Balance and the slow quasi-manifold: some explicit results

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June 17, 1998,

revised March 22, 1999

J. Atmos. Sci., in press.

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§The Centre for Atmospheric Science is a joint initiative of the Department of Chemistry and the Department of Applied Mathematics and Theoretical Physics in the University of Cambridge.

Abstract

The ultimate limitations of the balance, slow-manifold and potential-vorticity-inversion concepts are investigated. These limitations are associated with the weak but nonvanishing spontaneous-adjustment emission, or Lighthill radiation, of inertia–gravity waves by unsteady, two-dimensional or layerwise-two-dimensional vortical flow (the wave emission mechanism sometimes being called "geostrophic" adjustment even though it need not take the flow toward geostrophic balance). Spontaneous-adjustment emission is studied in detail for the case of unbounded f-plane shallow-water flow, in which the potential-vorticity anomalies are confined to a finite-sized region, but whose distribution within the region is otherwise completely general. The approach assumes that the Froude number F and Rossby number R satisfy $\mathsf{F} \ll 1$ and $\mathsf{R} \gtrsim 1$ (implying, incidentally, that any balance would have to include gradient-wind and other ageostrophic contributions). The method of matched asymptotic expansions is used to obtain a general mathematical description of spontaneous-adjustment emission in this parameter regime. Expansions are carried out to $O(\mathsf{F}^4)$, which is a high enough order to describe not only the weakly emitted waves but also, explicitly, the correspondingly weak radiation reaction upon the vortical flow, accounting for the loss of vortical energy. Exact evolution on a slow manifold, in its usual strict sense, would be incompatible with the arrow of time introduced by this radiation reaction and energy loss. The magnitude, $O(\mathsf{F}^4)$, of the radiation reaction may thus be taken to measure the degree of "fuzziness" of the entity that must exist in place of the strict slow manifold. That entity must, presumably, be not a simple invariant manifold but rather an $O(\mathsf{F}^4)$-thin, multi-leaved, fractal "stochastic layer" like those known for analogous but low-order coupled oscillator systems. It could more appropriately be called the 'slow quasi-manifold'.

1 Introduction

The ideas of balanced flow and slow manifold for stratified, rotating fluid systems (e.g., Charney 1948, Leith 1980, Lorenz 1980) are among the most useful, important and arguably central ideas in dynamical meteorology and oceanography, for well known reasons. In one form or another, these ideas underlie practically all our theoretical knowledge about vortical eddy flow in such systems, including shear instabilities, teleconnections, blocking, vortex isolation and other phenomena dependent on Rossby-wave 'quasi-elasticity' (e.g., Armi *et al.* 1988, Manney *et al.* 1994, McIntyre 1993, Polvani & Plumb, 1992 Simmons and Hoskins 1979, Hoskins *et al.* 1985, Chang and Orlanski 1993, Thorncroft *et al.* 1993, & refs). The same ideas have recently led to a new interpretation of helioseismic data (Gough and McIntyre 1998) that promises to revolutionize thinking about the behaviour and possible variability of the Sun's stratified, rotating interior.

The ideas are central because of their conceptual simplifying power. Not only do these ideas, when justifiable, permit one to confine attention to a far smaller phase space of possible states but they also make explicit — and keep conceptually separate — important aspects of the dynamics that are hard to disentangle from Newton's laws or from the primitive equations of motion. These aspects include prognostic versus diagnostic, advective versus nonadvective, and local versus nonlocal aspects. The resulting viewpoint is able to treat the central difficulty of fluid dynamics, the advective nonlinearity, with maximum possible simplicity by representing it solely in terms of the advection of potential vorticity and near-surface potential temperature. The nonlocal aspects are made explicit through the idea of "potential-vorticity inversion", helped by the various analogies such as the membrane analogy and the electrostatic analogy (Obukhov 1962; Hoskins *et al.* 1985 & refs; McIntyre 1993 & refs; Bishop and Thorpe 1994).

It is therefore of great interest to know the accuracy and limitations, if any, of the ideas of balanced flow, slow manifold, and potential-vorticity inversion. These ideas have a long history, going back to the classic papers of Charney (1948), Kleinschmidt (1950a, 1950b, 1951), Lorenz (1980), and Leith (1980); and it has long been appreciated that the slow manifold idea entails mathematical subtlety. Strictly speaking, "slow manifold" means an invariant manifold or single, smooth hypersurface within phase space on which freely-propagating inertia–gravity waves are altogether absent from the corresponding physical states. What this means mathematically will be made precise shortly. The implied picture has always been attractive since, if correct, it means that balanced flow is an exact and uniquely defined concept having the same qualitative character as quasigeostrophic or other approximately balanced flow, and that potential-vorticity inversion is, in principle, exactly and uniquely definable.

Indeed, so attractive is this picture that there have been attempts to argue for the existence of such a slow manifold, in the strict mathematical sense just referred to, via expansion or iteration procedures that are presumed to converge. However, there are two strong lines of evidence against such convergence, supplementing the arguments already in the literature (Errico 1981, Warn 1997, Warn and Ménard 1986, Vautard & Legras 1986 Lorenz & Krishnamurthy 1987, Lorenz 1992).

The first line of evidence comes from the theory of coupled oscillators, which provides examples of simpler systems, such as the perturbed simple pendulum, to which fundamentally similar arguments might seem to apply but in fact, as has been rigorously proven, do not. The expansion or iteration procedures do not converge because, as it turns out, there is nothing for them to converge to. An isolated, unperturbed simple pendulum, for instance, has what might be called a "slowest" invariant manifold, namely, the homoclinic orbit corresponding to the unperturbed pendulum motion of infinite period, motion on which orbit can reasonably be considered analogous to quasigeostrophic or other strictly balanced motion in fluid systems. But this homoclinic orbit ceases to exist, as a uniquely defined invariant manifold, under even the smallest perturbation, for instance weak coupling to another oscillator of higher frequency (e.g., Berry 1978, Bokhove and Shepherd 1996). The single homoclinic orbit is replaced by a "stochastic layer" of chaotic orbits, having small but finite thickness, and fractal dimensionality. There is no single orbit to converge to, and it is this, rather than any purely technical difficulty with the expansion procedure, that accounts for the failure of convergence. Such coupled oscillator problems are reasonable analogues for our purposes because the fluid system can be regarded as an infinite set of nonlinear coupled oscillators; indeed, that it how it is routinely regarded in the literature on numerical weather prediction.

The second, and for our purpose even stronger, line of evidence — related to the above but applying directly to the fluid systems of interest here, which are dynamical systems with infinite-dimensional phase spaces — comes from a large body of theoretical and experimental work on the spontaneous emission or radiation of sound waves by unstratified, three-dimensional vortical flows beginning with the Lighthill theory for the case of small Mach number (Lighthill 1952; also e.g., Crighton 1981, Kambe 1986, Webster 1970). A generalized version of the Lighthill theory, with inertia–gravity waves in the role of sound waves, can be developed (e.g., section 2 below), and strongly suggests that the unsteady stratified, rotating, vortical flows of interest here, which can be regarded as "layerwise-two-dimensional", must generally emit inertia–gravity waves. Here, the Froude number corresponds to the Mach number of the original theory. In a strongly justifiable sense, at least for small Froude number, the waves can be said to be spontaneously emitted or radiated *by* the vortical flow. This again implies non-existence of a slow manifold in the strict sense of complete absence of freely-propagating inertia–gravity waves. Coriolis effects can be expected to weaken the emission still further, but not to make it exactly zero, even for arbitrarily small Rossby number. This is because of the expectation that typical vortical flows, being chaotically unsteady (e.g. Aref and Pomphrey 1982), will have a frequency spectrum with no high-frequency cutoff (Errico 1981). We say "spontaneously emitted or radiated" to emphasize the distinction between such emission or radiation on the one hand, and the radiation due to an imbalance in the initial conditions, as in the well known Rossby adjustment problem, on the other (e.g., Gill 1982). A further example of spontaneous emission, confirming unequivocally that it is liable to take place, though very weakly, at arbitrarily small Rossby number, has been studied by Ford (1994a).

We may summarize the essence of the foregoing, and make it more precise, by defining what a strict slow manifold is — or, rather, what it would have to be if it existed — as follows:

> *Definition:* A **strict slow manifold** is an invariant manifold within the full phase space, such that the full flow at each instant can be deduced, uniquely and exactly, by potential-vorticity inversion.

Here potential-vorticity inversion is understood in its usual sense. That is, it signifies that the full flow at each instant, including velocity and geopotential fields, can be deduced diagnostically and unambiguously from the potential vorticity field alone where "potential vorticity field", in the case of stratified flow, has to include surface potential temperature, and where it is understood that the basic stratification has been prescribed in terms of the mass under each isentropic surface (e.g., Hoskins *et al.* 1985) or, for shallow-water systems, in terms of the total mass or mean depth of the fluid layer. Potential-vorticity inversion, as understood here, also has the usual time-symmetry or "sign-reversal property". That is, if the potential vorticity field has its sign reversed everywhere (which includes changing the sign of the Coriolis parameter f if we view the system in a rotating frame, but keeping surface potential temperature unchanged), then the velocity field produced by inversion also changes sign everywhere.

Notice that the definition just given is consonant with the standard idea of slow manifold as normally used in connection with approximate *balanced models*, such as quasigeostrophy, semigeostrophy and Bolin–Charney balance (e.g. Whitaker 1993). Apart from being approximate, all these balanced models permit potential-vorticity inversion in the sense just described, including the sign-reversal property.

It is plain that the existence of a slow manifold, in the strict sense just defined, requires the complete absence of freely-propagating inertia–gravity waves, or, more precisely, that it requires the corresponding vortical flows never to emit such waves. Any such emission would be incompatible with the sign-reversal property. The example of Ford (1994a) is a sufficient illustration. There, a circular vortex patch spontaneously develops sinusoidal Rossby-wave undulations of its edge, growing exponentially in time and emitting inertia–gravity waves with a spiral pattern of phase lines. For the corresponding case in which the sign of the potential vorticity is reversed, we get an inertia–gravity wave spiral in the opposite sense (because the waves are still being emitted). It follows that, in this example, not all contributions to the velocity field have their sign reversed. In particular, the radial velocity components in the outgoing wave field keep the same sign. This is enough to show that the strict slow manifold cannot, in fact, exist.

There is still, however, a question as to whether there could be a "generalized slow manifold" that includes the spontaneously-emitted waves (J. Tribbia, personal communication), such as the spiral wave field in Ford's example. This motivates an additional

> *Definition*: A **generalized slow manifold** is an invariant manifold on which the full flow is known in terms of the instantaneous potential-vorticity field, but which does not respect the sign-reversal property and may therefore include inertia–gravity waves satisfying a causality or radiation condition

If such a generalized slow manifold were to exist, then finding it would have to involve backward integration in time from the instant at which the potential vorticity is given. We postpone further discussion until after the detailed analysis to be presented here.

Working with the simplest fluid system for which the foregoing issues are non-trivial, namely the shallow water system on an unbounded f-plane, we investigate in detail the fluid dynamics of spontaneous emission in a way that complements, and in a sense generalizes, the example presented in Ford (1994a). The outcome is new mechanistic insight plus a clear confirmation that non-zero spontaneous emission is the usual state of things, even for arbitrarily small Froude number, verifying the expectation from the Lighthill theory and adding to the evidence against the existence of a strict slow manifold. It also becomes clear how the notion of high-order potential-vorticity inversion (McIntyre and Norton, 1999) fits into the picture, and what its limitations are in the parameter regime studied here. Furthermore, we make progress toward answering the question of whether a generalized slow manifold might exist for this fluid system.

The plan of the paper is as follows. Section 2 presents the generalization of Lighthill's theory to include Coriolis effects, and points to why a more detailed analysis is necessary. Section 3 presents the detailed analysis. The approach assumes that the Froude number F and Rossby number R satisfy $\mathsf{F} \ll 1$ and $\mathsf{R} \gtrsim 1$ (implying, incidentally, that any balance will be strongly ageostrophic). The method of matched asymptotic expansions is used (Van Dyke 1964) to obtain a general mathematical description of spontaneous-adjustment emission in this parameter regime. Expansions are carried out to $O(\mathsf{F}^4)$, which is high enough to describe not only the emitted waves but also the correspondingly weak radiation reaction upon the vortical flow. The radiation reaction is described by a contribution $\boldsymbol{u}_4^{(\mathrm{rad})}$ to the advecting velocity field. This shows explicitly how time symmetry is violated and the existence of the strict slow manifold precluded. In section 4 we offer some concluding remarks on the implications for the concepts of slow manifold, generalized slow manifold, and related concepts.

2 Lighthill's argument in a rotating reference frame

In order to generalize Lighthill's argument, we write the f-plane shallow-water equations in flux form:

$$\frac{\partial}{\partial t}(hu_i) + \frac{\partial}{\partial x_j}(hu_iu_j) - \varepsilon_{ij} f hu_j + \frac{g}{2}\frac{\partial}{\partial x_i}(h^2) = 0 \ , \tag{1a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i}(hu_i) = 0 \ . \tag{1b}$$

where h is the layer depth, $(u_i) = (u_1, u_2)$ are the components of the velocity field in Cartesian coordinates $(x_i) = (x_1, x_2)$, $f > 0$ is the constant Coriolis parameter or inertia frequency, g the gravitational force per unit mass, and ε_{ij} is the two-dimensional alternating tensor, defined by $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. As in Lighthill's original work, one can derive from these equations a single equation with the appropriate linear wave operator on the left, and all the nonlinear terms on the right. The most convenient dependent variable for the left hand side is $\partial h/\partial t$; so we proceed by subtracting $\partial/\partial t$ of the divergence of Eq. (1a) from $-f$ times the curl of Eq. (1a), then eliminating $\nabla.(h\boldsymbol{u})$ and its second time derivative using Eq. (1b) and its second time derivative. The result is

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - c_0^2\nabla^2\right)\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x_i \partial x_j}T_{ij} \ , \tag{2}$$

where

$$T_{ij} = \frac{\partial}{\partial t}(hu_iu_j) + \frac{f}{2}\left(\varepsilon_{ik}hu_ju_k + \varepsilon_{jk}hu_iu_k\right) + \frac{g}{2}\frac{\partial}{\partial t}(h-h_0)^2\delta_{ij} \ . \tag{3}$$

The wave operator on the left is the linear Poincaré or inertia–gravity wave operator, as distinct from the classical wave operator with no f^2 term, which appears in Lighthill's original theory. Crucially, however, the nonlinear terms on the right-hand side retain the second-derivative form found in the original theory.

