Abstract

In this mini-project, the Vlasov–Poisson system is studied in its form which is used to describe time evolution of a galaxy. The main properties are carefully revisited and a few results on nonlinear stability are given. The major part of the report is dedicated to the concentration-compactness principle, which can be used in more general context in order to study stability of ground states in partial differential equations. Also, the concentration-compactness principle applied to the nonlinear Schrödinger equation is given. Finally, a few important recent results on the nonlinear stability of the Vlasov–Poisson system are revisited, which use the notion of rearrangements.

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Introduction

Describing galaxy configurations which are stable over time is an interesting and mathematically challenging problem from astrophysics. There are around $10^{11}$ gravitationally interacting stars in a typical galaxy and keeping track of the dynamics of each of them is not feasible. Instead, a statistical description of a galaxy is employed so that one can equivalently study an evolution of the statistical distribution of a $N$-body system for large $N$, i.e. an evolution of a galaxy for large number of stars.

This statistical approach is characteristic for the kinetic theory and it was first applied to the problem of describing a collisional gas, which gave rise to the Boltzmann equation in 1860’s. However, as opposed to the particles in a gas, collisions among stars in a galaxy are sufficiently rare so that they can be neglected. Here, bodies of a system, i.e. stars, interact at the distance and only by the gravitational field which they create collectively. Therefore, to such a system, the non-collisional or so-called the mean-field limit can be applied, discovered by Vlasov in 1930’s. As a result, the Vlasov–Poisson equation is obtained, which is a nonlinear reversible partial-differential equation typically used to describe time evolution of a plasma or galaxy.

In this mini-project, we study the Vlasov–Poisson system in its gravitational form. We are interested in the stability of its stationary solutions, which model the steady galaxy states. It was shown in [LMR08] that some of the galaxy models can be characterized as the ground states of a certain energy functional and therefore, we are interested in the stability of the minimizers of such a functional. As we explain later, this can be derived as a direct consequence of a result that all such minimizing sequences are compact. However, first, one had to find a tool to prove the desired compactness.

Such question of the orbital stability is typical for any PDE system. In particular, a very similar problem of analyzing the energy functional in order to derive the stability of its minimizers is recognized in some nonlinear Schrödinger equations, [CLS2]. The compactness of minimizing sequences in such circumstances was successfully proved in the Schrödinger and then in the Vlasov–Poisson equations by using the concentration-compactness principle introduced by Lions in [Lio84a, Lio84b]. More generally, one can use this principle to minimize a functional on a bounded and closed subset of an infinite-dimensional function space defined on an unbounded domain such as $\mathbb{R}^N$, and therefore, this is a very useful tool in variety of problems.

Let us mention here that the concentration-compactness was successfully used to prove only the orbital stability of certain group of steady galaxy models, so-called polytopic models, while for more general models different approach had to be applied. In 1960’s [Ant61, Ant65], Antonov resolved the problem of stability for more general models, but only in a linearized framework, and posed the conjecture of nonlinear stability of spherically-symmetric, isotropic and decreasing models. This conjecture of nonlinear stability was finally resolved by Lemou, Méhats and Raphaël in 2011 [LMR11b]. Still, in the absence of a spherical symmetry, the question of stability is open even for the linearized problem.

In Section 1, we revised the main features of the gravitational Vlasov–Poisson system, mainly by following the recent review paper [Mou12]. Section 2 is devoted to the concept of the concentration-compactness principle, which is firstly described in a simple setting and then a general framework is given. Also, we summarize how this method can be used to show the stability of solitary waves of some Schrödinger equation [CLS2] and the stability of polytropic states of the Vlasov–Poisson system [LMR08]. Finally, in the Section 3, we sketch the proof of the resolution of the conjecture given in [LMR11b], also by following the review paper [Mou12].
1 The gravitational Vlasov–Poisson system

1.1 From the Newtonian N-body problem to the Vlasov–Poisson system

Consider a system of $N$ bodies of the same mass normalized to 1. Each body is associated with an index $1 \leq i \leq N$, a position $x_i$ and a velocity $v_i$. Binary interactions are assumed through an interaction potential $\phi$, without an external field. For such a system, the Hamiltonian formulation of $N$-body problem is

$$
\begin{align*}
x_i'(t) &= v_i(t) = \frac{\partial H}{\partial v_i}(X(t), V(t)), \\
v_i'(t) &= -\sum_{i \neq j} \nabla_x \psi(x_i - x_j) = -\frac{\partial H}{\partial x_i}(X(t), V(t)),
\end{align*}
$$

for $1 \leq i \leq N$ and for the Hamiltonian

$$
H(X, V) = \sum_{i=1}^{N} \frac{|v_i|^2}{2} + \sum_{i<j} \psi(x_i - x_j),
$$

where $X = (x_1, \ldots, x_N)$, $V = (v_1, \ldots, v_N)$ for each $x_i, v_i \in \mathbb{R}^3$. This Hamiltonian $H$ is called the microscopic Hamiltonian. It is preserved along the evolution of the Newton equations (1.1), which we denote with $S_t$. Hence, one has

$$
\forall t \in \mathbb{R}, \quad H(S_t(X, V)) = H(X, V).
$$

Instead of this microscopic viewpoint, one can shift to a more statistical point of view and consider a joint distribution function for all particles $f^{(N)} = f^{(N)}(t, X, V)$. The claim is that $f^{(N)}$ evolves according to the following $N$-body Liouville equation

$$
\partial_t f^{(N)} + \sum_{i=1}^{N} \left( \frac{\partial H}{\partial v_i} \cdot \frac{\partial f^{(N)}}{\partial x_i} - \frac{\partial H}{\partial x_i} \cdot \frac{\partial f^{(N)}}{\partial v_i} \right) = \partial_t f^{(N)} + \{ f^{(N)}, H \} = 0,
$$

where $\{ \cdot, \cdot \}$ denotes Poisson’s brackets on $(\mathbb{R}^3)^N \times (\mathbb{R}^3)^N$ defined as

$$
\{ f^{(N)}, H \} = \nabla_X f^{(N)} \cdot \nabla_V g^{(N)} - \nabla_X g^{(N)} \cdot \nabla_V f^{(N)}.
$$

The equation (1.2) is derived from the ODE system (1.1) by imposing that the distribution $f^{(N)}$ is preserved when following the trajectories of each particle, i.e.

$$
\forall t \geq 0, \quad f^{(N)}(t, S_t(X, V)) = f^{(N)}(0, X, V).
$$

The solution of (1.2) satisfies the following conservation laws:

- the conservation of the sign

$$
f^{(N)}(0, X, V) \geq 0 \implies f^{(N)}(t, X, V) \geq 0, \quad \forall t \geq 0;
$$

- the conservation of the mass

$$
\int_{(\mathbb{R}^3)^N \times (\mathbb{R}^3)^N} f^{(N)}(t, X, V) \, dx \, dv = \int_{(\mathbb{R}^3)^N \times (\mathbb{R}^3)^N} f^{(N)}(0, X, V) \, dx \, dv, \quad \forall t \geq 0;
$$


- the conservation of any Casimir functional

\[
\int_{(\mathbb{R}^3)^N \times (\mathbb{R}^3)^N} C \left( f^{(N)} (t, X, V) \right) \, dx \, dv = \int_{(\mathbb{R}^3)^N \times (\mathbb{R}^3)^N} C \left( f^{(N)} (0, X, V) \right) \, dx \, dv, \quad \forall t \geq 0,
\]

where \( C \in C^1 (\mathbb{R}_+, \mathbb{R}_+) \), \( C (0) = 0 \).

To simplify the description of the system, one can introduce a distribution of one body

\[
f (t, x, v) := \int_{(\mathbb{R}^3)^{N-1} \times (\mathbb{R}^3)^N} f^{(N)} (t, X, V) \, dx_2 \ldots dx_N \, dv_2 \ldots dv_N,
\]

which is determined as a marginal distribution of \( f^{(N)} \). To obtain the equation which describes how the one-body distribution evolves, one integrates the Liouville equation (1.2) in \( x_2, v_2, \ldots, x_N, v_N \) variables and gets

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f - (N - 1) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_x \psi (x - y) \cdot \nabla_v f^{(2)} (x, y, v, v_*) \, dy \, dv_* = 0. \quad (1.3)
\]

This equation depends on the two-particle distribution \( f^{(2)} \), i.e the second marginal

\[
f^{(2)} (t, x_1, x_2, v_1, v_2) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} f^{(N)} \, dx_3 \ldots dx_N \, dv_3 \ldots dv_N,
\]

which contains the information on the correlations in the system.

Now, the goal is to take the limit as \( N \to \infty \) and obtain an equation on the first marginal only. The two following assumptions are required when taking the mean field limit, also called the Vlasov limit:

- each binary interaction is of the order \( O (1/N) \) and therefore the appropriate scaling of the interaction potential is

\[
\psi = \frac{\bar{\psi}}{N},
\]

which means that the influence of one star to another star is negligible across the entire galaxy, while the collective action of the entire galaxy on a given star is important and it is of the order \( O ((N - 1) / N) = O (1) \);

- the “chaos property”

\[
f^{(2)} \sim f \times f, \quad \text{as } N \to \infty,
\]

which indicates weak correlations in the system.

Under these assumptions, the following equation is obtained from (1.3) when \( N \to \infty \)

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \nabla_x \bar{\psi} (x - y) \cdot \nabla_v f (y, v_*) \, dy \, dv_* = 0,
\]

which is closed on the distribution \( f \) and can be rewritten as

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \left( \int_{\mathbb{R}^3} \nabla_x \bar{\psi} (x - y) \left( \int_{\mathbb{R}^3} f (y, v_*) \, dv_* \right) \, dy \right) \cdot \nabla_v f (x, v) = 0. \quad (1.4)
\]
In (1.4), if the microscopic potential is chosen such that it corresponds to the gravitational interactions

\[ \psi = -\frac{1}{4\pi|x|}, \]

one obtains the gravitational Vlasov–Poisson system

\[ \begin{align*}
\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f &= 0, & t \in \mathbb{R}_+, x \in \mathbb{R}^3, v \in \mathbb{R}^3, \\
f(t = 0, x, v) &= f_{in}(x, v),
\end{align*} \tag{1.5} \]

with the gravitational mean field \( \phi_f \) defined by

\[ \phi_f := -\frac{1}{4\pi|x|} \ast \rho_f, \quad \rho_f := \int_{\mathbb{R}^3} f(t, x, v) \, dv. \tag{1.6} \]

where \( \rho_f(t, x) \) gives the spatial density. Observe that the field \( \phi_f \) solves the Poisson elliptic equation

\[ \Delta \phi_f = \rho_f, \tag{1.7} \]

which explains the name of the Vlasov–Poisson system. We note here that the minus sign in the definition of the potential in (1.6) is due to the attractive interactions in the gravitational models. In the case when the Vlasov–Poisson system is used to describe the plasma, here we would have the plus sign, which corresponds to the repulsive interactions.

