Geophysical and Astrophysical Fluid Dynamics
Vol. 00, No. 00, submitted September 2019, revised February 2020, accepted April 2020, 1–12

Wave–vortex interactions and effective mean forces: three basic problems

Michael Edgeworth McIntyre†*

†Department of Applied Mathematics and Theoretical Physics, Cambridge CB3 0WA, UK

(*Submitted to the Special Issue on Geophysical and Astrophysical Vortex Interactions*)

Three examples of wave-vortex interaction are studied, in analytically-tractable weak refraction regimes with attention to the mean recoil forces, local and remote, that are associated with refractive changes in wave pseudomomentum fluxes. Wave-induced mean forces of this kind can be persistent, with cumulative effects, even in the absence of wave dissipation. In each example, a single wavetrain propagates past a single vortex. In the first two examples, in a two-dimensional, non-rotating acoustic or shallow-water setting, the focus is on whether or not the wavetrain overlaps the vortex core. In the overlapping case, the recoil has a local contribution given by the Craik–Leibovich force on the vortex core, the vector product of Stokes drift and mean vorticity. (For a quantum vortex this contribution is called the Iordanskii force arising from the Aharonov–Bohm effect on a phonon current.) However, in all except one special limiting case there are additional 'remote' contributions, mediated by Stokes-drift-induced return flows that can intersect the vortex core well away from locations where the waves are refracted.

The third example is a non-overlapping, remote-recoil-only example in a rapidly rotating frame, in which the waves are deep-water gravity waves and the mean flow obeys shallow-water quasigeostrophic dynamics. Contrary to what might at first be thought, the Ursell 'anti-Stokes flow' induced by the rotation – an Eulerian-mean flow tending to cancel the Stokes drift – fails to suppress remote recoil. There are nontrivial open questions about extending these results to regimes of stronger refraction, especially regarding the scope of the 'pseudomomentum rule' for the wave-induced recoil forces.

Keywords: waves; momentum; pseudomomentum; wave–vortex interactions; mean recoil forces; missing forces in gravity-wave parametrizations; Iordanskii force; Aharonov–Bohm effect

1. Introduction

The following is a shortened, mainly descriptive version of a longer paper (McIntyre 2019, hereafter M19), to which the reader is referred for full technical details. M19 explores analytically tractable, precisely soluble versions of the wave–vortex interaction problems to be discussed, with careful attention to asymptotic validity, and with cross-checks via independent analyses in complementary, but overlapping, asymptotic regimes. The present, shorter paper tries to add value to M19 by providing a relatively concise summary of the main results together with some additional discussion. The problems look simple at first sight but have proved to be surprisingly tricky – conceptually as well as technically. They are fundamental, moreover, to any attempt to complete our understanding of the $O(a^2)$ wave-induced mean forces arising from wave-induced momentum transport, where a is wave amplitude defined such that $a \ll 1$ validates linearization.

It hardly needs saying that mean forces of this kind are scientifically important. They have long been recognized as playing a key role in, for instance, global-scale atmospheric dynamics, as recalled in M19 and in greater detail in the reviews by Fritts (1984), Holton *et al.* (1995), and Baldwin *et al.* (2001). See also Dritschel and McIntyre (2008, & refs.). Examples include what used to be the enigma of the quasi-biennial oscillation or 13-monthly reversal of the east-west winds in the equatorial lower stratosphere (*e.g.* Baldwin *et al.* 2001), and the enigma of the cold summer mesopause with its noctilucent clouds (*e.g.* Fritts 1984). (A global-scale circulation, gyroscopically pumped by wave-induced mean forces, turns the

*Email: mem@damtp.cam.ac.uk

summer mesosphere into a giant refrigerator.) Some of the waves involved in these phenomena are internal gravity waves whose scales are too small to be resolved in weather and climate forecasting models, and whose mean effects are therefore routinely represented in the models via so-called gravity-wave parametrization schemes (*e.g.* Garcia *et al.* 2017, & refs.).

The parametrization schemes and the associated theoretical literature have always, however, neglected the nondissipative wave-induced mean recoil forces associated with the deflection of waves by vortices, and other horizontal-refraction effects. Here, the word 'refraction' will be used in its most general sense – which is the sense that is relevant here – to mean not just the bending of rays but any distortion of the wave field by the vortex flow. Examples include those illustrated in equations (2)–(5) below, as well as in studies like those of, for instance, Sakov (1993), Sonin (1997), Coste *et al.* (1999) and Ford and Llewellyn Smith (1999, hereafter FLS99). In various ways those examples include, but also go beyond, standard ray theory (JWKB theory). The present work makes use of ray theory but also goes beyond it in significant ways.

The nondissipative recoil forces in question are potentially important because they can be persistent, in the same sense that the more familiar dissipative wave-induced forces are persistent. They can act cumulatively over an arbitrary number of wave periods. And one of the conceptually tricky questions about them is the question of *where* such forces are exerted.

Even the simplest problems, or thought-experiments, in which a single wavetrain is refracted by a single vortex, illustrate what is involved. Consider the problem sketched schematically in figure 1(a). A steady train of gravity waves or sound waves passes to one side of a vortex, in an inviscid, two-dimensional, non-rotating shallow water or homentropic gas dynamical system. The vortex flow has small Froude or Mach number

$$\epsilon = U/c_0 \ll 1\,, \tag{1}$$

and the wave refraction is correspondingly weak (and left invisible in the figure, but see section 2 below). Here U is a vortex flow speed and c_0 an intrinsic wave speed. For definiteness, c_0 will be taken as the wave speed at $r = \infty$ and U as the flow speed at the edge of the vortex core, $r = r_0$, say, where $r^2 = x^2 + y^2$ in the notation of the figure.

The question of where the mean recoil force is exerted is ambiguous. It can be asked and answered in more than one way. Simplest and most useful is to ask the question in the way that is relevant to gravity-wave parametrization. What force would be required if the waves were removed, in order to have the same effect on the mean flow? For the problem sketched in figure 1(a) the answer was found in an earlier study by Bühler and McIntyre (2003, hereafter BM03). The answer may seem surprising at first sight. The force has to be exerted not where the waves are refracted, within the wavetrain as it passes the vortex, but, rather, on the vortex core. Because the core can be at an arbitrary distance from the wavetrain, BM03 called this effective mean force a 'remote recoil'.

Of course there is no mystery here – no violation of Newton's Third Law – because a fluid medium has a mean pressure field that can mediate actions and reactions continuously, across substantial distances, just as in ordinary vortex–vortex interactions. The point may be obvious, but is sometimes overlooked when problems like these are discussed from a particle-physics perspective. And it is perfectly reasonable to say, alternatively, that when the waves are present the mean force is exerted where they are refracted. However, to make sense of the resulting picture one would then have to solve for the $O(a^2)$ mean pressure field and explicitly describe how it transmits the force across the gap between the wavetrain and the vortex core.