Re-phrased in terms of the present problem, Lighthill's main point is that when $\mathsf{F} \ll 1$ the right-hand side of (2) is known to good approximation from the vortical flow alone, approximated as nondivergent. The right-hand side can therefore be regarded as a given source of inertia–gravity waves — a source that is known as soon as the nondivergent barotropic vortex flow is known — with two important implications. First, with few if any exceptions, unsteady vortical flows will emit freely propagating inertia–gravity waves, implying nonexistence of a strict slow manifold as defined above. Second, however, the emission is very weak for $\mathsf{F} \ll 1$ (with radiated power $O(\mathsf{F}^4)$, details below) — corresponding to very weak coupling in the coupled-oscillator analogy — helping to explain why balance and potential-vorticity inversion, though inherently approximate, can be far more accurate than might be suggested by the standard order-of-magnitude considerations and filtered balanced models (Norton 1988, McIntyre and Norton 1999). As Lighthill pointed out, it is the second-derivative form of the right-hand side of (2), rather than the precise form of T_{ij} itself, that makes the emission weak.

More precisely, if we assume that T_{ij} is "compact" in the sense of having significant magnitude only over a finite region in space that is of small length scale in comparison with the wavelength of radiated waves, then the emission is of quadrupole type in the sense that it suffers two orders of destructive interference, where 'order' corresponds to powers of the small parameter F. It is also quadrupole in the sense that far from the source the waves are of form $r^{-1/2}e^{i(kr-\omega t)}e^{2i\theta}$ to leading order (as will be shown in section 3.6), where ω is the frequency of the emitted wave, k the radial wavenumber, $r = (x_1^2 + x_2^2)^{1/2}$ the radial coordinate, and θ the azimuthal coordinate. By contrast, monopole radiation has the form $r^{-1/2}e^{i(kr-\omega t)}$ and dipole radiation the form $r^{-1/2}e^{i(kr-\omega t)}e^{i\theta}$.

That this picture of weak, quadrupole radiation is justified is confirmed by the detailed analysis to be given next. In the analysis, nontrivial technical difficulties emerge. Circumventing them is one of the motivations for this paper. Some of these difficulties are related to the dispersive character of the wave operator on the left of (2) when $f \neq 0$ and to understanding in detail how the presence of rotation weakens the spontaneous emission, despite the factors f in some of the terms on the right of (2). Other difficulties are related to the nonconvergence of certain integrals in two space dimensions, to the occurrence of logarithmic terms in the analysis, and to the need to maintain, if possible, well-ordered expansions over a large time interval. The way in which this last difficulty is overcome connects the present work with the results of the companion paper and those of Warn *et al.* (1995).

3 Detailed Analysis

3.1 Preliminaries

Attention will be restricted to cases in which the vortical flow takes place within some relatively small region of the f-plane, so that we can clearly see whether freely-propagating waves are being emitted therefrom. By assumption, this vortical region, as we shall call it, is characterized by a horizontal scale L. We shall assume that the potential-vorticity anomaly $q - q_{\rm b}$ decays exponentially in space on the length scale L: i.e. $q - q_{\rm b} = O(r^{-\infty})$ as $r \to \infty$ in the vortical region, where the notation $O(r^{-\infty})$ means smaller than any inverse power of r, and $q_{\rm b}$ is the uniform background value. To the order of asymptotic analysis that we require, the vortical region is then the only region in which the potential vorticity q differs significantly from $q_{\rm b}$. We further assume that the Froude number F of the flow in the vortical region is small. This implies that the layer depth everywhere takes a uniform value D_0, with departures from this uniform value at $O(\mathsf{F}^2)$. The value of $q_{\rm b}$ is thus f/D_0.

The Froude number is defined here as $\mathsf{F} = U/c_0 \ll 1$, where U is the typical velocity associated with the vortical region, and $c_0 = (gD_0)^{1/2}$ is the non-rotating gravity wave phase speed. The Rossby number R is defined by $\mathsf{R} = U/fL$ and we assume that $\mathsf{R} \gtrsim 1$. Inertia–gravity wavelengths λ for waves of frequency ω are given by $\lambda = 2\pi c_0/(\omega^2 - f^2)^{1/2}$. If the waves are generated by the vortical flow then we expect $\omega \sim U/L$, hence $\lambda \gtrsim 2\pi L/\mathsf{F}$. This means that wavelengths of emitted inertia–gravity waves are long in comparison with (2π times) the length scale L of the vortical region. The larger scale L/F characterizes a second region, the wave region, as we shall call it, in which waves propagate freely away from the vortical region. The regions are shown in Fig. 1.

The technique to be used, following and extending the work of Crow (1970), is the method of matched asymptotic expansions (see also Van Dyke 1964). Here the small expansion parameter is the Froude number F. For the present purposes, it is necessary to take the expansions to higher order in F than in Crow's pioneering work. It is also necessary to treat the vortical evolution in the special way suggested by the work of Norton (1988), and more explicitly by that of Warn *et al.* (1995), in order to avoid secular disordering of the expansions and a corresponding restriction of their time interval of validity.

To cope with the technicalities, we need to introduce some special notation. In particular, we replace the total layer depth h of (1) by D where

$$D = D_0(1 + \mathsf{F}^2 h) \ . \tag{4}$$

Here D_0, a constant, is the undisturbed layer depth as before, and h now represents a dimensionless measure of the departure therefrom.

With velocity $\boldsymbol{u} = (u_1, u_2)$ scaled on U, horizontal coordinates $\boldsymbol{x} = (x_1, x_2)$ scaled on L, time t scaled on L/U (so that the dimensionless inverse Coriolis parameter $f^{-1} = \mathsf{R}$, the Rossby number), and h in its new sense defined by (4), the dimensionless shallow water equations appropriate to the vortical region, now more conveniently written in advective rather than flux form, are

$$\frac{D\boldsymbol{u}}{Dt} + f\boldsymbol{k} \times \boldsymbol{u} + \nabla_x h = 0 \tag{5a}$$

$$\mathsf{F}^2 \left(\frac{Dh}{Dt} + h\nabla_x.\boldsymbol{u} \right) + \nabla_x.\boldsymbol{u} = 0 \ . \tag{5b}$$

where ∇_x represents $\partial/\partial\boldsymbol{x}$, as distinct from $\partial/\partial\boldsymbol{X}$, to be used below. The dimensional potential vorticity is scaled on $U/(D_0L)$, so that the dimensionless potential vorticity q takes the form

$$q = \frac{f+\zeta}{1+\mathsf{F}^2 h} \ = f + \widehat{q} \ , \quad \text{say}, \tag{6}$$

where ζ is the dimensionless relative vorticity $\partial u_2/\partial x_1 - \partial u_1/\partial x_2$. Note that the uniform background potential vorticity q_{b} takes the dimensionless value f. The potential vorticity evolves according to

$$\frac{\partial q}{\partial t} + \boldsymbol{u}.\nabla_x q = 0 \ . \tag{7}$$

Thus the leading-order dynamics in the vortical region, (5a) with $\mathsf{F} = 0$ in (5b), is simply two-dimensional non-divergent barotropic vortex dynamics. The presence of background rotation plays no part in the leading-order evolution of the potential vorticity field, relative to the uniform background value. The rotation does affect the dimensionless height field h, but h affects the definition of q at $O(\mathsf{F}^2)$ only. Therefore the leading-order evolution of the potential vorticity field is unaffected by f, over short time intervals $t = O(1)$.

To take the analysis further, it proves convenient to use (7), (5b), and the divergence of (5a) as the evolution equations. We write the last of these as

$$\frac{\partial}{\partial t}\nabla_x.\boldsymbol{u} + \nabla_x.[(f+\zeta)\boldsymbol{k}\times\boldsymbol{u}] + \nabla_x^2 B = 0 \ , \tag{8}$$

where $B = h + \frac{1}{2}|\boldsymbol{u}|^2$ is the Bernoulli potential. This use of B rather than h will prove necessary in order to maintain convergence of certain integral representations that arise in the analysis, given by (15), (34), and (35) below. Note incidentally that such a device is not needed in the three-dimensional non-rotating problem (Crow 1970).

To obtain equations for the surrounding wave region, valid in the limit of small F, we must rescale the equations using the length scale L/F. We therefore introduce the wave-region spatial variable $\boldsymbol{X}$, defined such that $\boldsymbol{X} = \mathsf{F}\boldsymbol{x}$. Similarly, other scaled fields in the wave region are represented by capital letters in place of the corresponding lower-case letters for the corresponding field in the vortical region. Thus h is replaced by H, the departure from undisturbed layer depth in the wave region. The velocity is one order in F smaller in the wave region, so $\boldsymbol{u}$ is replaced by $\mathsf{F}\boldsymbol{U}$. Under these scalings, (5a) and (5b) are replaced by

$$\frac{\partial \boldsymbol{U}}{\partial t} + f\boldsymbol{k}\times\boldsymbol{U} + \nabla_X H + \mathsf{F}^2\left((\boldsymbol{k}.\nabla_X\times\boldsymbol{U})\boldsymbol{k}\times\boldsymbol{U} + \tfrac{1}{2}\nabla_X\boldsymbol{U}^2\right) = 0 \tag{9a}$$

$$\frac{\partial H}{\partial t} + \nabla_X.\boldsymbol{U} + \mathsf{F}^2\nabla_X.(H\boldsymbol{U}) = 0 \ , \tag{9b}$$

where ∇_X represents $\partial/\partial \boldsymbol{X}$. Equations (9a) and (9b) admit propagating inertia–gravity waves as leading-order solutions for small F. In the wave region it is the nonlinear terms, rather than the leading-order divergence terms, that are of small order in F. The potential vorticity in the wave region is given by

$$Q = \frac{f + \mathsf{F}^2 \boldsymbol{k}.\nabla_X \times \boldsymbol{U}}{1 + \mathsf{F}^2 H} \ . \tag{10}$$

By assumption, the potential vorticity anomaly $\widehat{q} = q - f$ is $O(r^{-\infty})$ in the vortical region as $r \to \infty$. Therefore, in the wave region, the potential vorticity Q takes its uniform value f to all algebraic orders in F. This fact will be used in (20) below.

The flow in each of the two regions is expressed as an asymptotic expansion, with small expansion parameter F. The expansions in the two regions must be "matched" i.e., made mutually consistent in the region where they overlap. More precisely, the "inner limit" $|\boldsymbol{X}| \to 0$ of the flow in the wave region, when re-expressed in terms of $\boldsymbol{x}$, must be the same as the "outer limit" $|\boldsymbol{x}| \to \infty$ of the flow in the vortical region (Van Dyke, 1964). This is known as the asymptotic matching condition. It will be used repeatedly throughout section 3. The entire solution, in both regions, with matching, is referred to as a pair of "matched asymptotic expansions". We impose a radiation condition on the wave region in the limit $|\boldsymbol{X}| \to \infty$.

3.2 Perturbation expansion for the flow in the vortical region

The expansion for the flow in the vortical region takes the form

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{u}_0 &+ \mathsf{F}^2 \ln \mathsf{F}\ \boldsymbol{u}_{21} + \mathsf{F}^2\ \boldsymbol{u}_2 \\ &+ \mathsf{F}^4 \ln^2 \mathsf{F}\ \boldsymbol{u}_{42} + \mathsf{F}^4 \ln \mathsf{F}\ \boldsymbol{u}_{41} + \mathsf{F}^4\ \boldsymbol{u}_4 + \cdots \end{aligned} \tag{11a}$$

$$h = \ln \mathsf{F}\ h_{01} + h_0 + \mathsf{F}^2 \ln^2 \mathsf{F}\ h_{22} + \mathsf{F}^2 \ln \mathsf{F}\ h_{21} + \mathsf{F}^2\ h_2 + \cdots \ . \tag{11b}$$

The logarithmic terms, which cannot be predicted from (5), i.e., from consideration of the vortical region alone, arise from the asymptotic matching condition between the flow in the vortical region and the flow in the wave region. Although the expansion for h starts at $O(\ln \mathsf{F})$, which becomes unbounded in the limit $\mathsf{F} \to 0$, we should recall from (4) that the actual non-dimensional layer depth is given by $1 + \mathsf{F}^2 h$, and so the actual layer depth remains finite as $\mathsf{F} \to 0$. Moreover, we will show that $\nabla_x h_{01} = 0$, so there is no unbounded term in the momentum equation (5a).

Now in order to maintain validity over times $t \gg 1$, we need to pay special attention to the potential vorticity field $q(\boldsymbol{x}, t)$. Given q everywhere in space, it proves possible to determine not only the leading-order velocity $\boldsymbol{u}_0$ in the vortical region but also, by inverting a sequence of Poisson equations, the higher-order terms in the velocity and height fields up to the highest orders displayed in (11a) and (11b), with the exception of a contribution $\boldsymbol{u}_4^{(\mathrm{rad})}$

to $\boldsymbol{u}_4$ associated with the radiation reaction. Note well that q itself is not expanded, nor is its evolution equation (7). Rather, q is evolved according to (7) with the advecting velocity $\boldsymbol{u}$ replaced by its full expansion up to whatever order is required, which in our case is just that displayed in (11a).

As explained by Warn *et al.* (1995), this is necessary in order to maintain extended validity in time. Such validity requires that the expansion remain well ordered over some suitable time interval. In particular, the terms arising from the sequence of Poisson equations need to remain well ordered, in turn requiring that they do not increase secularly in time. The terms do remain well ordered provided that we evolve q in the way just described. This part of the procedure is equivalent to a high-order potential vorticity inversion, done by asymptotic expansion (Warn *et al.* 1995) rather than by numerical iteration (Norton 1988, McIntyre and Norton 1999).