### 1.2 Properties of the gravitational Vlasov–Poisson system

The Vlasov–Poisson system (1.5)–(1.6) is nonlinear due to the mean-field term, i.e. due to \( \nabla_x \phi_f \). Nevertheless, it shares a similar structure with the Liouville equation (1.2), which is a linear PDE, in the following way. The Vlasov–Poisson equation can be written in the form

\[ \partial_t f + \{f, E_f\} = 0, \]

where \( \{\cdot, \cdot\} \) denotes the Poisson brackets on \( \mathbb{R}^3 \times \mathbb{R}^3 \) and

\[ E_f = E_f(t, x, v) = \frac{|v|^2}{2} + \phi_f(t, x) \]

defines the microscopic mean-field Hamiltonian. Also, note here, that the system (1.5)–(1.6) is associated with the solutions to the differential system

\[ \begin{align*}
x'(t) &= v(t) = \frac{\partial E_f}{\partial v}, \\
v'(t) &= -\nabla_x \phi_f(x(t)) = -\frac{\partial E_f}{\partial x}.
\end{align*} \tag{1.8} \tag{1.9} \]

Next, it is shown that the quantity called macroscopic Hamiltonian and given by

\[ H(f) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v|^2}{2} f(t, x, v) \, dx \, dv + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\phi_f(t, x)}{2} f(t, x, v) \, dx \, dv \]

is preserved along the evolution of the Vlasov–Poisson system (1.5)–(1.6). First, observe that it can be rewritten as follows

\[ H(f) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v|^2}{2} f(t, x, v) \, dx \, dv - \int_{\mathbb{R}^3} \frac{\left|\nabla_x \phi_f(t, x)\right|^2}{2} \, dx, \tag{1.10} \]
by using the definition of \( \rho_f \), the Poisson equation (1.7) and by applying the partial integration on \( \frac{1}{2} \int_{S^3} \phi f (t, x) \Delta \phi_f (t, x) \, dx \). Now, by differentiating the macroscopic Hamiltonian in time, one gets the following

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v|^2}{2} f \, dv \, dx + \int_{\mathbb{R}^3} \frac{\nabla_x \phi_f}{2} \, dx \right) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v|^2}{2} \frac{\partial f}{\partial t} \, dv \, dx - \int_{\mathbb{R}^3} \nabla_x \phi_f \cdot \nabla_x \frac{\partial \phi_f}{\partial t} \, dx
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v|^2}{2} \frac{\partial f}{\partial t} \, dv \, dx + \int_{\mathbb{R}^3} \phi_f \left( \nabla_x \frac{\partial \phi_f}{\partial t} \right) \, dx
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|v|^2}{2} \frac{\partial f}{\partial t} \, dv \, dx + \int_{\mathbb{R}^3} \phi_f \left( \frac{\partial f}{\partial t} \right) \, dx
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \frac{|v|^2}{2} + \phi_f \right) \frac{\partial f}{\partial t} \, dv \, dx
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} E_f (-v \cdot \nabla_x f + \nabla_x \phi_f \cdot \nabla_x f) \, dv \, dx
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\nabla_x \phi_f v f - v \cdot \nabla_x \phi_f f) \, dv \, dx = 0.
\]

Note that the Hamiltonian \( \mathcal{H} \) can be written as the difference of two positive terms, which we will denote as follows

\[
\mathcal{H}(f) = \mathcal{H}_{\text{kin}}(f) - \mathcal{H}_{\text{pot}}(f).
\]

Therefore, it is possible to have the situation when the total energy of the system \( \mathcal{H} \) stays constant over time, but the kinetic energy \( \mathcal{H}_{\text{kin}} \) and potential energy \( \mathcal{H}_{\text{pot}} \) diverge. That means, it is possible to have a transfer between kinetic and potential energy, which may diverge while balancing. This creates the mathematical difficulties when analyzing the stability of the system (1.5)–(1.6). It is not enough to have a control over the total energy, but the kinetic and the potential energy have to be controlled separately as well. Luckily, it is possible to prove a key interpolation inequality, which gives the control of the potential energy by the kinetic energy. Thus, it is sufficient to bound the kinetic energy. Let us mention here that by the analogy with the dispersive partial differential equations, in particular with the focusing nonlinear Schrödinger equation, where the Hamiltonian is also composed of two terms of opposite sign, the Vlasov-Poisson equation is called a focusing equation.

Before commenting more on these energy terms and proving the key interpolation inequality, let us mention another very important conservation law for the Vlasov–Poisson system (1.5)–(1.6). Namely, that for all Casimir functionals it is true that

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} C(f(t, x, v)) \, dv \, dx = \int_{\mathbb{R}^3 \times \mathbb{R}^3} C(f(0, x, v)) \, dv \, dx \quad \forall \; t \geq 0.
\]

This is verified by the following computation

\[
\frac{\partial}{\partial t} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} C(f(t, x, v)) \, dv \, dx \right) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} C'(f) \frac{\partial f}{\partial t} \, dv \, dx
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} C'(f) (-v \cdot \nabla_x f + \nabla_x \phi_f \cdot \nabla_x f) \, dv \, dx
\]

\[
= - \int_{\mathbb{R}^3} v \left( \int_{\mathbb{R}^3} \nabla_x C(f) \, dx \right) \, dv + \int_{\mathbb{R}^3} \nabla_x \phi_f \left( \int_{\mathbb{R}^3} \nabla_v C(f) \, dv \right) \, dx = 0,
\]
where one uses that the function $C$ vanishes when $f$ is equal to zero, i.e. when $x$ and $v$ go to infinity. This property of the conservation of a Casimir functional leads to the preservations of all $L^p$ norms for the solution $f$, i.e.

$$\|f(t,x,v)\|_{L^p} = \|f_{in}(x,v)\|_{L^p} \forall t \geq 0.$$  

The following classical result, gives is the key interpolation inequality, which says that the potential energy can be controlled by the kinetic energy and the invariant of the system, i.e. the $L^p$ norm of a solution $f$.

**Theorem 1.1.** For every $p \in [1, \infty]$, the integrability of the spatial density is estimated as follows

$$\|\rho f\|_{L^{\frac{2p-3}{p-1}}(\mathbb{R}^3)} \leq C \|f\|_{L^{p}(\mathbb{R}^6)}^{\frac{2p}{p-1}} \mathcal{H}_{kin}(f)^{\frac{3p-3}{p-1}},$$

and in the limiting case when $p = \infty$

$$\|\rho f\|_{L^\infty(\mathbb{R}^3)} \leq C \|f\|_{L^\infty(\mathbb{R}^6)} \mathcal{H}_{kin}(f)^{\frac{3}{2}},$$

for some constant $C > 0$.

The following control of the potential energy for $p > p_c = 9/7$ is deduced

$$\mathcal{H}_{pot}(f) \leq C \|f\|_{L^{\frac{6p-9}{2p-17}}(\mathbb{R}^6)} \|f\|_{L^p(\mathbb{R}^6)}^{\frac{3p-1}{p-1}} \mathcal{H}_{kin}(f)^{\frac{1}{2}},$$

for some constant $C > 0$.

**Proof.** For any $p > 1$ and $R > 0$, it is possible to decompose the spatial density in the following way

$$\int_{\mathbb{R}^3} f \, dv = \int_{|v| \leq R} f \, dv + \int_{|v| > R} f \, dv$$

$$\leq \|f\|_{L^p} \left( \int_{|v| \leq R} dv \right)^{\frac{p-1}{p}} + R^{-2} \int_{\mathbb{R}^3} f |v|^2 \, dv$$

$$\leq \|f\|_{L^p} \left( \frac{4\pi R^3}{3} \right)^{\frac{p-1}{p}} + R^{-2} \int_{\mathbb{R}^3} f |v|^2 \, dv.$$  

By the optimization over the parameter $R$, one obtains

$$\int_{\mathbb{R}^3} f \, dv \leq C \|f(x,\cdot)\|_{L^p(\mathbb{R}^3)}^{\frac{2p}{(p-1)} \left( \int_{\mathbb{R}^3} f |v|^2 \, dv \right)^{\frac{3p-3}{(p-1)}}},$$

and then by the Cauchy–Schwarz inequality

$$\|\rho f\|_{L^{\frac{5p-3}{5p-1}}(\mathbb{R}^3)} \leq C \left( \int_{\mathbb{R}^3} \|f(x,\cdot)\|_{L^p(\mathbb{R}^3)}^{\frac{2p}{(p-1)} \left( \int_{\mathbb{R}^3} f |v|^2 \, dv \right)^{\frac{3p-3}{(p-1)}}} \, dx \right)^{\frac{3p-1}{(p-1)}}$$

which gives the first inequality from the statement of the theorem.

To obtain the control on the potential energy, we need the inequality

$$\|\nabla x f\|_{L^2(\mathbb{R}^3)} \leq C \|\rho f\|_{L^{6/5}(\mathbb{R}^3)},$$
which can be justified from the Poisson equation (1.7), by using the formula \( \nabla_x \phi_f = \frac{x}{4 \pi |x|^3} \ast \rho_f \) and the Hardy–Littlewood–Sobolev inequality. Therefore, one has
\[
\mathcal{H}_{\text{pot}}(f) = \|\nabla_x \phi_f\|_{L^2(\mathbb{R}^3)}^2 \leq C\|\rho_f\|_{L^{\theta/5}(\mathbb{R}^3)}^2.
\]
For any \( p > p_c = 9/7 \), one has \((5p - 3)/(3p - 1) > 6/5 \) and then it is possible to write the following
\[
\|\rho_f\|_{L^{\theta/5}(\mathbb{R}^3)} \leq \|\rho_f\|_{L_1^6(\mathbb{R}^3)} \|\rho_f\|_{L^{5p-3}_{5p-12}(\mathbb{R}^3)} \\
\leq \|f\|_{L_1^6(\mathbb{R}^3)} \|\rho_f\|_{L^{5p-9}_{5p-12}(\mathbb{R}^3)}.
\]
By combining all this, together with the first inequality from the statement, one gets
\[
\mathcal{H}_{\text{pot}}(f) \leq C\|f\|_{L_1^6(\mathbb{R}^6)}^2 \|f\|_{L^p_{\text{kin}}(\mathbb{R}^6)} \mathcal{H}_{\text{kin}}(f)^{\frac{1}{2}},
\]
for any \( p > p_c = 9/7 \), which finishes the proof.}
1.3 Steady galaxy configurations and orbital stability

A stationary solution $f^0$ of the Vlasov–Poisson system (1.5)-(1.6) satisfies

$$\forall x, v \in \mathbb{R}^3, \quad v \cdot \nabla_x f^0 - \nabla_x U(x) \cdot \nabla_v f^0 = 0.$$  \hspace{1cm} (1.13)

Additionally, if a stationary solution $f^0$ is spherical symmetric, it is invariant with respect to the referential rotations, i.e.

$$\forall (x, v) \in \mathbb{R}^6, \quad f^0(Ax, Av) = f^0(x, v)$$

for all orthogonal matrices $A \in \mathbb{R}^{3 \times 3}$. Equivalently, one has that

$$f^0 = f^0(|x|, |v|, x \cdot v)$$

must hold.

