Wind-generated water waves: two overlooked mechanisms?

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Abstract

From research on stratospheric dynamics there has emerged what might prove to be some useful fresh ideas about the problem of water-wave generation by wind. As pointed out in an earlier contribution (1993), water waves can be systematically amplified by two irreversible, ratchet-like mechanisms that depend on spatio-temporal inhomogeneities, such as wind gustiness and wave groupiness. Neither mechanism is represented in the models used for wind-wave forecasting. The first mechanism is simply the drag from what might be called Rossby or vorticity lee waves in the airflow downstream of water-wave groups. The second is intermittent Rossby-wave breaking or vertical mixing of spanwise horizontal vorticity in the airflow, a highly nonlinear, non-Fourier-superposable mechanism. Both mechanisms can amplify non-breaking water waves, as well as contributing to the amplification of breaking water waves.

1. Introduction

I want to suggest that the mechanisms by which wind generates water waves can be illuminated by today's understanding of the large-scale fluid dynamics of the Earth's stratosphere. Persistent, irreversible momentum transport, involving a phase-coherent interaction between waves and turbulence, enters both the wind-wave problem and the stratospheric problem in an essential and fundamentally similar way. The ideas — which both subsume, and also go beyond, recent extensions of the Miles theory, including that of Belcher *et al.* (1999) presented in the predecessor to this wind-wave conference proceedings — can be traced back to G.I. Taylor's classic paper on eddy motion in the atmosphere (Taylor 1915) and its nonlinear extension by Killworth & McIntyre (1985), hereafter KM. The relevance of these ideas to the wind-wave problem were first pointed out in a Sectional Lecture (McIntyre 1993) to the Minisymposium *Sea Surface Mechanics and Air–Sea Interaction* at the 18th International Congress of Theoretical and Applied Mechanics.

In the case of the stratosphere we have a secure and well-developed understanding of the highly inhomogeneous 'wave–turbulence jigsaw puzzle' with which the stratosphere confronts us when viewed on a global scale. For the history of ideas leading to that understanding the reader may consult, for instance, the review by Hoskins *et al.* (1985) and two recent reviews of mine (2000, 2002). The history takes on a new interest when

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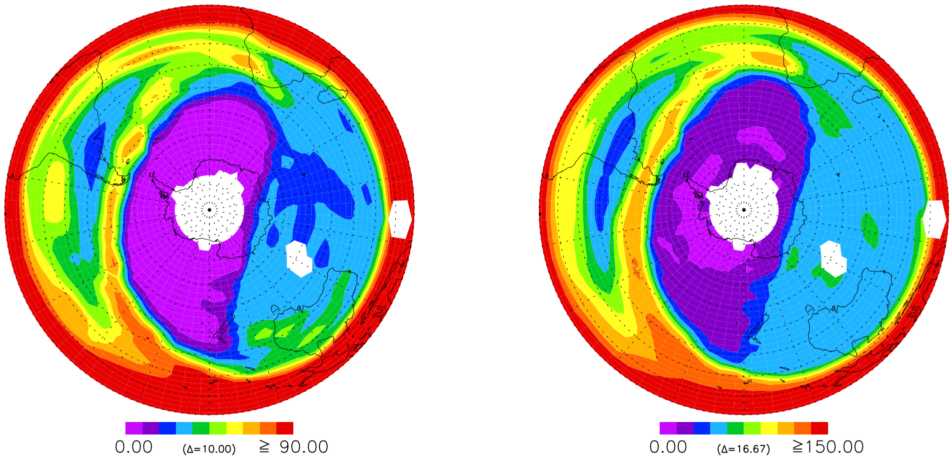

FIGURE 1. Nitrous oxide (N_2O) mixing ratios observed at two stratospheric altitudes on 11 August 1997 by the CRISTA instrument, from Riese *et al.* (2002). White regions are data gaps. On Rossby-wave timescales of days and weeks N_2O is an accurate passive tracer, though destroyed photochemically on Brewer–Dobson timescales of years (see end of §2 below). In the right half of each picture the N_2O mixing ratios increase equatorward nearly monotonically or stepwise monotonically (being nearly constant over the large blue regions on the right). The mixing ratios increase equatorward from polar-vortex values close to zero (purple) to large tropical values imported from the troposphere by the Brewer–Dobson upwelling (red). **At left and right respectively,** pressure-altitudes are 4.64 hPa and 10 hPa, roughly 37 km and 31 km; ranges of mixing ratios in parts per billion by volume are 0–90+ and 0–150+ with contour intervals 10 and 16.67, where '+' signifies that maximum values may slightly overshoot the plotted range. The light band at the subtropical edge of the surf zone (yellow) highlights the ranges 60–70 and 100–116.67 ppbv. CRISTA (Cryogenic Infrared Spectrometers and Telescopes for the Atmosphere) detects a number of chemical species through their infrared spectral signatures and is a large (1350 kg) helium-cooled instrument flown from the Space Shuttle.

