Shouting at a Staircase:
Acoustic Scattering off a Plane with a Step

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Introduction

- 1st-year PhD student at DAMTP in the Waves group, supervised by David Abrahams (Isaac Newton Institute) and Nigel Peake (DAMTP)
- PhD in area of Structural Acoustics
  - What is the internal and external response of a structure under heavy fluid loading?
- Use both solid and fluid mechanics in work — fluid mechanics for the fluid loading and solid mechanics for response of structure
Motivation

- My supervisors told me to do it!
  (arguably no further motivation is needed)
- To optimise design of submarine hulls to reduce sound emitted by internal sources (e.g. vibrations from engine)
  - Defense purposes
  - Ecological purposes
  - Environmental purposes
- Also interesting from mathematical point of view — essentially singular perturbation problem in acoustics
- Analytical approach is useful as problem is difficult for computational approach — need very fine-grained mesh to simulate at sufficient detail but this requires heavy computational resources
Question: What is the optimal design and placement of the bulkheads to minimise vibrations and hence noise?

Initially model hull of submarine as a cylinder (neglecting complex end effects) with bulkheads as reinforcing ribs.

Could also treat surface with noise-reducing coating (e.g. via tiles of layered metamaterials).
Motivation

- Further simplify by assuming wavelength of wave is very large compared to width of constraint ($a/\lambda \ll 1 \implies ka \ll 1$)

\[ F = F \delta(x) \]

- Simplify even further by considering the ribs to be sufficiently far from each other so that we can consider each rib separately — this is a classical elastostatic ‘punch’ problem
Motivation

- These define an ‘outer’ and ‘inner’ problem respectively, necessitating use of Matched Asymptotic Expansions:
  - Outer problem is on the scale of the wave
  - Inner problem is on the scale of the rib
- Solve each problem separately and then determine unknown constants in each problem by matching
- Hence form a composite solution that is valid for both the inner and outer problem
- A problem of this type was tackled by Abrahams et al.\(^1\)
- Aim of PhD is to extend results to different geometries and in-plane vibrations
- Started by considering scattering off plane with step — topic of this talk!

Matched Asymptotic Expansions Example

- Easiest to illustrate concept of MAEs with a simple example
- Consider a small sphere oscillating in still fluid; what is the acoustic field generated by the oscillations?

Find the acoustic field \( \phi \) by solving the Helmholtz equation
\[
(\nabla^2 + k^2)\phi = 0,
\]
where \( k \) is the acoustic wavenumber

- Have two boundary conditions:
  - Sommerfeld condition at infinity: no waves incoming from infinity
  - No-flux boundary condition: velocity of the fluid is equal to velocity of sphere on boundary of sphere
Inner Problem Example

- On scale of sphere, it only ‘sees’ incompressible flow within wavelength of oscillations, so only need to solve Laplace’s equation $\nabla^2 \phi = 0$ in this ‘inner’ region — hence terminology
- We still have the no-flux boundary condition, but now cannot apply the Sommerfeld condition
- Hence have a second-order PDE with one boundary condition — this leaves one constant undetermined in solution!
Outer Problem Example

- Far away from sphere, see it as a point at the origin with singular behaviour
- Now need to solve full Helmholtz equation with Sommerfeld boundary condition, but this time we cannot apply no-flux
- Again end up with second-order PDE with one boundary condition and so have one undetermined constant
Matching Process

- We have two solutions with two undetermined constants: one for the inner problem and one for the outer problem.
- How do we find these constants?
- Logic: there must be some kind of intermediate region in between the inner and outer regions.
- In this intermediate region, both solutions are valid and so can ‘match’ solutions using a matching process to find the constants.
- There are multiple matching processes, but the ‘Asymptotic Matching Principle’ pioneered by Van Dyke and improved by Crighton & Leppington is the most straightforward to use.
- Add the inner and outer solutions to obtain the composite solution which is uniformly valid throughout whole space of problem.
Intermediate Region

outer region

intermediate region

inner region
Given $\phi_t = \phi_i + \phi$, where $\phi_i$ is known time-harmonic incident wave plus a reflected time-harmonic wave so that $\phi$ is then purely scattering potential — think of it as a correction to $\phi_i$ for small step of non-dimensional size $\epsilon \equiv ka$.

Wave equation for $\phi$ therefore simplifies to Helmholtz equation: $(\nabla^2 + k^2)\phi = 0$.

