Paparella and Young published in 2002 [19], hereafter PY02, puts important and mathematically rigorous constraints on our thinking about these questions. It does so by placing a severe upper bound on the average turbulent dissipation rates ε attainable in a horizontal-convection scenario for small molecular diffusivities. If one accepts the usual empirical (Ellison–Britter–Osborn) relation between ε values and diapycnal mixing rates

$$K_z \sim \gamma\varepsilon/N^2 \tag{1}$$

where K_z is the vertical eddy diffusivity describing the mixing, and $\gamma \lesssim 0.2$, see e.g. [17], also [11], then one can strengthen the arguments for the importance of me-

[1] For instance "very gradually" would mean a downward drift of just under a kilometre per millennium for every ten sverdrups of downward flow that remained shut off. This downward drift of the stratification surfaces would not, however, detach the main thermocline from its Southern Ocean outcrop and therefore would not shut off the wind-driven mechanical stirring and Ekman transport across the outcropping stratification surfaces, contributing to the "upward" branch [25, & refs.]. This is because of the way in which the speed of the Antarctic Circumpolar Current is regulated internally, almost independently of wind stress but very much favouring the outcropping, e.g. [22] & refs. – a very different circumstance from that in the hothouse climate of the early Eocene 50 million years ago, when the Sun was about half a percent cooler but the Drake Passage still closed.

chanical stirring to the upward branch [14; 15; 28], including the superficial stirring and Ekman transport across the Southern Ocean outcrop [25, & refs.], and hence to the transports of nutrients across the ocean's stratification surfaces into the upper ocean, along with other questions about the oceanic general circulation.

However, PY02's epsilon theorem applies only to a highly idealized Boussinesq ocean model, in a rectangular domain, with a linear equation of state that neglects salinity altogether. In my talk at the international Workshop on 10 December 2008 I described a generalization in which all these restrictions were lifted except the Boussinesq approximation. (A video of the talk, "Beyond Lighthill...", was made publicly available via `http://sms.cam.ac.uk/media/518985/formats` later that month.) In particular, the new epsilon theorem allowed for arbitrary bottom topography, curved geopotentials and an equation of state that included not only salinity, but also the two main nonlinearities of real seawater, the thermal and thermobaric nonlinearities. This was work in collaboration with Francesco Paparella and William R. Young. Young had earlier reported our first breakthrough, the first result for a nonlinear equation of state, to the Stockholm Sandström Centennial Meeting on 3 November 2008.

As shown in Sects. 3 and 4 below, the key to proving these generalized epsilon theorems, and further generalizations arrived at after the Workshop, was to exploit Young's formulation of Boussinesq energetics. The formulation is a simple way of handling thermobaricity and is essentially that given, with due acknowledgement, on p. 73 of the textbook by Vallis [26]. Young (personal communication and ref. [30]) beautifully clarifies the way in which a vestige of the thermodynamic energies survives in the Boussinesq limit, adding to the gravitational potential energy.

Armed with Young's formulation, we were able to go on to prove epsilon theorems for still more realistic equations of state, including those being standardized by the international SCOR/IAPSO Working Group 127 on the *Thermodynamics and Equation of State of Seawater*. The accuracies are state of the art, comparable to those in recent publications such as [3; 7]. In addition, we were able to lift the Boussinesq restriction and thereby shed a fundamentally new light on the subject. A full report is in preparation, whose first part is to be submitted to the *Journal of Fluid Mechanics*, hereafter "JFM".

A few days before submission of the present paper, we learned that Jonas Nycander [16] had independently arrived at a generalized epsilon theorem based on Young's Boussinesq energy equation, (13) below, and a nonlinear equation of state almost the same as (20) below. Nycander's Boussinesq result almost exactly parallels the result to be described in Sect. 4 below, and reported in my December Workshop talk.

