XXI. *Stability of Fluid Motion (continued from the May and June numbers).—Rectilineal Motion of Viscous Fluid between two Parallel Planes**. *By* Sir W. Thomson, *LL.D., F.R.S.*

27. Since the communication of the first of this series of articles to the Royal Society of Edinburgh in April, and its publication in the Philosophical Magazine in May and June, the stability or instability of the steady motion of a viscous fluid has been proposed as subject for the Adams Prize of the University of Cambridge for 1888 †. The present communication (§§ 27–40) solves the simpler of the two cases specially referred to by the Examiners in their announcement, and prepares the way for the investigation of the less simple by a preliminary laying down, in §§ 27–29, and equations (7) to (12) below, of the fundamental equations of motion of a viscous fluid kept moving by gravity between two infinite plane boundaries inclined to the horizon at any angle I, and given with any motion deviating infinitely little from the determinate steady motion which would be the unique and essentially stable solution if the viscosity were sufficiently large. It seems probable, almost certain indeed, that analysis similar to that of §§ 38 and 39 will demonstrate that the steady motion is stable for any viscosity, however small; and that the practical unsteadiness pointed out by Stokes forty-four years ago, and so admirably investigated experimentally five or six years ago by Osborne Reynolds, is to be explained by limits of stability becoming narrower and narrower the smaller is the viscosity.

Let OX be chosen in one of the bounding planes, parallel to the direction of the rectilineal motion; and OY perpendicular to the two planes. Let the x-, y-, z-, component velocities, and the pressure, at (x, y, z, t), be denoted by $U+u$, v, w, and p respectively; U denoting a function of (y, t). Then, calling the density of the fluid unity, and the viscosity μ, we have, as the equations of motion ‡,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad (1);$$

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* Communicated by the Author, having been read before the Royal Society of Edinburgh, July 18, 1887.

† See Phil. Mag. July 1887, p. 142.

‡ Stokes's Collected Papers, vol. i. p. 93.

$$\left.\begin{aligned}
\frac{d}{dt}(U+u)+(U+u)\frac{du}{dx}+v\frac{d}{dy}(U+u)+w\frac{dw}{dz}&=\mu\nabla^2(U+u)-\frac{dp}{dx}+g\sin I,\\
\frac{dv}{dt}+(U+u)\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}&=\mu\nabla^2 v-\frac{dp}{dy}-g\cos I,\\
\frac{dw}{dt}+(U+u)\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}&=\mu\nabla^2 w-\frac{dp}{dz},
\end{aligned}\right\}\quad (2);$$

where ∇^2 denotes the "Laplacian" $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$.

28. If we have $u=0$, $v=0$, $w=0$; $p=C-g\cos I y$; the four equations are satisfied identically; except the first of (2), which becomes

$$\frac{dU}{dt}=\mu\frac{d^2U}{dy^2}+g\sin I \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

This is reduced to

$$\frac{dv}{dt}=\mu\frac{d^2v}{dy^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

if we put

$$U=v+\tfrac{1}{2}g\sin I/\mu\,.\,(b^2-y^2) \quad . \quad . \quad . \quad (5).$$

For terminal conditions (the bounding planes supposed to be $y=0$ and $y=b$), we may have

$$\left.\begin{aligned} v&=F(t) \text{ when } y=0\\ v&=\mathfrak{F}(t) \quad ,, \quad y=b\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6),$$

where F and $\mathfrak{F}$ denote arbitrary functions. These equations (4) and (6) show (what was found forty-two years ago by Stokes) that the diffusion of velocity in parallel layers, *provided it is exactly in parallel layers*, through a viscous fluid, follows Fourier's law of the "linear" diffusion of heat through a homogeneous solid. Now, towards answering the highly important and interesting question which Stokes raised,—Is this laminar motion unstable in some cases?—go back to (1) and (2), and in them suppose u, v, w to be each infinitely small: (1) is unchanged; (2), with U eliminated by (5), become

$$\frac{du}{dt}+[v+\tfrac{1}{2}c(b^2-y^2)]\frac{du}{dx}+v\left(\frac{dv}{dy}-cy\right)=\mu\nabla^2u-\frac{dp}{dx} \quad . \quad . \quad (7),$$

$$\frac{dv}{dt}+[v+\tfrac{1}{2}c(b^2-y^2)]\frac{dv}{dx}=\mu\nabla^2v-\frac{dp}{dy} \quad . \quad . \quad (8),$$

$$\frac{dw}{dt}+[v+\tfrac{1}{2}c(b^2-y^2)]\frac{dw}{dx}=\mu\nabla^2w-\frac{dp}{dz} \quad . \quad . \quad (9);$$

where

$$c=g\sin I/\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and, for brevity, p now denotes, instead of as before the pressure, the pressure $+g \cos \mathrm{I}\, y$.

We still suppose v to be a function of y and t determined by (4) and (6). Thus (1) and (7), (8), (9) are four equations which, with proper initial and boundary conditions, determine the four unknown quantities u, v, w, p; in terms of x, y, z, t.

29. It is convenient to eliminate u and w; by taking $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ of (7), (8), (9), and adding. Thus we find, in virtue of (1),

$$2\left(\frac{dv}{dy}-cy\right)\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad (11).$$

This and (8) are two equations for the determination of v and p. Eliminating p between them, we find

$$\frac{d\nabla^2 v}{dt}-\left(\frac{d^2 v}{dy^2}-c\right)\frac{dv}{dx}+\left[v-\tfrac{1}{2}c(b^2-y^2)\right]\frac{d\nabla^2 v}{dx}=\mu\nabla^4 v \quad . \quad . \quad (12),$$

a single equation which, with proper initial and boundary conditions, determines the one unknown, v. When v is thus found, (8), (7), (9) determine p, u, and w.