BM03 also found that the mean force complies with what is now called the 'pseudomomentum rule' (*e.g.* Bühler 2014, hereafter B14). Its validity is tacitly assumed in, for instance, parts of the literature on gravity-wave parametrization, and on quantum vortices as well (*e.g.* Sonin 1997). When valid, it avoids any consideration of the $O(a^2)$ mean equations. It says that the magnitude and direction of the mean force can be calculated from linearized

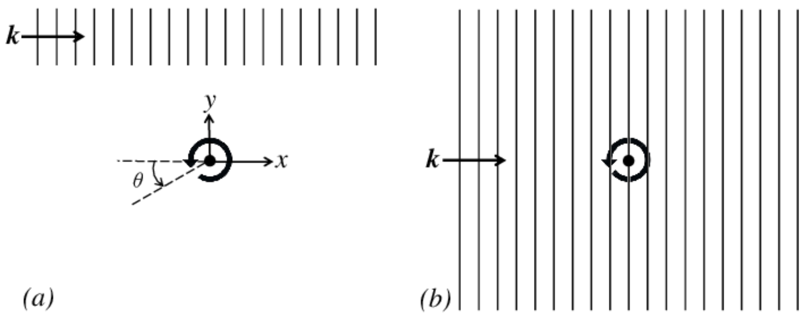

Figure 1. Panels (a) and (b) are schematics of wave–vortex interaction problems (i) and (ii) respectively. Waves of wavenumber $\boldsymbol{k}$ are incident from the left and are weakly refracted by the vortex. The refraction effects are left invisible to emphasize their extreme weakness (but see section 2 below). The azimuthal angle θ is defined unconventionally but in a way that will be convenient when discussing the Aharonov–Bohm effect, which turns out to be one of the significant wave refraction effects.

wave theory alone as if pseudomomentum were momentum, and as if the fluid medium were absent.

(Pseudomomentum, also called quasimomentum or wave momentum, or phonon momentum, is the $O(a^2)$ linear-theoretic wave property whose nondissipative conservation depends, through Noether's theorem, on translational invariance of the mean or background state on which the waves propagate, as distinct from translational invariance of the entire physical system, background plus waves, which implies conservation of momentum (*e.g.* McIntyre 1981, Peierls 1991, & refs.). In a linearized ray-theoretic description the pseudomomentum $\mathbf{p}$ per unit mass is $\mathcal{A}\boldsymbol{k}$, where $\boldsymbol{k}$ is the wavenumber vector and $\mathcal{A}$ the wave-action, the intrinsic wave energy divided by the intrinsic frequency, per unit mass. B14 gives more general expressions for $\mathbf{p}$ outside the scope of linearized ray theory.)

Another way to state the pseudomomentum rule is to say that the magnitude and direction of the recoil can be calculated as if the problem were one of particles such as photons hitting an obstacle in a vacuum, *i.e.* ignoring the $O(a^2)$ pressure field and treating the waves as bullet-like, with momentum equal to their pseudomomentum. Such ideas are implicit in the phraseology sometimes encountered in which the waves are described as exchanging 'their momentum' with the mean state. A tendency to conflate momentum with pseudomomentum can be found scattered throughout the physics literature under headings such as 'Abraham–Minkowski controversy' (*e.g.* Peierls 1991).

The problem sketched in figure 1(a) is the first of a set of three problems, or thought-experiments, considered here. It is shown in M19 that the pseudomomentum rule holds in all three of them, at least to leading order in ϵ. The first two problems are in the two-dimensional, non-rotating setting and the third involves rotation:

(i) As in figure 1(a). The vorticity $\omega_0(r)$ is zero outside the vortex core.

(ii) As in figure 1(b). The wavetrain overlaps the vortex core. This brings in an additional refraction effect, familiar in the quantum literature as the so-called Aharonov–Bohm topological phase jump (figure 2 below).

(iii) As in figure 1(a) but in an inviscid, unstratified, incompressible, rapidly-rotating system of finite depth H with a free upper surface, under gravity g. The quasi-geostrophic potential vorticity is uniform outside the vortex core. The waves are surface gravity waves with kH large enough to make $\exp(-kH)$ negligible, where k is the magnitude of the wavenumber vector $\boldsymbol{k}$. The mean-flow Rossby number is small, but the intrinsic wave frequency $(gk)^{1/2} \gg f$, the Coriolis parameter.

In all three problems it is assumed that $a \ll \epsilon \ll 1$, allowing linearized wave theory to be used to describe the weak wave refraction. In problem (iii) we take the mean-flow Rossby number to be of the same order as ϵ. In some but not all cases, the wavenumber k is assumed large enough to permit the use of ray theory.

The plan of the paper is as follows. In section 2 and figure 2 we present and discuss a simple asymptotic solution for the $O(a)$ wave field far from the vortex core, applicable to problems (i) and (ii) and describing the Aharonov–Bohm phase jump and other relevant wave-refraction effects. In a significant sense this solution encompasses ray theory but also goes beyond it.

In section 3 we introduce what turns out to be the simplest way to compute the mean recoil forces and to understand their origin. It is to compute the complete nondissipative $O(a^2)$ mean flows associated with the wavetrains, sometimes called 'Bretherton flows'. Such a flow consists of the Stokes drift within the wavetrain together with the return flow required by mass conservation. For a narrow wavetrain the return flow takes place mostly outside it. Figure 3 shows an example. Section 4 presents the very simple mean-flow equations derived in M19. They govern the Bretherton flows that are relevant to leading order in our three problems. As noted in BM03 and in M19, this route to the results avoids any need to analyse the wave refraction explicitly, a remarkable simplifying feature. To leading order in ϵ it is enough to compute Bretherton flows for the *unrefracted* wavetrains, *i.e.* correct to $O(a^2\epsilon^0)$. And that in turn shows the leading-order results to be robust, in that they hold outside the ray-theoretic and other regimes within which the wave refraction can be computed explicitly.

In section 5 we present the resulting formulae for the recoil forces in all three problems, correct to leading order, in limiting cases for which the formulae become very simple. In problem (ii) the simplest results are for wavetrains whose width W and length L are both infinite. But immediately we encounter a surprise. The results depend strongly on the limiting value of W/L. The limits $W \to \infty$ and $L \to \infty$ are noninterchangeable. Problem (iii) is interesting in a different way. It exhibits remote recoil just as in problem (i), contrary to what might be suggested by the effects of rapid rotation. Rotation produces a tendency for the Stokes drift to be cancelled by the well-known 'anti-Stokes flow' (Ursell 1950). Nevertheless, the cancellation is incomplete such that there is still a significant Bretherton flow giving rise to a non-vanishing remote recoil, which moreover satisfies the pseudomomentum rule to leading order in ϵ.