compared to the history of ideas in wind-wave generation, especially if one takes as a conceptual starting point the linear-theoretic wave ideas embodied in the Miles theory — as further developed by, for instance, Lighthill (1962), Janssen (1982, 1991) and Belcher *et al.* (1999) — and then goes on to recognize various forms of strong nonlinearity that operate in reality (e.g. Banner 1990, McIntyre 1993 & refs., Belcher & Hunt 1998 & refs.), especially those nonlinearities associated with the critical layer or matched layer in the airflow above the water waves, with the implication that Fourier superposition must generally fail in descriptions of water-wave generation. Over the years, our understanding of stratospheric dynamics has undergone a closely analogous transition from linear to nonlinear thinking.

2. A glimpse of the real stratosphere

Concerns about stratospheric ozone have produced highly sophisticated observing systems that allow us to see the real stratosphere in considerable detail. Figure 1, of which an animated version can be seen on my website,† is reproduced here by courtesy of Dirk Offermann and Martin Riese of the CRISTA space-based remote-sensing project; see Riese *et al.* (2002). It shows the distributions at about 31 and 37 km altitude of a chemical

† at www.atm.damtp.cam.ac.uk/people/mem/papers/LIM/index.html#crista-movie

tracer, nitrous oxide, which for present purposes is accurately an inert material tracer — the pictures are like laboratory dye pictures — because photochemical timescales for nitrous oxide are of the order of years, far longer than the relevant fluid-dynamical timescales. White regions are data gaps.

Aside from the overall pole-to-equator gradient, with high equatorial tracer values (red) and low polar values (purple), the pattern seen in both pictures is shaped by nearly-horizontal, layerwise-two-dimensional air motion on timescales of days to weeks. It reveals distinct airmasses such as the well-mixed region (blue) on the right, with almost perfectly uniform tracer values, sandwiched between relatively isolated polar and tropical airmasses with very different tracer values, and with steep gradients in transition zones between. It is clear, especially from the animated version of figure 1 and from many other data studies and from numerical model simulations of the air motion, all of which tend to produce generically similar tracer pictures (e.g. Norton 1994, Lahoz *et al.* 1996), that horizontal fluid motion is causing strong mixing in an extensive midlatitude region between the polar and tropical airmasses.

The early stages of a typical mixing event can be seen on the left. A long tongue of subtropical air (light orange–yellow) is being drawn around the edge of the vortex, crossing the tip of South America, and the animation clearly shows that air is circulating within the part of the midlatitude region overlying the Atlantic and South America. Air is also circulating and mixing over the right-hand part overlying the Indian and Southern Oceans and Australia but, because that part has already been fairly well mixed, the circulating motion there does not show up clearly in the animation.

We shall see that the midlatitude mixing region, which for reasons to be explained is called the *stratospheric surf zone*, is the stratospheric counterpart of the critical layer in the airflow above water waves, turned on its side. It is only the dimensionless parameters that matter and not, of course, the absolute spatial scale, nor the superficial geometrical differences. But one point to note is that the stratospheric surf zone is anything but narrow. Critical layers have often been thought of as narrow or thin, a mental picture that tended to feature prominently in the early history of thinking about stratospheric fluid dynamics (e.g. Dickinson 1969), just as has tended to be the case in other parts of the research literature on theoretical fluid dynamics.

In the real stratosphere the situation shown in figure 1, with its broad and extensive surf zone, is very typical. The whole picture is not only well simulated by numerical models, but is also well understood dynamically, using ideas like that of 'potential-vorticity inversion' (§4 below; also, e.g., Hoskins *et al.* 1985, Ford *et al.* 2000, 2002). There is an intimate interplay of wavelike and quasi-turbulent dynamics leading, as we shall see, to a persistent momentum transport, outside the scope of classical turbulence theory. It gives rise to a westward force on the stratosphere and an eastward reaction on the denser tropospheric air below. It is this momentum transport that is fundamentally similar to the momentum transport involved in the wind-wave problem.