30. An interesting and practically important case is presented by supposing one or both of the bounding planes to be kept oscillating in its own plane; that is, F and $\mathfrak{F}$ of (6) to be periodic functions of t. For example, take

$$\mathrm{F}=a\cos\omega t,\quad \mathfrak{F}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The corresponding periodic solution of (4) is

$$v=a\frac{\epsilon^{(b-y)\sqrt{\frac{\omega}{2\mu}}}-\epsilon^{-(b-y)\sqrt{\frac{\omega}{2\mu}}}}{\epsilon^{b\sqrt{\frac{\omega}{2\mu}}}-\epsilon^{-b\sqrt{\frac{\omega}{2\mu}}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2\mu}}\right) \quad . \quad . \quad (14).$$

In connexion with this case there is no particular interest in supposing a current to be maintained by gravity; and we shall therefore take $c=0$, which reduces (7), (8), (9), (11), (12), to

$$\frac{du}{dt}+v\frac{du}{dx}+\frac{dv}{dy}v=\mu\nabla^2 u-\frac{dp}{dx} \quad . \quad . \quad . \quad (15),$$

$$\frac{dv}{dt}+v\frac{dv}{dx}=\mu\nabla^2 v-\frac{dp}{dy} \quad . \quad . \quad . \quad (16),$$

$$\frac{dw}{dt}+v\frac{dw}{dx}=\mu\nabla^2 w-\frac{dp}{dz} \quad . \quad . \quad . \quad (17),$$

$$2\frac{dv}{dy}\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$\frac{d\nabla^2 v}{dt}-\frac{d^2 v}{dy^2}\frac{dv}{dx}+v\frac{d\nabla^2 v}{dx}=\mu\nabla^4 v \quad . \quad . \quad . \quad . \quad . \quad . \quad (19);$$

in all of which v is the function of (y, t) expressed by (14).

These equations (15) ... (19) are of course satisfied by $u=0$, $v=0$, $w=0$, $p=0$. The question of stability is, Does every possible solution of them come to this in time? It seems to me probable that it does; but I cannot, at present at all events, enter on the investigation. The case of $b=\infty$ is specially important and interesting.

31. The present communication is confined to the much simpler case in which the two bounding planes are kept moving relatively with constant velocity; including as sub-case, the two planes held at rest, and the fluid caused by gravity to move between them. But we shall first take the much simpler sub-case, in which there is relative motion of the two planes, and no gravity. This is the very simplest of all cases of the general question of the Stability or Instability of the Motion of a Viscous Fluid. It is the second of the two cases prescribed by the Examiners for the Adams Prize of 1888. I have ascertained, and I now give (§§ 32...39 below) the proof, that in this sub-case the steady motion is wholly stable, however small or however great be the viscosity; and this without limitation to two-dimensional motion of the admissible disturbances.

32. In our present sub-case, let βb be the relative velocity of the two planes; so that in (6) we may take $\mathrm{F}=0$, $\mathfrak{F}=\beta b$; and the corresponding steady solution of (4) is

$$v = \beta y \quad \text{. . . . . . . (20).}$$

Thus equation (19) becomes reduced to

$$\left.\begin{aligned} \frac{d\sigma}{dt} + \beta y \frac{d\sigma}{dx} &= \mu \nabla^2 \sigma, \\ \text{where} \qquad \sigma &= \nabla^2 v \end{aligned}\right\} \quad \text{. . . . (21);}$$

and (18), (15), (16), (17) become

$$2\beta \frac{dv}{dx} = -\nabla^2 p \quad \text{. . . . . (22),}$$

$$\frac{du}{dt} + \beta y \frac{du}{dx} + \beta v = \mu \nabla^2 u - \frac{dp}{dx} \quad \text{. . . . (23),}$$

$$\frac{dv}{dt} + \beta y \frac{dv}{dx} \qquad = \mu \nabla^2 v - \frac{dp}{dy} \quad \text{. . . . (24),}$$

$$\frac{dw}{dt} + \beta y \frac{dw}{dx} \qquad = \mu \nabla^2 w - \frac{dp}{dz} \quad \text{. . . . (25).}$$

It may be remarked that equations (22) ... (25) imply (1), and that any four of the five determines the four quantities u, v, w, p. It will still be convenient occasionally to use (1).

We proceed to find the complete solution of the problem before us, consisting of expressions for u, v, w, p satisfying (22) ... (25) for all values of x, y, z, t; and the following initial and boundary conditions :—

$$\left.\begin{array}{r}\text{when } t=0:\ u, v, w \text{ to be arbitrary functions} \\ \text{of } x, y, z, \text{ subject only to (1)}\end{array}\right\} \quad . \quad (26);$$

$$\left.\begin{array}{l}u=0,\ v=0,\ w=0, \text{ for } y=0 \text{ and all values of } x, z, t \\ u=0,\ v=0,\ w=0, \text{ for } y=b \qquad ,, \qquad ,,\end{array}\right\} \quad (27).$$

33. First let us find a particular solution $\mathrm{u}, \mathrm{v}, \mathrm{w}, \mathrm{p}$, which shall satisfy the initial conditions (26), irrespectively of the boundary conditions (27), except as follows :—

$$\left.\begin{array}{l}\mathrm{v}=0, \text{ when } t=0 \text{ and } y=0 \\ \mathrm{v}=0, \text{ when } t=0 \text{ and } y=b\end{array}\right\} \quad . \quad . \quad . \quad (28).$$