Section 6 summarizes M19's independent and more lengthy, and indeed more delicate, derivations of the same results from refraction calculations. Those derivations make use of an appropriate 'impulse–pseudomomentum theorem', which justifies the pseudomomentum rule independently, alongside direct calculations of the pseudomomentum fluxes in the refracted wave fields correct to $O(a^2\epsilon^1)$. Ray theory is used in some of these calculations, and the non-ray-theoretic results of FLS99 in others. Section 7 offers brief concluding remarks in which some challenges for future work are noted, particularly regarding what happens at higher orders in ϵ, for which the impulse–pseudomomentum theorem fails. In some but not all circumstances the pseudomomentum rule still holds, but the precise circumstances remain to be clarified.

2. Wave refraction in problems (i) and (ii)

Before considering the $O(a^2)$ mean flows, we note some key wave-refraction effects in problems (i) and (ii). As shown in M19, the linearized equations – see (2.1)–(2.3) of M19 – have the following far-field solution. For sufficiently large r, the velocity potential ϕ' describing the waves has the asymptotic form $\phi' = A\exp(\mathrm{i}\Phi)$ where the $O(a)$ amplitude envelope A is slowly-varying and where the phase Φ is given by

$$\Phi \;=\; k_0(x - c_0 t) \;-\; \alpha\theta \;+\; \text{const.} \;+\; O(\epsilon^2 r_0^2/r^2)\ . \tag{2}$$

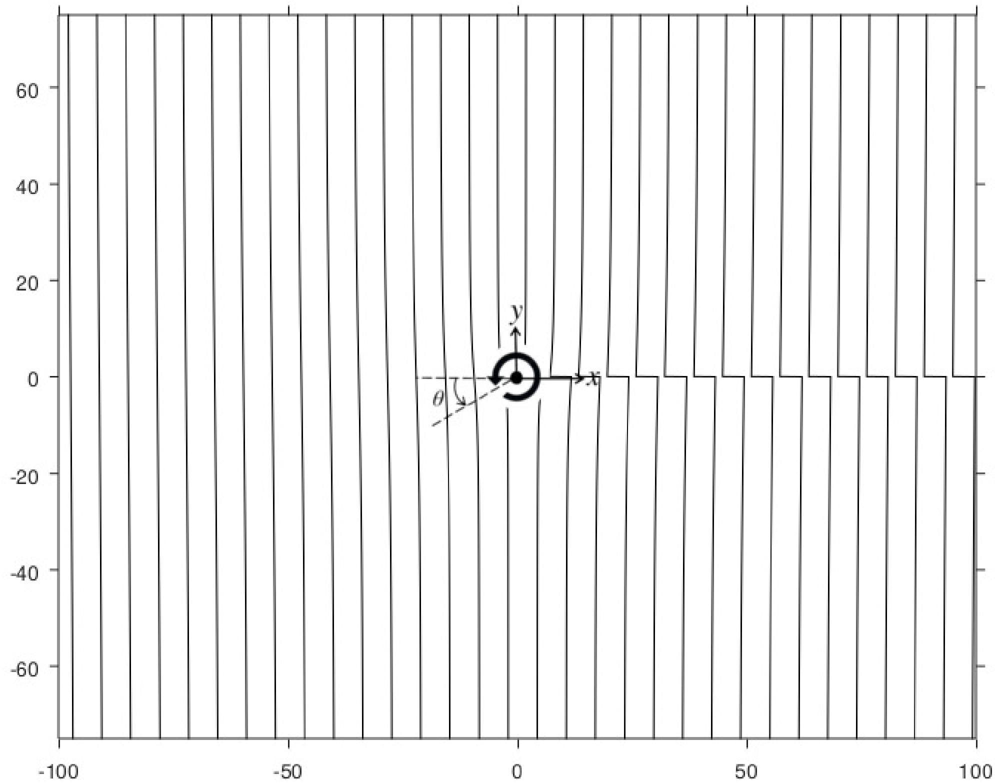

Figure 2. Wavecrests plotted from the far-field solution (2), with $\alpha = 0.75$. The unit of length is taken as k_0^{-1} so that the unrefracted wavelength is 2π. The Aharonov–Bohm phase jump appears as a phase discontinuity on the positive x axis. In a full solution this discontinuity is smoothed out across a relatively narrow 'wake' region. The other relevant refraction effect is the very slight rotation of the wavecrests that can be seen, for instance, by careful inspection of the left-hand edge of the plot.

The incident wavenumber k_0 and far-field phase speed c_0 are constants, and α is another constant, to be defined in (3). The azimuthal angle θ is defined as in the figures. It ranges from $-\pi$ to π.

In problem (ii), with the main focus on a wavetrain that is infinitely wide and infinitely long, we can take A to be a real constant. The constant α in (2) is defined by

$$\alpha \;=\; \Gamma k_0/2\pi c_0 \;=\; U k_0 r_0/c_0 \;=\; k_0 r_0 \epsilon \tag{3}$$

where Γ is the Kelvin circulation of the vortex, equal to $\iint \omega_0 \,\mathrm{d}x\mathrm{d}y$. The phase jump $2\pi\alpha$ across the positive x axis is the Aharonov–Bohm phase jump, a topological defect or dislocation of Φ. In a full solution it is smoothed out across a relatively narrow 'wake' region surrounding the positive x axis. The phase jump measures the effect of the vortex flow outside the core, $\boldsymbol{u}_0(r) = U r_0 r^{-1}\hat{\boldsymbol{\theta}} = \epsilon c_0 r_0 r^{-1}\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector in the θ direction, in compressing the wavetrain on one side while stretching it on the other, at positive and negative y respectively.