The contribution $\boldsymbol{u}_4^{(\mathrm{rad})}$ depends on a certain "history integral", appearing in Eq. (67) below and involving $q(\boldsymbol{x}, t)$ for all past t. On the assumption that the same pattern of Poisson solutions and history integrals continues to higher order, with no secular behavior, the expansion (11) gives us an approximate solution valid out to times of order $\mathsf{F}^{-4} \gg 1$.

3.3 The leading-order flow in the vortical region

We now take the first step in the solution procedure just described, by obtaining from the potential vorticity field expressions for the velocity field and height field at leading order in F in the vortical region. Because (5b) reduces to $\nabla_x . \boldsymbol{u}_0 = 0$ at leading order, we may write $\boldsymbol{u}_0 = \boldsymbol{k} \times \nabla_x \psi_0$. Thus, neglecting $\mathsf{F}^2 h$ in (6), we have

$$\nabla_x^2 \psi_0 = q - f = \widehat{q} \; . \tag{12}$$

The potential vorticity anomaly $\widehat{q}$ is assumed, as mentioned earlier, to be $O(r^{-\infty})$ as $r \to \infty$. It follows that

$$\psi_0(\boldsymbol{x}, t) = \frac{1}{2\pi} \iint_{R^2} \widehat{q}(\boldsymbol{x}', t) \ln |\boldsymbol{x} - \boldsymbol{x}'| d^2\boldsymbol{x}' + \psi_0^{\mathrm{c}} \; , \tag{13}$$

where the complementary function ψ_0^{c} is a nonsingular solution of Laplace's equation $\nabla_x^2 \psi_0^{\mathrm{c}} = 0$, i.e., ψ_0^{c} is a linear combination of terms of the form $r^n e^{in\theta}$, where $r = |\boldsymbol{x}|$, and $n = 0, 1, 2, \ldots$. However, we will show from the asymptotic matching conditions as $r \to \infty$ that $\psi_0^{\mathrm{c}} =$ constant. In other words, the leading-order velocity $\boldsymbol{u}_0$ is unaffected by the flow in the wave region, and is obtained by inversion as in two-dimensional non-divergent barotropic vortex dynamics with relative vorticity $\widehat{q}(\boldsymbol{x}, t)$. The condition $\widehat{q} = O(r^{-\infty})$ as $r \to \infty$ is sufficient to guarantee convergence of the integral in (13).

A simple example, with three Gaussian vortices in the vortical region, is shown in Fig. 2. For maximal simplicity in this example we have taken $f = 0$. The potential vorticity q is shown in Fig. 2a, and the corresponding streamfunction ψ_0 is shown in Fig. 2b.

To obtain the corresponding expression for h_0, we use the leading approximation to (8), which is

$$\nabla_x^2 B_0 = \nabla_x.(\widehat{q}\nabla_x\psi_0) + f\nabla_x^2\psi_0 \ . \tag{14}$$

Inverting the Laplacian in (14), and using $h_0 = B_0 - \frac{1}{2}|\nabla_x\psi_0|^2$, and $|\nabla_x\psi_0|^2 = \frac{1}{2}\nabla_x^2(\psi_0^2) - \psi_0\nabla_x^2\psi_0$, we obtain

$$\begin{aligned} h_0(\boldsymbol{x},t) = {} & f\psi_0(\boldsymbol{x},t) + \tfrac{1}{2}\widehat{q}(\boldsymbol{x},t)\psi_0(\boldsymbol{x},t) - \tfrac{1}{4}\nabla_x^2\left\{\psi_0^2(\boldsymbol{x},t)\right\} \\ & + \frac{1}{2\pi}\iint_{R^2}\nabla_{x'}.(\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t))\ln|\boldsymbol{x}-\boldsymbol{x}'|d^2\boldsymbol{x}' + h_0^{\mathrm{c}}(\boldsymbol{x},t) \ , \end{aligned} \tag{15}$$

where $\nabla_{x'} = \partial/\partial\boldsymbol{x}'$, and h_0^{c} satisfies $\nabla_x^2 h_0^{\mathrm{c}} = 0$. Again, h_0^{c} will be determined from the asymptotic matching conditions as $r \to \infty$. By contrast with ψ_0^{c}, however, we shall show that h_0^{c} is nontrivially different from zero.

The field h_0 corresponding to the Gaussian vortices in Fig. 2a is shown in Fig. 2c. Note that, as $r \to \infty$, h_0 takes a quadrupolar pattern, with amplitude decreasing as r^{-2}, consistent with matching to a quadrupolar wave in the surrounding wave region.

3.4 The flow in the wave region at $O(1)$ and $O(\mathsf{F})$

The expansion for the flow in the wave region takes the form

$$\boldsymbol{U} = \boldsymbol{U}_0 + \mathsf{F}\,\boldsymbol{U}_1 + \mathsf{F}^2\ln\mathsf{F}\,\boldsymbol{U}_{21} + \mathsf{F}^2\,\boldsymbol{U}_2 + \cdots \tag{16}$$

$$H = H_0 + \mathsf{F}\,H_1 + \mathsf{F}^2\ln\mathsf{F}\,H_{21} + \mathsf{F}^2\,H_2 + \cdots \ . \tag{17}$$

These must be matched to (13) and (15) as $r \to \infty$. Taking the limit $r \to \infty$ in (13), and recalling $\widehat{q} = O(r^{-\infty})$ as $r \to \infty$, we obtain

$$\begin{aligned} \psi_0(\boldsymbol{x},t) = {} & \frac{\ln r}{2\pi}\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}' && \text{(18a)} \\ & - \frac{1}{2\pi r}\,\mathrm{Re}\left\{e^{i\theta}\iint_{R^2}(x'-iy')\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}'\right\} && \text{(18b)} \\ & + O(r^{-2}) \ . \end{aligned}$$

The two integrals (18a) and (18b) have vanishing time derivatives at leading order, because at this order the dynamics looks the same as two-dimensional non-divergent barotropic vortex dynamics with relative vorticity $\widehat{q}$ (see, for example, Batchelor, 1967). From this and from matching to be done shortly, we shall show that the flow in the wave region may be taken to satisfy $\partial H/\partial t = O(\mathsf{F}^2)$ and $\partial\boldsymbol{U}/\partial t = O(\mathsf{F}^2)$. Equation (9b) then implies that $\nabla_X.\boldsymbol{U} = 0$ at $O(1)$ and at $O(\mathsf{F})$, and this means that we may write $\boldsymbol{U}_0 = \boldsymbol{k}\times\nabla_X\Psi_0$ and $\boldsymbol{U}_1 = \boldsymbol{k}\times\nabla_X\Psi_1$. Moreover, since $\partial\boldsymbol{U}/\partial t = O(\mathsf{F}^2)$, we have from (9a), neglecting $O(\mathsf{F}^2)$, that

$$f\{\Psi_0,\Psi_1\} = \{H_0,H_1\} \ . \tag{19}$$

Now, at all algebraic orders in F the potential vorticity in the wave region takes its uniform background value f, and so (10) implies that

$$(\nabla_X^2 - f^2)\{\Psi_0, \Psi_1\} = 0 \ . \tag{20}$$

The match to the vortical region, see (18), implies that Ψ_0 and H_0 must be independent of θ, and that Ψ_1 and H_1 must have the θ-dependence $e^{i\theta}$. Solving (20) with decaying boundary conditions as $|\boldsymbol{X}| \to \infty$, we get

$$\Psi_0 = C_0 \mathrm{K}_0(fR) \ , \tag{21}$$

$$\Psi_1 = C_1 \mathrm{K}_1(fR) e^{i\theta} \ , \tag{22}$$

where $\mathrm{K}_n(z)$ is the modified Bessel function of order n that decays as $z \to \infty$, and it is understood that the real part of (22) is taken, as in (18b). Here we have denoted $|\boldsymbol{X}|$ by R. The values of C_0 and C_1 are determined by matching conditions onto the flow in the vortical region in the limit $R \to 0$, as follows. In the limit $R \to 0$, we have

$$\mathrm{K}_0(fR) \sim -(\ln R + \ln f + \gamma - \ln 2)\left(1 + \tfrac{1}{4} f^2 R^2\right) + \tfrac{1}{4} f^2 R^2 + O(R^4 \ln R) \ , \tag{23}$$

where $\gamma \approx 0.5772$ is Euler's constant. If this is rewritten in terms of $r = R/\mathsf{F}$, and expanded for small F, we obtain from (21) the expression for Ψ_0 in terms of the coordinates of the vortical region:

$$\Psi_0 \sim -C_0(\ln r + \ln \mathsf{F} + \ln f + \gamma - \ln 2)\left(1 + \tfrac{1}{4}\mathsf{F}^2 f^2 r^2\right) + \tfrac{1}{4}\mathsf{F}^2 f^2 r^2 + O(\mathsf{F}^4 \ln \mathsf{F}) \ . \tag{24}$$

The asymptotic matching condition requires that, at each order in F, the limit of the velocity field $\mathsf{F}\boldsymbol{U}$ in the wave region as $R \to 0$ must agree with that of the velocity field $\boldsymbol{u}$ in the vortical region as $r \to \infty$. In particular, at $O(1)$, this means that the coefficients of the $\ln r$ term in (18a) and (24) must agree, and hence

$$C_0 = -\frac{1}{2\pi} \iint_{R^2} \widehat{q}(\boldsymbol{x}, t)\, d^2\boldsymbol{x} \ . \tag{25}$$

Notice that, since the asymptotic matching condition applies only to the pair of physical variables $\boldsymbol{u}$ and $\boldsymbol{U}$, and not to ψ and Ψ directly, the expressions for ψ and Ψ may differ in the matching region by a global constant. This implies that we can satisfy all matching and boundary conditions by taking $\psi_0^{\mathrm{c}} = 0$.

The asymptotic matching condition applies also to the height field. From (15), we can see that in the vortical region $h_0 \sim f\psi_0 + O(r^{-2}) + h_0^{\mathrm{c}}$ as $r \to \infty$, and this must agree with the expression for h in the wave region in the limit $R \to 0$. Now, in the wave region, (19), (21), and (23) imply that

$$H_0 \sim -fC_0(\ln R + \ln f + \gamma - \ln 2) + O(R^2 \ln R) \tag{26}$$

as $R \to 0$. If this is rewritten in terms of $r = R/\mathsf{F}$, we see that, at $O(1)$ in the matching condition, the terms of form $\ln r$ agree, provided C_0 is given by (25). However, the full expression for h in the vortical region in the limit $r \to \infty$ can be made to agree, up to $O(1)$, with the expression for H in the wave region in the limit $R \to 0$ only if we can find h_0^{c} and h_{01} such that $\ln \mathsf{F}\, h_{01} + h_0^{\mathrm{c}} \sim -fC_0(\ln \mathsf{F} + \ln f + \gamma - \ln 2)$ as $r \to \infty$. Recall now that h_0^{c} satisfies $\nabla_x^2 h_0^{\mathrm{c}} = 0$. Also, h_{01} satisfies $\nabla_x h_{01} = 0$ for consistency with (5a). The only solution is $h_0^{\mathrm{c}} =$ constant and $h_{01} =$ constant, throughout the vortical region. Matching now implies

$$h_0^{\mathrm{c}}(\boldsymbol{x}, t) = -fC_0(\ln f + \gamma - \ln 2) \ , \qquad h_{01}(\boldsymbol{x}, t) = -fC_0 \ , \tag{27}$$

where C_0 is given by (25). Note that it is the $\ln \mathsf{F}$ term in h that contributes the largest term to the height field in the vortical region. As a check on the sign of h_{01}, we see that a cyclonic vortex ($fC_0 < 0$) corresponds to a depression in the free surface of $O(\mathsf{F}^2 \ln \mathsf{F})$.

An expression for C_1 is similarly obtained as follows. In the limit $R \to 0$ we have

$$\mathrm{K}_1(fR) \sim \frac{1}{fR} + \tfrac{1}{2} fR \ln(fR) + \left(\tfrac{1}{2}\gamma + \tfrac{9}{4} - \tfrac{1}{2}\ln 2\right) fR + O(R^3 \ln R) \ . \tag{28}$$

Again writing this in terms of $r = R/\mathsf{F}$ and expanding for small F, we obtain from (22) the expression for Ψ_1 in terms of the coordinates of the vortical region. The asymptotic matching condition on the velocity requires that the coefficients of the r^{-1} terms at $O(1)$ from (18b) and (28) agree, and this implies

$$C_1 = -\frac{f}{2\pi} \iint_{R^2} (x - iy)\widehat{q}(\boldsymbol{x}, t)\, d^2\boldsymbol{x} \ . \tag{29}$$

It can readily be verified that, with C_1 given by (29), the expressions for the height also match at this order.

3.5 The flow in the vortical region at $O(\mathsf{F}^2 \ln \mathsf{F})$ and $O(\mathsf{F}^2)$

Now that we have determined the flow in the vortical region up to $O(1)$, we can proceed to obtain the velocity field in the vortical region at $O(\mathsf{F}^2 \ln \mathsf{F})$ and $O(\mathsf{F}^2)$. Here we write $\boldsymbol{u}_{21} = \boldsymbol{k} \times \nabla_x \psi_{21}$ and $\boldsymbol{u}_2 = \nabla_x \phi_2 + \boldsymbol{k} \times \nabla_x \psi_2$. Expanding (5b) and (6) to $O(\mathsf{F}^2)$, we obtain equations for ϕ_2, ψ_2, and ψ_{21}:

$$\nabla_x^2 \phi_2 = -\frac{\partial_0 h_0}{\partial t} - \nabla_x.(\boldsymbol{u}_0 h_0) = -\frac{\partial_0 h_0}{\partial t} - \boldsymbol{u}_0.\nabla_x h_0 \ , \tag{30}$$

$$\nabla_x^2 \psi_2 = (f + \widehat{q}) h_0 \ , \tag{31}$$

and

$$\nabla_x^2 \psi_{21} = (f + \widehat{q}) h_{01} \ . \tag{32}$$

The symbol $\partial_0/\partial t$ is a leading-order diagnostic estimate of $\partial/\partial t$, defined as follows. First, $\partial_0\widehat{q}/\partial t$ is defined to be $-\boldsymbol{u}_0.\nabla_x\widehat{q}$. Second, $\partial_0\psi_0/\partial t$ is defined to be the result of applying $\partial_0/\partial t$ to (13), with $\partial_0\psi_0^{\mathrm{c}}/\partial t = 0$ by definition, so that

$$\frac{\partial_0}{\partial t}\psi_0(\boldsymbol{x},t) = -\frac{1}{2\pi}\iint_{R^2}\boldsymbol{u}_0.\nabla_x\widehat{q}(\boldsymbol{x}',t)\ln|\boldsymbol{x}-\boldsymbol{x}'|d^2\boldsymbol{x}' \ . \tag{33}$$

Finally, $\partial_0 h_0/\partial t$ is defined to be the result of applying $\partial_0/\partial t$ to (15). Note that $\partial_0 h_0^{\mathrm{c}}/\partial t = 0$ by the first of (27) and the fact that $\partial_0 C_0/\partial t = 0$, by (25) and the remarks below (18b).