Next, find another particular solution, $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{p}$, satisfying the following initial and boundary equations :—

$$\mathfrak{u}=0,\ \mathfrak{v}=0,\ \mathfrak{w}=0, \text{ when } t=0 \quad . \quad . \quad . \quad (29);$$

$$\left.\begin{array}{r}\mathfrak{u}+\mathrm{u}=0,\ \mathfrak{v}+\mathrm{v}=0,\ \mathfrak{w}+\mathrm{w}=0, \text{ when } y=0 \\ \text{and when } y=b\end{array}\right\} \quad . \quad . \quad (30).$$

The required complete solution will then be

$$u=\mathfrak{u}+\mathrm{u}, \quad v=\mathfrak{v}+\mathrm{v}, \quad w=\mathfrak{w}+\mathrm{w} \quad . \quad . \quad . \quad (31).$$

34. To find $\mathrm{u}, \mathrm{v}, \mathrm{w}$, remark that, if μ were zero, the complete integral of (21) would be

$$\sigma = \text{arb. func. } (x-\beta yt) \ ;$$

and take therefore as a trial for a type-solution with μ not zero,

$$\sigma = \mathrm{T}\epsilon^{\iota[mx+(n-m\beta t)y+qz]} \quad . \quad . \quad . \quad . \quad (32);$$

where T is a function of t, and ι denotes $\sqrt{-1}$. Substituting accordingly in (21), we find

$$\frac{d\mathrm{T}}{dt} = -\mu[m^2+(n-m\beta t)^2+q^2]\mathrm{T} \quad . \quad . \quad . \quad (33);$$

whence, by integration,

$$\mathrm{T}=\mathrm{C}\epsilon^{-\mu t\left[m^2+n^2+q^2-nm\beta t+\frac{m^2}{3}\beta^2t^2\right]} \quad . \quad . \quad . \quad (34).$$

By the second of (21), and (32), we find

$$v=-\mathrm{T}\frac{\epsilon^{\iota[mx+(n-m\beta t)y+qz]}}{m^2+(n-m\beta t)^2+q^2} \quad . \quad . \quad . \quad . \quad (35);$$

whence, by (22),

$$p = -2\beta m\iota T \frac{\epsilon^{\iota[mx+(n-m\beta t)y+qz]}}{[m^2+(n-m\beta t)^2+q^2]^2} \quad . \quad . \quad . \quad (36).$$

Using this in (25), and putting

$$w = W\epsilon^{\iota[mx+(n-m\beta t)y+qz]} \quad . \quad . \quad . \quad . \quad . \quad (37),$$

we find

$$\frac{dW}{dt} = -\mu[m^2+(n-m\beta t)^2+q^2]W - \frac{2\beta mqT}{[m^2+(n-m\beta t)^2+q^2]} \quad . \quad (38),$$

which, integrated, gives W.

Having thus found v and w, we find u by (1), as follows:—

$$u = -\frac{(n-m\beta t)v+qw}{m} \quad . \quad . \quad . \quad . \quad . \quad (39).$$

35. Realizing, by adding type-solutions for $\pm\iota$ and $\pm n$, with proper values of C, we arrive at a complete real type-solution with, for v, the following—in which K denotes an arbitrary constant:

$$v = \tfrac{1}{2}K\left\{\frac{\epsilon^{-\mu t[m^2+n^2+q^2-nm\beta t+\frac{1}{3}m^2\beta^2t^2]}}{m^2+(n-m\beta t)^2+q^2}\,{}^{\cos}_{\sin}\,[mx+(n-m\beta t)y+qz]\right.$$

$$\left.-\frac{\epsilon^{-\mu t[m^2+n^2+q^2+nm\beta t+\frac{1}{3}m^2\beta^2t^2]}}{m^2+(n+m\beta t)^2+q^2}\,{}^{\cos}_{\sin}\,[mx-(n+m\beta t)y+qz]\right\} \quad (40).$$

This gives, when $t=0$,

$$v = \frac{\mp K}{m^2+n^2+q^2}\sin ny\,{}^{\sin}_{\cos}\,(mx+qz) \quad . \quad . \quad . \quad (41),$$

which fulfils (28) if we make

$$n = i\pi y/b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42);$$

and allows us, by proper summation for all values of i from 1 to ∞, and summation or integration with reference to m and q, with properly determined values of K, after the manner of Fourier, to give any arbitrarily assigned value to $v_{t=0}$ for every value of x, y, z,

$$\left.\begin{array}{llll} \text{from} & x=-\infty & \text{to} & x=+\infty, \\ \text{,,} & y=0 & \text{,,} & y=b, \\ \text{,,} & z=-\infty & \text{,,} & z=+\infty. \end{array}\right\} \quad . \quad . \quad . \quad (43).$$

The same summation and integration applied to (40) gives **v** for all values of t, x, y, z; and then by (38), (37), (39) we find corresponding determinate values of w and u.