Figure 2 plots the far-field wavecrest shapes $\Phi =$ constant described by (2) with the error term neglected. We have taken $\alpha = 0.75$, fixing the phase jump at three quarters of a wavelength to make it clearly visible. Also visible, less clearly, is another refraction effect that nevertheless has comparable importance. Except on the y axis, the wavecrests are slightly rotated away from the y direction, through angles $O(\epsilon r_0 r^{-1})$. The effect can be seen by careful inspection of the left-hand edge of the plot. The local wavenumber vector $\boldsymbol{k} = \boldsymbol{\nabla}\Phi$ has a refractive contribution $-\alpha r^{-1}\hat{\boldsymbol{\theta}} = -\epsilon k_0 r_0 r^{-1}\hat{\boldsymbol{\theta}}$ directed against the vortex flow:

$$\boldsymbol{k} \;=\; \boldsymbol{\nabla}\Phi \;=\; k_0\{\hat{\boldsymbol{x}} - \epsilon r_0 r^{-1}\hat{\boldsymbol{\theta}} + O(\epsilon^2 r_0^2 r^{-2})\}\;, \tag{4}$$

where $\hat{\boldsymbol{x}}$ is the unit vector in the x direction. As in M19 we note that (2) and (4) are consistent with ray theory, but also go beyond it in the sense that phase changes over long distances of order r are represented accurately enough to describe the Aharonov–Bohm phase jump.

Despite the rotation of the wavecrests and of $\boldsymbol{k}$, the absolute group velocity $\boldsymbol{C}^{\text{abs}}$ remains parallel to the x axis correct to $O(\epsilon r_0 r^{-1})$, as can be checked from its leading-order expression

$$\boldsymbol{C}^{\text{abs}} = = \frac{c_0 \boldsymbol{k}}{|\boldsymbol{k}|} + \boldsymbol{u}_0(r) + O(\epsilon^2 c_0 r_0^2 r^{-2}) \tag{5}$$

by taking the y components of (4) and of $\boldsymbol{u}_0(r) = \epsilon c_0 r_0 r^{-1} \hat{\boldsymbol{\theta}}$. Propagation due to the y component of $\boldsymbol{k}$ cancels advection due to the y component of $\boldsymbol{u}_0$. The cancellation follows alternatively from the vanishing of the vorticity outside the vortex core, in virtue of the curl-curvature formula of ray theory, B14 p. 86, which was first derived by Landau and Lifshitz (1959) and generalized to dispersive waves by Dysthe (2001). Dysthe's result is made use of in problem (iii).

The property of $\boldsymbol{C}^{\text{abs}}$ just noted means that (2) can also be applied to problem (i), with a y-dependent amplitude envelope A, as long as the wavetrain passes the vortex at a distance great enough for the expression (2) and ray theory to be asymptotically valid, as was assumed in BM03. Then A can be taken to depend on y alone in a way that restricts the wavetrain as sketched in figure 1(a). The width scale W of this envelope $\gg k_0^{-1}$ for consistency with ray theory, yet small by comparison with the distance to the vortex core.

The phase function (2) is well known in the quantum literature. It applies not only to the vortex problem but also to the original Aharonov–Bohm problem, in which the waves represent nonrelativistic electrons going past a thin magnetic solenoid, as recalled in M19 Appendix A, with the magnetic vector potential in the role of the vortex flow $\boldsymbol{u}_0(r)$.

What is not apparent from (2) is the character of the wake region that smooths out the Aharonov–Bohm phase jump. In the original Aharonov–Bohm problem the wake is symmetric about the x axis, for arbitrary α, and within it the discontinuous structure (2) is replaced by a smooth Fresnel-diffractive structure with angular size tending toward zero like $(k_0 x)^{-1/2}$ as $x \to \infty$. In the vortex problem, by contrast, the wake generally has small but non-vanishing angular size $O(\epsilon)$ and an asymmetry about the x axis of the same order, except in an extreme long-wave limiting case with both ϵ and $k_0 r_0$ tending to zero. In that case the wake structure tends toward a Fresnel-diffractive structure symmetric about the x axis, as shown in Sakov (1993) and in FLS99. In cases of stronger refraction at finite ϵ, the wave field becomes more complicated and the wake asymmetry increased (Coste *et al.* 1999). In all cases, however, the phase jump seen in figure 2 is smoothed out in some manner.

3. Bretherton flow and Kelvin impulse

Following the past literature including the pioneering work of Bretherton (1969), we use the term Bretherton flow to denote the entire $O(a^2)$ wave-induced Lagrangian-mean flow. For instance in cases with relatively narrow wavetrains such as that of figure 1(a) the mean flow includes not only Stokes drifts but also any sideways return flows required by mass conservation. An example from BM03 is shown in figure 3. This represents schematically a version of problem (i) analysed in section 5.1 of BM03, in a particular formal limit, namely that of an infinitely long wavetrain slightly deflected by the vortex.

Within the wavetrain (which again is considered wide by comparison with k_0^{-1}, like a laser beam, even though narrow by comparison with the distance to the vortex core), the Stokes drift is toward the right. Therefore the return part of the Bretherton flow advects the vortex core toward the left. The core translates leftward at velocity $\boldsymbol{u}_{\text{tr}}$ say. The resulting rate of change in the Kelvin impulse $\boldsymbol{I}$ of the vortex, eq. (7) below, is the same as if the waves

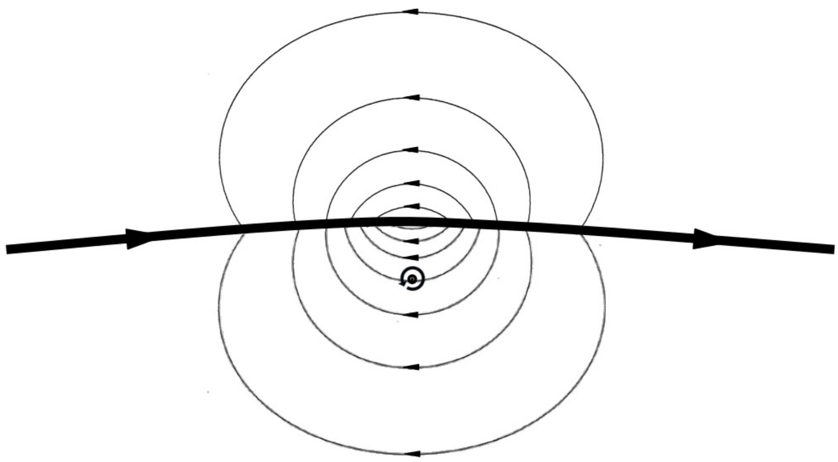

Figure 3. Schematic of the Bretherton flow arising in a version of problem (i) studied in BM03. The $O(a^2)$ mean flow within a narrow wavetrain, whose ray path is shown by the heavy curve, is dominated by the Stokes drift. A small portion of its mass flux, $O(a^2\epsilon^1)$ in this case, leaks sideways as a consequence of wave refraction. To describe this situation the refraction problem must be considered correct to two orders in ϵ, as was done in section 5.1 of BM03. Refraction effects enter at both orders, not only the $O(\epsilon)$ effects illustrated in figure 2 but also an $O(\epsilon^2)$ change in the direction of the absolute group velocity, exaggerated in this schematic.

were removed and a suitably tailored body force field $\boldsymbol{F}$, pointing in the $+y$ direction, were artificially applied to the vortex core. As already indicated, this is the effective mean recoil force in the sense considered here. It is exactly the sense required by – though, in fact, so far neglected in – gravity-wave parametrizations in weather and climate forecasting models.