In the Vlasov–Poisson system with the spherical symmetry, the Poisson equation (1.7) becomes

$$\frac{1}{r^2} (r^2 U')' = \int_{\mathbb{R}^3} f^0(|x|, |v|, x \cdot v) \, dv,$$

where the potential $U$ depends only on $r = |x|$. In this case, the equations of the motion of the system (1.13) are

$$x'(t) = v,$$

$$v'(t) = -\nabla_x U = -\frac{x}{|x|} U'(|x|),$$

and therefore, it follows that

$$\frac{d}{dt} (x(t) \wedge v(t)) = (x'(t) \wedge v(t)) + (v'(t) \wedge x(t))$$

$$= (v(t) \wedge v(t)) - \left( \frac{U'(|x|)}{|x|} x(t) \wedge x(t) \right)$$

$$= 0.$$

Beside the macroscopic Hamiltonian, this is another invariant of the system. Since the phase space of the spherical system has dimension three, i.e. $(|x|, |v|, x \cdot v) \in \mathbb{R}^3$, and since the system now possesses two independent invariants, it is left with one degrees of freedom and hence it is completely integrable. Having the completely integrable system, the Jeans theorem, see \cite[Section 4]{BT87}, shows that $f^0$ is a function depending only on the invariants of that system. Thus, the stationary solution can be written as

$$f^0(x, v) = F(E, L),$$

where

$$E(x, v) := E_{f^0}(x, v) = \frac{|v|^2}{2} + \phi_{f^0}(t, x),$$

$$L(x, v) := |x \wedge v|^2.$$
Therefore, here, one has
\[ \Delta_x \phi = \int_{\mathbb{R}^3} F(E, L) \, dv. \]

If \( F = F(E) \) depends only on \( E \), it represents a isotropic spherical model, and if \( F = F(E, L) \) it is a anisotropic spherical model.

Among all spherical steady galaxy models, we mention two isotropic ones, which are the most common in the literature. The first one is the (normalized) polytrope
\[ f^0(x, v) = F(E) = (E_0 - E)^n, \]
where \( (\cdot)_+ := \max\{\cdot, 0\} \), \( E_0 \) is a non-positive constant which represents some energy threshold and \( 0 < n \leq \frac{7}{2} \). For \( n > \frac{7}{2} \), it leads to the solutions of an infinite mass and a noncompact support by adjusting the parameter \( E_0 \). For the limiting case when \( n = \frac{7}{2} \), it has a finite mass but non-compact support and it is called the Plummer model. The other model that we mention is the King model, which is given by the following function
\[ f^0(x, v) = F(E) = (e^{E_0 - E} - 1)_+, \]
for a given constant \( E_0 < 0 \). The King model describes isothermal galaxies, formally corresponding to \( n = \infty \) in the polytropes, which is widely used in astronomy. For more galaxy models see [BT87].

One of the central question when describing the Vlasov–Poisson system, or any PDE system in general, is the stability of its stationary solutions, or in our case, the stability of steady galaxy models \( f^0 \). Here, we introduce the notion of the orbital stability, which will be discussed in more details in the following sections. A stationary solution \( f^0 \) is nonlinearly stable if the solutions \( f_t \) of the nonlinear Vlasov–Poisson system remain arbitrarily close to the stationary solution for all times \( t \geq 0 \), provided that they start sufficiently close to the stationary solution, i.e. provided that \( f_{in} \) is close to \( f^0 \). The distance is measured by the system invariants. This is usually expressed as follows: for all \( \epsilon > 0 \), there is \( \nu > 0 \) such that
\[ D(f_{in}, f^0) := \|f_{in} - f^0\|_{L^1(\mathbb{R}^3)} + |\mathcal{H}(f_{in}) - \mathcal{H}(f^0)| \leq \nu \implies \forall t \geq 0, D(f_t, f^0) \leq \epsilon. \]

However, one must include the transformations under which the Vlasov–Poisson system is invariant and then the notion of the orbital stability is reformulated in the following manner: there is a translation function \( z = z(t) \in \mathbb{R}^3 \) such as
\[ D(f_{in}, f^0) \leq \nu \implies \forall t \geq 0, D(f_t, f^0(\cdot - z(t), \cdot)) \leq \epsilon. \]

In the case when spherically symmetric perturbations are considered, the parameter \( z(t) \) is necessarily always zero.

On the other hand, when one talks about linear stability, the solutions \( f_t \) are replaced by the solutions of the corresponding linearized problem. The question of the linear stability for the spherically-symmetric decreasing models, \( F' < 0 \), is resolved in the 1960s and 1970s as a result of the pioneering work by Antonov [Ant61, Ant65]. For the systematical review on the results in linear stability see [Mon12] and [Guo08].

Concerning nonlinear stability, there are different variational and non-variational (direct) approaches. Also, for the detail review see [Mon12] and [Guo08]. Here, we will discuss the variational approach by concentration-compactness technique, which is successfully applied to the polytropic models. After that, we discuss the method by rearrangement used to resolve the question of orbital stability of more general models, precisely the spherically-symmetric, isotropic
and decreasing models under general perturbations. Also, this method is successfully applied to prove orbital stability of spherically-symmetric, anisotropic and decreasing models under the spherically symmetric perturbations. Let us add that in the absence of spherical symmetry in the stationary solution, the question of stability is open even for the linearized problem.

2 Nonlinear stability by concentration-compactness

It can be shown, that the minimizers of the microscopic Hamiltonian (1.10) are described by the polytropic models (1.14). Later, we will state this result formally in the Theorem 2.9. Therefore, considering the stability of the gravitational Vlasov–Poisson system (1.5)-(1.6), one is interested in the orbital stability of the states which are characterized as the minimizers of the functional $H$. Since those states are of the lowest energy, they are called the ground states. Thus, essentially, one deals with a problem of minimizing a functional $F$ on the bounded and closed subset $E$ of an infinite-dimensional function space $H$, defined on an unbounded domain, such as $\mathbb{R}^N$, a typical problem from calculus of variations. In what follows we deal with a more general framework which motivates the writing $F$ instead of $H$.

In the infinite-dimensional function space $H$, we are faced with the difficulty that Bolzano–Weierstrass theorem does not hold, i.e. a bounded sequence in such space does not need to have a convergent subsequence, in the sense of the strong topology. Equivalently, a closed and bounded subset in such space does not have to be compact. Hence, it is not possible easily to show the existence of the solution of such minimization problem.

There are different ways to get around this difficulty of the loss of compactness, depending on a certain variational problem and additional properties of the functional $F$. Often, one wants to weaken the topology in order to recover the compactness of $E$, but this makes the continuity, or semicontinuity, of $F$ harder to establish. The technique applicable in this situation is the concentration-compactness principle, [Lio84a, Lio84b]. The idea here is to make the topology weaker so that the functional $F$ is continuous, but for which $E$ is still not quite compact. It is not quite compact in the sense that a sequence $x^{(n)}$ in $E$ still does not have to have a subsequence converging to a constant element $x \in E$ in this weaker topology. However, the concentration-compactness principle fully describes the failure of the compactness, which then can be eliminated with the additional assumption on the functional $F$.

Basically, when a sequence $x^{(n)}$ does not have a subsequence converging to a constant element $x \in E$, it may have a subsequence converging to another non-constant sequence $y^{(n)}$ in this weaker topology. This sequence $y^{(n)}$ is some kind of an approximating sequence and it is fully described by the concentration-compactness principle. It can be converging to a point or going off to infinity, or it can be a superposition of such sequences.

This principle is carefully described in simple settings in the next subsection, following what is written in Terence Tao’s book [Tao09, Section 1.16]. Afterwards, the general framework is given. As this method was successfully applied to many problems, we chose one which is structurally close to the problem of orbital stability of the polytropes in the gravitational Vlasov–Poisson system. In particularly, the orbital stability of ground solitary states of some nonlinear Schrödinger equation is shown first. Then, the similar procedure is applied in the Vlasov–Poisson context.

2.1 The concentration-compactness principle in a simple setting

Instead of working in a function space such as Sobolev space, first we demonstrate the concentration-compactness principle in simple settings. The function space $H$ is taken to be the space of
absolutely summable two-sided infinite sequences

\[ l^1(\mathbb{Z}) = \left\{ (x_m)_{m \in \mathbb{Z}} : \sum_{m \in \mathbb{Z}} |x_m| < \infty \right\} \]

with the translation group \( \mathbb{Z} \). It is a well-known result, which will be used a lot here, that for every two-sided sequence \( x = (x_m)_{m \in \mathbb{Z}} \in l^1(\mathbb{Z}) \), one has \( x_m \to 0 \), as \( m \to \pm \infty \). There are several topologies defined on this space, and we will use the ones given in the following definition.

**Definition 2.1.** Let \( x^{(n)} \), \( n = 1, 2, \ldots \) be a sequence in \( l^1(\mathbb{Z}) \), such that elements of this sequence are two-sided sequences denoted as \( (x_m^{(n)})_{m \in \mathbb{Z}} \), and let \( x = (x_m)_{m \in \mathbb{Z}} \) be another element in \( l^1(\mathbb{Z}) \).

(i) (Strong topology) The sequence \( x^{(n)} \) converges to \( x \), as \( n \to \infty \), in the strong topology (or \( l^1 \) topology) if the \( l^1 \) distance \( \|x^{(n)} - x\|_{l^1(\mathbb{Z})} := \sum_{m \in \mathbb{Z}} |x_m^{(n)} - x_m| \) converges to zero.

(ii) (Intermediate topology) The sequence \( x^{(n)} \) converges to \( x \), as \( n \to \infty \), in the intermediate topology (or \( l^\infty \) topology) if the \( l^\infty \) distance \( \|x^{(n)} - x\|_{l^\infty(\mathbb{Z})} := \sup_{m \in \mathbb{Z}} |x_m^{(n)} - x_m| \) converges to zero.

(iii) (Weak topology) The sequence \( x^{(n)} \) converges to \( x \), as \( n \to \infty \), in the weak topology (or pointwise topology) if \( x_m^{(n)} \to x_m \), as \( n \to \infty \), for each \( m \).

Here, the translation group is given by \( \mathbb{Z} \) and the translation action is defined by the shift operator \( T^h : l^1(\mathbb{Z}) \to l^1(\mathbb{Z}) \), for \( h \in \mathbb{Z} \), such that

\[ T^h(x_m)_{m \in \mathbb{Z}} = (x_{m-h})_{m \in \mathbb{Z}}. \]

For \( h > 0 \), this is the right shift operator, which moves the two-sided infinite sequence as follows:

\[
\begin{align*}
  h = 1 & \quad \ldots \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \ldots \\
  h = 2 & \quad \ldots \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad \ldots \\
\end{align*}
\]

In the infinite-dimensional space \( l^1(\mathbb{Z}) \), we take a closed and bounded set \( E \) to be

\[ E = \left\{ (x_m)_{m \in \mathbb{Z}} \in l^1(\mathbb{Z}) : \sum_{m \in \mathbb{Z}} |x_m| = 1 \right\}. \tag{2.1} \]

This set is invariant under the above defined translation action. Also, note that this set is closed and bounded in strong topology, but it is not closed in weak or intermediate topology. This is because we can find a sequence in \( E \) converging to zero in weak and intermediate topology, but zero is not in \( E \). For example, the sequence of the basis vectors \( e^{(n)} = \frac{1}{2n+1} \sum_{k=-n}^{n} \delta_k^{(n)} \), with the Kronecker symbol \( \delta \), which is a sequence of two-sided sequences from \( E \), converges to zero in the weak and intermediate topology.

The failure of the closure in the weak and intermediate topology causes the loss of compactness of \( E \) in the intermediate and strong topology. To demonstrate this, let us suppose that \( E \) is compact in the strong or intermediate topologies. That would mean that every sequence in \( E \) has a convergent subsequence, in that particular topology. Also, the limit of the convergent subsequence in these topologies has to be equal to its weak limit. On the other hand, there is, for example, the sequence \( e^{(n)} \), which has the weak limit zero, but it does not converge to zero in other two topologies, what gives the contradiction.