36. To give now an arbitrary initial value, $\mathbf{w}_0$, to the

z-component of velocity, for every value of x, y, z, add to the solution (u, v, w), which we have now found, a particular solution (u', v', w') fulfilling the following conditions:—

$$\left.\begin{array}{l} v'=0 \text{ for all values of } t, x, y, z; \\ w'=\mathrm{w}_0-w_0 \text{ for } t=0, \text{ and all values of } x, y, z \end{array}\right\} \quad . \quad (44),$$

and to be found from (25) and (1), by remarking that $v'=0$ makes, by (22), $p'=0$, and therefore (23) and (25) become

$$\frac{du'}{dt}+\beta y\frac{du'}{dx}=\mu\nabla^2 u' \quad . \quad . \quad . \quad . \quad . \quad (45),$$

$$\frac{dw'}{dt}+\beta y\frac{dw'}{dx}=\mu\nabla^2 w' \quad . \quad . \quad . \quad . \quad . \quad (46).$$

Solving (46); just as we solved (21), by (32), (33), (34); and then realizing and summing to satisfy the arbitrary initial condition, as we did for v in (40), (41), (42), we achieve the determination of w'; and by (1) we determine the corresponding u', *ipso facto* satisfying (45). Lastly, putting together our two solutions, we find

$$\mathfrak{u}=u+u', \quad \mathrm{v}=\mathrm{v}, \quad \mathrm{w}=w+w' \quad . \quad . \quad . \quad (47)$$

as a solution of (26) without (27), in answer to the first requisition of § 33. It remains to find $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, in answer to the second requisition of § 33.

37. This we shall do by first finding a real (simple harmonic) periodic solution of (21), (22), (23), (25), fulfilling the condition

$$\left.\begin{array}{l}\left.\begin{array}{l} u=\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t \\ v=\mathrm{C}\cos\omega t+\mathrm{D}\sin\omega t \\ w=\mathrm{E}\cos\omega t+\mathrm{F}\sin\omega t \end{array}\right\} \text{ when } y=0 \\ \left.\begin{array}{l} u=\mathfrak{A}\cos\omega t+\mathfrak{B}\sin\omega t \\ v=\mathfrak{C}\cos\omega t+\mathfrak{D}\sin\omega t \\ w=\mathfrak{E}\cos\omega t+\mathfrak{F}\sin\omega t \end{array}\right\} \text{ when } y=b \end{array}\right\} \quad . \quad . \quad (48),$$

where A, B, C, D, E, F, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ are twelve arbitrary functions of (x, z). Then, by taking $\int_0^\infty d\omega f(\omega)$ of each of these after the manner of Fourier, we solve the problem of determining the motion produced throughout the fluid, by giving to every point of each of its approximately plane boundaries an infinitesimal displacement of which each of the three components is an arbitrary function of x, z, t. Lastly, by taking these functions each $=0$ from $t=-\infty$ to $t=0$, and each equal to minus the value of u, v, w for every point of each boundary, we find the $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ of § 33. The solution of our problem of § 32 is then completed by equations (31). To do all this is a mere routine after an imaginary type solution is provided as follows.

38. To satisfy (21) assume

$$v=\epsilon^{\iota(\omega t+mx+qz)}\mathcal{V}$$

$$=\epsilon^{\iota(\omega t+mx+qz)}\{H\epsilon^{y\sqrt{(m^2+q^2)}}+K\epsilon^{-y\sqrt{(m^2+q^2)}}+Lf(y)+MF(y)\} \quad . \quad (49),$$

where H, K, L, M are arbitrary constants and f, F any two particular solutions of

$$\iota(\omega+m\beta y)\sigma=\mu\left[\frac{d^2\sigma}{dy^2}-(m^2+q^2)\sigma\right] \quad . \quad . \quad (50).$$

This equation, if we put

$$m\beta/\mu=\gamma, \text{ and } m^2+q^2+\iota\omega/\mu=\lambda \quad . \quad . \quad . \quad (51),$$

becomes

$$\frac{d^2\sigma}{dy^2}=(\lambda+\iota\gamma y)\sigma \quad . \quad . \quad . \quad . \quad . \quad (52);$$

which, integrated in ascending powers of $(\lambda+\iota\gamma y)$, gives two particular solutions, which we may conveniently take for our f and F, as follows :—

$$\left.\begin{aligned} f(y)&=1-\frac{\gamma^{-2}(\lambda+\iota\gamma y)^3}{3.2}+\frac{\gamma^{-4}(\lambda+\iota\gamma y)^6}{6.5.3.2}-\frac{\gamma^{-6}(\lambda+\iota\gamma y)^9}{9.8.6.5.3.2}+\&c. \\ F(y)&=\lambda+\iota\gamma y-\frac{\gamma^{-2}(\lambda+\iota\gamma y)^4}{4.3}+\frac{\gamma^{-4}(\lambda+\iota\gamma y)^7}{7.6.4.3}-\frac{\gamma^{-6}(\lambda+\iota\gamma y)^{10}}{10.9.7.6.4.3}+\&c. \end{aligned}\right\}(53).$$

39. *These series are essentially convergent for all values of y.* Hence in (49) we have a solution continuous from $y=0$ to $y=b$; and by its four arbitrary constants we can give any prescribed values to $\mathcal{V}$, and $\frac{d\mathcal{V}}{dy}$, for $y=0$ and $y=b$. This done, find p determinately by (24); and then integrate (25) for w in *an essentially convergent series* of ascending powers of $\lambda+\iota\gamma y$, which is easily worked out, but need not be written down at present, except in abstract as follows :—

$$w=\mathcal{W}\epsilon^{\iota(\omega t+mx+qz)} \quad . \quad . \quad . \quad . \quad . \quad (54);$$

where

$$\left.\begin{aligned} \mathcal{W}=&H\mathfrak{F}_1(\lambda+\iota\gamma y)+K\mathfrak{F}_2(\lambda+\iota\gamma y)+L\mathfrak{F}_3(\lambda+\iota\gamma y) \\ &+M\mathfrak{F}_4(\lambda+\iota\gamma y)+P\epsilon^{y\sqrt{(m^2+q^2)}}+Q\epsilon^{-y\sqrt{(m^2+q^2)}} \end{aligned}\right\}(55).$$

Here P and Q are the two fresh constants, due to the integration for w. By these we can give to $\mathcal{W}$ any prescribed values for $y=0$ and $y=b$. Lastly, by (1), with (49), we have

$$\left.\begin{aligned} u &= \mathcal{U}\epsilon^{\iota(\omega t+mx+gz)} \\ \text{where}\quad \mathcal{U} &= -\left(\frac{1}{m\iota}\frac{d\mathcal{V}}{dy}+\frac{g}{m}\mathcal{W}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (56).$$

Our six arbitrary constants, H, K, L, M, P, Q, clearly allow us to give any prescribed values to each of $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, for $y=0$ and for $y=b$. Thus the completion of the realized problem with real data of arbitrary functions, as described in § 37, becomes a mere affair of routine.