The 'remoteness' of the recoil can now be seen to be related to the fact that the return flow extends well outside the wavetrain in cases like this. The Stokes drift does not directly contribute to $\boldsymbol{u}_{\rm tr}$, but only the return part of the Bretherton flow. In problem (ii), by contrast, the wavetrain overlaps the vortex core so that the local Stokes drift $\overline{\boldsymbol{u}}^{\rm S}$ contributes to $\boldsymbol{u}_{\rm tr}$, as well as remote contributions from other parts of the wavetrain.

For our core with vorticity $\omega_0(r)$ it is easy to verify that the effective force is just $\boldsymbol{F} = -\omega_0\,\hat{\boldsymbol{z}}\times\boldsymbol{u}_{\rm tr}$ where the unit vector $\hat{\boldsymbol{z}}$ points out of the paper.[1] Being transverse to the vortex motion, the resultant force $\boldsymbol{R}$ has the character of a Magnus force, namely

$$\boldsymbol{R} \;=\; \iint \boldsymbol{F}\,{\rm d}x{\rm d}y \;=\; -\hat{\boldsymbol{z}}\times\boldsymbol{u}_{\rm tr}\iint \omega_0\,{\rm d}x{\rm d}y \;=\; -\Gamma\,\hat{\boldsymbol{z}}\times\boldsymbol{u}_{\rm tr}\,. \tag{6}$$

We note that the Kelvin circulation Γ is an $O(\epsilon)$ quantity and that, in the case of figure 3, BM03 found that $\boldsymbol{u}_{\rm tr}$ is $O(a^2\epsilon^1)$ so that $\boldsymbol{R}$ is $O(a^2\epsilon^2)$. It is readily shown (M19, eq. (3.7)) that ${\rm d}\boldsymbol{I}/{\rm d}t = \boldsymbol{R}$ for our translating vortex core, with vorticity $\omega_0(r')$ where $r' = |\boldsymbol{x} - \boldsymbol{u}_{\rm tr}t|$. The two-dimensional Kelvin impulse $\boldsymbol{I}$ is defined by

$$\boldsymbol{I} \;=\; \iint (y,\,-x)\,\omega_0\,{\rm d}x{\rm d}y \;=\; \iint -\hat{\boldsymbol{z}}\times\boldsymbol{x}\;\omega_0\,{\rm d}x{\rm d}y \tag{7}$$

(*e.g.* Batchelor 1967, equation (7.3.7)). In the case of figure 3, BM03 also found that $\boldsymbol{u}_{\rm tr}$ is just such that $\boldsymbol{R}$ satisfies the pseudomomentum rule.

The effect on the vortex, continually moving it parallel to the x axis, is persistent and cumulative, over an arbitrary number of wave periods. In that respect the wave-induced recoil

[1]The curl of this two-dimensional force field $\boldsymbol{F}$ is just that required to move the vortex core leftward through the fluid at velocity $\boldsymbol{u}_{\rm tr}$, while the divergence of $\boldsymbol{F}$ sets up the dipolar pressure field required to produce the corresponding changes outside the core, where the velocity field is irrotational. Thus defined, $\boldsymbol{F}$ has the dimensions of acceleration, length/(time)^2, *i.e.* force per unit mass, since it is a forcing term on the right-hand side of the standard momentum equation having $\partial\boldsymbol{u}/\partial t$ on the left, whose curl is the standard vorticity equation. So for instance the resultant force on a two-dimensional vortex core of depth H is $\rho H\iint\boldsymbol{F}\,{\rm d}x{\rm d}y$ where ρ is fluid density. The factor ρH will be ignored in what follows. Strictly speaking, therefore, 'resultant force' and 'impulse' in the main text should be read as ρ^{-1} times resultant force and impulse per unit core depth.

is like the wave-induced mean forces that arise from wave dissipation, even though in our three problems there need not be any such dissipation. Even the wave sink need not be dissipative. It can be a wavemaker whose amplitude and phase are contrived to give perfect absorption, as for instance in the thought-experiments used by Léon Brillouin in his classic works on radiation stress (*e.g.* Brillouin 1936, & refs.).

Two further points to note are first that the wave field can be taken as steady only as an approximation, for small a and ϵ, and second that the weakness, $O(a^2)$, of the return flow and its strain-rate means that the vortex core is advected bodily without significant distortion (Kida 1981) as indeed was already assumed below (6), simplifying the calculation of $\mathrm{d}\boldsymbol{I}/\mathrm{d}t$ from (7).

In problem (ii), as already said, there is an additional, local contribution to $\boldsymbol{u}_{\mathrm{tr}}$ and therefore to $\boldsymbol{R}$, from the Stokes drift $\overline{\boldsymbol{u}}^{\mathrm{S}}$ of the wavetrain where it overlaps the vortex core. This local contribution is just the Craik–Leibovich vortex force as usually defined, $\boldsymbol{F}_{\mathrm{CL}} = \overline{\boldsymbol{u}}^{\mathrm{S}} \times \boldsymbol{\omega}_0$, where $\boldsymbol{\omega}_0 = \omega_0\,\hat{\boldsymbol{z}}$. The remote or return-flow contribution, from other parts of the wavetrain, varies with W/L. So it is the remote and not the local contribution that gives rise to the noninterchangeability of limits already mentioned.

4. Mean-flow equations at leading order

From here on we restrict attention to leading-order, $O(a^2\epsilon^1)$ recoil forces, thus excluding further consideration of cases like that of figure 3 in which the recoil is $O(a^2\epsilon^2)$ or smaller. Then (6) can be used with $\boldsymbol{u}_{\mathrm{tr}}$ correct to $O(a^2\epsilon^0)$ only, because of the factor $\Gamma = O(\epsilon)$. So as said earlier we need only compute Bretherton flows for unrefracted wavetrains.