2 Spontaneous Imbalance

The recent progress in this area concerns spontaneous imbalance of the kind discovered in the 1990s by O'Sullivan and Dunkerton [18], in continuously stratified flows. In particular, it is now clear why these continuously stratified scenarios – let us call them OSD-type scenarios – are at an opposite extreme to what would be expected from the Lighthill theory [9], which applies not only to Lighthill's original case of acoustic imbalance but also to typical shallow-water scenarios [4]. The Lighthill theory is the classic milestone in the field. When it was published in 1952 it offered profound new insights. These were the first generic insights into the nature of spontaneous imbalance, even though not all-embracing, as it now turns out. A review and historical perspective may be found in [12].[2]

In brief, the continuously stratified, OSD-type scenarios differ drastically from Lighthill-type scenarios in three respects. First, the radiation reaction on the wave generation region is substantial. In a Lighthill scenario, by contrast, the radiation reaction is weak, permitting non-iterative computation of the spontaneous imbalance and the resulting wave emission after first computing the vortical motion using PV inversion, i.e. altogether neglecting the wave emission. This was Lighthill's most fundamental point. The vortical motion can be regarded as known before computing the wave emission.

Such non-iterative computation is impossible in an OSD-type scenario. The vortical motion and wave emission are intimately part of each other throughout the wave source region. As pointed out in [12], the wave emission process is fundamentally similar to mountain-wave generation except in one crucial respect. In order for the analogy to be accurate, recognizing the substantial radiation reaction, one must consider the notional mountain to be made of an elastic substance so pliable that the wave emission process substantially changes the shape of the mountain, hence substantially changes the vertical velocity field.

The second and related respect is the lack of scale separation in the wave source region, in an OSD-type scenario. The reason why the spontaneous wave emission is weak in a Lighthill scenario is the destructive interference arising from scale separation. The emitted waves have typical scales, or reciprocal wavenumbers, that greatly exceed typical vortex scales. In an OSD scenario, by contrast, the waves arise in the source region with reciprocal wavenumbers indistinguishable from typical scales of the vortex motion. As they propagate away from the source region the waves undergo "wave capture", or strain-enhanced critical-layer absorption, with wavenumbers increasing exponentially fast [20, & refs.]. This ensures that back-reflection and resonance phenomena are negligible and that the radiation reaction on the source is similar to that of waves satisfying a radiation condition.

The third respect is unsteadiness versus quasi-steadiness of the wave emission process. In a Lighthill scenario, unsteadiness of the vortical motion is a crucial part of the spontaneous-imbalance mechanism. In an OSD scenario, with its mountain-wave-like character, it is now clear that one can have inertia–gravity wave emission from a steady vortical flow. Perhaps the first work to point clearly to that fact was the study reported in [24], in which a surface temperature front was prevented from collapsing by applying an artificial diffusivity, holding the front approximately steady. Recently, examples have come to light for which perfectly steady flow is a natural

[2] Also available from `www.atm.damtp.cam.ac.uk/people/mem/#imbalance`

idealization. These are the propagating vortex dipoles described in several recent papers including [23] and [27]. They were discovered through high-resolution numerical experiments. The review in [12] gives a careful description and comparison between the two best-resolved cases, including the evidence for a substantial radiation reaction.

3 Epsilon theorems for realistic ocean models

PY02's epsilon theorem [19] constrains the turbulent dissipation rates ε attainable in an idealized horizontal-convection scenario. A Boussinesq liquid in a rigid, thermally insulating, box-shaped container of depth H, with gravity uniform and the top surface exactly horizontal, is set in motion purely by maintaining a nonuniform temperature ϑ at the top surface. The buoyancy acceleration b is a linear function of ϑ alone. No mechanical stirring is allowed. A statistically steady state is assumed. For this scenario PY02 rigorously established a bound proportional to the range Δb of buoyancy-acceleration values at the top surface,

$$\langle\langle\varepsilon\rangle\rangle \;\leqslant\; \kappa\Delta b/H \tag{2}$$

where the double angle brackets denote the domain and time average and where κ is the thermal molecular diffusivity. Therefore, in particular, $\langle\langle\varepsilon\rangle\rangle$ goes to zero as the first power of κ in the limit of small molecular diffusivities, for instance holding the Prandtl number ν/κ constant where ν is the molecular diffusivity of momentum. The vanishing of $\langle\langle\varepsilon\rangle\rangle$ in that limit was at first called an “anti-turbulence theorem”. However, we now prefer to call it an “epsilon theorem” for two reasons, first because the theorem does not rule out locally finite ε values in the limit, in shrinking subvolumes of the domain, and second because, even if ε were to go to zero at the same rate as $\langle\langle\varepsilon\rangle\rangle$ in all locations, it would still be possible to have weak yet fully-developed turbulence in the sense of having a Richardson cascade and a vanishingly small Kolmogorov scale $(\nu^3/\varepsilon)^{1/4} \propto \kappa^{1/2}$.