40. Now remark that the (u, v, w) solution of § 34 comes essentially to nothing, asymptotically as time advances, as we see by (33), (34), and (38). Hence the (𝔲, 𝔳, 𝔴) of § 37, which rise gradually from zero at $t=0$, comes asymptotically to zero again as t increases to ∞. We conclude that the steady motion is stable.

[To be continued.]

XXII. *On Evaporation and Dissociation.*—Part VI. (*continued*). *On the Continuous Change from the Gaseous to the Liquid State at all Temperatures.* By WILLIAM RAMSAY, *Ph.D.*, *and* SYDNEY YOUNG, *D.Sc.**

[Plates III.-V.]

THE following pages give a further proof of the correctness of the relation $p=bt-a$, where $v=$constant, applicable both to gases and liquids. The data for methyl alcohol apply solely to the gaseous state, for the very high pressures which its vapour exerts precluded measurements at temperatures above its critical point. With ethyl alcohol the determinations of the compressibility of the liquid are more complete than with ether; the experimental observations in the neighbourhood of the critical volume are, however, not very numerous, for the highest temperature for which an isothermal was constructed is 246°, the critical temperature being 243°·1. The values of a and b at volumes near the critical are consequently somewhat uncertain. The data for the gaseous condition are, however, pretty full. We have also a considerable number of data for acetic acid (Trans. Chem. Soc. 1886, p. 790). Here the temperature at which the highest isothermal was measured was the highest conveniently attainable by our method, viz. 280°. But as the critical temperature

* Communicated by the Physical Society: read April 23, 1887.

lies much higher, the pressures were in no case very great. The behaviour of acetic acid, however, contrasted with that of the alcohols, ether, and carbon dioxide is very striking. The equation $p=bt-a$ does not apply; in other words lines of equal volume are not straight, but are curves of double flexure. We shall consider the meaning of this peculiarity after adducing data.

1. *Methyl Alcohol.*—The data are at present in the hands of the Royal Society. The values of b were, as before, determined by reading points on the isothermal curves at equal volumes; constructing isochors graphically, and having thus obtained approximate values of b, these were smoothed by plotting them as abscissæ, the ordinates being the reciprocals of the volumes. The values of b given in the Table which follows were those read from this curve. The values of a were calculated from the equation $a=bt-p$, the mean value obtained from all the readings at each volume being taken as correct.

TABLE I.

Vol.	*b.*	log *b.*	*a.*
c. c. per gram.			
7	626·5	2·79692	263430
8	569·0	2·75511	235370
9	509·0	2·70672	206290
10	452·5	2·65562	179090
11	405·3	2·60778	156660
12	365·0	2·56229	137810
14	299·0	2·47567	107460
16	248·5	2·39533	84730
18	211·0	2·32428	68364
20	184·8	2·26670	57474
25	139·8	2·14551	39724
30	112·5	2·05115	29834
40	80·0	1·90309	18913
50	62·0	1·79239	13401
70	43·5	1·63849	8631
100	26·95	1·43056	4092
135	18·95	1·27761	2398
170	14·50	1·06137	1624
200	12·00	1·07918	1187
240	9·65	0·98453	794
280	8·00	0·90309	455
340	6·30	0·79934	307
400	5·15	0·71181	159
450	4·57	0·65992	130

The following Table gives the pressures read from the isothermals from which these values of a and b were obtained; and we have added, for the sake of comparison, the pressures recalculated by help of these values.

XXXIV. *Stability of Motion (continued from the May, June, and August Numbers).—Broad River flowing down an Inclined Plane Bed.* *By* Sir WILLIAM THOMSON, *F.R.S.**

41. CONSIDER now the second of the two cases referred to in § 27—that is to say, the case of water on an inclined plane bottom, under a fixed parallel plane cover (ice, for example), both planes infinite in all directions and gravity everywhere uniform. We shall include, as a sub-case, the icy cover moving with the water in contact with it, which is particularly interesting, because, as it annuls tangential force at the upper surface, it is, for the steady motion, the same case as that of a broad open river flowing uniformly over a perfectly smooth inclined plane bed. It is not the same, except when the motion is steadily laminar, the difference being that the surface is kept rigorously plane, but not free from tangential force, by a rigid cover, while the open surface is kept almost but not quite rigorously plane by gravity, and rigorously free from tangential force. But, provided the bottom is smooth, the smallness of the dimples and little round hollows which we see on the surface, produced by turbulence (when the motion is turbulent), seems to prove that the motion must be very nearly the same as it would be if the upper surface were kept rigorously plane, and free from tangential force.