The failure of compactness of \( E \) in the intermediate topology is described by the following concentration-compactness lemma.
Lemma 2.2. Let \( x^{(n)} \) be a sequence in \( E \). Then, it has a subsequence, which is still denoted \( x^{(n)} \), such that there exist \( x_j \in l^1(\mathbb{Z}) \), satisfying
\[
\sum_j \|x_j\|_{l^1(\mathbb{Z})} \leq 1, \tag{2.2}
\]
and there exist sequences \( h_j^{(n)} \) of integers obeying the asymptotic orthogonality condition
\[
|h_k^{(n)} - h_j^{(n)}| \to \infty, \text{ as } n \to \infty, \tag{2.3}
\]
for all \( k > j \), so that the following decomposition of \( x^{(n)} \) holds
\[
x^{(n)} = \sum_j T^{h_j^{(n)}} x_j + w^{(n)}, \tag{2.4}
\]
where the remainder \( w^{(n)} \) converges to zero in the intermediate topology.

Remark 2.3. If there is the equality in (2.2), it is possible to make \( w^{(n)} \) converging in the strong topology as well.

Remark 2.4. This lemma says that every sequence \( x^{(n)} \) in \( E \), i.e. every bounded sequence of two-sided sequences from \( l^1(\mathbb{Z}) \), has a subsequence converging to the sum of sequences of very structured form, in the intermediate topology. Analyzing the decomposition (2.4), we see that the mentioned subsequence can converge to zero in the intermediate topology, in which case we take \( j \) to go over the empty set. Further, it can converge to any other element in \( l^1(\mathbb{Z}) \) with the \( l^1 \) norm less or equal to 1, in which case we have exactly one \( x_j \neq 0 \) and the corresponding sequence \( h_j^{(n)} \) does not go to infinity. Otherwise, in the case when \( x_j \neq 0 \) and \( h_j^{(n)} \) goes off to infinity for at least one \( j \), the sequence can not be compact, i.e. it can not have a convergent subsequence. We know that every subsequence of a compact sequence in \( E \) has to have the intermediate limit equal to its weak limit. However, the weak limit of such decomposed sequence \( x^{(n)} \) is equal to zero, but the intermediate and strong limit are not. Therefore, it has no convergent subsequence in the intermediate (or strong) topology.

Proof. If the sequence \( x^{(n)} \) converges to 0 in the intermediate topology, then it is enough to take \( j \) to go over the empty set, as mentioned in the Remark 2.4. So, let us suppose that \( x^{(n)} \) does not converge to 0 in this \( l^\infty \) topology. This means that there exist a subsequence, which we denote the same, and \( \epsilon_1 > 0 \) such that \( \|x^{(n)}\|_{l^\infty} > \epsilon_1 \) for all \( n \). Therefore, for each two-sided sequence in the sequence \( x^{(n)} \), it is possible to find an integer \( h_1^{(n)} \) such that \( |x^{(n)}_{h_1^{(n)}}| > \epsilon_1 \). Now we want to move those "large" elements at the origin position, in each two-sided sequence. If we pick all those integers \( h_1^{(n)} \) and shift the corresponding elements of \( x^{(n)} \) to the left, we get the rearranged sequence \( T^{-h_1^{(n)}} x^{(n)} \) such that all its two-sided sequences, at the origin position \( m = 0 \), have a real number bounded below in magnitude by \( \epsilon_1 \).

The sequence \( x^{(n)} \) is a sequence of elements in \( E \) and that still holds for the shifted sequence \( T^{-h_1^{(n)}} x^{(n)} \). The set \( E \) is bounded and closed in \( l^1 \), and therefore, after passing to another subsequence which we still denote the same, we can conclude that shifted sequence converge weakly to some limit \( x_1 \in l^1(\mathbb{Z}) \). Since this shifted sequence has those "large" elements at the origin, for the weak limit the inequality \( \|x_1\|_{l^1(\mathbb{Z})} \geq \epsilon_1 \) has to hold. Now, by taking the inverse shift, it is obvious that we have obtained the following decomposition
\[
x^{(n)} = T^{h_1^{(n)}} x_1 + w_1^{(n)}, \tag{2.5}
\]
where the remainder $w_1^{(n)}$ is such that $T^{-h_1^{(n)}}w_1^{(n)}$ converges weakly to zero. Both of these sequences on the right hand side in \((2.5)\) can be understood as asymptotically vanishing. Hence, it is possible to show

$$\|x^{(n)}\|_{l^1(\mathbb{Z})} = \|T^{h_1^{(n)}}x_1\|_{l^1(\mathbb{Z})} + \|w_1^{(n)}\|_{l^1(\mathbb{Z})} + o(1),$$

where $o(1)$ goes to zero as $n \to \infty$, and since $l^1$ norm is invariant on the shift operator, one has

$$1 = \|x_1\|_{l^1(\mathbb{Z})} + \lim_{n \to \infty} \|w_1^{(n)}\|_{l^1(\mathbb{Z})}.$$  

Also, note here that "mass" of the remainder $w_1^{(n)}$ is eventually strictly less then that of the original sequence, since

$$\|w_1^{(n)}\|_{l^1(\mathbb{Z})} \leq 1 - \epsilon_1 + o(1).$$

Then, the same procedure is applied to the remainder $w_1^{(n)}$. If it converges to zero in the intermediate topology, the proof is done. Otherwise, as before, it is possible to find $\epsilon_2 > 0$ and sequence of integers $h_2^{(n)}$ such that the shifted sequence $T^{-h_2^{(n)}}w_1^{(n)}$ is bounded from below by $\epsilon_2$ at the origin positions. Therefore, after passing to another subsequence, one has that elements of $T^{-h_2^{(n)}}w_1^{(n)}$ converge weakly to some non-trivial limit $x_2 \in l^1(\mathbb{Z})$, such that $\|x_2\|_{l^1(\mathbb{Z})} \geq \epsilon_2$.

Now, it is possible to conclude that the orthogonality condition \((2.3)\) must hold. Suppose it does not hold, i.e. $|h_1^{(n)} - h_2^{(n)}|$ does not go to infinity. Then, there is a subsequence for which $h_1^{(n)} - h_2^{(n)}$ is equal to some constant $c$. In this case, one has

$$T^{-h_1^{(n)}}w_1^{(n)} = T^{-c} \left( T^{-h_2^{(n)}}w_1^{(n)} \right),$$

which gives a contradiction because $T^{-h_1^{(n)}}w_1^{(n)}$ converges weakly to zero and $T^{-h_2^{(n)}}w_1^{(n)}$ does not.

By undoing the shift, the following decompositions is obtained

$$w_1^{(n)} = T^{h_2^{(n)}}x_2 + w_2^{(n)},$$

where the remainder $w_2^{(n)}$ is such that $T^{-h_2^{(n)}}w_2^{(n)}$ weakly converges to zero. Combining this with \((2.5)\), one gets

$$x^{(n)} = T^{h_1^{(n)}}x_1 + T^{h_2^{(n)}}x_2 + w_2^{(n)}.$$  

Therefore,

$$T^{-h_1^{(n)}}x^{(n)} = x_1 + T^{h_2^{(n)} - h_1^{(n)}}x_2 + T^{-h_1^{(n)}}w_2^{(n)}$$

and one can see that $T^{-h_1^{(n)}}w_2^{(n)}$ also converges weakly to zero. Moreover, by the asymptotic orthogonality, the following is concluded

$$1 = \|x_1\|_{l^1(\mathbb{Z})} + \|x_2\|_{l^1(\mathbb{Z})} + \lim_{n \to \infty} \|w_2^{(n)}\|_{l^1(\mathbb{Z})}.$$  

Again, note that the norm of the remainder $w_2^{(n)}$ has decreased as follows

$$\|w_2^{(n)}\|_{l^1(\mathbb{Z})} \leq 1 - \epsilon_1 - \epsilon_2 + o(1).$$

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By proceeding in this manner, one gets finer and finer decomposition on further subsequences. The subsequences depend on \( j \), but one can work with a single sequence throughout, by Cantor diagonalization argument. One stops if the remainder \( w_j^{(n)} \) converges to zero in the intermediate topology. However, the remainders are getting smaller and smaller, and one can observe that their asymptotic \( l^\infty \) norm must decay to zero as \( j \to \infty \), in the following way.

Notice that, at each step, we identified the large piece of \( x^{(n)} \) and those pieces were exactly the obstructions for the sequence to converge in the intermediate topology. Moreover, at each step, we decreased the total mass in play by "extracting the mass" of \( \epsilon_j \). One observes here that, since the starting sequence is in \( E \), the extracted mass can not exceed 1, i.e. \( \sum_j \epsilon_j \leq 1 \) and \( 2.2 \) holds. In particular, the extracted mass \( \epsilon_j \) must go to zero as \( j \to \infty \). It is important to chose \( \epsilon_j \) in a "greedy" manner and then the asymptotic \( l^\infty \) norm of the remainders \( w_j^{(n)} \) for \( n \to \infty \) must decay to zero as \( j \to \infty \). For example, one can choose \( \epsilon_j > \frac{1}{2} \lim_{n \to \infty} \| w_j^{(n)} \|_{l^\infty} \), where \( w_0^{(n)} \) is just the starting sequence \( x^{(n)} \) and where one passes to another subsequence of \( w_j^{(n)} \) to provide the existence of \( \lim_{n \to \infty} \| w_j^{(n)} \|_{l^\infty} \). Since \( \epsilon_j \) depends on this limit, one also passes to another subsequence, so that the arguments \( \| w_j^{n} \|_{l^\infty} > \epsilon_j \) for all \( n \) hold. Hence, at each step \( j \), the asymptotic \( l^\infty \) norm of the remainder is reduced by \( \frac{1}{2} \), so it has to converge to zero as \( j \to \infty \). This gives the wanted decomposition \( 2.4 \) and completes the proof.

The result of this lemma is then used to establish the existence of the minimum of the functional \( \mathcal{F} \) defined on \( E \). This is essentially the concentration-compactness principle. The failure of the compactness has to be compensate with additional assumptions on \( \mathcal{F} \). Usually, it is the case that the assumptions are adapted to the particular problem. In the following theorem, which can be found in [Tao09], the typical hypotheses on \( \mathcal{F} \) are taken.

**Theorem 2.5.** Let the set \( E \) be the subset in \( l^1(\mathbb{Z}) \) given by \( 2.1 \), and let \( \mathcal{F} : l^1(\mathbb{Z}) \to \mathbb{R} \) be a functional with the negative infimum \( I := \inf_{x \in E} \mathcal{F}(x) < 0 \) and with the following properties:

(a) (Continuity) \( \mathcal{F} \) is continuous in the intermediate topology on \( E \).

(b) (Homogeneity) \( \mathcal{F} \) is homogeneous of some degree \( 1 < p < \infty \), i.e.

\[
\mathcal{F}(\alpha x) = \alpha^p \mathcal{F}(x)
\]

for all \( \alpha > 0 \) and \( x \in l^1(\mathbb{Z}) \).

(c) (Invariance) \( \mathcal{F} \) is invariant to the translation action defined by the shift operator \( T^h \), i.e. \( \mathcal{F}(T^h x) = \mathcal{F}(x) \) for all \( h \in \mathbb{Z} \) and \( x \in l^1(\mathbb{Z}) \).