Now the key to proving epsilon theorems is to avoid considering the complete energetics. Indeed, one must dissect the complete energetics in a certain way. This happens automatically for the Boussinesq equations and is one of the facts implicitly exploited in PY02's proof, along with the linearity of the equation of state, $b \propto \vartheta$. In the Boussinesq equations the internal and chemical energies of seawater are relegated to an almost passive role. For the full equations the proof is harder to spot because the complete energetics must, of course, take account of the full thermodynamics including the internal and chemical energies.

Even within the Boussinesq equations there are nontrivial technical obstacles to be overcome, beyond making the geometry and the centrifugal–gravitational field more realistic. They concern the nonlinearities in realistic equations of state for seawater. The first steps toward overcoming these obstacles were taken by Francesco Paparella, William R. Young and myself, working together last year. The key was to recognize first that a mathematical device used in PY02, consisting of two successive integrations with respect to altitude z – the steps leading from (2.1b,c) to (2.3) and then to (3.3), in ref. [19] – could be replaced by a single integration after multiplication by z. The second step was to recognize that this integration was a pointer toward using the Boussinesq energy formulation discovered by Young (personal communication and ref. [30]), working on a different problem after PY02 was published. For a general, nonlinear equation of state the crucial step is to introduce a quantity most aptly called the *dynamic enthalpy*, which in this context takes on the superficial appearance of a buoyancy-associated potential energy, the quantity denoted by Π in Eq. (2.116) of Vallis [26] but here denoted by $h^{\ddagger}$ in order to flag the connection with enthalpy. That connection is carefully explained in [30]. It clarifies the "almost passive" role of the internal and chemical energies.

Consider a domain like that in Fig. 1 with arbitrary topography and curved geopotentials. It is now easiest to take the Boussinesq equations in coordinate-independent form

$$\mathrm{D}\mathbf{u}/\mathrm{D}t + 2\boldsymbol{\Omega}\times\mathbf{u} + \boldsymbol{\nabla}p - b\boldsymbol{\nabla}Z \;=\; \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\,, \tag{3}$$

$$\mathrm{D}\vartheta/\mathrm{D}t \;=\; -\boldsymbol{\nabla}\cdot\mathbf{J}_{\vartheta}\,, \tag{4}$$

$$\mathrm{D}S/\mathrm{D}t \;=\; -\boldsymbol{\nabla}\cdot\mathbf{J}_{S}\,, \tag{5}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} \;=\; 0\,, \tag{6}$$

where $\boldsymbol{\Omega}$ is the Earth's angular velocity, $\mathrm{D}/\mathrm{D}t$ the material derivative $\partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$, $\mathbf{u}(\mathbf{x},t)$ the relative velocity, $p(\mathbf{x},t)$ the pressure anomaly with the reference density $\rho = \rho_0 = 1$ in suitable units, $Z = Z(\mathbf{x})$ a scaled geopotential height to be defined below, with its zero level at the top surface, $\boldsymbol{\sigma}$ the viscous stress tensor with components $\sigma_{ij} = \sigma_{ji}$ while S and ϑ are salinity and conservative temperature [10], and $\mathbf{J}_S$

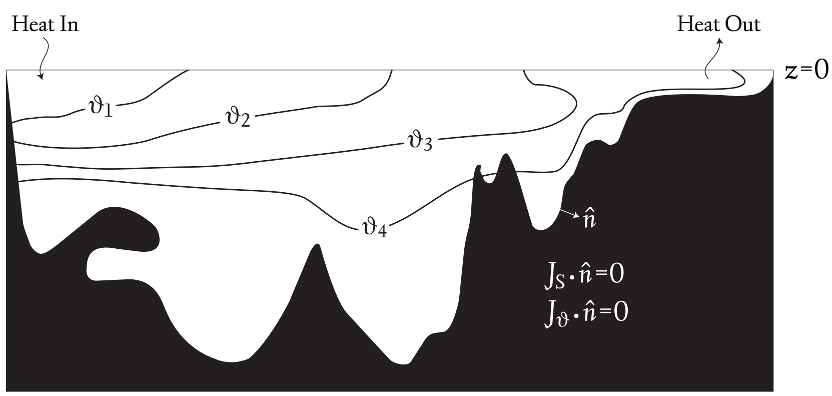

Fig. 1 Schematic of a model ocean. Instead of PY02's idealized rectangular domain, arbitrary topography is allowed, across which there are no salt or heat fluxes. The geothermal heat flux is assumed negligible. Salinity as well as temperature variations are allowed, as is a fully nonlinear equation of state. The effective geopotentials (gravitational plus centrifugal) are allowed to curve realistically, with gravity plus centrifugal force nonuniform, and Z the geopotential altitude.