42. The sub-case described in § 31 having been disposed of in §§ 32–40, we now take the including case, described in the first half-sentence of § 31; for which we have, as steady solution, according to (5),

$$U=\beta y-\tfrac{1}{2}cy^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

if we reckon y from the bottom upwards. Thus (7), (8), (9), (11), (12) become

$$\frac{du}{dt}+(\beta y-\tfrac{1}{2}cy^2)\frac{du}{dx}+(\beta-cy)v=\mu\nabla^2u-\frac{dp}{dx} \quad . \quad (58),$$

$$\frac{dv}{dt}+(\beta y-\tfrac{1}{2}cy^2)\frac{dv}{dx} \qquad =\mu\nabla^2v-\frac{dp}{dy} \quad . \quad (59),$$

$$\frac{dw}{dt}+(\beta y-\tfrac{1}{2}cy^2)\frac{dw}{dx} \qquad =\mu\nabla^2w-\frac{dp}{dz} \quad . \quad (60),$$

$$2(\beta-cy)\frac{dv}{dx}= \qquad -\nabla^2p \quad . \quad (61),$$

$$\frac{d\nabla^2v}{dt}+c\frac{dv}{dx}+(\beta y-\tfrac{1}{2}cy^2)\frac{d\nabla^2v}{dx}=\mu\nabla^4v \quad . \quad . \quad . \quad (62).$$

* Communicated by the Author.

43. We have not now any such simple partial solution as that of §§ 34, 35, 36 for the sub-case there dealt with; and we proceed at once to the virtually inclusive* investigation specified in § 37, and, as in § 38, assume

$$v=\epsilon^{\iota(\omega t+mx+qz)}\mathcal{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad (63).$$

This gives

$$\frac{d}{dt}=\iota\omega, \quad \frac{d}{dx}=\iota m, \quad \text{and} \quad \nabla^2=\frac{d^2}{dy^2}-m^2-q^2 \quad . \quad (64):$$

and (62) becomes therefore

$$\mu\frac{d^4\mathcal{V}}{dy^4}-\{2\mu(m^2+q^2)+\iota[\omega+m(\beta y-\tfrac{1}{2}cy^2)]\}\frac{d^2\mathcal{V}}{dy^2}$$
$$+\{\mu(m^2+q^2)^2+\iota[\omega+m(\beta y-\tfrac{1}{2}cy^2)](m^2+q^2)-\iota cm\}\mathcal{V}=0 \quad . \quad (65),$$

or, for brevity,

$$\mu\frac{d^4\mathcal{V}}{dy^4}+(e+fy+gy^2)\frac{d^2\mathcal{V}}{dy^2}+(h+ky+ly^2)\mathcal{V}=0 \quad . \quad (66).$$

To integrate this, assume

$$\mathcal{V}=c_0+c_1y+c_2y^2+c_3y^3+c_4y^4+\&c. \quad . \quad . \quad . \quad (67);$$

and, by equating to zero the coefficient of y^i in (66), we find

$$(i+4)(i+3)(i+2)(i+1)\mu c_{i+4}+(i+2)(i+1)ec_{i+2}$$
$$+(i+1)ifc_{i+1}+[i(i-1)g+h]c_i+kc_{i-1}+lc_{i-2}=0 \quad . \quad (68).$$

Making now successively $i=0$, $i=1$, $i=2$, . . . , and remembering that c with any negative suffix is zero, we find

$$\left.\begin{array}{l}
4.3.2.1.\mu c_4+2.1.ec_2+hc_0=0,\\
5.4.3.2.\mu c_5+3.2.ec_3+2.1.fc_2+hc_1+kc_0=0,\\
6.5.4.3.\mu c_6+4.3.ec_4+3.2.fc_3+(2.1.g+h)c_2+kc_1+lc_0=0,\\
7.6.5.4.\mu c_7+5.4.ec_5+4.3.fc_4+(3.2.g+h)c_3+kc_2+lc_1=0,\\
\quad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{array}\right\} (69)$$

These equations, taken in order, give successively c_4, c_5, c_6, . . ., each explicitly as a linear function of c_0, c_1, c_2, c_3; and by

* The Fourier-Sturm-Liouville analysis (Fourier, *Théorie de la Chaleur*; Sturm and Liouville, Liouville's Journal for the year 1836, and Lord Rayleigh's 'Theory of Sound,' § 142, vol. ii. shows how to express an arbitrary function of x, y, z by summation of the type solutions of §§ 37, 39 above and § 43 (63), (67), (70) here, and so to complete, whether for our present case or former sub-case, the fulfilment of the conditions (26), (27), without using the method of §§ 34, 35, 36.

using in (67) the expressions so obtained, we find

$$\mathfrak{V}=c_0\mathfrak{F}_0(y)+c_1\mathfrak{F}_1(y)+c_2\mathfrak{F}_2(y)+c_3\mathfrak{F}_3(y) \quad . \quad . \quad (70),$$

where c_0, c_1, c_2, c_3 are four arbitrary constants, and $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ four functions, each wholly determinate, expressed in a series of ascending powers of y which by (68) we see to be *convergent for all values of* y, unless μ be zero. The essential convergency of these series proves (as in § 39 for the case of no gravity) that *the steady motion* ($u=0$, $v=0$, $w=0$) *is stable, however small be* μ, *provided it is not zero.*

44. The less is μ, the less the convergence. When μ is very small there is divergence for many terms, but ultimate convergence.