The solutions must be regular for all finite $\boldsymbol{x}$, and this means that ϕ_2, ψ_2 and ψ_{21} are determined up to nonsingular solutions of Laplace's equation. Using the integral expression (15) for h_0 obtained above, we can write down the solutions:

$$\begin{aligned}
\phi_2(\boldsymbol{x},t) = &-\frac{f}{8\pi}\frac{\partial_0}{\partial t}\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)|\boldsymbol{x}-\boldsymbol{x}'|^2(\ln|\boldsymbol{x}-\boldsymbol{x}'|-1)d^2\boldsymbol{x}' \\
&-\frac{1}{4\pi}\frac{\partial_0}{\partial t}\iint_{R^2}\psi_0(\boldsymbol{x}',t)\widehat{q}(\boldsymbol{x}',t)\ln|\boldsymbol{x}-\boldsymbol{x}'|d^2\boldsymbol{x}' \\
&+\frac{1}{2}\psi_0(\boldsymbol{x},t)\frac{\partial_0\psi_0(\boldsymbol{x},t)}{\partial t} \\
&-\frac{1}{8\pi}\frac{\partial_0}{\partial t}\iint_{R^2}\nabla_{x'}[\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t)]|\boldsymbol{x}-\boldsymbol{x}'|^2(\ln|\boldsymbol{x}-\boldsymbol{x}'|-1)d^2\boldsymbol{x}' \\
&-\frac{1}{2\pi}\iint_{R^2}\nabla_{x'}.[(h_0(\boldsymbol{x}',t)-f\psi_0(\boldsymbol{x}',t))\boldsymbol{u}_0(\boldsymbol{x}',t)]\ln|\boldsymbol{x}-\boldsymbol{x}'|d^2\boldsymbol{x}' \\
&+\phi_2^{\mathrm{c}}(\boldsymbol{x},t)\ ,
\end{aligned}\tag{34}$$

$$\begin{aligned}
\psi_2(\boldsymbol{x},t) = &-\tfrac{1}{4}f\psi_0^2(\boldsymbol{x},t) \\
&+\frac{1}{2\pi}\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)\left[h_0(\boldsymbol{x}',t)+\tfrac{1}{2}f\psi_0(\boldsymbol{x}',t)\right]\ln|\boldsymbol{x}-\boldsymbol{x}'|d^2\boldsymbol{x}' \\
&+\frac{f^2}{8\pi}\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)|\boldsymbol{x}-\boldsymbol{x}'|^2(\ln|\boldsymbol{x}-\boldsymbol{x}'|-1)d^2\boldsymbol{x}' \\
&+\frac{f}{8\pi}\iint_{R^2}\nabla_{x'}[\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t)]|\boldsymbol{x}-\boldsymbol{x}'|^2(\ln|\boldsymbol{x}-\boldsymbol{x}'|-1)d^2\boldsymbol{x}' \\
&+\tfrac{1}{4}fr^2h_0^{\mathrm{c}}+\psi_2^{\mathrm{c}}(\boldsymbol{x},t)\ ,
\end{aligned}\tag{35}$$

and

$$\psi_{21}(\boldsymbol{x},t) = \frac{f}{4}h_{01}r^2 + h_{01}\psi_0(\boldsymbol{x},t) + \psi_{21}^{\mathrm{c}}(\boldsymbol{x},t)\ , \tag{36}$$

where ϕ_2^{c}, ψ_2^{c} and ψ_{21}^{c} are solutions of Laplace's equation, the boundary conditions for which must be determined by asymptotic matching conditions. The factors $\widehat{q}$ appear in all but one of the integrands, because of the use of (8) via (15).

It can readily be verified that (34), (35) and (36) satisfy (30), (31) and (32) respectively. It is also necessary to establish that all the integrals in (34) and (35) converge. To see this, recall that $\widehat{q} = O(r^{-\infty})$ as $r \to \infty$. This is sufficient to establish that all integrals converge, except the last integral in (34). To show that this integral converges, we observe that $h_0 - f\psi_0 = O(r^{-2})$, and $\boldsymbol{u}_0 = O(r^{-1})$, as $r \to \infty$. Thus the integrand is $O(r^{-4} \ln r)$ as $r \to \infty$, and hence the integral converges. For further details, see Ford (1993).

As in the case of the $O(1)$ flow, we must use asymptotic matching conditions to determine $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$. The full details of the analysis are given in appendix A. However, we can readily see that such matching conditions cannot affect the time-dependent flow in the vortical region at $O(\mathsf{F}^2)$, as follows.

First, recall that, since $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$ satisfy Laplace's equation, and must be finite for all finite r, they can be expressed as a sum of terms of form $r^n e^{in\theta}$, for $n = 0, 1, 2 \ldots$. Moreover, since ϕ and ψ are undetermined up to a global constant, we need only consider the matching conditions for $n = 1, 2 \ldots$. By considering the order, in F, of such terms, when $r = O(\mathsf{F}^{-1})$, we see that they will be determined by the flow in the wave region at $O(\mathsf{F}), O(1), \ldots$ respectively. There is no flow in the wave region at orders greater than $O(1)$, so we conclude that we need only consider the addition of terms with $n = 1, 2$ to $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$, and that any such terms will be determined by asymptotic matching conditions to the flow in the wave region at $O(1)$ and $O(\mathsf{F})$. However, we know from section 3.4 that the flow in the wave region at these two orders has vanishing time derivative at these orders, and therefore so will $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$. Consequently, they will not affect the details of the wave radiation at $O(\mathsf{F}^2)$.

The complete analysis to determine $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$, in the way just sketched, is presented in appendix A. The result is

$$\psi_2^{\rm c} = \tfrac{1}{2} C_1 f (5 + \gamma + \ln f - \ln 2) r e^{i\theta} , \qquad \psi_{21}^{\rm c} = \tfrac{1}{2} C_1 f r e^{i\theta} , \qquad \phi_2^{\rm c} = 0 , \tag{37}$$

where C_1 is given by (29).

The field ϕ_2 corresponding to the Gaussian vortices in Fig. 2a is shown in Fig. 2d. Note that, as $r \to \infty$, ϕ_2 takes a quadrupolar pattern, with amplitude approaching a constant value as $r \to \infty$ (see Eq. 56 below). This is consistent with matching to a quadrupolar wave in the surrounding wave region (see Eq. 61 below). With $f = 0$ the corresponding ψ_2 field (not shown) has no quardupolar far-field form, and is qualitatively similar to the ψ_0 field.

No actual time derivatives $\partial/\partial t$ have yet appeared in the analysis, but only the leading-order diagnostic estimate $\partial_0/\partial t$. The entire analysis so far, including the Poisson equations (30)–(32), has been purely diagnostic, and corresponds, as mentioned in section 3.2, to part of a sequence of Poisson equations representing a high-order potential-vorticity inversion operator.

Now that we have a complete description to $O(\mathsf{F}^2)$ of the flow in the vortical region, we can determine the flow to $O(\mathsf{F}^3)$ in the wave region. Propagating waves arise at $O(\mathsf{F}^2)$. The terms representing the waves are found in section 3.6. Then section 3.7 finds the radiation reaction on the vortical region, which requires taking the expansion in the vortical region as far as $O(\mathsf{F}^4)$.

3.6 Propagating waves in the wave region at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$

At $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$, the flow in the wave region is conveniently decomposed into two parts: one part, e.g. $\overline{\Psi}_2$, for which $\partial_0\overline{\Psi}_2/\partial t = 0$, and a second part, e.g. Ψ_2', for which $\partial_0\Psi_2'/\partial t \neq 0$. We shall refer to the first as the quasi-steady part of the flow, and the second as the wavelike part. The division is made unique, and useful, in the following way.

We expand the equations for the flow in the wave region, (9), to $O(\mathsf{F}^3)$. The flow in the wave region at $O(1)$ and $O(\mathsf{F})$ makes quasi-steady, i.e., not wavelike, contributions to the equations at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$ through the nonlinear terms in (9). The quasi-steady flow in the wave region at $O(1)$ and $O(\mathsf{F})$ has non-vanishing time-derivative at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$. However, C_0 and C_1 represent leading-order approximations to the circulation and momentum in the vortical region. Since the circulation in the vortical region must be conserved to all orders, and the momentum conserved to $O(\mathsf{F}^2)$, it follows that by making corrections $O(\mathsf{F}^2)$ to the coefficients C_0 and C_1, the time-derivatives of the $O(1)$ and $O(\mathsf{F})$ flow in the wave region can be made $O(\mathsf{F}^4)$ and $O(\mathsf{F}^5)$ respectively. Therefore, the quasi-steady nonlinear contributions to (9) at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$ have time derivatives $O(\mathsf{F}^4)$ and $O(\mathsf{F}^5)$ respectively. By retaining these quasi-steady terms in the equations for the quasi-steady fields at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$, (and therefore removing these quasi-steady terms from the equations for the wavelike fields at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$), we find that the equations for $\overline{\Psi}_2$ and $\overline{\Psi}_3$ are

$$(\nabla_X^2 - f^2)\overline{\Psi}_2 = -\tfrac{1}{2}f|\nabla_X\Psi_0|^2 + \tfrac{1}{2}f^3\Psi_0^2 \ . \tag{38}$$

and

$$(\nabla_X^2 - f^2)\overline{\Psi}_3 = -f\nabla_X\Psi_0.\nabla_X\Psi_1 + f^3\Psi_0\Psi_1 \ , \tag{39}$$

with

$$f\{\overline{\Psi}_2, \overline{\Psi}_3\} = \{\overline{H}_2, \overline{H}_3\} \ . \tag{40}$$

None of these quasi-steady contributions will be considered further, because they do not represent propagating waves in the description up to $O(\mathsf{F}^3)$, nor do they interact with the propagating waves that are emitted at these orders.

We turn now to determining the wavelike part of the flow in the wave region, Ψ' etc., at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$. In particular, we shall determine explicit expressions for the waves emitted from the vortical region at $O(\mathsf{F}^2)$, and also the nature of the next correction at $O(\mathsf{F}^3)$. Since, as just explained, the nonlinear terms in (9) do not enter the equations for the wavelike part of the flow in the wave region at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$, the equations for the wavelike part of the flow, which can be obtained from (9), are

$$\left(\frac{\partial^2}{\partial t^2} + f^2 - \nabla_X^2\right)\{\Phi_2', \Psi_2', H_2', \Phi_3', \Psi_3', H_3'\} = 0\ , \tag{41}$$

where $\boldsymbol{U}_2' = \nabla_X\Phi_2' + \boldsymbol{k}\times\nabla_X\Psi_2'$ and $\boldsymbol{U}_3' = \nabla_X\Phi_3' + \boldsymbol{k}\times\nabla_X\Psi_3'$, and where, without loss of generality, we take

$$\frac{\partial}{\partial t}\{\Psi_2', \Psi_3'\} + f\{\Phi_2', \Phi_3'\} = 0\ . \tag{42}$$

The boundary conditions are the radiation condition at infinity and the matching to the vortical region as $R \to 0$. Because our concern is entirely with the wavelike part of the flow at $O(\mathsf{F}^2)$, we drop the primes from Ψ_2', Φ_2' and H_2'. The solutions of these equations are freely-propagating waves, the amplitude and phase of which must be determined by matching to the wave region, as we shall now demonstrate.

For the purposes of applying the asymptotic matching conditions, it is convenient to work in the frequency domain, in which the general solutions of (41) are $\mathrm{H}_m^{(k)}\left((\omega^2-f^2)^{1/2}R\right)e^{i(m\theta-\omega t)}$, provided that ω represents a radian frequency. Here, $\mathrm{H}_m^{(k)}$ represents a Hankel function of the kth kind, exactly as defined by Abramowitz & Stegun (1964); m is the order of the Hankel function, determined by the θ dependence of the match to the vortical region; k is determined by the radiation condition, and $R = |\boldsymbol{X}|$ as before. The order $m = 0$ corresponds to a monopole wave, $m = 1$ to a dipole, $m = 2$ to a quadrupole, and so on.

Throughout, we shall represent the Fourier transform of a function $g(t)$ by $\widetilde{g}(\omega)$, where

$$\widetilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}dt\ . \tag{43}$$

Consequently, it can be shown that solutions of form $\mathrm{H}_m^{(1)}\left((\omega^2-f^2)^{1/2}R\right)e^{i(m\theta-\omega t)}$ satisfy the radiation condition as $R\to\infty$, provided we define an analytic branch of $(\omega^2-f^2)^{1/2}$ in the complex ω-plane such that $\mathrm{Re}(\omega^2-f^2)^{1/2} > 0$ for $\omega > f$, and $\mathrm{Re}(\omega^2-f^2)^{1/2} < 0$ for $\omega < -f$. The solution also satisfies an evanescence condition for $\omega^2 < f^2$, provided we take $\mathrm{Im}(\omega^2-f^2)^{1/2} > 0$ for real ω between $-f$ and f, i.e., for real ω,

$$(\omega^2-f^2)^{1/2} = \begin{cases} |\omega^2-f^2|^{1/2} & \omega > f \\ i|f^2-\omega^2|^{1/2} & -f < \omega < f \\ -|\omega^2-f^2|^{1/2} & \omega < -f \end{cases}\ . \tag{44}$$

We now turn to the asymptotic matching condition, which completely determines the $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$ waves in the wave region. To do this, we must examine the time-dependent flow in the vortical region in the limit $r\to\infty$, which we obtain by expansion of (13), (34), (35) and (36) for $|\boldsymbol{x}'| \ll |\boldsymbol{x}|$.