3. Breaking Rossby waves

The animated version of figure 1 makes the wavelike aspect conspicuous. It shows the long axis of the central purple region rotating clockwise through an angle of about 70° longitude in 5 days, 10–15 August 1997. The purple region is the so-called 'polar vortex', more precisely, the core of the polar vortex, characterized by large negative values of the relevant measure of vorticity — in this case the so-called Rossby–Ertel 'potential vorticity' (e.g. Hoskins *et al.* 1985) — which behaves approximately like an advected tracer and has a distribution like that of the nitrous oxide, up to an additive constant. The rate at which the long axis of the purple region rotates is determined by a competition between the mean winds on the one hand — which blow clockwise, in high latitudes such as 60°S, at speeds of the order of $80\,\mathrm{m\,s^{-1}}$, about nine times faster than 70° longitude in 5 days — and a powerful wave propagation mechanism on the other, which rotates the long axis anticlockwise relative to the mean winds. Today this is usually called the 'Rossby-wave' mechanism, even though (by the usual etiquette honoured more in the breach than in the observance) it 'should', probably, be called the Kelvin–Kirchhoff–Rayleigh–Rossby mechanism.†

The wave mechanism operates, as is well known, whenever the relevant measure of vorticity, Q say, has a cross-stream mean gradient $\bar{Q}_y$, such as the atmosphere's strong pole-to-equator gradient caused by the Earth's rotation. The counterpart in the airflow above water waves is the vertical gradient of ordinary vorticity, associated with the curvature $\bar{u}_{yy}$ of the mean velocity profile. Material displacements across that gradient $\bar{Q}_y$, whatever its cause, give rise to a pattern of fluctuating Q anomalies, Q' say, that alternate in sign downstream. They do so every 90° of longitude in the case of figure 1. Inversion of those Q anomalies to obtain the fluctuating velocity field produces north–south velocities a quarter wavelength out of phase with the displacements, hence phase propagation. This is *one-way* phase propagation, and the signs make it anticlockwise in figure 1, and upstream in the airflow above water waves. For recent work on the concept of potential-vorticity inversion and its ultimate limitations see Ford *et al.* (2000, 2002).

The resulting wavelike, quasi-elastic resilience of the edge of the vortex core shows up in figure 1 in another way, through the lack of mixing across the edge, as evidenced by the steep tracer gradients at the edge and the very different tracer values inside and outside the vortex, now routinely observed by stratospheric researchers and much studied because of the significance for ozone-layer chemistry. 'Shear sheltering' is also involved (Juckes & McIntyre 1987; Hunt & Durbin 1999). The vortex core is largely isolated chemically from its surroundings, as evidenced by its purple colour in figure 1 (as with the smoke in traditional smoke rings familiar to Kelvin, Kirchhoff and Rayleigh). Yet just next to the edge, in the midlatitude surf-zone region, horizontal mixing is strong, as already noted. It is strong for well-understood reasons associated with flow unsteadiness, hyperbolic points, and so on (e.g. Polvani & Plumb 1992). We may say that the surf zone is chaotically advective. It is often described as two-dimensionally or 'geostrophically' turbulent, more aptly 'layerwise-two-dimensionally turbulent', under the constraint of the strong stable stratification. The stratification is strong in the sense that its timescale, i.e. the timescale of internal gravity waves or buoyancy oscillations, minutes to hours, is far shorter than the days, weeks and years already mentioned.

The evidence for horizontal mixing shows that the Rossby waves must be considered to be *breaking.* The flow unsteadiness, hyperbolic points, etc., hence the turbulence itself, are due to the Rossby-wave motion, just as the turbulence in ocean-beach surf can be said to be due to the water-wave motion. That is the reason for the term 'stratospheric surf zone'. The Rossby-wave breaking is part of what makes the momentum transport persistent and irreversible, in the same sense as the momentum transport generating longshore currents on ocean beaches is rendered persistent and irreversible, by the breaking of incoming water waves. The next section will analyse the Rossby-wave case in more detail, in the simplest possible way, showing the essential role of nonlinearity.