45. In the case of $\mu=0$, the differential equation (66), or (67), becomes reduced from the 4th to the 2nd order, and may be written as follows:—

$$\frac{d^2\mathfrak{V}}{dy^2}=\left\{m^2+q^2-\frac{cm}{\omega+m(\beta y-\frac{1}{2}cy^2)}\right\}\mathfrak{V} \quad . \quad . \quad (71).$$

This, for the case of two-dimensional motion ($q=0$), agrees with Lord Rayleigh's result, expressed in the last equation of his paper on "The Stability or Instability of certain Fluid Motions" (Proc. Lond. Math. Soc. Feb. 12, 1880). The integral, but now with only two arbitrary constants (c_0, c_1), is still given in ascending powers of y by (67) and (68), which, with $\mu=0$, and the thus-simplified values of e, f, g put in place of these letters, becomes

$$-\iota[(i+2)(i+1)\omega c_{i+2}+(i+1)im\beta c_{i+1}]$$
$$+\left[\frac{\iota}{2}i(i-1)mc+h\right]c_i+kc_{i-1}+lc_{i-2}=0 \quad . \quad (72).$$

For very great values of i this gives

$$\omega c_{i+2}+m\beta c_{i+1}-\tfrac{1}{2}mcc_i=0 \quad . \quad . \quad . \quad . \quad (73),$$

which shows that ultimately, except in the case of one particular value of the ratio c_1/c_0,

$$c_{i+1}/c_i=\zeta^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74),$$

where ζ denotes the smaller root of the equation

$$\omega+m\beta y-\tfrac{1}{2}mcy^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

Hence there is certainly not convergence for values of y exceeding the smaller root of (75), and thus the proof of stability is lost.

46. But the differential equation, simplified in (71) for the case of no viscosity, may no doubt be treated more appro-

priately in respect to the question of stability or instability, by writing it as follows [ζ', ζ denoting the two roots of (75)],

$$\frac{d^2\mathcal{V}}{dy^2}=\left\{m^2+q^2+\frac{2}{\zeta'-\zeta}\left(\frac{1}{\zeta-y}-\frac{1}{\zeta'-y}\right)\right\}\mathcal{V} \quad . \quad (76),$$

and integrating with special consideration of the infinities at $y=\zeta$ and $y=\zeta'$. One way of doing this, which I merely suggest at present, and do not follow out for want of time, is to assume

$$\mathcal{V}=\mathrm{C}\{\zeta-y+c_2(\zeta-y)^2+c_3(\zeta-y)^3+\&c.\},$$
$$+\mathrm{C}'\{\zeta'-y+c_2'(\zeta'-y)^2+c_3'(\zeta'-y)^3+\&c.\} \quad . \quad . \quad (77),$$

where C and C′ are two arbitrary constants, and $c_2, c_3, \ldots$, $c_2', c_3', \ldots$ coefficients to be determined so as to satisfy the differential equation. This is very easily done; and when done shows that each series converges for all values of y less, in absolute magnitude, than $\zeta'-\zeta$, and diverges for values of y exceeding $\zeta'-\zeta$. The working out of this in detail would be very interesting, and would constitute the full mathematical treatment of the problem of finding sinuous stream-lines (curves of sines) throughout the space between two "cat's-eye" borders (corresponding to $y=\zeta$ and $y=\zeta'$) which I proposed in a short communication to Section A of the British Association at Swansea, in 1880*, "On a Disturbing Infinity in Lord Rayleigh's solution for Waves in a plane Vortex stratum." It is to be remarked that this disturbing infinity vitiates the seeming proof of stability contained in Lord Rayleigh's equations (56), (57), (58).

47. Realizing (63), and interpreting the result in connexion with (57), we see that

(a) The solution which we have found consists of a wave-disturbance travelling in any (x, z) direction, of which the propagational velocity in the x-direction is $-\omega/m$.

(b) The roots (ζ, ζ') of (75) are values of y at places where the velocity of the undisturbed laminar flow is equal to the x-velocity of the wave-disturbance.

Hence, supposing the bounding-planes to be plastic, and force to be applied to either or both of them so as to produce an infinitesimal undulatory corrugation, according to the formula $\cos(\omega t+mx+qz)$, this surface-action will cause throughout the interior a corresponding infinitesimal wave-motion if *ω/m is not equal to the value of* U *for any plane of*

* Of which an abstract is published in 'Nature' for November 11, 1880, and in the British Association volume Report for the year. In this abstract cancel the statement "is stable," with reference to a certain steady motion described in it.

the fluid between its boundaries. But the infinity corresponding to $y=\zeta$ or $y=\zeta'$ will vitiate this solution *if* ω/m *is equal to the value of* U *for some one plane of the fluid or for two planes of the fluid*; and the true solution will involve the "cat's-eye pattern" of stream-lines, and the enclosed elliptic whirls*, at this plane or these planes.

48. Now let the fluid be given moving with the steady laminar flow between two parallel boundary planes, expressed by (57), which would be a condition of kinetic equilibrium (proved stable in § 43) under the influence of gravity and viscosity; and let both gravity and viscosity be suddenly annulled. The fluid is still in kinetic equilibrium; but is the equilibrium stable? To answer this question, let one or both bounding-surfaces be infinitesimally dimpled in any place and made plane again. The Fourier synthesis of this surface-operation is

$$\int_0^\infty \int_0^\infty \int_0^\infty d\omega\, dm\, dq\, f(\omega)\, \mathrm{F}(m)\, \mathfrak{F}(q) \cos \omega t \cos mx \cos qz \quad (78),$$

or

$$\tfrac{1}{2}\int_0^\infty \int_0^\infty \int_0^\infty d\omega\, dm\, dq\, f(\omega)\, \mathrm{F}(m)\, \mathfrak{F}(q) \{\cos(\omega t - mx) - \cos(\omega t + mx)\} \cos qz \,. \quad (79),$$

which implies harmonic surface-undulations travelling in opposite x-directions, with all values from 0 to ∞ of (ω/m), the $\pm x$ of wave-velocity. Hence (§ 47) the interior disturbance essentially involves elliptic whirls. Thus we see that the given steady laminar motion is *thoroughly* unstable, being ready to break up into eddies in every place, on the occasion of the slightest shock or bump on either plastic plane boundary. The slightest degree of viscosity, as we have seen, makes the laminar motion stable; but the smaller the viscosity with a given value of $g \sin \mathrm{I}$, or the greater the value of $g \sin \mathrm{I}$ with the same viscosity, the narrower are the limits of this stability. Thus we have been led by purely mathematical investigation to a state of motion agreeing perfectly with the following remarkable descriptions of observed results by Osborne Reynolds (Phil. Trans. March 15, 1883, pp. 955, 956):—