As shown in M19, at this order the Bretherton flows $\overline{\boldsymbol{u}}^{\mathrm{L}}_{\mathrm{B}}$ are nondivergent, with streamfunction

$$\tilde{\psi}_{\mathrm{B}} \;=\; \tilde{\psi} \;-\; \tilde{\psi}_0 \tag{8}$$

say, where $\tilde{\psi}_0 = \tilde{\psi}_0(r')$ is the streamfunction for the nondivergent velocity field $\boldsymbol{u}_0(r')$ of the vortex flow, and where the complete Lagrangian-mean flow $\overline{\boldsymbol{u}}^{\mathrm{L}}$, vortex flow $\boldsymbol{u}_0$ plus Bretherton flow $\overline{\boldsymbol{u}}^{\mathrm{L}}_{\mathrm{B}}$, has x and y components

$$\overline{u}^{\mathrm{L}} \;=\; -\frac{\partial\tilde{\psi}}{\partial y} \qquad \text{and} \qquad \overline{v}^{\mathrm{L}} \;=\; \frac{\partial\tilde{\psi}}{\partial x}\;. \tag{9}$$

M19 showed that correct to $O(a^2\epsilon^0)$ the mean-flow equations can be written in the very simple forms

$$\nabla^2_{\mathrm{H}}\tilde{\psi}_{\mathrm{B}} \;=\; \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\mathbf{p} \qquad \text{in problems (i) and (ii)} \tag{10}$$

and

$$(\nabla^2_{\mathrm{H}} - L_{\mathrm{D}}^{-2})\tilde{\psi}_{\mathrm{B}} \;=\; \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\times\langle\mathbf{p}\rangle \qquad \text{in problem (iii)}\;, \tag{11}$$

where $\mathbf{p}$ is the wavetrain's pseudomomentum per unit mass, as before, ∇^2_{H} is the Laplacian in the xy plane, and L_{D} in problem (iii) is the Rossby deformation length-scale $L_{\mathrm{D}} = f^{-1}(gH)^{1/2}$. The angle brackets denote vertical averaging, needed in problem (iii) because of the strong dependence of $\mathbf{p}$ upon the vertical coordinate z, namely $\mathbf{p} \propto \exp(2k_0z)$. The streamfunction $\tilde{\psi}_{\mathrm{B}}$ need not be averaged vertically, in problem (iii), because at small Rossby number the Taylor–Proudman effect makes it z-independent.

The simplicity of eqs. (10) and (11) comes from their close relation to Kelvin's circulation theorem as expressed most succinctly by GLM (generalized Lagrangian-mean) theory; see for instance section 10.2.7 of B14, and equations (2.9)–(2.11) of M19.

Because the wavetrains are unrefracted, they can be taken to have the simple sinusoidal structure $A\exp(\mathrm{i}\Phi)$ with $\Phi = k_0(x - c_0t)$, and constant or slowly-varying A. From this, and from the irrotationality of the wavemotion in all three problems, we have $\mathbf{p} = \overline{\boldsymbol{u}}^{\mathrm{S}}$. See for instance B14, eqs. (10.15) and (10.17). The elliptic operators on the left of (10) and (11) show why Bretherton flows extend well outside any relatively narrow wavetrain.

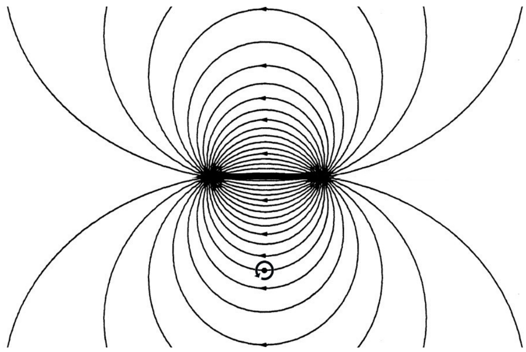

Figure 4. Schematic of Bretherton-flow streamlines in problem (i), as analysed in BM03 correct to lowest order $O(a^2\epsilon^0)$ for the finite wavetrain whose ray path is shown by the heavy straight line. At this order the Stokes drift is nondivergent except within the wave source and sink regions. The waves propagate from a source on the left to a sink on the right.

5. Bretherton flows and recoil forces at leading order

To take advantage of the simplifications just noted, with Bretherton flows computed from unrefracted wavetrains, we need to consider a wavetrain of finite length in the case of problem (i) as was done in BM03. In the formal limit of an infinitely long wavetrain, in that problem, to leading order in ϵ, the recoil and net pseudomomentum flux vanish because the rays remain straight and parallel to the x axis thanks to the cancellation already noted in the absolute group velocity (5), between the y components of the leading terms. (The bending of rays indicated in figure 3 takes place at the next order in ϵ, with the mean-flow equations becoming less simple, though still elliptic, as shown in section 5.1 of BM03.)

So for problem (i) at leading order we consider the situation sketched in figure 4, with an unrefracted wavetrain of finite length marked by the heavy straight line, along with its surrounding return flow satisfying eq. (10). The heavy straight line corresponds to the wavetrain sketched in figure 1(a), whose width scale $W \gg k_0^{-1}$ to permit the use of ray theory as in BM03. So once again the wavetrain is wide by comparison with k_0^{-1}, like a laser beam, even though narrow by comparison with the distance to the vortex core.

The wave source and sink are modelled as irrotational body-force fields, with k_0 large enough to allow the wave source and sink to be considered approximately localized near positions $(x, y) = (\pm X, Y)$, say, with $X, Y \gg W$. The Stokes drift is toward the right, straight along the wavetrain where the right-hand side of (10) is nonzero, and the return flow with right-hand side zero is irrotational, emanating from the wave sink and returning through the wave source. Its streamlines are mirror-symmetric about the wavetrain, because the wavetrain is unrefracted at this order. As in BM03 and M19 the $O(a^2\epsilon^0)$ flow advecting the vortex core at $(x,\, y) = (0, 0)$ can then be shown to be

$$\boldsymbol{u}_{\rm tr} \;=\; \overline{\boldsymbol{u}}_{\rm B}^{\rm L}(0,0) \;=\; \frac{S}{\pi}\,\frac{X}{X^2+Y^2}(-\hat{\boldsymbol{x}})\ , \tag{12}$$

where

$$S = \int \mathsf{p}_1(y)\,dy = \int \overline{u}^{\rm S}(y)\,dy\ . \tag{13}$$

The pseudomomentum per unit mass within the wavetrain has been written as $\mathbf{p} = \mathsf{p}_1(y)\hat{\boldsymbol{x}} = \overline{u}^{\rm S}(y)\hat{\boldsymbol{x}}$, and the integral is taken across the wavetrain. Applying the Magnus formula (6) we see that the corresponding recoil force is

$$\boldsymbol{R} \;=\; \frac{\Gamma S}{\pi}\,\frac{X}{X^2+Y^2}(+\hat{\boldsymbol{y}}) \tag{14}$$

correct to $O(a^2\epsilon^1)$, where $\hat{\boldsymbol{y}}$ is the unit vector in the y direction. As shown in BM03 and in B14 there is a corresponding imbalance between the mean forces exerted by the wave source and sink, as implied by Newton's third law.