† Of course we can't call these vorticity waves 'Kelvin waves', because that means something else in today's terminology, namely Coriolis-trapped gravity waves. 'Kirchhoff waves' would suggest a restriction to the wave-2, quasi-elliptical case. Carl-Gustaf Rossby was a great pioneer in atmospheric dynamics, discovering among other things one of its most crucial concepts, that of potential vorticity (Rossby 1936, 1940), which in stratified systems plays a role like that of ordinary vorticity in aerodynamics through the concept of 'potential-vorticity inversion'.

As for the stratosphere's reponse to the momentum transport under discussion, that is a separate question, and irrelevant to the wind-wave problem; but the essence of what happens is noted briefly because it explains the pole-to-equator gradient that makes nitrous oxide such an effective material tracer in figure 1. The response depends on Coriolis forces, which are immensely strong on the timescales of interest; the reader interested in the mathematical details may consult the review by Holton *et al.* (1995). Because stratospheric air is persistently pushed westward, Coriolis forces persistently deflect it poleward, giving rise to a pumping action. This may be called 'gyroscopic pumping', the familiar 'Ekman pumping' being just the special case of it where the persistent force happens to be frictional rather than wave-induced.

The gyroscopic pumping in the stratosphere, due chiefly to momentum transport by breaking Rossby waves like those seen in figure 1, drives a slow but very persistent global-scale circulation, known as the Brewer–Dobson circulation, in which tropospheric air is drawn up into the tropical stratosphere and is then pumped poleward and eventually downward while undergoing photochemical transformation, on timescales of years. That is part of why northern pollution causes a southern ozone hole, and is also the reason for the strong pole-to-equator gradient of nitrous oxide seen in figure 1. Nitrous oxide is produced by biological processes and is well mixed throughout the troposphere. It is drawn up into the tropical stratosphere by the Brewer–Dobson circulation and then destroyed photochemically as it drifts poleward.

4. The dynamics of the wave–turbulence jigsaw

How does the wave–turbulence jigsaw suggested by figure 1 actually work to produce the persistent momentum transport? Part of the answer is that there are highly specific phase relations and phase shifts among the wavelike and quasi-turbulent elements, and that the spatial inhomogeneity conspicuous in figure 1, the juxtaposition of wavelike and turbulent regions, is an essential feature. The spatial inhomogeneity takes the problem well outside the scope of classical turbulence theories, as does the wave propagation itself. A thorough discussion of this last point, from a wider historical and theoretical-physics perspective, can be found in the first of my recent pair of reviews (2000). Here we sketch only what is most basic and essential for the Rossby-wave case and the wind-wave problem. The essentials include the advective nonlinearity.

All the essentials are contained in a simple and elegant model whose workings are fully understood, and which illuminates both the stratospheric problem and the wind-wave problem. This is the Rossby-wave critical layer problem first solved in the complementary papers of Stewartson (1978) and Warn & Warn (1978), hereafter 'SWW problem'. Fundamentally relevant, too, are the relations concerning momentum transport found by G. I. Taylor (1915), displayed in (4.6) below, and extended by KM, who also carefully reviewed and extended the SWW work. Our understanding of the SWW problem was brought to completion in the definitive work of Haynes (1989).

The starting point is the simplest relevant dynamical system, two-dimensional frictionless, incompressible motion. Here Q is the ordinary vorticity, so that inverting Q to obtain the velocity field demands nothing more than solving a Poisson equation of the form $\nabla^2\psi = Q$ under suitable boundary conditions, such as evanescence of $|\nabla\psi|$ at infinity. Coriolis effects are being ignored for the moment. Symbolically,

$$\psi \;=\; \nabla^{-2}Q \;, \tag{4.1}$$

where the stream function $\psi(x, y, t)$ is defined such that

$$\mathbf{u} \;=\; (u, v) \;, \qquad u \;=\; -\,\partial\psi/\partial y \;, \quad v \;=\; \partial\psi/\partial x \;, \tag{4.2}$$

(x, y) being Cartesian coordinates and (u, v) the corresponding components of the velocity $\mathbf{u}(x, y, t)$. The two-dimensional Laplacian $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and the Green's function to invert it has a logarithmic kernel. In this dynamical system there is just one evolution equation,

$$\mathrm{D}Q/\mathrm{D}t \;=\; 0 \;, \tag{4.3}$$

where $\mathrm{D}/\mathrm{D}t$ is the two-dimensional material derivative,

$$\mathrm{D}/\mathrm{D}t \;=\; \partial/\partial t + \mathbf{u}\cdot\nabla \;=\; \partial/\partial t + u\partial/\partial x + v\partial/\partial y \;. \tag{4.4}$$

The single time derivative in (4.3) is the essential reason for the one-way character of Rossby phase propagation. Equation (4.3) with its single time derivative is common to this simple system and its more realistic generalizations, including realistic models of the stratosphere, with suitable redefinitions of Q and the associated inversion operator.