and $\mathbf{J}_\vartheta$ their molecular-diffusive fluxes in suitable units. These may include cross-diffusivities arising from the Soret and Dufour effects and other thermodynamic dependences. The nonlinear equation of state has the generic form

$$b = b(\vartheta, Z, S)\,. \tag{7}$$

To help formalize the Boussinesq limit, it is convenient to define Z as the actual geopotential relative to the top surface divided by a constant reference value g_0 of the gravity acceleration g, so that Z is approximately the geometric altitude and ∇Z approximately a unit vertical vector. Then b is $-g_0$ times the fractional density anomaly. Thermobaric nonlinearities are represented within the Boussinesq framework by the dependence of b on Z, since the background reference pressure $= -\rho_0 g_0 Z +$ (surface pressure). Defining $W := \mathrm{D}Z/\mathrm{D}t$ and $\varepsilon := \nabla\mathbf{u}{:}\boldsymbol{\sigma} = u_{i,j}\sigma_{ij}$, the local per-unit-mass viscous rate of conversion of mechanical energy into thermal energy, and taking the scalar product of (3) with $\mathbf{u}$ we have, using incompressibility (6),

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\mathbf{u}|^2\right) - Wb + \nabla\cdot\left\{\mathbf{u}\left(\tfrac{1}{2}|\mathbf{u}|^2 + p\right) - \mathbf{u}\cdot\boldsymbol{\sigma}\right\} = -\varepsilon \tag{8}$$

where $\nabla\cdot(\mathbf{u}\cdot\boldsymbol{\sigma}) = (u_i\sigma_{ij})_{,j}$. The problem now is what to do with the buoyancy term Wb. Standard ways to turn it into a rate of change of potential energy fail because of thermobaricity. The difficulty can be overcome by introducing Young's dynamic enthalpy $h^\ddagger$ (personal communication and ref. [30]), whose definition is

$$h^\ddagger(\vartheta, Z, S) := \int_Z^0 b(\vartheta, Z', S)\,\mathrm{d}Z'\,. \tag{9}$$

Then, by the chain rule,

$$\frac{\mathrm{D}h^\ddagger}{\mathrm{D}t} = -Wb + \mathscr{D}(\vartheta, Z, S) \tag{10}$$

where the dissipative contribution

$$\mathscr{D}(\vartheta, Z, S) := \frac{\partial h^\ddagger}{\partial\vartheta}\frac{\mathrm{D}\vartheta}{\mathrm{D}t} + \frac{\partial h^\ddagger}{\partial S}\frac{\mathrm{D}S}{\mathrm{D}t} \tag{11}$$

$$= -\frac{\partial h^\ddagger}{\partial\vartheta}\nabla\cdot\mathbf{J}_\vartheta - \frac{\partial h^\ddagger}{\partial S}\nabla\cdot\mathbf{J}_S\,. \tag{12}$$

Then from (6), (8) and (10) we have an equation whose left-hand side is in conservation form,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\mathbf{u}|^2 + h^\ddagger\right) + \nabla\cdot\left\{\mathbf{u}\left(\tfrac{1}{2}|\mathbf{u}|^2 + p + h^\ddagger\right) - \mathbf{u}\cdot\boldsymbol{\sigma}\right\} = -\varepsilon + \mathscr{D}(\vartheta, Z, S)\,, \tag{13}$$

and whose domain and time average is, for the statistically steady state,

$$\langle\!\langle \varepsilon \rangle\!\rangle - \langle\!\langle \mathscr{D}(\vartheta, Z, S) \rangle\!\rangle = 0\,, \tag{14}$$