We note that (12) and (14) tend toward zero in the formal limit of an infinitely long wavetrain, $X \to \infty$. More precisely, since (13) implies the scaling $S \sim W|\overline{\boldsymbol{u}}^{\rm S}|$, we have

$$|\boldsymbol{u}_{\rm tr}| \;\sim\; W|\overline{\boldsymbol{u}}^{\rm S}|/X \quad \text{and} \quad |\boldsymbol{R}| \;\sim\; \Gamma W|\overline{\boldsymbol{u}}^{\rm S}|/X \qquad \text{as } X \to \infty\,, \tag{15}$$

going to zero like X^{-1}. The irrotational return part of the Bretherton flow becomes increasingly spread out in the y direction, diluting its effect at $(x,\,y) = (0,0)$. The vanishing of $\boldsymbol{R}$ in this formal limit in problem (i) is consistent with the straightness of the rays and the vanishing of all refractive distortions as $|x| \to \infty$, at fixed $y = Y$ in (2). In the limit the incoming and outgoing pseudomomentum fluxes become equal, and the wave source and sink exert equal and opposite mean forces.

The dilution effect summarized by (15) is key to understanding the noninterchangeability of limits in problem (ii). For instance, the same dilution effect occurs for any unrefracted wavetrain whose width W is given an arbitrary fixed value while its length $L = 2X \to \infty$, whether or not it overlaps the vortex core. When it does overlap, the local Stokes drift $\overline{\boldsymbol{u}}^{\rm S}$ contributes to $\boldsymbol{u}_{\rm tr}$ while the diluted return flow is still governed by (15), going to zero in the limit. It remains zero if the limit $W \to \infty$ is taken subsequently. Therefore, for problem (ii) in the limit $L \to \infty$ followed by $W \to \infty$, the formulae (12) and (14) are replaced by

$$\boldsymbol{u}_{\rm tr} \;=\; \overline{\boldsymbol{u}}^{\rm S}(0,0) \;=\; \mathsf{p}_1(0)(+\hat{\boldsymbol{x}}) \tag{16}$$

and

$$\boldsymbol{R} \;=\; \Gamma\,\mathsf{p}_1(0)(-\hat{\boldsymbol{y}})\;. \tag{17}$$

Not only the magnitudes but also the signs have changed. Notice again that (17) is equal to the Craik–Leibovich vortex force $\boldsymbol{F}_{\rm CL} = \overline{\boldsymbol{u}}^{\rm S}\times\boldsymbol{\omega}_0$ integrated over the vortex core, corresponding to what is called the Iordanskii force in the quantum vortex literature (*e.g.* Sonin 1997, Stone 2000), with $\overline{\boldsymbol{u}}^{\rm S} = \mathbf{p}$ corresponding to the phonon current per unit mass.

If we take the limits in the opposite order, $W \to \infty$ followed by $L \to \infty$, it is easy to see that $\boldsymbol{u}_{\rm tr}$ and $\boldsymbol{R}$ both go to zero. For an infinitely wide wavetrain of finite length, with p_1 uniform across the wavetrain, the dilution effect is banished to $|y| = \infty$ so that the return flow at each finite $|y|$ is just $-\overline{\boldsymbol{u}}^{\rm S}$. Thus $\boldsymbol{u}_{\rm tr} = \overline{\boldsymbol{u}}^{\rm S} - \overline{\boldsymbol{u}}^{\rm S} = 0$. For intermediate cases in which W/L has a finite limiting value, and in which the wavetrain is uniform and symmetric about the $|x|$ axis, we obtain the intermediate values

$$\boldsymbol{R} \;=\; -\left\{1 - \frac{2}{\pi}\,\lim\,\arctan\left(\frac{W}{L}\right)\right\}\Gamma\,\mathsf{p}_1\hat{\boldsymbol{y}}\;, \tag{18}$$

as shown in M19, where lim denotes the limit $W \to \infty$ and $L \to \infty$ with W/L tending to a constant. In cases where the constant has a value of order unity, the local and remote contributions have comparable importance.

In problem (iii) there is no dilution effect as $L \to \infty$, because the Bretherton flow satisfies (11) and therefore, for a long wavetrain, decays sideways like $\exp(-|y|/L_{\rm D})$ on the fixed lengthscale $L_{\rm D}$. In the formal limit $L \to \infty$, and with a narrow wavetrain, $W \ll Y$ and $W \ll L_{\rm D}$, we have $\overline{\boldsymbol{u}}^{\rm L}_{\rm B} \;=\; \overline{\boldsymbol{u}}^{\rm L}_{\rm B}(y) \;=\; (S/2L_{\rm D})\exp(-|y-Y|/L_{\rm D})(-\hat{\boldsymbol{x}})$ for $|y-Y| > W$, *i.e.* outside the wavetrain, so that, for a small vortex core $r_0 \ll L_{\rm D}$, the vortex-core advection velocity and recoil force are

$$\boldsymbol{u}_{\rm tr} \;=\; \overline{\boldsymbol{u}}^{\rm L}_{\rm B}(0) \;=\; (S/2L_{\rm D})\exp(-|Y|/L_{\rm D})(-\hat{\boldsymbol{x}}) \tag{19}$$

and

$$\boldsymbol{R} \;=\; (\Gamma S/2L_{\rm D})\exp(-|Y|/L_{\rm D})(+\hat{\boldsymbol{y}}) \tag{20}$$

with Γ evaluated at the edge of the core and with vertical averaging understood in (13). Notice that the signs have reverted to those in problem (i).

6. Cross-checks from refraction calculations

The foregoing recoil formulae were rederived in M19 via a completely independent route, in two stages. Firstly, it was shown that an appropriate 'impulse-pseudomomentum theorem' holds for any vortical flow and any wave field with irrotational sources and sinks, provided that the mean flows comply with either (10) or (11). This means that the pseudomomentum rule is guaranteed to hold in any such situation. Then, secondly, explicit linear-theoretic calculations of wave refraction were carried out to yield the incoming and outgoing pseudomomentum fluxes in our three problems. They were found to be precisely consistent with the foregoing results (14), (17), (18), and (20).

The calculations used ray theory in all three problems and, in addition, in problem (ii), used the long-wavelength asymptotics and Fresnel-diffractive wake structure analysed in FLS99. FLS99's careful analysis completes the picture described by (2) in an elegant way when both $k_0 r_0$ and ϵ are small, making the Aharonov–Bohm phase jump $2\pi\alpha$ even smaller, being the product of *two* small quantities as seen in (3).