A step toward such generalizations is to introduce the Coriolis parameter $f(y)$. This is the vertical component of the Earth's absolute vorticity picked out by the atmosphere's stable stratification, and is a strong function of latitudinal distance y. Then (4.1) is replaced by

$$\psi \;=\; \nabla^{-2}(Q - f) \;. \tag{4.5}$$

When f is made to depend linearly on y, with, say, $df/dy = \beta =$ constant, (4.2)–(4.5) becomes the Rossby 'beta plane' or 'flat earth' model. With its constant background vorticity gradient $\bar{Q}_y = \beta$, this well known model provides the simplest textbook examples of Rossby-wave propagation, and in addition the framework for the SWW problem.

If we assume that averaging in the x-direction is well defined, with vanishing mean pressure gradient $\partial\bar{p}/\partial x$, then it is straightforward to derive from (4.2)–(4.5) and the associated x-momentum equation the following relations for the airflow:

$$\frac{\partial\bar{u}}{\partial t} \;=\; -\frac{\partial}{\partial y}\left(\overline{u'v'}\right) \;=\; \overline{v'Q'} \;=\; -\frac{\partial}{\partial t}\left(\tfrac{1}{2}\bar{Q}_y\overline{\eta'^2}\right) \;, \tag{4.6}$$

where the overbars denote the (Eulerian) x-average and the primes fluctuations about it, and where $\eta'(x, y, t)$ is the fluctuating material displacement in the y-direction, satisfying $Q' = -\bar{Q}_y\eta'$. The middle and last equalities are Taylor's relations. The middle equality is an exact consequence of (4.2) and ∇^2(4.5) alone, and is often referred to as *Taylor's identity.*† The last equality holds in general for small disturbances only.

Taylor's expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ has an exact, finite-amplitude counterpart, which KM discovered and exploited in their work on the SWW problem. Not surprisingly, the exact expression, omitted here for brevity, implies that the approximate expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ remains qualitatively correct when a disturbance grows from small to finite amplitude, as long as the mean vorticity profile $\bar{Q}(y)$ is monotonic, i.e. $\bar{Q}_y(y)$ is one-signed, and is not too much changed from the profile in the initial state. The exact expression has been shown to be related to a Hamiltonian 'Casimir invariant' involved in the stability theorems of V. I. Arnol'd, and is often referred to today as the Rossby-wave 'activity' or minus the 'pseudomomentum' per unit mass (e.g. McIntyre 1981, Shepherd 1990).

Notice, now, the special case of a small disturbance growing exponentially, with all fluctuating quantities proportional to $\exp(\sigma t)$, say, with constant growth rate σ. This is precisely the case analysed in Belcher *et al.* (1999), in which the above equations represent two-dimensional frictionless airflow over two-dimensional frictionless water waves, with y vertical and with $\beta = 0$ and $\bar{Q}_y = -\bar{u}_{yy} > 0$. The last equality in (4.6) holds

† Taylor's identity generalizes to more realistic models of the stratosphere, with $-\partial\left(\overline{u'v'}\right)/\partial y$ replaced by the divergence of the so-called Eliassen–Palm flux (e.g. Andrews *et al.* 1987), which has an extra term representing vertical momentum transport.

quantitatively, because of the small-disturbances assumption, and $\partial/\partial t$ is replaced simply by 2σ. We see immediately that if $\bar{Q}_y > 0$ then there is a momentum-transport divergence, $\partial\left(\overline{u'v'}\right)/\partial y > 0$, and, with the assumption of vanishing mean pressure gradient $\partial\bar{p}/\partial x$, an associated mean rate of change of momentum $\partial\bar{u}/\partial t < 0$ in the airflow. This momentum change has to be accompanied by an equal and opposite reaction on, and momentum change in, the water, manifesting itself (in this frictionless model) entirely as an amplification of the water waves driven by a nonvanishing correlation between surface-pressure fluctuations and surface displacement gradients $\partial\eta'/\partial x$.