there being no mass flow across the boundary, $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ where $\hat{\mathbf{n}}$ is the outward normal, and no work done by surface viscous stresses since there is no mechanical stirring e.g. by wind stress at the top surface. Using (12) we now have

$$\langle\!\langle \varepsilon \rangle\!\rangle - \langle\!\langle\, \mathbf{J}_\vartheta\cdot\boldsymbol{\nabla}(\partial h^\ddagger/\partial\vartheta) + \mathbf{J}_S\cdot\boldsymbol{\nabla}(\partial h^\ddagger/\partial S)\, \rangle\!\rangle = 0 \tag{15}$$

after integrating by parts. This last step produces no boundary terms – the most crucial step in proving an epsilon theorem – because of (a) the vanishing of $\mathbf{J}_\vartheta\cdot\hat{\mathbf{n}}$ and $\mathbf{J}_S\cdot\hat{\mathbf{n}}$ on the topography and (b) the vanishing of $\partial h^\ddagger/\partial\vartheta$ and $\partial h^\ddagger/\partial S$, thanks to (9), on the top surface where $Z = 0$. On the top surface $\mathbf{J}_\vartheta\cdot\hat{\mathbf{n}}$ and $\mathbf{J}_S\cdot\hat{\mathbf{n}}$ need not vanish, indeed cannot both vanish if the system is to be forced into motion without mechanical stirring.

Now the most accurate models of seawater all assume, with strong justification, that conditions are everywhere close to local thermodynamic equilibrium. Thus the temperature T is well defined, and $\mathbf{J}_\vartheta$ and $\mathbf{J}_S$ can be expressed as linear combinations of $\boldsymbol{\nabla}T$ and $\boldsymbol{\nabla}\mu$ where μ is the chemical potential of salt relative to water [8]. The expressions for $\mathbf{J}_\vartheta$ and $\mathbf{J}_S$ include the Dufour and Soret cross-diffusive terms. Respectively, those terms represent a contribution to $\mathbf{J}_\vartheta$ from $\boldsymbol{\nabla}\mu$ and to $\mathbf{J}_S$ from $\boldsymbol{\nabla}T$. Since T and μ are functions of state we may write

$$T = T(\vartheta, Z, S) \qquad \text{and} \qquad \mu = \mu(\vartheta, Z, S) \tag{16}$$

where the functional dependence, expressing local thermodynamic equilibrium, may reasonably be assumed smooth. Then $\mathbf{J}_\vartheta$ and $\mathbf{J}_S$ become linear combinations of $\boldsymbol{\nabla}\vartheta$, $\boldsymbol{\nabla}Z$ and $\boldsymbol{\nabla}S$, and the second term in (15) a quadratic form in the components of $\boldsymbol{\nabla}\vartheta$, $\boldsymbol{\nabla}Z$ and $\boldsymbol{\nabla}S$. That is, (15) has the form

$$\begin{aligned}\langle\!\langle \varepsilon \rangle\!\rangle + \kappa_0 \langle\!\langle\, A\,|\boldsymbol{\nabla}\vartheta|^2 + B\boldsymbol{\nabla}\vartheta\cdot\boldsymbol{\nabla}S + C\,|\boldsymbol{\nabla}S|^2 + {}&\\ + D\boldsymbol{\nabla}\vartheta\cdot\boldsymbol{\nabla}Z + E\,\boldsymbol{\nabla}S\cdot\boldsymbol{\nabla}Z + F\,|\boldsymbol{\nabla}Z|^2 \rangle\!\rangle &= 0\,,\end{aligned} \tag{17}$$

where the coefficients A, B,.. are smooth functions of ϑ, Z, and S. The coefficient κ_0 is a reference diffusivity, whose ratio to each actual molecular diffusivity and cross-diffusivity (including ν, the molecular diffusivity of momentum) will be held constant in the small-diffusivities limit $\kappa_0 \to 0$. This is the natural generalization of PY02's constant Prandtl number. The coefficients $A, B, \ldots F$ are bounded as $\kappa_0 \to 0$.