We first consider (13). In the limit $r \to \infty$, this yields, going beyond (18),

$$\psi(\boldsymbol{x},t) = \ln r \left\{ \frac{1}{2\pi} \iint_{R^2} \widehat{q}(\boldsymbol{x}',t)\, d^2\boldsymbol{x}' \right\} \tag{45a}$$

$$- \frac{1}{r} e^{i\theta} \left\{ \frac{1}{2\pi} \iint_{R^2} (x' - iy') \widehat{q}(\boldsymbol{x}',t)\, d^2\boldsymbol{x}' \right\} \tag{45b}$$

$$- \frac{1}{r^2} e^{2i\theta} \left\{ \frac{1}{4\pi} \iint_{R^2} (x' - iy')^2 \widehat{q}(\boldsymbol{x}',t)\, d^2\boldsymbol{x}' \right\} \tag{45c}$$

$$+ O(r^{-3}) \; .$$

Recall that (45a) and (45b) are identical to (18a) and (18b), which have vanishing time derivatives at leading order. Thus, the time-dependent form of ψ_0 in the vortical region as $r \to \infty$ is given in (45c), which has the spatial form $r^{-2} e^{2i\theta}$.

To match this to a flow in the wave region, we note that the $O(\mathsf{F}^2)$ wave field must have $e^{2i\theta}$ angular dependence, and R^{-2} radial dependence as $R \to 0$. Hence we choose mode $m = 2$ in the Hankel function, and write the general solution for the $O(\mathsf{F}^2)$ wave fields in the frequency domain:

$$\widetilde{\Phi}_2 = i\widetilde{A}(\omega)(\omega^2 - f^2) \mathrm{H}_2^{(1)} \left((\omega^2 - f^2)^{1/2} R \right) e^{2i\theta} \tag{46a}$$

$$\widetilde{\Psi}_2 = i\widetilde{B}(\omega)(\omega^2 - f^2) \mathrm{H}_2^{(1)} \left((\omega^2 - f^2)^{1/2} R \right) e^{2i\theta} \tag{46b}$$

$$\widetilde{H}_2 = i\widetilde{C}(\omega)(\omega^2 - f^2) \mathrm{H}_2^{(1)} \left((\omega^2 - f^2)^{1/2} R \right) e^{2i\theta} \; . \tag{46c}$$

To determine expressions for $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, we must examine (46a, 46b, 46c) in the limit $R \to 0$, re-expressed in terms of the variable r. With the Hankel function defined as in Abramowitz & Stegun (1964), we have that

$$\begin{aligned} \mathsf{F}^2 i(\omega^2 - f^2) \mathrm{H}_2^{(1)}((\omega^2 - f^2)^{1/2} R) &= \mathsf{F}^2 \left(\frac{4}{\pi R^2} + \frac{1}{\pi}(\omega^2 - f^2) + O(R^2 \ln R) \right) \\ &= \frac{4}{\pi r^2} + \mathsf{F}^2 \frac{1}{\pi}(\omega^2 - f^2) + O(\mathsf{F}^4 \ln \mathsf{F}) \end{aligned} \tag{47}$$

as $R \to 0$.

We consider first the terms that are $O(r^{-2})$ in ψ. The corresponding velocity field in the wave region, in the limit $R \to 0$, re-expressed in terms of the coordinate r, is

$$\mathsf{F}^3 \widetilde{\boldsymbol{U}}_2 = \mathsf{F}^3 \left(\nabla_X \widetilde{\Phi}_2 + \boldsymbol{k} \times \nabla_X \widetilde{\Psi}_2 \right) = \frac{4}{\pi r^3} \left[(-2\widetilde{A} - 2i\widetilde{B}) \widehat{\boldsymbol{r}} + (2i\widetilde{A} - 2\widetilde{B}) \widehat{\boldsymbol{\theta}} \right] e^{2i\theta} + O(r^{-1}) \; , \tag{48}$$

where $\widehat{\boldsymbol{r}}$ and $\widehat{\boldsymbol{\theta}}$ are unit vectors in the radial and aximuthal directions respectively. The condition (42) gives us

$$- i\omega \widetilde{B} + f\widetilde{A} = 0 \; , \tag{49}$$

and hence

$$\mathsf{F}^3\widetilde{\boldsymbol{U}}_2 = \frac{8}{\pi r^3}\left[-(f/\omega+1)\widetilde{A}\widehat{\boldsymbol{r}} + i(f/\omega+1)\widetilde{A}\widehat{\boldsymbol{\theta}}\right]e^{2i\theta} + O(r^{-1}) \ . \tag{50}$$

The velocity field in the wave region at $O(\mathsf{F}^2)$, in the limit $R \to 0$, is thus equal to the velocity associated with an effective streamfunction

$$\widetilde{\psi}_{\text{eff}} = \frac{4}{i\pi r^2}\left(\frac{f}{\omega}+1\right)\widetilde{A}\,e^{2i\theta} \ . \tag{51}$$

The asymptotic matching condition, applied to the velocity field, requires that $\widetilde{\psi}_{\text{eff}}$ be equal to the time-dependent part of ψ_0 in the limit $r \to \infty$. This we know from (45c) to be given by

$$\psi_{0_{\text{time-dependent}}}(\boldsymbol{x},t) = -\frac{1}{4\pi r^2}\iint_{R^2}(x'-iy')^2\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}'\ e^{2i\theta} + O(r^{-3}). \tag{52}$$

Equating these two asymptotic forms (51) and (52), as required by the asymptotic matching condition, we obtain an expression for $\widetilde{A}$:

$$\widetilde{A} = -\frac{i\omega}{\omega+f}\widetilde{\alpha}(\omega) \ , \tag{53}$$

where $\widetilde{\alpha}(\omega)$ is the Fourier transform of $\alpha(t)$, and $\alpha(t)$ is defined by

$$\alpha(t) = \frac{1}{16}\iint_{R^2}(x'-iy')^2\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}' \ . \tag{54}$$

The expression for $\widetilde{B}$ follows directly from (53) and (49). The expression for $\widetilde{C}$ can be obtained by applying the asymptotic matching condition to the height field directly, but it is more readily obtained by using an identity in the wave region — $\widetilde{H} = i\omega\widetilde{\Phi} + f\widetilde{\Psi}$ — which implies that $\widetilde{C} = i\omega\widetilde{A} + f\widetilde{B}$. The final result is:

$$\widetilde{A} = -\frac{i\omega\widetilde{\alpha}}{\omega+f} \ , \quad \widetilde{B} = -\frac{f\widetilde{\alpha}}{\omega+f} \ , \quad \widetilde{C} = (\omega-f)\widetilde{\alpha} \ . \tag{55}$$

Note that, despite the singular appearance of the expressions for $\widetilde{A}$ and $\widetilde{B}$, the velocity fields remain regular, even for $\omega = -f$.

Taking Φ_2, Ψ_2, and H_2 defined by (46), with $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ defined by (55), ensures matching between the flow at $O(1)$ in the vortical region and the flow at $O(\mathsf{F}^2)$ in the wave region. To complete the asymptotic matching for the time-dependent flow in the wave region at $O(\mathsf{F}^2)$, we must also ensure matching between the flow in the vortical region at $O(\mathsf{F}^2)$ and the flow in the wave region at $O(\mathsf{F}^2)$.

To perform this matching, we must obtain the time-dependent parts of $\mathsf{F}^2\phi_2$, $\mathsf{F}^2\psi_2$ and $\mathsf{F}^2\ln\mathsf{F}\psi_{21}$, in the limit $r \to \infty$, retaining those terms which at $O(\mathsf{F}^2)$ or larger when $r = (\mathsf{F}^{-1})$. This must then match to terms in the time-dependent flow in the wave region to (F^2).

Consider $\mathsf{F}^2\phi_2$, $\mathsf{F}^2\psi_2$ and $\mathsf{F}^2\ln\mathsf{F}\psi_{21}$ in the limit $r \to \infty$. From (34), (35) and (36) we have

$$
\begin{aligned}
\phi_2(\boldsymbol{x},t) = &-\frac{f}{8\pi}\left(\frac{x_i x_j}{r^2}-\frac{\delta_{ij}}{2}\right)\frac{d}{dt}\iint_{R^2} x_i' x_j' \widehat{q}(\boldsymbol{x}',t)\,d^2\boldsymbol{x}' \\
&-\frac{1}{8\pi}\left(\frac{x_i x_j}{r^2}-\frac{\delta_{ij}}{2}\right)\frac{d}{dt}\iint_{R^2} x_i' x_j' \nabla_{x'}.(\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t))\,d^2\boldsymbol{x}' + \phi_2^{\mathrm{c}} \\
&+ O(r^{-1})\ ,
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\psi_2(\boldsymbol{x},t) = &+\frac{f^2}{8\pi}r^2(\ln r - 1)\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)\,d^2\boldsymbol{x}' + \frac{f}{4}r^2 h_0^{\mathrm{c}} \\
&-\frac{f^2}{8\pi}(2\ln r - 1)x_i\iint_{R^2} x_i'\widehat{q}(\boldsymbol{x}',t)\,d^2\boldsymbol{x}' \\
&+\frac{f^2}{8\pi}\left(\frac{x_i x_j}{r^2}-\frac{\delta_{ij}}{2}\right)\iint_{R^2} x_i' x_j'\widehat{q}(\boldsymbol{x}',t)\,d^2\boldsymbol{x}' \\
&+\frac{f}{8\pi}\left(\frac{x_i x_j}{r^2}-\frac{\delta_{ij}}{2}\right)\iint_{R^2} x_i' x_j' \nabla_{x'}.(\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t))\,d^2\boldsymbol{x}' \\
&+\frac{1}{2\pi}\ln r\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)h_0(\boldsymbol{x}',t)\,d^2\boldsymbol{x}' \\
&+\left[\frac{f^2}{4\pi}\iint_{R^2} r'^2\widehat{q}(\boldsymbol{x}',t)\,d^2\boldsymbol{x}' + \frac{f}{4\pi}\iint_{R^2} r'^2\nabla_{x'}.(\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t))\,d^2\boldsymbol{x}'\right]\ln r \\
&-\frac{f}{16\pi^2}\left[\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)\,d^2\boldsymbol{x}'\right]^2\ln^2 r \\
&+ O(r^{-1}\ln r)\ ,
\end{aligned} \tag{57}
$$

and

$$
\psi_{21} \sim \frac{f}{4}h_0^{\mathrm{c}} r^2 - h_{01}C_0\ln r + \psi_{21}^{\mathrm{c}} + O(r^{-1})\ . \tag{58}
$$

In obtaining the expression for (56) from (34), we have used the fact that $\partial_0/\partial t$ of $\iint_{R^2} r^2\widehat{q}(\boldsymbol{x},t)d^2\boldsymbol{x}$, $\iint_{R^2} r^2\nabla_x.(\widehat{q}(\boldsymbol{x},t)\nabla_x\psi_0(\boldsymbol{x}))d^2\boldsymbol{x}$, and $\iint_{R^2}\psi_0(\boldsymbol{x},t)\widehat{q}(\boldsymbol{x})d^2\boldsymbol{x}$ are all zero. This shows that (56) repesents the leading-order time-dependent part of ϕ_2 in the limit $r\to\infty$. The expressions for h_0^{c}, h_{01}^{c} and C_0 also have vanishing time derivatives at leading order, and so ψ_{21} has vanishing time at leading order in F, in the limit $r\to\infty$.

This fact proves useful in the discussion of the time-dependent behavior of (57), which follows. The first two integrals in (57) have vanishing time derivatives at leading order, i.e., $\partial_0/\partial t$ of each of them is zero, as remarked in section 3.4. Therefore, these terms are of no interest in this section, and are not considered further.

The next two integrals in (57) are time-dependent at leading order, i.e., $\partial_0/\partial t \neq 0$, and so they must be retained in our representation of the the leading-order time-dependent part of ψ_2 in the limit $r\to\infty$.