As Belcher *et al.* point out, the original Miles theory is merely one case of this, in which σ is the growth rate of the Miles instability. The relations (4.6) generalize the result still further: it is now plain that the growth does not need to be exponential either. The result is far more robust than that. Any growth will do. As long as we stay in the realm of small-amplitude disturbances, the expression on the right of (4.6) tells us that the result is not at all dependent on the analytical devices associated with Cauchy's theorem and the calculus of residues, nor on any other analytical device applicable only to disturbances that grow precisely exponentially. The airflow suffers a momentum deficit, and the water waves are amplified — in this frictionless, two-dimensional model — whenever Taylor's wave-activity expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ increases for any reason at all, such as the arrival of a wave group.

However, as figure 1 reminds us, it is essential to take into account the finite-amplitude effects of the advective nonlinearity if one wants to describe what happens beyond early growth. Then it is not enough to restrict attention to infinitesimal amplitude. This point is well illustrated by the SWW and related solutions, and by numerical simulations of situations like that of figure 1 (e.g. Haynes 1989, Norton 1994), in which the constant-Q contours in the critical-layer or surf-zone region wrap round in complicated ways, grossly violating the condition of small sideways slope necessary for the validity of small-disturbance theory. This again is Rossby-wave breaking. Taylor's wave-activity expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ applies during the early stages of the process, when the sideways slopes of the constant-Q contours are still small. Then the expression becomes increasingly inaccurate as the slopes steepen and the contours begin to wrap round, leading to the irreversible mixing of the Q distribution. The Miles theory predicts its own breakdown in a fundamentally similar way.

The defining property of wave breaking — if one wants a concept general enough to apply to wave-induced momentum transport for a wide variety of transverse waves, including Rossby waves as well as gravity waves — is the rapid and irreversible deformation of those material contours that would be described by linear wave theory as sloping gently and undulating reversibly, if the linear theory is self-consistently applied. A careful justification of this definition was given in McIntyre and Palmer (1984, 1985). A central consideration is the role of Kelvin's circulation theorem in wave–mean interaction theory, as further discussed in the abovementioned review (2000). Examples of the relevant material contours include the constant-Q contours for Rossby waves, and, for water waves, material contours lying in the free surface.

A side benefit of the definition is that it avoids using "zoological" ideas of wave breaking, based on features of special cases such as whether or not there is air entrainment, whether or not instabilities are involved, and to what extent and in what sense the resulting flow should be considered turbulent.

Solutions to the SWW and related problems provide not only examples of Rossby-wave breaking as a phenomenon but also, in exquisite detail, descriptions of precisely how wave breaking mediates momentum transport. The SWW solution precisely represents, through the technique of matched asymptotic expansions, the interplay between a

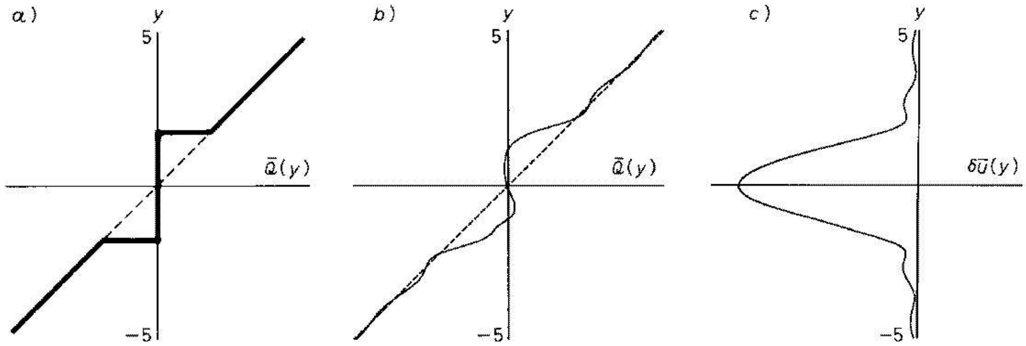

FIGURE 2. The relation between nonlinear Q-rearrangement and momentum transport in the simplest relevant model system, the dynamical system (4.2)–(4.5), when used to describe a breaking Rossby wave. Courtesy P. H. Haynes; for mathematical details see KM and Haynes (1989). Plot (a) shows idealized Q distributions before and after perfect mixing in some y-interval or latitude band $-\frac{1}{2}b < y < \frac{1}{2}b$. Plot ($b$) shows the x-averaged Q distribution, $\bar{Q}(y)$, in an actual model simulation using equations (4.2)–(4.5) within the SWW matched-asymptotics framework. Plot (c) shows the resulting mean momentum deficit, equation (4.7), whose profile would assume the simple parabolic shape (4.8) in the idealized case corresponding to (a).