(d) (Asymptotic additivity) If \( h_j^{(n)} \), is a collection of sequences obeying the asymptotic orthogonality condition \( 2.3 \), and if \( x_j \in l^1(\mathbb{Z}) \) are such that \( \sum_j \| x_j \|_{l^1(\mathbb{Z})} < \infty \), then

\[
\sum_j \mathcal{F}(x_j) < \infty \quad \text{and} \quad \mathcal{F}\left( \sum_j T^{h_j^{(n)}} x_j \right) = \sum_j \mathcal{F}(x_j) + o(1),
\]

where the quantity \( o(1) \) goes to zero as \( n \to \infty \). More generally, if \( w^{(n)} \) is bounded in \( l^1 \) and converges to zero in the intermediate topology, then

\[
\mathcal{F}\left( \sum_j T^{h_j^{(n)}} x_j + w^{(n)} \right) = \sum_j \mathcal{F}(x_j) + o(1).
\]
Then $\mathcal{F}$ attains its minimum on $E$.

Proof. By the definition of the infimum, one can find the minimizing sequence $x^{(n)} \in E$, i.e. the sequence in $E$ such that $\mathcal{F}(x^{(n)}) \to I$, as $n \to \infty$. After applying the Lemma 2.2 to this minimizing sequence, one passes to the subsequence and obtains the decomposition (2.4). Since the asymptotic additivity assumption (d) holds, from the decomposition (2.4) one gets

$$\mathcal{F}(x^{(n)}) = \sum_j \mathcal{F}(x_j) + o(1).$$

Letting $n$ to go to infinity, one obtains

$$I = \sum_j \mathcal{F}(x_j).$$

From this, we have that $I$ is finite, and now we want to show that it is equal to $\mathcal{F}(x_0)$, for some $x_0 \in E$. Since $I$ is the infimum of $\mathcal{F}$ on the set $E$, one has $I \leq \mathcal{F}(x)$ when $\|x\|_{l^1(\mathbb{Z})} = 1$. And since $l^1$ norm of $\frac{x_j}{\|x_j\|}$ equals to 1, by using the homogeneity assumption (b), for those $x_j \neq 0$ one obtains

$$I \leq \left( \frac{1}{\|x_j\|_{l^1(\mathbb{Z})}} \right)^p \mathcal{F}(x_j).$$

From here, by summing over all $j$ and by using (2.6), one gets the inequality

$$\left( \sum_j \|x_j\|_{l^1(\mathbb{Z})}^p \right) I \leq I.$$ 

Since $I < 0$, one gets the following

$$1 \leq \sum_j \|x_j\|_{l^1(\mathbb{Z})}^p \leq \left( \sum_j \|x_j\|_{l^1(\mathbb{Z})} \right)^p \leq 1,$$

where the last inequality is due to (2.2). Hence, all these inequalities must be the equality and, since $1 < p < \infty$, we conclude that all $x_j$ must vanish except one, which we denote $x_0$. The $l^1$ norm of $x_0$ must be equal to 1 and, therefore, we have $x_0 \in E$. Also, from the decomposition (2.4), we see that $w^{(n)}$ converges to zero in the strong topology and therefore $x^{(n)} \to x_0$ strongly.

Finally, from (2.6), it follows that $\mathcal{F}(x_0) = I$, which finishes the proof.

2.2 The general concentration-compactness principle

In this subsection, the concentration-compactness principle is given in more general settings, [Lio84a, Lio84b]. An analogous lemma to the one in the previous subsection is given below. The correspondence of possible cases given in the following lemma and the decomposition (2.4) is clear from the Remark 2.4.

Lemma 2.6. Let $\{\rho_n\}_{n \geq 1}$ be a sequence in $L^1(\mathbb{R}^N)$, satisfying

$$\rho_n \geq 0, \quad \int_{\mathbb{R}^N} \rho_n dx = 1.$$ 

Then there exists a subsequence, still denoted by $\{\rho_n\}_{n \geq 1}$, satisfying one of the three following possibilities:
(a) (Compactness) For each \( n \), there exists \( y_n \in \mathbb{R}^N \) such that \( \rho_n (\cdot + y_n) \) is tight, i.e.

\[
\forall \epsilon > 0, \exists R < \infty, \int_{y_n + B_R} \rho_n (x) \, dx \geq 1 - \epsilon;
\]

(b) (Vanishing) For all \( R < \infty \)

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{y + B_R} \rho_n (x) \, dx = 0;
\]

(c) (Dichotomy) There exists \( \alpha \in (0, 1) \), such that for all \( \epsilon > 0 \), there exists \( n_0 \geq 1 \), such that for all \( n \geq n_0 \), there exist non-negative functions \( \rho_n^1, \rho_n^2 \in L^1 (\mathbb{R}^N) \) satisfying

\[
\left| \int_{\mathbb{R}^N} \rho_n^1 \, dx - \alpha \right| \leq \epsilon, \quad \left| \int_{\mathbb{R}^N} \rho_n^2 \, dx - (1 - \alpha) \right| \leq \epsilon \quad \text{and}
\]

\[
\text{dist} (\text{supp} \rho_n^1, \text{supp} \rho_n^2) \to \infty \text{ as } n \to \infty,
\]

such that the following holds

\[
\| \rho_n - \left( \rho_n^1 + \rho_n^2 \right) \|_{L^1 (\mathbb{R}^N)} \leq \epsilon.
\]

Proof. Consider the functions

\[
Q_n (r) = \sup_{y \in \mathbb{R}^N} \int_{y + B_r} \rho_n (x) \, dx.
\]

\( (Q_n)_n \) is a sequence of nondecreasing, nonnegative, uniformly bounded functions on \( \mathbb{R}_+ \). Since \( \int_{\mathbb{R}^N} \rho_n \, dx = 1 \), one has that \( \lim_{r \to \infty} Q_n (r) = 1 \). This sequence has a pointwise convergent subsequence, so there exists a nondecreasing, nonnegative function \( Q \) such that for all \( r \geq 0 \)

\[
Q_{n_k} (r) \to Q (r), \quad k \to \infty.
\]

It is clear that

\[
\lim_{r \to \infty} Q (r) = \alpha \in [0, 1].
\]

If \( \alpha = 0 \), then obviously the case (b) happens. If \( \alpha = 1 \) it is possible to show that the case (a) happens and here we give a sketch of that argument. If one takes \( \mu > \frac{1}{2} \), then there exists \( R (\mu) \) such that for all \( k \geq 1 \) one has

\[
Q_{n_k} (R) = \sup_{y \in \mathbb{R}^N} \int_{y + B_R (\mu)} \rho_{n_k} (x) \, dx > \mu.
\]

Therefore, there exists a sequence \( y_k (\mu) \) satisfying

\[
\int_{y_k (\mu) + B_R (\mu)} \rho_{n_k} (x) \, dx > \mu.
\]

Also, there exist \( R \left( \frac{1}{2} \right) \) and a sequence \( y_k = y_k \left( \frac{1}{2} \right) \) for which

\[
\int_{y_k + B_R (\frac{1}{2})} \rho_{n_k} (x) \, dx > \frac{1}{2}.
\]
Since \( \mu > \frac{1}{2} \) and since one has that total mass of \( \rho_k \) for each \( k \) is equal to 1, i.e. \( \int_{\mathbb{R}^N} \rho_k \, dx = 1 \), one concludes that balls \( B(y_k(\mu), R(\mu)) \) and \( B(y_k, R(\frac{1}{2})) \) must intersect, i.e. \( |y_k(\mu) - y_k| \leq R(\frac{1}{2}) + R(\mu) \), for all \( \mu \geq \frac{1}{2} \). If one takes \( R'(\mu) = R(\frac{1}{2}) + 2R(\mu) \), then one gets that the sequence \( y_k \in \mathbb{R}^N \) is such that for all \( k \geq 1 \) and for all \( \mu \geq \frac{1}{2} \) the following holds

\[
\int_{y_k + B_R} \rho_{n_k}(x) \, dx > \mu,
\]

which gives the case (a). Finally, if \( \alpha \in (0, 1) \), the case (c) occurs. Let \( \epsilon > 0 \) and choose \( R \) such that \( Q(R) > \alpha - \epsilon \). Then for \( k \) large enough, one has \( \alpha - \epsilon < Q_{n_k}(R) < \alpha + \epsilon \). Moreover, one can find \( R_k \to \infty \) for \( k \to \infty \) such that \( Q_{n_k}(R_k) \leq \alpha + \epsilon \). So, there exists \( y_k \in \mathbb{R}^N \) such that

\[
\int_{y_k + B_R} \rho_{n_k}(x) \, dx \in (\alpha - \epsilon, \alpha + \epsilon).
\]

Then, if one takes \( \rho_k^1 = \rho_{n_k}1_{y_k + B_R} \) and \( \rho_k^2 = \rho_{n_k}1_{\mathbb{R}^N \setminus (y_k + B_R)} \), it is clear that one gets the case (c), since

\[
\int_{\mathbb{R}^N} (\rho_{n_k} - \rho_k^1 - \rho_k^2) \, dx = \int_{R \leq |x - y_k| \leq R_k} \rho_{n_k}(x) \, dx \\
\leq Q_{n_k}(R_k) - Q_{n_k}(R) + 2\epsilon \\
\leq (\alpha + \epsilon) - (\alpha - \epsilon) + 2\epsilon = 4\epsilon.
\]

By using this lemma, the goal is to minimize \( C^1 \) real-valued functional \( \mathcal{F} \) of the following type

\[
\mathcal{F}(u) = \int_{\mathbb{R}^N} f(Au(x)) \, dx,
\]

on some Hilbert function space \( H \) defined on \( \mathbb{R}^N \). The constraint is defined with the help of the other functional \( \mathcal{G} \), which is of similar type

\[
\mathcal{G}(u) = \int_{\mathbb{R}^N} g(Bu(x)) \, dx,
\]

such that \( \mathcal{G}(0) = 0 \). Here, \( f \) and \( g \) are some real-valued functions on \( \mathbb{R}^n \) and \( \mathbb{R}^m \), while \( A \) and \( B \) are some operators from \( H \) to another function space with values in \( \mathbb{R}^n \) and \( \mathbb{R}^m \), invariant by the translations of \( \mathbb{R}^N \). Therefore, the functionals \( \mathcal{F} \) and \( \mathcal{G} \) are also invariant by the translations of \( \mathbb{R}^N \). Having this, the following minimization problem is considered

\[
I_\mu = \inf \{ \mathcal{F}(u) : u \in H, \mathcal{G}(u) = \mu \}, \tag{2.7}
\]

for fixed \( \mu > 0 \), so that \( I_0 = 0 \).

It can be checked that the sub-additivity condition

\[
I_\mu \leq I_\alpha + I_{\mu - \alpha}, \quad \forall \alpha \in [0, \mu) \tag{2.8}
\]

always holds, see the argument \[Lio84a\] p. 113-114. However, the concentration-compactness principle states that all minimizing sequences of the problem \( \mathcal{F} \) \tag{2.7} are relatively compact up to a translation if and only if the following strict sub-additivity condition holds

\[
I_\mu < I_\alpha + I_{\mu - \alpha}, \quad \forall \alpha \in (0, \mu). \tag{2.9}
\]
Due to this relative compactness of all minimizing sequences, it is concluded that the solution of (2.7) is attained.

That this condition (2.9) is necessary for the relative compactness of all minimizing sequences is proved by the contraposition. If opposite of (2.9) is assumed, by (2.8) that means if $I_\mu = I_\alpha + I_{\mu-\alpha}$, $\forall \alpha \in (0,\mu)$, holds, then it is possible to construct a minimizing sequence of (2.7) which is not relatively compact.

The proof of the other direction, i.e. the sufficiency of the condition (2.9) for the compactness, is based on the concentration-compactness lemma given above. It is used in order to show that all minimizing sequences of (2.7) are relatively compact when the sub-additivity condition (2.9) holds. The lemma says that the only possible loss of compactness is due to the decomposition of the functions in $H$ into at least two parts which are going infinitely away from each other. Since it can be shown that this possibility of failure of compactness is ruled out by the given strict inequality (2.9), see [Lio84a], the stated form of compactness is obtained.