The fifth integral in (57) has nonvanishing $\partial_0/\partial t$. However, to determine the time-dependence of ψ at $O(\mathsf{F}^2)$, we must consider not only ψ_2 but also the $O(\mathsf{F}^2)$ time-dependence of ψ_0. Therefore, we combine $\iint_{R^2} \widehat{q}(\boldsymbol{x},t)d^2\boldsymbol{x}$ from ψ_0 and $\iint_{R^2} h_0(\boldsymbol{x},t)\widehat{q}(\boldsymbol{x},t)d^2\boldsymbol{x}$ from ψ_2. If we include also the contribution $\iint_{R^2} h_{01}\widehat{q}(\boldsymbol{x},t)d^2\boldsymbol{x}$ from ψ_{21}, which in any case has vanishing $\partial_0/\partial t$, then the total, $\iint_{R^2} \left[1+\mathsf{F}^2\ln\mathsf{F}\ h_{01}+\mathsf{F}^2\ h_0(\boldsymbol{x},t)\right]\widehat{q}(\boldsymbol{x},t)d^2\boldsymbol{x}$ represents the circulation of the vortical region to $O(\mathsf{F}^2)$. Since the circulation around any closed contour is conserved by the shallow water equations, it follows that $\iint_{R^2} \left[1+\mathsf{F}^2\ln\mathsf{F}\ h_{01}+\mathsf{F}^2\ h_0(\boldsymbol{x},t)\right]\widehat{q}(\boldsymbol{x},t)d^2\boldsymbol{x}$ must have vanishing time derivative to $O(\mathsf{F}^2)$. The remaining integrals in (57) have vanishing $\partial_0/\partial t$, and hence the time-dependent parts of ϕ and ψ, at $O(\mathsf{F}^2)$, in the limit $r\to\infty$, are

$$\begin{aligned}\phi_{2_{\text{time-dependent}}}(\boldsymbol{x},t) = &-\frac{f}{8\pi}\left(\frac{x_ix_j}{r^2}-\frac{\delta_{ij}}{2}\right)\frac{d}{dt}\iint_{R^2} x_i'x_j'\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}'\\ &-\frac{1}{8\pi}\left(\frac{x_ix_j}{r^2}-\frac{\delta_{ij}}{2}\right)\frac{d}{dt}\iint_{R^2} x_i'x_j'\nabla_{x'}.(\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t))d^2\boldsymbol{x}'\\ &+O(r^{-1})\ ,\end{aligned} \tag{59}$$

$$\begin{aligned}\psi_{2_{\text{time-dependent}}}(\boldsymbol{x},t) \sim &+\frac{f^2}{8\pi}\left(\frac{x_ix_j}{r^2}-\frac{\delta_{ij}}{2}\right)\iint_{R^2} x_i'x_j'\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}'\\ &+\frac{f}{8\pi}\left(\frac{x_ix_j}{r^2}-\frac{\delta_{ij}}{2}\right)\iint_{R^2} x_i'x_j'\nabla_{x'}.(\widehat{q}(\boldsymbol{x}',t)\nabla_{x'}\psi_0(\boldsymbol{x}',t))d^2\boldsymbol{x}'\\ &+O(r^{-1})\ ,\end{aligned} \tag{60}$$

with no $\partial_0/\partial t$ time-dependence at $O(\mathsf{F}^2\ln\mathsf{F})$ at $O(r^0)$. These terms must match to flow in the wave region at $O(\mathsf{F}^2)$.

In the wave region, we have

$$\widetilde{\Phi}_2\Big|_{O(1)} = \frac{1}{\pi}(\omega^2-f^2)\widetilde{A}e^{2i\theta}\ , \tag{61}$$

$$\widetilde{\Psi}_2\Big|_{O(1)} = \frac{1}{\pi}(\omega^2-f^2)\widetilde{B}e^{2i\theta}\ , \tag{62}$$

where $\widetilde{A}$ and $\widetilde{B}$ are given by (55), where the shorthand notation $\widetilde{\Phi}_2\Big|_{O(1)}$ etc. means the contribution to $\widetilde{\Phi}_2$ that is $O(1)$ in the limit $R\to 0$.

If (61) and (62) are re-expressed in the time domain, it can be shown that (59) and (61) are equivalent, and also that (60) and (62) are equivalent. Therefore, (46) represents the leading-order (i.e. $O(\mathsf{F}^2)$) radiating wave field that is spontaneously emitted by the flow in the vortical region. The flow in the vortical region at $O(\mathsf{F}^2)$, in the limit $r\to\infty$, provides further terms required for the asymptotic matching to the wave field (46), which was required by the matching conditions at lower order, but no further waves are required at $O(\mathsf{F}^2)$.

Note, however, that the expressions (34) and (35) will in general possess time-dependent dipoles, of form $\beta(t)r^{-1}e^{i\theta}$ as $r\to\infty$, for some time-dependent function $\beta(t)$, which can be computed in terms of integrals over the vortical region. (The details are omitted for brevity, but direct computation has shown that $d\beta/dt \neq 0$ for the potential-vorticity distribution shown in Fig. 2). These give rise to dipolar waves at $O(\mathsf{F}^3)$ in the wave region, reminding us that Lighthill's result — that the radiated wave field is of quadrupole or higher type — is true only at the leading order in F.

3.7 The effect of wave radiation on the vortical flow

We turn now to the radiation-reaction problem. That is, we calculate the effect of the wave radiation in the wave region on the flow that is generating it. In the vortical region, the radiation reaction must be felt via a velocity field that somehow causes the vortical region to lose energy, in compensation for the energy radiated by the waves. We shall show that this velocity field is $O(\mathsf{F}^4)$.

To obtain the flow in the vortical region at $O(\mathsf{F}^4)$, we can proceed as in section 3.5, solving a sequence of Poisson equations similar to (30)–(32). The radiation reaction enters not though the right-hand-sides of the Poisson equations, which are purely diagnostic in the same sense as before, but only through the boundary conditions in the limit $r \to \infty$. The boundary conditions are derived from the asymptotic matching condition with the wave region, just as before; and the radiation reaction enters at $O(\mathsf{F}^4)$.

In order to determine the asymptotic matching conditions implied by the $O(\mathsf{F}^2)$ waves in the wave region, in the limit $R \to 0$, we must consider the expansion of the Hankel function $\mathrm{H}_2^{(1)}((\omega^2 - f^2)^{1/2}R)$ in the limit $R \to 0$:

$$
\begin{aligned}
i\pi(\omega^2 - f^2)\mathrm{H}_2^{(1)}((\omega^2 - f^2)^{1/2}R) = \\
&\frac{4}{R^2} && \text{(63a)}\\
&+ (\omega^2 - f^2) && \text{(63b)}\\
&- \tfrac{1}{4}(\omega^2 - f^2)^2 R^2 \ln\left(\tfrac{1}{2}(\omega^2 - f^2)^{1/2}R\right) && \text{(63c)}\\
&+ \frac{1}{8}(\omega^2 - f^2)^2(\psi_\Gamma(1) + \psi_\Gamma(3))R^2 && \text{(63d)}\\
&+ i\pi\frac{1}{8}(\omega^2 - f^2)^2 R^2 && \text{(63e)}\\
&+ O(R^4 \ln R)\ ,
\end{aligned}
$$

where $\psi_\Gamma(\cdot)$ is the logarithmic derivative of the Γ function (Abramowitz & Stegun 1964, p.258), i.e. $\psi_\Gamma(\alpha) = (d/d\alpha)\ln\Gamma(\alpha)$. Expanding this in the variables r, θ, and substituting into (46a–46c) we get

$$\left\{\widetilde{\phi}, \widetilde{\psi}, \widetilde{h}\right\} = \left[\frac{4}{r^2}\right. \tag{64a}$$
$$+ \mathsf{F}^2(\omega^2 - f^2) \tag{64b}$$
$$- \mathsf{F}^4 \ln \mathsf{F} \tfrac{1}{4}(\omega^2 - f^2)^2 r^2 \tag{64c}$$
$$- \mathsf{F}^4 \tfrac{1}{4}(\omega^2 - f^2)^2 r^2 \ln r \tag{64d}$$
$$+ \mathsf{F}^4 \tfrac{1}{4}(\omega^2 - f^2)^2 r^2 \left(\ln 2 - \ln(\omega^2 - f^2)^{1/2}\right) \tag{64e}$$
$$+ \mathsf{F}^4 \frac{1}{8}(\omega^2 - f^2)^2 \left(\frac{3}{2} - 2\gamma\right) r^2 \tag{64f}$$
$$\left. + \, i\pi \frac{\mathsf{F}^4}{8}(\omega^2 - f^2)^2 r^2 \right] \tag{64g}$$
$$\times \frac{1}{\pi} \{\widetilde{A}, \widetilde{B}, \widetilde{C}\} e^{2i\theta}$$
$$+ O(\mathsf{F}^6 \ln \mathsf{F}) \ ,$$

where γ is Euler's constant, and the logarithm in (64e) is taken to be the principal branch, and with the branch of $(\omega^2 - f^2)^{1/2}$ taken according to (44). We now consider each of these terms, expressed in the time domain.

The term (64a) is consistent with the outer limit of the $O(1)$ flow in the vortical region by construction, since the amplitude of the flow in the wave region was determined by matching to the leading-order flow in the vortical region. The term (64b) can be shown to be consistent with the outer limit of the $O(\mathsf{F}^2)$ flow in the vortical region (56, 57, as discussed in section 3.6. We must now express terms (64c) – (64g) in the time domain, to provide asymptotic matching conditions on the flow in the vortical region up to $O(\mathsf{F}^4)$.

The coefficients of terms (64c) and (64d) can be expressed as a finite sum of integer powers of ω multiplying one of $\widetilde{A}, \widetilde{B}$, or $\widetilde{C}$. Upon taking the inverse Fourier tranform, we see that these coefficients, which depend on t, can be expressed in terms of a finite number of t derivatives acting on $\alpha(t)$, where $\alpha(t)$ is given by (54). Consequently, the straining flow at $O(\mathsf{F}^4 \ln \mathsf{F})$ at any time t, implied by (64c), can readily be expressed in terms of the leading-order flow in the vortical region, which can ultimately be determined from the potential vorticity at time t.

We must now consider (64e), (64f) and (64g). For (64e) we cannot represent the coefficients in terms of a finite sum of integral powers of ω multiplying $\widetilde{A}, \widetilde{B}$, or $\widetilde{C}$. Therefore, the inverse Fourier transform of these coefficients must be expressed as a convolution, the expression for which we obtain as follows:

First, it can readily be checked that taking the branch of $\ln(\omega^2 - f^2)^{1/2}$ defined by (44), we obtain

$$\ln(\omega^2 - f^2)^{1/2} = \ln|\omega^2 - f^2|^{1/2} - \frac{i\pi}{4}\mathrm{sgn}(\omega - f) - \frac{i\pi}{4}\mathrm{sgn}(\omega + f) + i\pi/2 \ . \tag{65}$$

Then, we combine the $\ln(\omega^2 - f^2)$ term in (64e) with (64g) and (for convenience) the γ term from (64f) to give

$$-\,\mathsf{F}^4(\omega^2 - f^2)^2 r^2 e^{2i\theta}\frac{1}{8\pi}\left(\ln|\omega - f| + \ln|\omega + f| + 2\gamma - \frac{i\pi}{2}((\mathrm{sgn}(\omega - f) + \mathrm{sgn}(\omega + f))\right)\widetilde{\sigma}(\omega)\ , \tag{66}$$

where $\widetilde{\sigma} = \{\widetilde{A}, \widetilde{B}, \widetilde{C}\}/\pi$.

The remaining terms in (64e) and (64f), in common with the terms (64c) and (64d) above, take the form of integral powers of ω multiplying known functions of time, and can therefore be obtained from the potential vorticity at time t.

The expression of (66) in the time domain now requires only some elementary manipulation of Fourier transforms. For details of the analysis, see appendix B. The result is that when (66) is converted back into the time domain, it produces an integral in the matching condition that depends on the past history of the dynamics.

When the terms (64c) – (64g) are taken all together, the asymptotic matching condition for ϕ_4, ψ_4 and h_4, and ϕ_{41}, ψ_{41} and h_{41} taken all together as $r \to \infty$ is

$$\begin{aligned}
&\mathsf{F}^4\{\phi_4,\, \psi_4,\, h_4\} + \mathsf{F}^4 \ln \mathsf{F}\{\phi_{41},\, \psi_{41},\, h_{41}\} = \\
&-\,\mathsf{F}^4\ \mathrm{Re}\left[r^2 e^{2i\theta}\int_{-\infty}^{t}\ln(t - t')\frac{d}{dt'}\left[\cos(f(t - t')) \times \{g_1(t'),\, g_2(t'),\, g_3(t')\}\right]dt'\right] \\
&-\,\mathsf{F}^4 \ln \mathsf{F}\ \mathrm{Re}\left[4r^2 e^{2i\theta}\{g_1(t'),\, g_2(t'),\, g_3(t')\}\right] \\
&+\,\mathsf{F}^4\ \mathrm{Re}\left[(3 - 4\ln(r/2))r^2 e^{2i\theta}\{g_1(t'),\, g_2(t'),\, g_3(t')\}\right] \\
&+\,O(r)\ ,
\end{aligned} \tag{67}$$

where

$$\{g_1(t), g_2(t), g_3(t)\} = \frac{1}{16\pi}\left(\frac{d^2}{dt^2} + f^2\right)^2\{A(t), B(t), C(t)\}\ . \tag{68}$$

and $A(t)$, $B(t)$ and $C(t)$ are defined by (55). The contribution $\boldsymbol{u}_4^{(\mathrm{rad})}$ mentioned in section 3.2 is that given by the contribution to $\nabla_x\phi_4 + \boldsymbol{k} \times \nabla_x\psi_4$ from the second line of (67).

Here at last, then, is the first nontrivial effect of gravity wave radiation, i.e., the first departure from high-order potential vorticity inversion anticipated in section 3.2. At this order, $O(\mathsf{F}^4)$, the effect of the radiation on the vortical flow is to induce a contribution to the large-scale straining flow described by the first line of (67). At time t this contribution cannot be computed from the potential vorticity at time t alone, but instead requires, as the time integral shows, knowledge of the potential vorticity field for the entire history of the vortical flow.

This $O(\mathsf{F}^4)$ straining flow in the vortical region can, and must, cause the energy in the vortical region to decrease in a way that is consistent with the radiation of energy by the waves. Recall that an $O(\mathsf{F}^2)$ wave radiates energy away from the vortical region at a rate $O(\mathsf{F}^2)^2 = O(\mathsf{F}^4)$. As a further check, the energy loss was explicitly calculated both ways for simple cases, that of a Kirchhoff ellipse (Lamb 1932; Ford 1994b; Zeitlin 1988, 1991) and a pair of point vortices. Details are omitted for brevity.

There is one further contribution to the radiation reaction, namely the next, $O(r)$ contribution to (67), again omitted for brevity, which matches to the the dipolar wave at $O(\mathsf{F}^3)$ in the wave region discussed at the end of section 3.6. This does not affect the energetics at $O(\mathsf{F}^4)$. It arises from matching the solution in the vortical region to the $O(\mathsf{F}^3)$ dipolar wave, and takes the form of a uniform flow across the vortical region. It may be computed in precisely the same way as the straining flow above, also requiring the evaluation of a history integral.

4 Concluding remarks

We have shown how the matched-asymptotic expansion method, based on the Froude number F as the small parameter, can be used to describe the spontaneous emission or radiation of inertia–gravity waves by unsteady vortical flow. The results add powerfully to the evidence against the existence of a strict slow manifold within the full phase space, and, furthermore, show explicitly how the spontaneous emission works and how, and at what order, the associated radiation reaction interferes with what would otherwise look like high-order potential-vorticity inversion.