nonlinear critical layer or surf zone and its wavelike surroundings in a background shear flow $\bar{u}(y)$, including a mathematical description of all the phase shifts involved and their impact on quantities like $\overline{u'v'}$ and $\overline{v'Q'}$ in the first two, exact, equalities in (4.6). Part of the interplay involves the fact that the wavefield, through the PV inversion operator, can remotely sense the nonlinear rearrangement of the Q distribution within the surf zone.

For a first analysis of what happens, it is possible to leave the phase-shift details implicit if one is prepared to assume what is plain from cases like that of figure 1, and from the SWW analysis as well, namely that the result of the Q contours deforming irreversibly in some surf zone $-\frac{1}{2}b < y < \frac{1}{2}b$, say, is an irreversible rearrangement of the Q distribution that weakens the mean gradient $\bar{Q}_y$ in that zone, as suggested in figures 2(a) and 2(b).

Figure 2(a) depicts an idealized strongly-nonlinear scenario in which the surf zone is perfectly mixed; figure 2(b) shows an actual $\bar{Q}(y)$ profile from a complete, dynamically consistent solution due to Haynes (1989 and personal communication), within the SWW matched-asymptotics framework but for a case less simple than the original SWW solution, and in some ways more typical. The detailed flow is more chaotic, because of the onset of secondary instabilities. Yet figure 2(b) is qualitatively similar to figure 2(a), even though the effects of peripheral weak Rossby-wave breaking, in the form of Kelvin 'sheared disturbances', involved in shear sheltering, are also noticeable. To get the corresponding zonally averaged momentum changes, we need the inversion implied by (4.5). In the x-averaged view, this inversion is trivial: one merely has to integrate the change $\delta\bar{Q}$ in $\bar{Q}$ once with respect to y, since $\delta\bar{Q} = -\delta\bar{u}_y$:

$$\delta\bar{u}(y) \;=\; \int_y^\infty \delta\bar{Q}(\tilde{y})\,\mathrm{d}\tilde{y}\;, \tag{4.7}$$

which is well defined because $\int_{-\infty}^{\infty}\delta\bar{Q}(y)\mathrm{d}y \;=\; 0$.

The result is shown in the right-hand graph, figure 2(c). The momentum change corresponding to the more idealized, left-hand $Q(y)$ profile in figure 2(a) is a simple parabolic shape (not shown) qualitatively similar to the right-hand graph, as is evident from a moment's consideration of (4.7):

$$\delta\bar{u}(y) \;=\; \begin{cases} \frac{1}{2}\bar{Q}_y(y^2 - \frac{1}{4}b^2) & (|y| < \frac{1}{2}b)\ , \\ 0 & (|y| > \frac{1}{2}b)\ \cdot \end{cases} \tag{4.8}$$

In this idealized case the total momentum change is precisely

$$\delta M \propto \int_{-\infty}^{\infty} \delta\bar{u}(y)\,\mathrm{d}y \;=\; -\tfrac{1}{12}\bar{Q}_y b^3\ . \tag{4.9}$$

It is plain from (4.7) that *any* Q rearrangement that creates a surf-zone-like feature with weakened $\bar{Q}_y$ will robustly, and irreversibly, give rise to a net momentum deficit. Integrating by parts, we have $\delta M = \int_{-\infty}^{\infty} y\delta\bar{Q}(y)\mathrm{d}y$; that is, δM is equal to the first moment of $\delta\bar{Q}(y)$. This is negative for any N-shaped $\delta\bar{Q}(y)$ profile representing a weakening of the gradient $\bar{Q}_y$ by mixing. Since a momentum deficit in the air corresponds to amplification of the water waves beneath, we may expect a very robust, ratchet-like, irreversible driving of the water waves whenever spanwise vorticity is mixed vertically in the airflow for whatever reason, perfectly or imperfectly. Such mixing is likely to result from, for instance, the arrival of a water-wave group, or the arrival of a wind gust over pre-existing water waves. It would be interesting to have more observational information about vorticity profiles in real gusts.