We note here that for some particular problems it can be shown that sub-additivity condition (2.9) always holds. Also, for some other problems, this condition holds whenever the functional $\mathcal{F}$ is homogeneous of some degree $p \in (1,\infty)$, the condition that we had in the Theorem 2.5 from the previous subsection.

### 2.3 The orbital stability and the nonlinear Schrödinger equation

Here, we will see how the concentration-compactness method is used to prove the orbital stability of solitary waves for the nonlinear Schrödinger equation

$$
i \frac{\partial \Phi}{\partial t} (t, x) + \Delta \Phi (t, x) + |\Phi (t, x)|^{p-1}\Phi (t, x) = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

$$\Phi (0, x) = \Phi_0 (x), \quad x \in \mathbb{R}^N,$$

where $\Phi_0$ is a complex-valued function in $H^1 (\mathbb{R}^N)$. If $p < 1 + \frac{4}{N}$, which is sub-critical case, it is known result [GV75] that all solutions to (2.10) are global and bounded in $H^1$. These solutions verify the following two conservation laws

$$\|\Phi (t, \cdot)\|_{L^2 (\mathbb{R}^N)} = \|\Phi_0\|_{L^2 (\mathbb{R}^N)},$$

$$\mathcal{F} (\Phi (t, \cdot)) = \mathcal{F} (\Phi_0),$$

for all $t \geq 0$, where

$$\mathcal{F} (\Phi) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \Phi (t, x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |\Phi (t, x)|^{p+1} dx.$$

We would like to check stability of some special solutions of the equation (2.10). Those solutions are so-called solitary waves and they take the form

$$\Phi (t, x) = e^{i\lambda t} u (x), \quad \lambda \in \mathbb{R},$$

where $u (x)$ solves the equation

$$-\Delta u + \lambda u = |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N.$$

Among these, we are interested in those of the lowest energy $\mathcal{F}$, so-called ground states, for which it is assumed that

$$\|u\|_{L^2 (\mathbb{R}^N)} = \mu,$$

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for a given $\mu > 0$. Therefore, we are interested in the solutions of the following minimization problem

$$I_\mu = \inf \left\{ F(u) : u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)} = \mu \right\}.$$  \hfill (2.13)

**Theorem 2.7.** For given $p < 1 + \frac{4}{N}$ and $\mu > 0$, every minimization sequence of \hfill (2.13)

is relatively compact in $H^1$ up to a translation in $\mathbb{R}^N$.

Moreover, every solution $u$ of the minimization problem \hfill (2.13)

is of the form $u(\cdot) = e^{i\theta} u_0(\cdot + y)$ for some $\theta \in \mathbb{R}$, $y \in \mathbb{R}^N$, where $u_0$ is the radially symmetric and positive solution of \hfill (2.13).

The proof of this theorem can be found in [CL82]. The first part says that $I_\mu$ from \hfill (2.13)

is attained. It is just a consequence of the concentration-compactness principle described in the

previous subsection, since it can be proved that sub-additivity condition \hfill (2.9)

for the problem \hfill (2.13) is always true. That is because for this problem it can be shown that $0 < I_\mu = \mu^{2/p} I_1$, for

more details see [Lio84b]. The second part describes the set of the ground solitary states, i.e.

the set of the solutions of \hfill (2.13), which we denote by $S_\mu$. The orbital stability of these solutions in the

set $S_\mu$ is a direct consequence of this result of relative compactness, and it is given in the following theorem, which can be also found in [CLS2].

**Theorem 2.8.** Let $p < 1 + \frac{4}{N}$ and $\mu > 0$. Then a ground state solitary $u \in S_\mu$ is orbitally

stable, i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that for any initial data $\Phi_0 \in H^1(\mathbb{R}^N)$ with

$$\inf_{u \in S_\mu} \|u - \Phi_0\|_{H^1(\mathbb{R}^N)} < \delta,$$

there exists $x(t) \in \mathbb{R}^N$, such that the corresponding solution $\Phi(t,x)$ of \hfill (2.10)

satisfies

$$\inf_{u \in S_\mu} \|u - \Phi(t, \cdot + x(t))\|_{H^1(\mathbb{R}^N)} < \epsilon,$$

for all $t \geq 0$.

**Proof.** This can be proved by the contradiction. If the claim were not true, then there would exist $\epsilon > 0$, $\Phi_0 \in H^1(\mathbb{R}^N)$ and $t_n \geq 0$ such that for all $\delta > 0$ and all $x(t_n) \in \mathbb{R}^N$

$$\inf_{u \in S_\mu} \|u - \Phi_0\|_{H^1(\mathbb{R}^N)} < \delta \quad \text{and} \quad \inf_{u \in S_\mu} \|u - \Phi_0(t_n, \cdot + x(t_n))\|_{H^1(\mathbb{R}^N)} > \epsilon_0$$  \hfill (2.14)

hold. Since $u \in S_\mu$, we have that $\mu - \delta < \|\Phi_0\|_{L^2(\mathbb{R}^N)} < \mu + \delta$, for all $\delta > 0$, and moreover that

$$F(\Phi_0) \to I_\mu = F(u) \quad \text{and} \quad \|\Phi_0\|_{L^2(\mathbb{R}^N)} \to \mu, \quad \text{as} \ n \to \infty.$$  \hfill (2.15)

Therefore, by using the conservation laws \hfill (2.11) and \hfill (2.12), we see that we have constructed the minimizing sequence \hfill \{\Phi^n(t_n, \cdot)\}

of the problem \hfill (2.13). This sequence is relatively compact in $H^1$ up to a translation, by the previous theorem, and hence

$$\inf_{u \in S_\mu} \|u - \Phi^n(t_n, \cdot + x(t_n))\|_{H^1(\mathbb{R}^N)} \to 0, \quad \text{as} \ n \to \infty,$$

which contradicts \hfill (2.14).  \hfill \square
2.4 The orbital stability and the Vlasov–Poisson system

Here, we will see how the concentration-compactness principle can be used to prove the orbital stability of some steady states of the gravitational Vlasov–Poisson system (1.5)–(1.6) in the space $E$ given by (1.11). Mainly, we follow the results of the work [LMR08], where they make a parallel with the nonlinear Schrödinger equation.

The orbital stability is shown for the isotropic polytropes defined as

$$f_0(x, v) = F(E) = \left(1 - \frac{|v|^2}{2} - \phi f_0\right)_{+}^{\frac{1}{p-1}}$$

(2.16)

for $\frac{9}{7} = p_{crit} < p \leq +\infty$. First, they are characterized as the ground states of the minimization problem given in the following theorem.

**Theorem 2.9.** Let $f_0$ be defined with (2.16). Then the minimization problem

$$\min_{f \in E, f \neq 0} \|\frac{\|v\|^2}{2} f \parallel_{L^2(R^6)} \|f\parallel_{L^1(R^6)}^{\frac{8}{3p-11}} \|f\parallel_{L^1(R^6)}^{\frac{7p-9}{8p-11}} \|\nabla_x \phi f\parallel_{L^2(R^3)}^2$$

is attained on the family with four parameters

$$\gamma f_0 \left(\frac{x-x_0}{\lambda}, \mu v\right), \quad \gamma \in \mathbb{R}_+, \lambda \in \mathbb{R}_+, \mu \in \mathbb{R}_+, x_0 \in \mathbb{R}^3.$$

By this theorem, the polytropic models (2.16) are characterized in terms of a best constant in the interpolation inequality given in the Theorem 1.1. Therefore, it is straightforward to see that the polytropic models are actually the ground states of the minimization problem of the form

$$\inf \{H(f) : f \in E, \|f\|_{L^1(R^6)} = M_1, \|f\|_{L^p(R^6)} = M_p\},$$

(2.17)

for given $M_1, M_p > 0$.

Next, by using the concentration-compactness principle, it is possible to show the compactness of the minimizing sequence of the problem (2.17), in even more general form. Namely, for $p_{crit} < p < +\infty$ considered is the minimization sequence of the the problem

$$I(M_1, M_j) = \inf \left\{H(f) : f \in E, \int_{R^6} f = M_1, \int_{R^6} j(f) = M_j \right\},$$

(2.18)

where $j$ is a strictly convex continuous non-negative function on $\mathbb{R}_+$ with

$$\forall t \geq 0, \quad j(t) \geq Ct^p \quad \text{and} \quad \lim_{t \to 0} \frac{j(t)}{t} = 0.$$

The claim is that if the following sub-additivity condition holds

$$I(M_1, M_j) < I(\alpha M_1, \beta M_j) + I((1 - \alpha) M_1, (1 - \beta) M_j),$$

(2.19)

for all $\alpha \in (0, 1)$ and $\beta \in (0, 1)$, then every minimizing sequence of (2.18) is relatively compact up to a translation in the energy space $E$. The result is a bit different for the case $p = \infty$, see [LMR08].

In the [LMR08], they also show the sufficient condition for (2.19) to happen. The sub-additivity inequality (2.19) is satisfied if either
(i) $j$ is a polytrope

$$j(f) = f^p, \quad \text{for } p_{\text{crit}} < p < +\infty,$$

or

(ii) there exists $\frac{2}{3} < p_1 < p_2 < +\infty$ such that

$$\forall t \geq 0, \forall b \geq 1, \quad b^{p_1} f(t) \leq j(bt) \leq b^{p_2} j(t).$$

Having the above mentioned relative compactness result, now it is possible to show the orbital stability in the energy space $E$ of the minimizers of (2.18). The derivation of this result is analogous to the one given in the Theorem 2.8 and it is stated in the following theorem from [LMR08].

**Theorem 2.10.** Let $p_{\text{crit}} < p < +\infty$. The polytrope $f^0$ defined by (2.16) is orbitally stable, i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that for any initial data $f_{in} \in E$ with

$$\mathcal{H}(f_{in}) - \mathcal{H}(f^0) < \delta, \quad \|f_{in}\|_{L^1} \leq \|f^0\|_{L^1} + \delta \quad \text{and} \quad \|f_{in}\|_{L^p} \leq \|f^0\|_{L^p} + \delta,$$

there exists a translation shift $x(t) \in \mathbb{R}^N$, such that the solution $f(t, x, v)$ of (1.5)–(1.6) with the initial data $f_{in}$ satisfies

$$\|f^0 - f(t, x + x(t), v)\|_{E} < \epsilon,$$

for all $t \geq 0$.

### 3 Nonlinear stability by rearrangements

In this section, we give the results from the work of [LMR11b], where they prove the nonlinear stability of spherically-symmetric, isotropic and decreasing galaxy models, under general perturbations. Previously, in [LMR11b], they resolved this question of stability in the case of anisotropic models under spherical perturbations. While in [LMR11b], they consider general perturbations but only for the isotropic case. The question of the nonlinear stability of anisotropic, spherically-symmetric and decreasing galaxy models, under general perturbations, remains open. They deal with the solutions of (1.5)–(1.6) which are in the energy space $E$, where the energy conservation is replaced by the inequality (1.12). In this space, we have already seen that nonlinear stability of specific subclasses of steady states, such as polytropes, can be proved as a direct consequence of concentration-compactness principle. However, here, the more general models are considered. The theorem they give in [LMR11b] is the following one.