In problem (ii), the two contributions to $\boldsymbol{R}$ in (18) can be attributed separately to the two refraction effects seen in figure 2. The first contribution in (18), which is the same as (17), arises solely from the wake and the Aharonov–Bohm phase change across it, while the second contribution arises solely from the other refraction effect noted earlier, the $O(\epsilon)$ rotation of wavecrests in the larger domain outside the wake. This second contribution becomes significant in problem (ii) when the width W of the wavetrain is large enough, accounting for the dependence on W/L. These attributions hold good not only when $k_0 r_0 \ll 1$ but also when $k_0 r_0 \gg 1$, allowing the use of ray theory. In that case the wake has a different structure involving a ray caustic, but continues to account solely for the first contribution in (18) while the $O(\epsilon)$ rotations of wavecrests outside the wake, which as noted earlier are consistent with ray theory, account solely for the second contribution. In particular, therefore, the Aharonov–Bohm effect is the only relevant refraction effect – as often assumed in the quantum vortex literature – only in the special limiting case for which $\lim (W/L) = 0$.

In problem (iii), ray theory is used. Because the vortex now has quasigeostrophic structure with finite scale L_{D} it is the potential vorticity, not the vorticity ω, that vanishes outside the vortex core. Therefore the curl-curvature formula of Dysthe (2001) implies that rays passing outside the vortex core, as in figure 1(a), are now deflected at leading order in ϵ. The formula tells us that the ray curvature is just ω/C where $C = \frac{1}{2}(g/k)^{1/2}$, the intrinsic group velocity for deep-water gravity waves. When the formula is used and the calculations carried out (M19 section 8), the result is found to agree perfectly with (20).

7. Concluding remarks

All the foregoing results depend on the restriction to leading order in ϵ, which is essential to the derivation of (10) and (11) and therefore essential to our computations of Bretherton flows, and to the proof of the impulse–pseudomomentum theorem. Yet cases are known in which the pseudomomentum rule holds at higher orders of accuracy in ϵ. One of them is the case of figure 3, which requires one further order of accuracy, and in which the mean-flow equations are less simple than (10) and (11). Indeed, in BM03 section 5.2 it was shown that there is a version of that case for which the rule holds to all orders in ϵ. On the other hand, as recalled in M19, exceptions to the rule have long been known.

There is a major unresolved puzzle here, and a challenge for future work. For now, one may speculate that the present restriction to leading order in ϵ may be a limitation of the Kelvin impulse concept, rather than of the pseudomomentum rule itself. Some further discussion of this issue is given in section 9 of M19.

Acknowledgements

Pavel Berloff provided the first stimulus to embark on this study. I thank him and Natalia Berloff, Oliver Bühler, David Dritschel, Victor Kopiev, Stefan Llewellyn Smith, Hayder Salman, Mike Stone, Jim Thomas, Jacques Vanneste, Bill Young, and the referees for M19 and for the present paper – all six of them – for their interest and for many useful and challenging comments, which led to extensive revisions of both papers.

References

Baldwin, M.P., Gray, L.J., Dunkerton, T.J., Hamilton, K., Haynes, P.H., Randel, W.J., Holton, J.R., Alexander, M.J., Hirota, I., Horinouchi, T., Jones, D.B.A., Kinnersley, J.S., Marquardt, C., Sato, K. and Takahashi, M., The quasi-biennial oscillation. *Revs. Geophys.*, 2001, **39**, 179–229.

Batchelor, G.K., *An Introduction to Fluid Dynamics*, 1967 (Cambridge University Press).

Bretherton, F.P., On the mean motion induced by internal gravity waves. *J. Fluid Mech.*, 1969, **36**, 785–803.

Brillouin, L., On radiation pressures and stresses (in French). *Revue d'Acoustique*, 1936, **5**, 99–111.

Bühler, O., *Waves and Mean Flows,* 2nd edn. (paperback), 2014 (Cambridge: University Press).

Bühler, O. and McIntyre, M.E., Remote recoil: a new wave–mean interaction effect. *J. Fluid Mech.*, 2003, **492**, 207–230.

Coste, C., Lund, F. and Umeki, M., Scattering of dislocated wave fronts by vertical vorticity and the Aharonov–Bohm effect. I. Shallow water. *Phys. Rev. E*, 1999, **60**, 4908–4916.

Dritschel, D.G. and McIntyre, M.E., Multiple jets as PV staircases: the Phillips effect and the resilience of eddy-transport barriers. *J. Atmos. Sci.*, 2008, **65**, 855–874.

Dysthe, K.B., Refraction of gravity waves by weak current gradients. *J. Fluid Mech.*, 2001, **442**, 157–159.

Ford, R. and Llewellyn Smith, S.G., Scattering of acoustic waves by a vortex. *J. Fluid Mech.*, 1999, **386**, 305–328.

Fritts, D.C., Gravity wave saturation in the middle atmosphere: a review of theory and observations. *Revs. Geophys. Space Phys.*, 1984, **22**, 275–308.

Garcia, R.R., Smith, A.K., Kinnison, D.E., de la Cámara, A. and Murphy, D.J., Modification of the gravity wave parameterization in the Whole Atmosphere Community Climate Model: motivation and results. *J. Atmos. Sci.*, 2017, **74**, 275–291.

Holton, J.R., Haynes, P.H., McIntyre, M.E., Douglass, A.R., Rood, R.B. and Pfister, L., Stratosphere–troposphere exchange. *Revs. Geophys.*, 1995, **33**, 403–439.

Kida, S., Motion of an elliptic vortex in a uniform shear flow. *J. Phys. Soc. Japan*, 1981, **50**, 3517–3520.

Landau, L.D. and Lifshitz, E.M., *Fluid Mech. 1st Engl. Edn*, 1959 (Pergamon).

McIntyre, M.E., On the "wave momentum" myth. *J. Fluid Mech.*, 1981, **106**, 331–347.

McIntyre, M.E., Wave–vortex interactions, remote recoil, the Aharonov–Bohm effect and the Craik–Leibovich equation. *J. Fluid Mech.*, 2019, **881**, 182–217 and http://arxiv.org/abs/1901.11525.

Peierls, R., *More Surprises in Theoretical Physics*, 1991 (Princeton University Press).

Sakov, P.V., Sound scattering by a vortex filament. *Acoust. Phys.*, 1993, **39**, 280–282.

Sonin, E., Magnus force in superfluids and superconductors. *Physical Review B*, 1997, **55**, 485–501.

Stone, M., Iordanskii force and the gravitational Aharonov–Bohm effect for a moving vortex. *Phys. Rev. B*, 2000, **61**, 11780–11786.

Ursell, F., On the theoretical form of ocean swell on a rotating Earth. *Mon. Not. Roy. Astron. Soc. Geophys. Suppl.*, 1950, **6**, 1–8.