As illustrated in the next section, the terms on the second line of (17) can be bounded as $\kappa_0 \to 0$, under reasonable assumptions, using the fact that Z is a smoothly-varying field independent of κ_0. An epsilon theorem $\langle\!\langle \varepsilon \rangle\!\rangle = O(\kappa_0)$ can then be proved whenever the first line of (17) is non-negative definite, which is true if

$$A \geqslant 0\,, \quad C \geqslant 0\,, \quad \text{and} \quad B^2 - 4AC \leqslant 0 \tag{18}$$

for all oceanographically relevant values of ϑ, Z, and S. For if κ_0 times the second line of (17) goes to zero in the limit then both $\langle\!\langle \varepsilon \rangle\!\rangle$ and the rest of the first line, when non-negative, must go to zero together in the limit. If (18) holds with A, B, and B^2-4AC bounded away from zero for fixed, nonzero Z, then (17) also puts significant constraints on the mean square gradients of ϑ and S, supplementing comparison-function constraints of the kind found by Balmforth and Young [1] and Winters and Young [29]. By themselves, the latter constraints would be insufficient to control the mean square gradients tightly enough to produce an epsilon theorem.

As will be noted in the next section, (18) is satisfied by the usual nonlinear models of seawater properties as described, for instance, in [26]. We are currently investigating whether (18) is satisfied by the still more accurate, state-of-the-art model currently being standardized by SCOR/IAPSO Working Group 127. The calculations are laborious but it seems clear that (18) is satisfied by this model as well, albeit by a slender margin at abyssal depths. The slender margin is mainly due to a contribution to the coefficient B not from the Dufour and Soret effects but from the interdependence of T, ϑ and S expressed by (16).

If (18) were violated, as appears thermodynamically possible, and realizable for conceivable fluid microstructures, then our proof would fail. Rather than signalling any dramatically different fluid behaviour, I suspect that this would merely widen the gap between what is true and what is provable, or what has so far been provable.

4 Specific examples

For illustrative purposes we simplify the expressions for $\mathbf{J}_\vartheta$ and $\mathbf{J}_S$ [8] to

$$\mathbf{J}_\vartheta \;=\; -\kappa\,(\boldsymbol{\nabla}\vartheta - \mathit{\Gamma}_\vartheta \boldsymbol{\nabla} Z) \quad \text{and} \quad \mathbf{J}_S \;=\; -\kappa_S\,(\boldsymbol{\nabla} S + \mathit{\Gamma}_S \boldsymbol{\nabla} Z)\;, \tag{19}$$

in units compatible with (4) and (5), where κ_S is the molecular salt diffusivity. The correction terms in $\mathit{\Gamma}_\vartheta$ and $\mathit{\Gamma}_S$ are necessary in order that $\mathbf{J}_\vartheta$ and $\mathbf{J}_S$ vanish when $\boldsymbol{\nabla} T = 0$ and $\boldsymbol{\nabla}\mu = 0$. Both $\mathit{\Gamma}_\vartheta$ and $\mathit{\Gamma}_S$ are positive. In the case of $\mathit{\Gamma}_S$ this allows the salinity S to find its natural scale height, with S diminishing upward under gravity at ~ 3‰ km^{-1} in a stagnant ocean with $\mathbf{J}_S = 0$ [6, & refs.]. Conservative temperature ϑ, being numerically close to the ordinary potential temperature [10], increases upward at $\sim 0.15\,\mathrm{K\,km^{-1}}$ in an isothermal stagnant ocean. Other small contributions to the fluxes arising from (16) and from the Dufour and Soret effects are neglected. They will be re-introduced and carefully discussed in our JFM paper.

For the equation of state we take a model similar to that in [26] except for the inclusion of a nonlinear term in salinity S,

$$b(\vartheta, Z, S) \;=\; g_0 \left\{ \beta_\vartheta (1-\gamma^* \rho_0 g_0 Z)\,\vartheta \;+\; \tfrac{1}{2}\beta_\vartheta^* \vartheta^2 \;-\; \beta_S S \;+\; \tfrac{1}{2}\beta_S^* S^2 \;+\; \frac{g_0 Z}{c_0^2} \right\}\,, \tag{20}$$

$$\Rightarrow\; h^\ddagger \;=\; -g_0 Z \left\{ \beta_\vartheta \left(1-\tfrac{1}{2}\gamma^*\rho_0 g_0 Z\right)\vartheta + \tfrac{1}{2}\beta_\vartheta^*\vartheta^2 - \beta_S S + \tfrac{1}{2}\beta_S^* S^2 + \frac{g_0 Z}{2c_0^2} \right\}\,. \tag{21}$$

