Appendix A. JWKB solution: advection-diffusion-dilution equation

For the purposes of section 3d and elswhere, we consider approximate solutions to (2) applied to $\hat{H}$, so that $S = 0$. Regarding the tape signal as a slowly modulated sinusoid, so that JWKB theory will be applicable, we assume a solution of the form $\chi(z,t) = \mathrm{Re}A(Z)\exp[i(\phi(z) - \omega t)]$ where $Z = \mu z$ is a slow height variable, $A(Z)$ is a slowly varying amplitude, and ϕ is now defined by $\phi_z = m(Z)$, so that m, though not ϕ, is assumed slowly varying. As in section 3c, we may take both A and ϕ to be real; note that, apart from restricting it to be a function of Z alone, hence slowly varying, A is the same as the $f(z)$ of section 3c. Defining $\hat{\phi} = \phi(z) - \omega t$, we have

$$\chi_z = \mathrm{Re}\left\{(\mu A' + imA)\exp i\hat{\phi}\right\} \tag{A.1a}$$

$$\chi_{zz} = \mathrm{Re}\left\{(\mu^2 A'' + 2im\mu A' + im'\mu A - m^2 A)\exp i\hat{\phi}\right\}\ , \tag{A.1b}$$

where primes denote differentiation with respect to Z, so that (2) with $S = 0$ becomes

$$-i\omega A + \tilde{w}(\mu A' + imA) = K(\mu^2 A'' + 2im\mu A' + im'\mu A - m^2 A) - \alpha A \tag{A.2}$$

Forward solution of this equation when $\tilde{w}$ and K are prescribed functions of Z alone proceeds with an expansion of A in powers of μ:

$$A = A_0 + \mu A_1 + \mu^2 A_2 + \cdots \tag{A.3}$$

so that, at zeroth order in μ,

$$-i\omega A_0 + im\tilde{w}A_0 = -(Km^2 + \alpha)A_0. \tag{A.4}$$

For real m it will be assumed that Km^2/ω and α/ω are $O(\mu)$ whence

$$m\tilde{w} = \omega \tag{A.5}$$

This is the *approximate* JWKB counterpart, at this order, of the *exact* relation (5). In general, the three terms of $\tilde{w} = w + (K/H) - \mu K'$ contribute to the apparent upwelling, including terms involving K, but under the JWKB approximation K is small so to leading order m is determined by w. At first order in μ, we then have

$$\tilde{w}A_0' = -(Km^2 + \alpha)A_0 \tag{A.6}$$

for the slowly varying amplitude, with $\tilde{w} \approx w$.

According to (A.5, A.6), vertical advection has two effects on the oscillatory solution. At zeroth order, it creates a wavy vertical structure, translating information from the lower boundary upward along characteristics

$$\left(\frac{dz}{dt}\right)_\chi \approx \tilde{w}. \tag{A.7}$$

At first order, advection maintains the amplitude envelope of χ against attenuation due to vertical diffusion of the wavy vertical structure and linear damping.