"The fact that the steady motion breaks down suddenly, shows that the fluid is in a state of instability for disturbances of the magnitude which cause it to break down. But the fact that in some conditions it will break down for a large disturbance, while it is stable for a smaller disturbance, shows

* See my former paper on the "Disturbing Infinity" already referred to.

that there is a certain residual stability, so long as the disturbances do not exceed a given amount.". . .

"And it was a matter of surprise to me to see the sudden force with which the eddies sprang into existence, showing a highly unstable condition to have existed at the time the steady motion broke down."

"This at once suggested the idea that the condition might be one of instability for disturbance of a certain magnitude, and stable for smaller disturbances."

49. The motion investigated experimentally by Reynolds, and referred to in the preceding statements, was that of water in a long straight uniform tube of circular section. It is to be hoped that candidates for the Adams Prize of 1888 may investigate this case mathematically, and give a complete solution for infinitesimal deviations from rectilineal motion. It is probable that for it, and generally for a uniform straight tube of any cross section, including the extreme, and extremely simplified, case of rectilinear motion of a viscous fluid between two parallel *fixed* planes, which I have worked out above, the same general conclusion as that stated at the end of § 26 and in §§ 43–48 will be found true.

50. In the case of no gravity ($g \sin I = 0$), and the viscous fluid kept in "shearing" or "laminar" motion by relative motion of the two parallel planes, there is, when viscosity is annulled, no disturbing instability in the steady uniform shearing motion, with its uniform molecular rotation throughout, which viscosity would produce; and therefore our reason for suspecting any limitation of the excursions within which there is stability, and for expecting possible *permanence* of any kind of turbulent or tumultuous motion between two *perfectly* smooth planes (or between two polished planes with any practical velocities) does not exist in this case. But a great variety of general observation (and particularly Rankine and Froude's doctrine of the "skin-resistance" of ships, and Froude's experimental determination of the resistance experienced by a very smooth, thin, vertical board, 19 inches broad and 50 feet long, moved at different uniform speeds* through water in a broad deep tank 278 feet

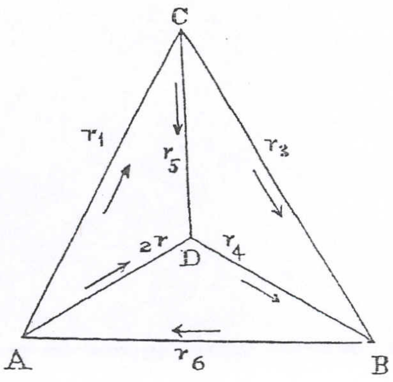
* 'Report to the Lords Commissioners of the Admiralty on Experiments for the Determination of the Frictional Resistance of Water on a Surface under various conditions, performed at Chelston Cross (Torquay), under the Authority of their Lordships.' By W. Froude. (London: Taylor and Francis. 1874.)

Froude found that, at a constant velocity of 600 feet per minute, the resistance of the water against one of his smoothest surfaces, at positions two feet abaft of the cutwater and 50 feet abaft of the cutwater, respectively, was ·295 of a pound per square foot, and ·244 of a pound per square long) makes it certain that if water be given at rest between two infinite planes both at rest, and if one of the planes be suddenly, *or not too gradually*, set in motion, and kept moving uniformly, the motion of the water will be at first turbulent, and the ultimate condition of uniform shearing will be approached by gradual reduction and ultimate annulment of the turbulence. I hope to make a communication on this subject to Section A of the British Association in Manchester, and to have it published in the October number of the Philosophical Magazine. Corresponding questions must be examined with reference to the corresponding tubular problem, of an infinitely long, straight, solid bar kept moving in water within an infinitely long fixed tube. It is to be hoped that the 1888 Adams Prize will bring out important investigations on this subject.

[To be continued.]

XXXV. *Note on an Elementary Proof of certain Theorems regarding the Steady Flow of Electricity in a Network of Conductors.* By ANDREW GRAY, *M.A., F.R.S.E., Professor of Physics in the University College of North Wales**.

THE following elementary proof of the principal theorems of a network of conductors may be of interest. It will be necessary to consider first the well known and, for our purpose, typical case of a network of five conductors, shown in the figure. We assume the so-called laws of Kirchhoff, namely the principle of continuity applied to the steady flow of electricity in a linear system; and the theorem (at once deducible from Ohm's law) that in any closed circuit of conductors forming part of a linear system, the sum of the products obtained by multiplying the current in each part taken in order round the circuit by its resistance, is equal to the sum of the electromotive forces in the circuit.



Let the wire joining A B contain

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foot. Remark that this astonishingly great force of a quarter of a pound per square foot (!!) is the resistance due to uniform laminar flow of water between two parallel planes $\frac{1}{30}$ of a centimetre ($\frac{1}{900}$ of a foot!) asunder, when one of the planes is moving relatively to the other at 10 feet (300 centimetres) per second, if the water be at the temperature 0° Cent., for which the viscosity, calculated from Poiseuille's observations on the flow of water in capillary tubes, is $1{\cdot}34 \times 10^{-5}$ of a gramme weight per square centimetre.

* Communicated by the Author.