Here ϑ and S are defined as increments relative to reference values such as $T_0 = 273\text{K}$, $S_0 = 35‰$, while β_ϑ, γ^*, ρ_0, β^*_ϑ, β_S, β^*_S, and c_0 are positive constants. To a first approximation, (20) represents the the two principal nonlinearities of seawater, the thermobaricity γ^* and the temperature nonlinearity β^*_ϑ, along with the weaker salinity nonlinearity β^*_S, as well as compressibility.[3] Then

$$\frac{\partial h^\ddagger}{\partial\vartheta} \;=\; -g_0 Z\big(\beta_\vartheta - \tfrac{1}{2}\gamma^*\beta_\vartheta\rho_0 g_0 Z + \beta^*_\vartheta\vartheta\big) \quad \text{and} \quad \frac{\partial h^\ddagger}{\partial S} \;=\; g_0 Z(\beta_S - \beta^*_S S)\,, \tag{22}$$

so that

$$\begin{aligned}\nabla\frac{\partial h^\ddagger}{\partial\vartheta} \;&=\; -g_0 Z\beta^*_\vartheta\nabla\vartheta - g_0\big(\beta_\vartheta - \gamma^*\beta_\vartheta\rho_0 g_0 Z + \beta^*_\vartheta\vartheta\big)\nabla Z \\ \text{and}\quad \nabla\frac{\partial h^\ddagger}{\partial S} \;&=\; -g_0 Z\beta^*_S\nabla S + g_0(\beta_S - \beta^*_S S)\nabla Z\,.\end{aligned} \tag{23}$$

The expression $g_0\big(\beta_\vartheta - \gamma^*\beta_\vartheta\rho_0 g_0 Z + \beta^*_\vartheta\vartheta\big)$ multiplying ∇Z is always positive-valued for seawater (even well below freezing temperatures), because at realistic salinities the ratio $\beta_\vartheta/\beta^*_\vartheta$ is typically well above 10 K, for instance 16.7 K with the values in footnote 3. Substituting (19) and (23) into (15) we recover (17) with

$$A \;=\; \hat\kappa_\vartheta\beta^*_\vartheta g_0|Z|\,, \tag{24a}$$

$$B \;=\; 0\,, \tag{24b}$$

$$C \;=\; \hat\kappa_S\beta^*_S g_0|Z|\,, \tag{24c}$$

$$D \;=\; -\hat\kappa_\vartheta g_0\big(\beta_\vartheta + \gamma^*\beta_\vartheta\rho_0 g_0|Z| + \beta^*_\vartheta\vartheta\big) - \hat\kappa_\vartheta\Gamma_\vartheta\beta^*_\vartheta g_0|Z|\,, \tag{24d}$$

$$E \;=\; \hat\kappa_S g_0(\beta_S - \beta^*_S S) + \hat\kappa_S\Gamma_S\beta^*_S g_0|Z|\,, \tag{24e}$$

$$F \;=\; \hat\kappa_\vartheta g_0\Gamma_\vartheta\big(\beta_\vartheta + \gamma^*\beta_\vartheta\rho_0 g_0|Z| + \beta^*_\vartheta\vartheta\big) + \hat\kappa_S g_0\Gamma_S\big(\beta_S - \beta^*_S S\big)\,, \tag{24f}$$

where $\hat\kappa_\vartheta := \kappa/\kappa_0$ and $\hat\kappa_S := \kappa_S/\kappa_0$, both order-unity quantities. Thus all the coefficients $A, B, \ldots, F$ are order-unity quantities in the limit $\kappa_0 \to 0$.

We can now see PY02's result in perspective. It is the case $A = B = C = E = F = 0$; the bound (2) comes from the D term in the second line of (17) with $D =$ con-