**Theorem 3.1.** Suppose $f^0 = F(E) \geq 0$ is a continuous stationary solution, nonzero, with compact support, of the system (1.5)–(1.6), for which there exists $E_0 < 0$ such as $F(E) = 0$ for $E \geq E_0$, $F$ is $C^1$ on $(-\infty, E_0)$ and $F' < 0$ on $(-\infty, E_0)$. Then $f^0$ is orbitally stable, i.e. for all $M > 0$ and $\varepsilon > 0$ there is $\nu > 0$ such as, for any initial data

$$f_{in} \in E = \{ g \geq 0, g \in L^1 \cap L^\infty(\mathbb{R}^6), |v|^2 g \in L^1(\mathbb{R}^6) \}$$

such that

$$\|f_{in} - f^0\|_{L^1(\mathbb{R}^6)} \leq \nu, \quad \mathcal{H}(f_{in}) \leq \mathcal{H}(f^0) + \nu, \quad \|f_{in}\|_{L^\infty(\mathbb{R}^6)} \leq \|f^0\|_{L^\infty(\mathbb{R}^6)} + M,$$

then any weak solution for given initial data like this verifies

$$\forall t \geq 0, \int_{\mathbb{R}^6} (1 + |v|^2) (f(t, x, v) - f^0(x - z(t), v)) \, dx \, dv \leq \varepsilon.$$
We will give the detail outline of the proof of this theorem as it is given in [Mou12]. However, before going into details, let us give just the strategy of the proof. The proof is based on the notion of rearrangements, which we discuss in the next subsection. The first step of the proof is to show that the Hamiltonian has a monotonicity property under the general symmetric rearrangements. Then, the reduced Hamiltonian is introduced and for the corresponding reduced minimization problem, certain coercivity inequality is shown. This coercivity inequality is then used to prove compactness of «generalized» minimizing sequences. Finally, the compactness argument gives the base for the contradiction used to prove Theorem 3.1, which we give in the last subsection.

3.1 Rearrangement according to the microscopic energy

First, let us define the symmetric rearrangements of a set and of a function, as they are defined in [LL01, Chapter 3].

**Definition 3.2.** Given a measurable set $A \in \mathbb{R}^6$, its symmetric rearrangement $A^*$ is an open ball centered at zero

$$A^* = \{ x : |x| < r \}$$

with $r$ chosen such that they are of the same volume, i.e. $|A| = |A^*|$.

**Definition 3.3.** Given a non-negative integrable function $f : \mathbb{R}^6 \to \mathbb{R}_+$, its symmetric rearrangement $f^*$ is a non-negative function whose upper level sets are obtained by the symmetrical rearrangements of the corresponding level sets of $f$, which gives the following formula

$$f^*(x,v) = \int_0^{+\infty} 1_{\{ f(x,v) \geq s \}}(x,v) \, ds, \quad \text{where} \quad 1_A^* = 1_{A^*}. \quad (3.1)$$

The rearrangement $f^*$ is radially symmetric, decreasing and equimeasurable to $f$. Next, the elementary property of the symmetrical rearrangements is given.

**Lemma 3.4.** For $f \geq 0$ integrable on $\mathbb{R}^6$ we have

$$\int_{\mathbb{R}^6} f^*(x,v) \, |(x,v)|_{\mathbb{R}^6} \, dx \, dv \leq \int_{\mathbb{R}^6} f(x,v) \, |(x,v)|_{\mathbb{R}^6} \, dx \, dv,$$

where $|(x,v)|_{\mathbb{R}^6}$ denotes the norm on $\mathbb{R}^6$.

**Proof.** Since for every measurable set $A \in \mathbb{R}^6$ one has $|A|_{\mathbb{R}^6} = \int_{\mathbb{R}^6} 1_A \, dx \, dv$, if there is another measurable set $B \in \mathbb{R}^6$, such that $|A| \leq |B|$, the following inequality holds

$$\int_{\mathbb{R}^6} 1_A 1_B \, dx \, dv = |A \cap B|_{\mathbb{R}^6} \leq |A|_{\mathbb{R}^6} = |A^*|_{\mathbb{R}^6} = |A^* \cap B^*|_{\mathbb{R}^6} = \int_{\mathbb{R}^6} 1_A^* 1_B^* \, dx \, dv.$$

By using this, the definition of the rearrangement (3.1) and the layer-cake representation of a function given by the formula

$$f(x,v) = \int_0^{\infty} 1_{\{ f(x,v) \geq s \}}(x,v) \, ds, \quad (3.2)$$

one gets

$$\int_{\mathbb{R}^6} f 1_{|(x,v)|_{\mathbb{R}^6} \leq s} \, dx \, dv \leq \int_{\mathbb{R}^6} f^* 1_{|(x,v)|_{\mathbb{R}^6} \leq s} \, dx \, dv,$$
and since $1_{[(x,v)|_{\mathbb{R}^6} \leq s]}(x,v) = 1_{[(x,v)|_{\mathbb{R}^6} \leq s]}$ for every $s \geq 0$ given, one also has
\[
\int_{\mathbb{R}^6} f 1_{[(x,v)|_{\mathbb{R}^6} \leq s]} \, dx dv \leq \int_{\mathbb{R}^6} f^* 1_{[(x,v)|_{\mathbb{R}^6} \leq s]} \, dx dv.
\]
From $\int_{\mathbb{R}^6} f \, dx dv = \int_{\mathbb{R}^6} f^* \, dx dv$ and by rewriting the expression $\int_{\mathbb{R}^6} f^* 1_{[(x,v)|_{\mathbb{R}^6} \leq s]} \, dx dv$ as
\[
\int_{\mathbb{R}^6} f^* \left( 1 - 1_{[(x,v)|_{\mathbb{R}^6} \leq s]} \right) \, dx dv,
\]
one obtains
\[
\int_{\mathbb{R}^6} f^* 1_{[(x,v)|_{\mathbb{R}^6} \geq s]} \, dx dv \leq \int_{\mathbb{R}^6} f 1_{[(x,v)|_{\mathbb{R}^6} \geq s]} \, dx dv.
\]
Finally, by integrating this with respect to $s \in [0, +\infty]$ and using (3.2), one deduces the result.

Now, in the similar manner, we introduce the rearrangement with respect to the microscopic energy
\[
E_\phi := \left( \frac{|v|^2}{2} + \phi(x) \right),
\]
for a given potential $\phi$ on $\mathbb{R}^3$.

**Definition 3.5.** Given a measurable set $A \subset \mathbb{R}^6$, its symmetric rearrangement $A^*\phi$ with respect to the microscopic energy $E_\phi$ is the open (energy) ball
\[
A^*\phi = \{(x,v) : E_\phi(x,v) < E_A\},
\]
where $E_A$ is selected such that $|A^*\phi| = |A|$.

**Definition 3.6.** Given a non-negative integrable function $f$ with a compact support on $\mathbb{R}^6$, its symmetric rearrangement $f^*\phi$ with respect to the microscopic energy $E_\phi$ is the non-negative function $f^*\phi$ on $\mathbb{R}^6$ whose level sets are obtained by rearrangement with respect to the microscopic energy of corresponding upper level set of $f$, which gives the following formula
\[
f^*\phi(x,v) = \int_{0}^{+\infty} 1_{\left\{ f \geq s \right\}}^* ds \quad \text{with} \quad 1_{A}^* = 1_A^*\phi.
\]
From the definition, the function $f^*\phi$ is a function of $E_\phi$, with the compact support, decreasing in the microscopic energy $E_\phi$, and equimeasurable to $f$. In the following lemma, given is the property which proof follows by the similar reasoning as we had in the previous lemma.

**Lemma 3.7.** For the function $f \geq 0$ integrable on its compact support in $\mathbb{R}^6$, the following is true
\[
\int_{\mathbb{R}^6} f^*\phi E_\phi(x,v) \, dx dv \leq \int_{\mathbb{R}^6} f(x,v) E_\phi(x,v) \, dx dv.
\]
We will use the rearrangement of $f$ with respect to the microscopic energy created by the function $f$ itself, which we denote $\tilde{f} = f^*\phi f$ where, as before, $\phi f = -\left(1/(4\pi|x|)\right) * \rho_f$. We see that we obtain an rearrangement operation $f \rightarrow \tilde{f}$ highly non-linear. Note immediately that the spherical stationary solution $f^0$ is a fixed point of this non-linear rearrangement, i.e.
\[
\tilde{f}^0 = (f^0)^*\phi f^0 = f^0.
\]
3.2 Monotonicity of the Hamiltonian and the reduced Hamiltonian

Firstly, the following functional is introduced

$$J_f(\phi) := \mathcal{H}(f^\ast \phi) + \frac{1}{2} \|\nabla_x \phi - \nabla_x \phi_{f^\ast} \|_{L^2(\mathbb{R}^3)}^2$$

and then the following monotonicity of Hamiltonian with respect to rearrangements is proved.

**Theorem 3.8.** If $f \in \mathcal{E}$ and $\hat{f} = f^\ast \phi_f$, then

$$\mathcal{H}(f) \geq J_f(\phi_f) \geq \mathcal{H}(\hat{f})$$

with equality if and only if $f = \hat{f}$.

**Proof.** The right inequality follows from the definition of the reduced Hamiltonian. By the simple calculation, for any two functions $f, g \in \mathcal{E}$ we have

$$\mathcal{H}(f) = \mathcal{H}(g) + \frac{1}{2} \|\nabla_x \phi_f - \nabla_x \phi_g \|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^6} \left( \frac{|v|^2}{2} + \phi_f \right) (f - g) \, dx \, dv. \quad (3.3)$$

Therefore, for $g = \hat{f}$

$$\mathcal{H}(f) = J_f(\phi_f) + \int_{\mathbb{R}^6} \left( \frac{|v|^2}{2} + \phi_f \right) (f - \hat{f}) \, dx \, dv. \quad (3.4)$$

The equality case is clear and the left inequality is concluded just by applying the previous lemma to

$$\int_{\mathbb{R}^6} \left( \frac{|v|^2}{2} + \phi_f \right) (f - \hat{f}) \, dx \, dv,$$

which then has to be non-negative, since every rearrangement is non-negative function.

We introduce now the functional $\mathcal{J}(\phi) := J_{f^\ast}(\phi)$, which we call the reduced Hamiltonian. Note that it just depends on the potential. It turns out that

$$\mathcal{J}(\phi_{f^\ast}) = \mathcal{H}(f^\ast),$$

and that

$$\mathcal{H}(f) - \mathcal{H}(f^\ast) + C \|f^\ast - (f^\ast)_{f^\ast}\|_{L^1} \geq \mathcal{J}(\phi) - \mathcal{J}(\phi_{f^\ast}). \quad (3.5)$$

Hence, it is reasonable to pursuit the local minimum $\phi_{f^\ast}$ of the reduced Hamiltonian $\mathcal{J}$, in order to show that $f^\ast$ is the local minimum of $\mathcal{H}$ when $\|f^\ast - (f^\ast)_{f^\ast}\|_{L^1} = 0$. Therefore, we have the reduced variational problem, which involves the functional $\mathcal{J}$ depending on the potential only.