The radiated inertia–gravity wave field is of quadrupole type, as expected from Lighthill's heuristic arguments (recall section 2). To justify this conclusion, however, a rather extensive analysis was needed, essentially because of the long-range character of the interactions as reflected, for instance, in the spatial divergence of certain integrals, and in the appearance of logarithmic terms in the expansions. The technicalities become more delicate than one would expect, because of having to go from 3 dimensions to 2, and from a non-rotating to a rotating system. The radiation reaction on the vortical region takes the form of a large-scale straining flow at $O(\mathsf{F}^4)$, the first term on the right of (67), the "history integral", which is the first term that affects the energetics of the flow in the vortical region and interferes with PV invertibility.

Potential-vorticity invertibility holds up to $O(\mathsf{F}^4 \ln \mathsf{F})$. That is, the terms in the velocity field at $O(1), O(\mathsf{F}^2 \ln \mathsf{F}), O(\mathsf{F}^2), O(\mathsf{F}^4 \ln^2 \mathsf{F})$ and $O(\mathsf{F}^4 \ln \mathsf{F})$ at any time t can be determined diagnostically from the potential vorticity at time t, and the result has all the usual properties of potential-vorticity inversion including the sign-reversal property.

So far we have said nothing about the possible existence of a generalized slow manifold, as distinct from an strict slow manifold. However, the nature of the problem is hinted at by the history integral in (67). The interpretation of the integral requires some care. If we proceed on the assumption that the functions $g_i(t')$ will remain $O(1)$ over all times $t' < t$, then the integrand in (67) does not generally decay, as $t' \to -\infty$, and, at first sight, there is no reason to suppose that the integral will converge.

This difficulty can be avoided as follows. Let us suppose that the source, represented by $g_i(t)$, was somehow "turned on" at time $t' = t_0$, so that we set $g_i(t') = 0$ for $t' < t_0$. Note in particular that this means that $g_i(t') = 0$ as $t' \to t_0^-$. Furthermore, for $t' > t_0$, $g_i(t')$ is a smooth function of t', given by (68), and will not necessarily be zero as $t' \to t_0^+$. Therefore, when we evaluate $(d/dt')(\cos(f(t-t'))g_i(t'))$, we must include a Dirac delta function at $t' = t_0$. More precisely, we write

$$\frac{d}{dt'}\cos(f(t-t'))g_i(t') = (\cos(f(t-t'))g_i(t'))' + \cos(f(t-t_0))g_i(t_0)\delta(t'-t_0)\ . \tag{69}$$

Thus, the integral in (67) can be re-written as

$$\begin{aligned}
&\int_{t_0}^{t} \ln(t-t')\frac{d}{dt'}\left[\cos(f(t-t')g_i(t'))\right]\, dt' \\
&= \mathrm{Lim}_{\epsilon\to 0^+}\left[\ln(t-t')\cos(f(t-t'))g_i(t')\right]_{t'=t_0}^{t'=t-\epsilon} \\
&+ \ln(t-t_0)\cos(f(t-t_0))g_i(t_0) + \mathrm{Lim}_{\epsilon\to 0^+}\int_{t_0}^{t-\epsilon}\frac{dt'}{t-t'}\cos(f(t-t'))g_i(t') \\
&= \mathrm{Lim}_{\epsilon\to 0^+}\left\{\ln\epsilon\cos(f\epsilon)g_i(t-\epsilon) + \int_{t_0}^{t-\epsilon}\frac{dt'}{t-t'}\cos(f(t-t'))g_i(t')\right\}.
\end{aligned} \tag{70}$$

Now, as $t_0 \to -\infty$, the last integral converges, on the assumption that $\cos(f(t-t'))g_i(t')$ has zero mean. This assumption might appear to be very restrictive. However, we recall that the $g_i(t)$ are given by (68). The form of (68) implies that none of the $g_i(t)$ will have any contribution at the Coriolis frequency f, and therefore $\cos(f(t-t'))g_i(t')$ will have zero mean, unless A, B and C exhibit secular growth in time. The absence of such secularity can be proven if the relative vorticity in the vortical region is single-signed, but not otherwise. In the latter case it is possible for vortex pairs to form and propagate out of the region, violating the assumption that the vortical region is confined to a finite-sized region on the scale L. Therefore, the last line in (70), an alternative form of the history integral, is a convergent integral representation of the effect of gravity wave radiation on the flow that generates that radiation, provided that the assumption of a finite vortical region remains valid for all times.

The last integral in (70) can evidently be computed if the full time-history of the evolving potential vorticity field is known. Moreover, if the time-history of the potential vorticity field is known for times $t > t_1 > t_0$, say, then last integral in (70) can be computed correct to $O(|t_1|^{-1})$, because the kernel $1/(t-t')$ implies that contributions from the more distant past (the integral from t_0 to t_1), are asymptotically smaller. This in turn might be thought to suggest the possible existence of a generalized slow manifold, because one can imagine obtaining past from present potential vorticity distributions by integrating backwards in time, advecting the potential vorticity with the direction of flow reversed. For instance, if we take $\mathsf{F}^{-1} \ll |t_1| \ll \mathsf{F}^{-2}$ then it is sufficient to use the leading-order velocity field given by (13) to advect the potential vorticity, because of the second condition; and the integral is obtained with error $O(\mathsf{F})$, because of the first condition. This means that the integral at time t can be computed to leading order, in principle, from the potential vorticity distribution at time t, provided that the backward time integration is feasible.

However, such backward time integration is not possible in practice, and may well not be possible in principle either, over asymptotically long time intervals $|t_1| \gg \mathsf{F}^{-1}$, owing to the chaotic nature of most unsteady vortex flows and to the time-reversed anti-cascade of enstrophy from small to large scales when time runs backwards. If the problem were to be modified by adding diffusion then the situation would be even worse: any diffusive process would, as is well known, make backwards time integration catastrophically ill-conditioned. These considerations, together with those about coupled oscillators mentioned in section 1, strongly suggest the non-existence of a single, well defined generalized slow manifold — despite the appearance of the last integral in (70).

The important point in the foregoing is that one cannot integrate the equations arbitrarily far backwards in time, as would be necessary for a unique and unambiguous evaluation of the history integral. In other words, there is no escape from evaluating the history integral using some finite turn-on time t_0. There are infinite numbers of choices for t_0. This is exactly what one expects from the fuzziness of stochastic layers in phase space: there are infinite numbers of neighboring "leaves" each of which looks locally like a generalized slow manifold but none of which has any privileged status — making potential-vorticity inversion correspondingly fuzzy.

We conclude that the stochastic-layer hypothesis is strongly supported by our analysis. In other words, it appears that neither a strict slow manifold nor a unique generalized slow manifold exist. As Warn (1997) and others have already argued in other ways, it seems practically certain, then, that the entity traditionally called the slow manifold — whose practical usefulness is not in question — is not, in fact, a manifold. We therefore suggest, for the sake of continuity with the traditional terminology, that this entity might be referred to as the slow quasi-manifold.

Appendices

A The determination of $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$

In this appendix, we show how to use asymptotic matching conditions to obtain the expressions (37) for $\phi_2^{\rm c}$, $\psi_2^{\rm c}$ and $\psi_{21}^{\rm c}$. We do this by matching the $\partial_0/\partial t$-wise time-independent part of the flow in the vortical region, at $O(\mathsf{F}^2)$, in the limit $r \to \infty$, to the $\partial_0/\partial t$ time-independent flow in the wave region.

In the limit $r \to \infty$, we see from analysis of (35) that ψ in the vortical region at $O(\mathsf{F}^2)$ is

$$\begin{aligned}\mathsf{F}^2\psi_2(\boldsymbol{x},t)_{\rm time-independent} \sim &+\frac{f^2}{8\pi}\mathsf{F}^2r^2(\ln r-1)\iint_{R^2}\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}' + \frac{f}{4}\mathsf{F}^2r^2h_0^{\rm c}\\ &-\frac{f^2}{8\pi}\mathsf{F}^2(2\ln r-1)x_i\iint_{R^2}x_i'\widehat{q}(\boldsymbol{x}',t)d^2\boldsymbol{x}' \\ &+\mathsf{F}^2\times O(\ln r)\ .\end{aligned} \tag{71}$$

This is required to match onto the $\partial_0/\partial t$ time-independent flow in the wave region, which must be expanded to $O(\mathsf{F}^2)$, and expressed in terms of the vortical coordinate r. We observe that, when $r = O(\mathsf{F}^{-1})$, the terms displayed in (71) are $O(\ln\mathsf{F})$, $O(1)$, $O(\mathsf{F}\ln\mathsf{F})$ and $O(\mathsf{F})$. We also have, at these orders, contributions from ψ_{21}:

$$\mathsf{F}^2\ln\mathsf{F}\psi_{21} \sim \tfrac{1}{4}h_0^{\rm c}\mathsf{F}^2\ln\mathsf{F}r^2 + O(\mathsf{F}^2\ln\mathsf{F})\times\ln r\ . \tag{72}$$

Here we observe that, when $r = O(\mathsf{F}^{-1})$, the term displayed in (72) is $O(\ln\mathsf{F})$. The remaining terms in (71) and (72) are asymptotically smaller, when $r = O(\mathsf{F}^{-1})$, than the terms displayed.

When $r = O(\mathsf{F}^{-1})$, the terms $O(\ln\mathsf{F})$ and $O(1)$ from (71) and (72), taken together, must match to the $O(R^2)$ and $O(R^2\ln R)$ terms from the $O(1)$-solution Ψ_0 in the wave region in the limit $R\to 0$. Similarly, the terms $O(\mathsf{F}\ln\mathsf{F})$ and $O(\mathsf{F})$ from (71) must match to the $O(R)$ term from the $O(\mathsf{F})$ solution Ψ_1 in the wave region. The remaining terms from (71) and (72) must match to term in the wave region that are $O(\mathsf{F}^2\ln^2\mathsf{F})$ or smaller.

We shall consider first match between the terms which are $O(R^2)$ and $O(R^2\ln R)$ in the wave region. We expand the $O(1)$ solution in the wave region in the limit $R\to 0$, and express it in terms of the vortical coordinate r. We find

$$\Psi_0 \sim C_0\tfrac{1}{4}f^2R^2\left(1+\ln 2-\ln R-\ln f-\gamma\right) = \mathsf{F}^2r^2C_0\tfrac{1}{4}f^2(1+\ln 2-\ln R-\ln\mathsf{F}-\ln f-\gamma)\ , \tag{73}$$

while in the vortical region, in the limit $r\to\infty$, the terms that are $O(\ln\mathsf{F})$ and $O(1)$ from (71) and (72) combined yield

$$\mathsf{F}^2\psi_2+\mathsf{F}^2\ln\mathsf{F}\psi_{21} \sim -\frac{f^2}{4}R^2C_0(\ln r-1)-\tfrac{1}{4}R^2f^2C_0(\ln f+\gamma-\ln 2)-\tfrac{1}{4}fR^2\ln\mathsf{F}fC_0\ . \tag{74}$$

It can be verified that (73) and (74) are identical, and hence the expressions for ψ_2 and ψ_{21}, combined, when $r = O(\mathsf{F}^{-1})$, match to the appropriate part of the solution in the wave region at $O(1)$. There is therefore no need to modify ψ_2 in the vortical region as a result of matching these terms.

We now consider the terms that are of $O(\mathsf{F}R)$ in the wave region as $R \to 0$. We expand the $O(\mathsf{F})$ solution in the wave region in the limit $R \to 0$, and express it in terms of the vortical coordinate r. We find

$$\mathsf{F}\Psi_1 \sim C_1 f \mathsf{F}^2 r \left(\tfrac{1}{2}\ln f + \tfrac{1}{2}\ln r + \ln \mathsf{F} + \tfrac{1}{2}\gamma + \tfrac{9}{4} - \tfrac{1}{2}\ln 2\right) e^{i\theta} \ . \tag{75}$$

From the $O(\mathsf{F}^2)$ flow in the vortical region we must consider the terms that are of $O(\mathsf{F}^2 r)$, $O(\mathsf{F}^2 r \ln r)$ and $O(\mathsf{F}^2 \ln \mathsf{F} r)$ in the limit $r \to \infty$, since it is these terms that are $O(\mathsf{F})$ when $r = O(\mathsf{F}^{-1})$. We find

$$\mathsf{F}^2\psi_2 + \mathsf{F}^2 \ln \mathsf{F}\psi_{21} \sim \tfrac{1}{4}C_1 f \mathsf{F}^2 r(2\ln r - 1)e^{i\theta} + \mathsf{F}^2\psi_2^{\mathrm{c}} + \mathsf{F}^2 \ln \mathsf{F}\psi_{21}^{\mathrm{c}} \ . \tag{76}$$

The asymptotic matching condition now requires that (75) and (76) agree, and we must add to ψ_{21} and ψ_2 terms to ensure this agreement. Since ψ_2^{c} and ψ_{21}^{c} both satisfy Laplace's equation, we can see from (75) and (76) that the added terms will be of form $Cre^{i\theta}$, where C is a constant. The asymptotic matching condition determines the value of C in each case. It is straightforward to show that

$$\psi_2^{\mathrm{c}} = \tfrac{1}{2}C_1 f(5 + \gamma + \ln f - \ln 2)re^{i\theta} \ , \qquad \psi_{21}^{\mathrm{c}} = \tfrac{1}{2}C_1 f r e^{i\theta} \ , \qquad \phi_2^{\mathrm{c}} = 0 \ , \tag{77}$$

where C_1 is given by (29). Notice that the terms of form $R\ln R$ in (75) and (76) agree, which is essential, since we cannot add to ψ_2 a term that satisfies Laplace's equation, with asymptotic form $r \ln r e^{i\theta}$ as $r \to \infty$. We take $\phi_2^{\mathrm{c}} = 0$, since there is no contribution to ϕ in either region at the order considered, and so the matching of the velocity is ensured by the matching of ψ.