[3] The positive constant c_0 is a nominal sound speed, notwithstanding that the actual Boussinesq sound speed is infinite because of the incompressibility condition (6); in any case the γ^* term makes the actual sound speed differ from the nominal c_0, typically by a few percent, unless $\vartheta = 0$. Table 1.2 of Vallis [26] gives a sufficient idea of the orders of magnitude: $c_0 = 1490\ \text{m}\,\text{s}^{-1}$, $\beta_\vartheta = 1.67\times10^{-4}\text{K}^{-1}$, $\gamma^* = 1.2\times10^{-8}\text{Pa}^{-1}$ (this is Vallis's γ'^*, his γ^* increased by a small increment $+0.07\times10^{-8}\text{Pa}^{-1}$), $\rho_0 = 1.027\times10^3\text{kg}\,\text{m}^{-3}$, $\beta^*_\vartheta = 1.0\times10^{-5}\text{K}^{-2}$, and $\beta_S = 0.78\times10^{-3}\,‰^{-1}$. Over the full range of oceanic conditions β^*_S is always positive, with order of magnitude $\beta^*_S \gtrsim 0.5\times10^{-6}\,‰^{-2}$ and reaching just over twice this value in some conditions, according to the accurate equation of state defined by Table B2 of [7]. Values of the molecular diffusivities κ and κ_S are respectively of the order of $10^{-7}\,\text{m}^2\text{s}^{-1}$ and $10^{-9}\,\text{m}^2\text{s}^{-1}$.

stant, allowing a trivial integration by parts, leaving only a term bounded by a geometric factor $\sim H^{-1}$ times $\kappa g_0 \beta_\vartheta \Delta\vartheta = \kappa \Delta b$, with Δ as in (2). The term $-\kappa\langle\!\langle b\nabla^2 \mathbf{Z}\rangle\!\rangle$ vanishes in PY02's case of uniform gravity.

In the more general case defined by (20)–(24), the crucial definiteness condition (18) is satisfied thanks to the lucky accident that β_ϑ^* and β_S^* are both positive for seawater (typical values in footnote 3). This enables us to prove a strong epsilon theorem. The bound is still $O(\kappa_0)$ in the limit, as in PY02. The independent work of Nycander [16] produced a similarly strong bound, $O(\kappa_0)$, albeit for the slightly less general case $A \geqslant 0$ with $B = C = F = 0$. There, the adiabatic gradients Γ_ϑ and Γ_S in (19) were taken to be zero along with β_S^*.

However, these epsilon theorems with $\langle\!\langle\varepsilon\rangle\!\rangle = O(\kappa_0)$ all rely on assuming constancy of the coefficients β_ϑ etc. in (20). For then we can still bound the D and E terms in the second line of (17) since the potentially dangerous factors $\vartheta\boldsymbol{\nabla}\vartheta$ and $S\boldsymbol{\nabla}S$ take the form of gradients, $\frac{1}{2}\boldsymbol{\nabla}(\vartheta^2)$ and $\frac{1}{2}\boldsymbol{\nabla}(S^2)$, so that we can again integrate by parts and use the boundedness of $|\nabla^2\mathbf{Z}|$. This eliminates the gradients of ϑ and S from the second line of (17), again giving a bound as the first power of κ_0.

When the coefficients are fully variable, however, such integrations by parts in the second line of (17) do not eliminate the gradients of ϑ and S, which if not integrated away can be dangerous because of the small scales in the ϑ and S fields. We then have to resort to a blunter tool, the Cauchy–Schwartz inequality. Despite this, it turns out to be possible to obtain bounds that are asymptotically nearly as good as in PY02, under reasonable assumptions. They are "nearly as good" in the sense that the bounds are proportional to the first power of κ_0 times only a logarithmic factor, $\langle\!\langle\varepsilon\rangle\!\rangle = O(\kappa_0 \ln \kappa_0)$. This comes from exploiting the constraints mentioned below (18).

5 Concluding remarks

Apart from lifting the Boussinesq restriction, the JFM paper will prove the $\langle\!\langle\varepsilon\rangle\!\rangle = O(\kappa_0 \ln \kappa_0)$ result and will push further toward the most realistic possible thermodynamics and equation of state. In a subsequent paper to the *Journal of Physical Oceanography* it is hoped to discuss the extent to which these epsilon theorems constrain our understanding of the real oceans' upper and lower meridional overturning circulations, taking account of the extent to which the bounds on $\langle\!\langle\varepsilon\rangle\!\rangle$ are further degraded by allowing a finite depth of radiative penetration of surface heating from the visible solar spectrum. The emphasis will then shift away from the asymptotic behaviour and will focus, rather, on the best numerical bounds on $\langle\!\langle\varepsilon\rangle\!\rangle$ obtainable for realistically small but finite values of κ_0.

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