Notice, finally, how misleading it would be to suppose that small-amplitude theory holds good, with Taylor's wave-activity expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ remaining valid on the right of (4.6), throughout the formation of a surf zone in the airflow caused by, for instance, the arrival and departure of a long water-wave group. Integrating (4.6) with respect to t, one would find zero net momentum change in the air and zero net driving of the water waves. The resolution of this paradox is, of course, the fact that the advective nonlinearity, responsible for producing the irreversible rearrangement of the Q distribution in the airflow, vitiates the small-amplitude expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ as KM showed in detail.

5. Concluding remarks

Of course the real airflow over real water waves is three-dimensional, and three-dimensionally turbulent. KM §7 discusses some of what is involved in adapting the above theory, and argues that a spanwise-averaged two-dimensional representation will still capture the essence of what happens, which is intrinsically robust.

One of the peculiar features of the wind–wave problem is that the quasi-linear theory presently used in wave forecasting models (Janssen 1982, 1991) gives, in a sense, the right answer for the wrong reasons. The quasi-linear theory assumes that real critical layers are infinitesimally thin and behave as in the Miles theory, and that an ensemble of them is distributed over a finite depth because there is a broad spectrum of water-wave phase speeds. The analysis summarized in figure 2, which applies equally well to a narrow, monochromatic spectrum — to say nothing of the stratospheric example with its single, monochromatic Rossby wave having exactly 2 full wavelengths around a latitude circle, and a nearly constant phase speed — underlines the point that real critical layers or surf zones may well be thick, even in a monochromatic case.

The right answer is obtained by the quasi-linear theory because it *diffuses* vorticity in the y-direction through a truly Fickian or Brownian process, in small steps via the ensemble of infinitesimally thin, linearized critical layers. The real process of mixing, in more or less one large step rather than in many small ones, is far more powerful, as Figure 1 reminds us, unless of course the diffusion coefficient D in the diffusion model is 'tuned' to be artificially big.

It is, of course, vorticity that is diffused or mixed, not mean velocity or momentum. An equation like $\bar{u}_t = D(y)\bar{u}_{yy}$ is not a diffusion equation; rather, it is the integral with respect to y of a diffusion equation. The difference is crucial for any case of nonuniform diffusion or mixing, and indeed crucial for the entire scenario summarized by equations (4.7)–(4.9) and figure 2 with its implication that net momentum is exported from the air to the water. Readers may recall the old argument, overlooking the role of wave propagation mechanisms (e.g. Stewart & Thomson 1977), that scenarios like that in figure 2 are 'impossible because they would violate momentum conservation'. For another telling example, see the footnote on page 482 of KM §7.

The wave breaking featuring in the above discussion is that of the Rossby or vorticity waves in the airflow only. If the water waves are also breaking, then flow separation will deepen the layer over which vorticity is mixed in the airflow, further enhancing the irreversible momentum transport and consistent with the discussion of Banner (1990).

As pointed out in my original (1993) discussion, the Rossby-wave-breaking mechanism is only one of *two* mechanisms suggested by the analogy with the global-scale stratosphere. Space precludes more than a brief mention of the second mechanism here [sorry, *first* in the abstract]. The ratcheting-up of the water waves by irreversible momentum transport can take place not only through local changes in airflow Rossby-wave activity — the increase in Taylor's wave-activity expression $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ in the early stages, made irreversible by Rossby-wave breaking — but also by any other process that causes Rossby-wave activity in the airflow to increase irreversibly. Another such process is the formation of Rossby lee waves in the airflow *downstream* of a water-wave group. Then the volume integral of $\frac{1}{2}\bar{Q}_y\overline{\eta'^2}$ increases simply through the rate of increase of the length of the lee-wave train. This may be relevant to explaining the 'energy front' of Chu *et al.* (1992). Further progress, which Stephen Belcher and I hope to pursue, will depend on the analysis of unpublished data for the actual $\bar{u}$ and $\bar{u}_{yy}$ profiles measured in the Chu *et al.* experiments.

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