3.3 Coercivity inequality for the reduced Hamiltonian

Now, given is the study of convexity properties of the functional $\mathcal{J}$ in the vicinity of $\phi_{f^\ast}$. By the following theorem, it is shown that $\phi_{f^\ast}$ is a local minimum of $\mathcal{J}$. But first, let us define a space of eligible potentials in which the statement will hold.

$$\mathcal{X} := \left\{ \phi \in C^0(\mathbb{R}^3) : \phi \leq 0, \lim_{\infty} \phi = 0, \nabla_x \phi \in L^2(\mathbb{R}^3) \right\} \inf_{x \in \mathbb{R}^3} (1 + |x|) |\phi(x)| > 0.$$
Theorem 3.9. There exist the constants $c_0, \delta_0 > 0$ and a continuous map $\phi \to z_\phi$ from $H^1(\mathbb{R}^3)$ into $\mathbb{R}^3$ such that for $\phi \in \mathcal{X}$ with

$$\inf_{z \in \mathbb{R}^3} \left[ \| \phi - \phi_f (\cdot - z) \|_{L^\infty(\mathbb{R}^3)} + \| \nabla_x \phi - \nabla_x \phi_f (\cdot - z_\phi) \|_{L^2(\mathbb{R}^3)} \right] < \delta_0,$$

the following holds

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_f) \geq c_0 \| \nabla_x \phi - \nabla_x \phi_f (\cdot - z_\phi) \|_{L^2(\mathbb{R}^3)}^2.$$  

Proof. Here, given is the outline of the proof. For more details see [LMR11b].

First step of the proof is to write the Taylor expansion of the functional $\mathcal{J}$ in the vicinity of the potential $\phi_f$. It turns out that the first variation of $\mathcal{J}$ is zero at $\phi_f$ and that Taylor expansion of order two is as follows

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_f) = \frac{1}{2} D^2 \mathcal{J}(\phi - \phi_f, \phi - \phi_f) + o\left( \| \phi - \phi_f \|_{L^\infty(\mathbb{R}^3)} \right) \| \nabla_x \phi - \nabla_x \phi_f \|_{L^2(\mathbb{R}^3)}^2$$

where

$$D^2 \mathcal{J}(\phi_f)(h,h) = \int_{\mathbb{R}^6} |\nabla_x h|^2 dx - \int_{\mathbb{R}^6} |F'(E)(h(x) - \langle \mathcal{P} h \rangle(x,v))|^2 dx dv,$$

with $E = E_{\phi_f} = \frac{|\phi|^2}{2} + \phi_f(x)$ and the projection operator $\mathcal{P}$ defined as

$$\langle \mathcal{P} h \rangle(x,v) := \frac{\int_{\mathbb{R}^3} (E - \phi_f(y))^{1/2} h(y) dy}{\int_{\mathbb{R}^3} (E - \phi_f(y))^{1/2} dy}.$$

Next step of the proof is to show that $D^2 \mathcal{J}(\phi_f)(h,h)$ is coercive, up to the degeneracy induced by the translation invariance. To do so, the following operator is defined

$$\mathcal{L} h := -\Delta_x h - \int_{\mathbb{R}^3} |F'(E)(1-\mathcal{P})| dv,$$

for which one has

$$\langle \mathcal{L} h, h \rangle_{L^2(\mathbb{R}^3)} = D^2 \mathcal{J}(\phi_f)(h,h).$$

Therefore, the problem is reduced to the study of the operator $\mathcal{L}$. There is the following Lemma describing this operator.

Lemma 3.10. The operator $\mathcal{L}$ is positive, it is a compact perturbation of laplacian on $H^1(\mathbb{R}^3)$, its kernel is given by

$$\text{Ker}(\mathcal{L}) = \text{Span}\{\partial_{x_i} \phi_f, 1 \leq i \leq 3\}$$

and the inequality

$$\langle \mathcal{L} h, h \rangle_{L^2(\mathbb{R}^3)} \geq c_0 \| \nabla_x h \|_{L^2(\mathbb{R}^3)}^2 - \frac{1}{c_0} \sum_{i=1}^3 \left( \int_{\mathbb{R}^3} h \Delta_x (\partial_{x_i} \phi_f) dx \right)^2$$

holds for all $h \in H^1(\mathbb{R}^3)$ and some constant $c_0 > 0$.

This lemma is proved by the decomposition of $h$ into radial part and its orthogonal complement

$$h = h_0 + h_1, \quad h_0 \in H^1_{\text{rad}}(\mathbb{R}^3), \quad h_1 \in \left( H^1_{\text{rad}}(\mathbb{R}^3) \right)^\perp.$$  

Note that in this case when $\mathcal{L}$ depends on the radial part, $\mathcal{P} h_1 = 0$ and $\mathcal{L}$ reduces to Schrödinger operator.

The completion of the proof of Theorem 3.9 is then done by canceling the defects of coercivity, i.e. the negative terms in (3.6).
3.4 Compactness of the minimizing sequence

It is possible to show the compactness of certain generalized minimizing sequences $f_n$ in the following sense.

**Theorem 3.11.** If $(f_n)_{n \geq 0}$ is the sequence in $E$ satisfying

$$
\sup_{n \geq 0} \inf_{z \in \mathbb{R}^3} \left[ \left\| \phi_f^n - \phi_f^0 (\cdot - z_{\phi_f^n}) \right\|_{L^\infty(\mathbb{R}^3)} + \left\| \nabla_x \phi_f^n - \nabla_x \phi_f^0 (\cdot - z_{\phi_f^n}) \right\|_{L^2(\mathbb{R}^3)} \right] < \delta_0
$$

and

$$
\lim_{n \to \infty} \left\| f_n^* - (f^0)^* \right\|_{L^1(\mathbb{R}^6)} = 0, \quad \liminf_{n \to \infty} \mathcal{H}(f_n) \leq \mathcal{H}(f^0),
$$

then

$$
\lim_{n \to \infty} \int_{\mathbb{R}^6} \left| (1 + |v|^2) (f_n(x, v) - f^0(x - z_{\phi_f^n}, v)) \right| dv = 0.
$$

**Proof.** First of all, notice that from the coercivity inequality proved in the Theorem 3.9, from the assumptions made here and the inequality (3.5), it follows that

$$
\lim_{n \to \infty} \left\| \nabla_x \phi_f^n - \nabla_x \phi_f^0 (\cdot - z_{\phi_f^n}) \right\|_{L^2(\mathbb{R}^3)} = 0.
$$

(3.7)

We introduce the notation $\bar{f}_n(x, v) = f_n(x + z_{\phi_f^n}, v)$ and return to the complete Hamiltonian by using the equality we already had

$$
\mathcal{H}(f) - \mathcal{H}(f^0) + \frac{1}{2} \left\| \nabla_x \phi_f^n - \nabla_x \phi_f^0 \right\|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^6} E_{\phi_f^0} \left( \bar{f}_n - f^0 \right) dv.
$$

(3.8)

Since the hypothesis on the potential and the convergence (3.7) hold, it is possible to conclude

$$
\limsup_{n \to \infty} \left( \mathcal{H} (\bar{f}_n) - \mathcal{H} (f^0) \right) = \limsup_{n \to \infty} \left( \mathcal{H} (f_n) - \mathcal{H} (f^0) \right) \leq 0
$$

and

$$
\left\| \nabla_x \phi_{\bar{f}_n} - \nabla_x \phi_{f^0} \right\|_{L^2(\mathbb{R}^3)}^2 = \left\| \nabla_x \phi_f^n - \nabla_x \phi_f^0 (\cdot - z_{\phi_f^n}) \right\|_{L^2(\mathbb{R}^3)}^2 \to 0, \quad \text{as } n \to \infty.
$$

Now, from (3.8) one derives

$$
\limsup_{n \to \infty} \int_{\mathbb{R}^6} E_{\phi_f^0} \left( \bar{f}_n - f^0 \right) dv \leq 0.
$$

By using the monotonicity of the rearrangement from the Lemma 3.7, the fact that the rearrangement is non-negative function and that $f^0$ is a fixed point of the operator $\hat{f}$, we get

$$
\lim_{n \to \infty} \int_{\mathbb{R}^6} E_{\phi_f^0} \left( f^0 - \bar{f}_n \right) dv = 0
$$

from where, by the monotonicity of the rearrangement, we deduce

$$
\limsup_{n \to \infty} \int_{\mathbb{R}^6} E_{\phi_f^0} \left( \bar{f}_n - f_n^{*\phi_f^0} \right) dv \leq 0.
$$
By using the monotonicity of the rearrangement once again, it is possible to conclude that the quantity above is also greater or equal to zero, which gives us equality
\[
\lim_{n \to \infty} \int_{\mathbb{R}^6} E_{\phi f^0} \left( f_n - f_n^* \right) \, dx \, dv = 0.
\]
Now, by the assumption \( \lim_{n \to \infty} \| f_n^* - (f^0)^* \|_{L^1(\mathbb{R}^6)} = 0 \) and by using again that \( f^0 \) is a fixed point of the operator \( \hat{f} \), we obtain
\[
\lim_{n \to \infty} \| f_n - f^0 \|_{L^1(\mathbb{R}^6)} = 0.
\]
To complete the claim of the proof, we still need to prove the convergence of the kinetic energy, i.e. to prove that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^6} |v|^2 f_n \, dx \, dv = \int_{\mathbb{R}^6} |v|^2 f^0 \, dx \, dv.
\]
But this follows from the convergence of the potential (3.7) and assumption on the Hamiltonian.

3.5 Completion of the proof of Theorem 3.1

Having all the lemmas and theorems from the previous subsections, finally we can prove the claim form the Theorem 3.1. The proof is given by the contradiction. Let us suppose opposite, i.e. \( f^0 = F(E) \) is not orbitally stable. Then, there would exist \( M > 0, \, \varepsilon > 0, \, t_n > 0 \) and \( f_n^{in} \in \mathcal{E} \), such that for all \( n \in \mathbb{N} \)
\[
\| f_n^{in} - f^0 \|_{L^1(\mathbb{R}^6)} \leq \frac{1}{n}, \quad \mathcal{H}(f_n^{in}) \leq \mathcal{H}(f^0) + \varepsilon, \quad \| f_n^{in} \|_{L^\infty(\mathbb{R}^6)} \leq \| f^0 \|_{L^\infty(\mathbb{R}^6)} + M \tag{3.9}
\]
and the weak solution \( f^n \) for the given initial data \( f_n^{in} \) verifies
\[
\int_{\mathbb{R}^6} \left( (1 + |v|^2) \left( f^n(t_n, x, v) - f^0(x - z(t_n), v) \right) \right) \, dx \, dv > \varepsilon. \tag{3.10}
\]

Since the conservation laws of Casimir functionals and Hamiltonian \( \mathcal{H} \) hold for the solution of Vlasov–Poisson system (1.5)–(1.6), we have constructed the sequence \( (f^n)_{n \in \mathbb{N}} \) in \( \mathcal{E} \) which satisfies the conditions of the Theorem 3.1. The condition on \( \| (f^n)^* - (f^0)^* \|_{L^1(\mathbb{R}^6)} \) is satisfied because the first inequality in (3.9) and since \( \| (f^n)^* - (f^0)^* \|_{L^1(\mathbb{R}^6)} \leq \| f^n - f^0 \|_{L^1(\mathbb{R}^6)} \) holds. The condition on Hamiltonian is satisfied because of the second inequality in (3.9), while the condition on the potential holds because of third inequality in (3.9). Therefore, we have the convergence stated in the Theorem 3.1 which is the contradiction with (3.10).

Let us note here, what also can be noted at the end of the Theorem 2.10 that we proved the orbital stability of \( f^0 \) in the sense given in the Theorem 3.1 respectively in the Theorem 2.10. However, the orbital stability given by (1.15) also holds, since the interpolation inequality from the Theorem 1.1 gives the control of the potential energy by the kinetic energy.
References