In principle, we could now continue to consider the t-independent terms at $O(\mathsf{F}^2)$ and $O(\mathsf{F}^3)$ in the wave region (i.e., to obtain equations for Ψ_2, Ψ_3, H_2 and H_3). However, since ψ_2 and ψ_{21} satisfy Laplace's equation, the asymptotic matching condition applied at $O(\mathsf{F}^2)$ can, at most, require constants to be added to ψ_2^{c} and ψ_{21}^{c}, and no further contributions to ψ_2^{c} and ψ_{21}^{c} can arise from application of the matching condition at any higher order. Moreover, since the matching condition applies to the velocity fields $\boldsymbol{u}$ and $\mathsf{F}\boldsymbol{U}$, and since the constants in ψ and Ψ are irrelevant to the determination of the corresponding velocity fields, the expressions given above for $\phi_2^{\mathrm{c}}, \psi_2^{\mathrm{c}}$ and ψ_{21}^{c}, when used in (34), (35) and (36), completely determine the flow in the vortical region up to $O(\mathsf{F}^2)$.

B Inverse Fourier transform of (66)

The purpose of this appendix is to obtain the inverse Fourier transform or

$$\left(\ln|\omega - f| + \ln|\omega + f| + 2\gamma - \frac{i\pi}{2}(\mathrm{sgn}(\omega + f) + \mathrm{sgn}(\omega - f))\right)\tilde{s}(\omega) \ , \tag{78}$$

where $s(t)$ is a arbitrary function of time, with Fourier transform $\widetilde{s}(\omega)$. We shall denote the Fourier transform operator by $\mathcal{L}_\omega$, so that $\widetilde{s}(\omega) = \mathcal{L}_\omega\{s(t)\}$.

First, it proves convenient to obtain the Fourier transform of $\ln tH(t)$, where $H(t)$ is the Heaviside step function. To do this, we first consider the Fourier transform of $t^\alpha H(t)$ for $0 < \alpha < 1$. Following Lighthill (1958), we find that the Fourier transform of $t^\alpha H(t)$ is $\Gamma(\alpha + 1)/(-i\omega)^{-(1+\alpha)}$, where Γ is the Gamma-function (see, for example, Abramowitz and Stegun, 1964). We then observe that

$$\frac{d}{d\alpha} t^\alpha H(t) = t^\alpha \ln tH(t) \ . \tag{79}$$

Thus, to obtain the Fourier transform of $\ln tH(t)$ we must compute the derivative, with respect to α, of the Fourier transform of $t^\alpha H(t)$, and evaluate this at $\alpha = 0$. After a few lines of algebra, we find that

$$\frac{d}{d\alpha}\left[\frac{\Gamma(\alpha+1)}{(-i\omega)^{(\alpha+1)}}\right] = \frac{\Gamma(\alpha+1)}{(-i\omega)^{\alpha+1}}\left(\psi(\alpha+1) - \ln|\omega| + \frac{i\pi}{2}\mathrm{sgn}(\omega)\right) \ . \tag{80}$$

Thus, setting $\alpha = 0$, we have

$$\mathcal{L}_\omega\{\ln tH(t)\} = \frac{1}{i\omega}\left(\ln|\omega| + \gamma - \frac{i\pi}{2}\mathrm{sgn}(\omega)\right) \ . \tag{81}$$

Using shifting formulae, we have the further results

$$\mathcal{L}_\omega\{e^{\pm ift}\ln tH(t)\} = \frac{1}{i(\omega \pm f)}\left(\ln|\omega \pm f| + \gamma - \frac{i\pi}{2}\mathrm{sgn}(\omega \pm f)\right) \ . \tag{82}$$

Thus, the inverse Fourier transform of

$$\left(\ln|\omega - f| + \ln|\omega + f| + 2\gamma - \frac{i\pi}{2}(\mathrm{sgn}(\omega+f) + \mathrm{sgn}(\omega - f))\right)\widetilde{s}(\omega) \tag{83}$$

is given by

$$\left\langle e^{ift}\ln tH(t), (-d_t + if)s\right\rangle + \left\langle e^{-ift}\ln tH(t), (-d_t - if)s\right\rangle \ , \tag{84}$$

where the angle brackets denote a convolution. Adding these together, we obtain

$$\begin{aligned} &\mathcal{L}_\omega\left\{\int_0^\infty 2\ln t' \frac{d}{dt'}(\cos(ft')s(t-t'))dt'\right\} \\ &= \left(\ln|\omega - f| + \ln|\omega + f| + 2\gamma - \frac{i\pi}{2}(\mathrm{sgn}(\omega+f) + \mathrm{sgn}(\omega - f))\right)\widetilde{s}(\omega). \end{aligned} \tag{85}$$

Acknowledgements

We thank J. S. Allen, O. Bühler, D. G. Crighton, M. J. P. Cullen, D. G. Dritschel, P. A. Glendinning, P. H. Haynes, R. S. MacKay, T. G. Shepherd, J. J. Tribbia, and V. Zeitlin for helpful comments and correspondence. This work received support through the US Office of

Naval Research, the UK Natural Environment Research Council through the UK Universities' Global Atmospheric Modelling Project (grant GR3/6516), the Isaac Newton Institute for Mathematical Sciences, a NERC studentship to RF, and a SERC/EPSRC Senior Research Fellowship to MEM. Two anonymous reviewers provided further perceptive and helpful comments.

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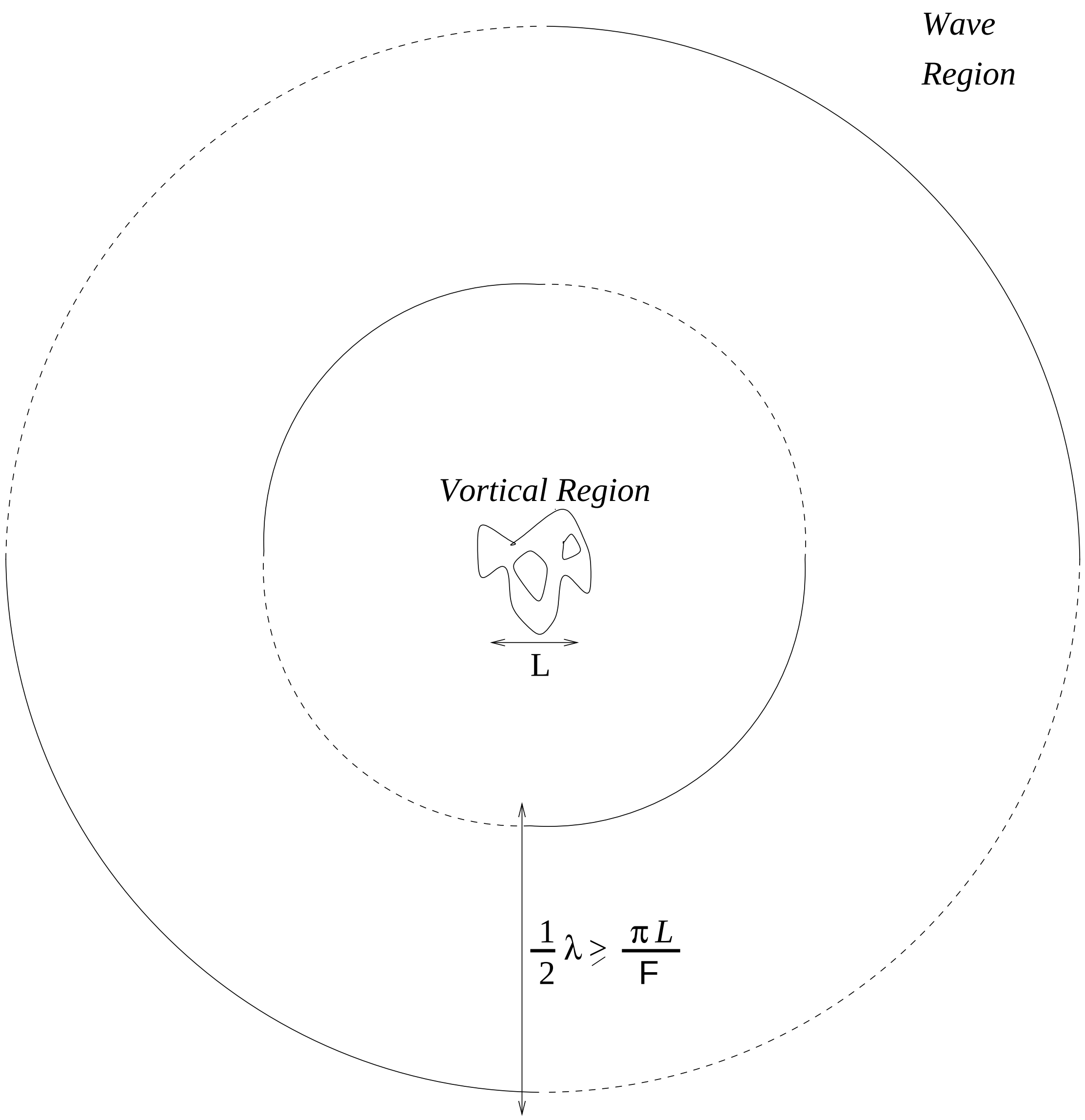
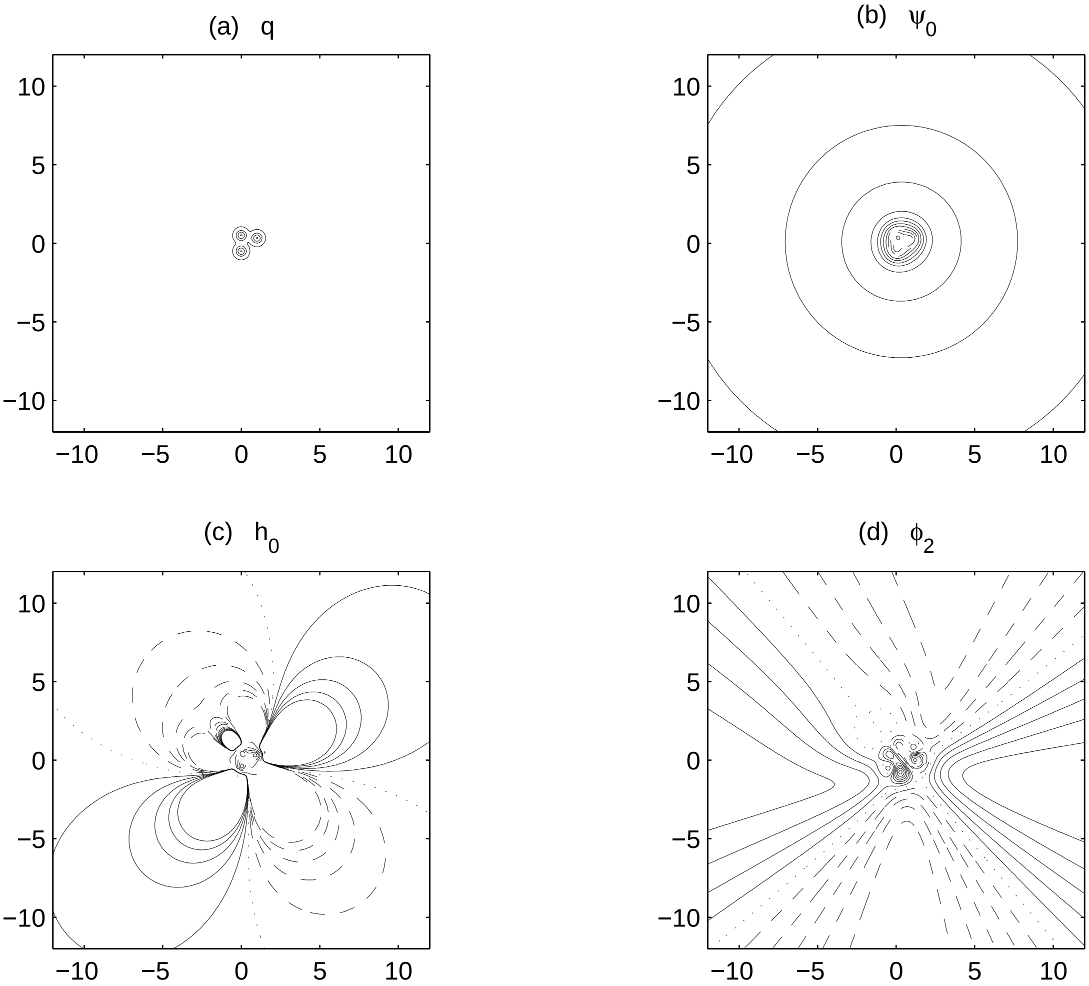
Figure Captions

Fig. 1. A schematic picture showing the vortical region and the surrounding wave region. The vortical region is characterized by length scale L, and the wave region by the larger scale L/F ($\mathsf{F} \ll 1$). The solid curves in the vortical region represent potential-vorticity contours. The circular arcs in the wave region represent wavecrests; solid and dashed lines are supposed to represent positive and negative divergence respectively. The 180° rotational symmetry of the wave field is consistent with radiation from a quadrupole source, as predicted by the Lighthill theory.

Fig. 2. Fields in the inner or vortical region for the simplest, non-rotating case (infinite Rossby number): (a) the PV q, Eq. (6), together with (b) the corresponding leading-order streamfunction ψ_0, Eq. (13), (c) height field h_0, Eq. (15), and (d) velocity potential ϕ_2, Eq. (34). The contour values are (a) 0.5 to 9.5 in increments of 3; (b) -0.75 to 0.55 in increments of 0.2, then 1 to 4 in increments of 1; (c) -2.25 to 0.25 in increments of 0.5, and -0.005 to 0.005 in increments of 0.001; (d) -0.07 to 0.07 in increments of 0.01. In all cases, solid lines indicate positive values, dashed lines indicate negative values, and dotted lines in (c) and (d) are the zero contour. The quadrupolar pattern in ϕ_2, see the far field expression (56), matches the velocity potential Φ of the radiating gravity waves in the surrounding outer or wave region, not shown.



Wave
Region
Vortical Region
L
$\frac{1}{2}\lambda \gtrsim \frac{\pi L}{F}$



(a) q
(b) ψ0
(c) h0
(d) φ2
10
5
0
−5
−10