Boundary conditions: Sommerfeld radiation condition and treat boundary surface as acoustically-hard surface.

Question: What is the far-field behaviour of $\phi$ and what is order of solution in terms of $\epsilon$?
Strategy

- Key assumption: $ka \equiv \epsilon \ll 1$; i.e. small step
- Presence of two distinct length scales ($a$ and $1/k$) implies singular perturbation problem
- Hence need to solve two problems: ‘outer’ and ‘inner problem’ — one for each length scale
- This is where method of Matched Asymptotic Expansions comes into use
- Will have undetermined constants in solutions to both problems due to impossibility of application of condition
  - $\partial \phi / \partial x = 0$ for outer problem (cannot meaningfully apply boundary condition to vanishingly thin edge)
  - Sommerfeld radiation condition for inner problem (cannot go to $\infty$)
For the outer problem, the plane appears uniformly flat with some sort of point inhomogeneity at the origin.

After scaling the problem appropriately for the outer problem, need to solve \((\nabla^2 + 1)\phi = 0\) with Sommerfeld boundary condition and acoustically-hard surface for flat plane.
Solution to Outer Problem

- Do so via Fourier transforms: end up with a fairly mathematical expression

$$\phi = -\epsilon \frac{i \sin^2 \theta}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha kx} - (\alpha^2 - 1)^{1/2} ky}{(\alpha^2 - 1)^{1/2}(\alpha + \cos \theta)} \, d\alpha + \epsilon a_1 H_0^{(1)}(kr)$$

- Here the Fourier integral is a representation of the small phase shift due to the step

- $H_0^{(1)}(r)$ is a Hankel function and represents outgoing cylindrical waves due to the point inhomogeneity which looks like a monopole source

- The undetermined constant here is $a_1$ which will be found later via the matching process
Inner Problem

- This time the inner problem looks like the step with height 1

\[
\begin{array}{c}
1 \\
\end{array}
\]

- After scaling for the inner problem, we get \((\nabla^2 + \epsilon^2)\phi = 0\) which can be approximated as \(\nabla^2 \phi = 0\) since \(\epsilon^2\) is much smaller than the \(\nabla^2\) term — like the inner problem in previous example

- We find there is a forcing of \(2i\cos \theta\) on the edge of the step which is the mechanism for generation of the cylindrical waves

\[
\begin{array}{c}
2i \cos \theta \\
\end{array}
\]
Solution to Inner Problem

- Solve via conformal mapping methods† or otherwise
- The solution ends up being
  \[ \phi = \frac{2i}{\pi} \cos \theta \log \left( \frac{\pi |x|}{2 \epsilon} \right) + B_1 \]

- Here \( B_1 \) is the undetermined constant, to be determined by matching
- Using the Asymptotic Matching Principle gives, after a lot of handwaving,

  \[ a_1 = \cos \theta, \quad B_1 = \cos \theta \left( 1 + \frac{2i}{\pi} \left\{ \gamma + \log \left( \frac{\epsilon}{\pi} \right) \right\} \right) \]

- But we only care about the far-field behaviour, so \( a_1 \) is what we are really interested in

† If we have time, the precise method will be discussed at the end
After much work, have found that

\[ \phi_t = e^{-ik(x \cos \theta + y \sin \theta)} + e^{-ik(x \cos \theta - y \sin \theta)} \]

\[ + \epsilon \left( -\frac{i \sin^2 \theta}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx} - (\alpha^2 - 1)^{1/2}ky}{(\alpha^2 - 1)^{1/2}(\alpha + \cos \theta)} d\alpha + \cos \theta H_0^{(1)}(kr) \right) \]

Could carry on to next order, but not much point since they would be much smaller than the leading-order and \( \epsilon \) terms

What does this tell us?
Far-Field Behaviour of Acoustic Field

- To zeroth-order, the far-field acoustic field is of a plane wave reflecting off a flat surface.
- The first-order $O(\epsilon)$ corrections are outgoing cylindrical waves represented by Hankel functions emanating from the inhomogeneity at the origin.

- Recent work showed that if the step is instead a sloping ramp, the $a_1$ term is still the same and so the far-field behaviour is unaffected by the precise profile of the inhomogeneity.
Future Work

- Aim to extend analysis to flexural structural waves within elastic structure as in figure to find reflection and transmission coefficients $R$ and $T$ respectively.

- Outer region appears as uniform plates joined together but cannot determine $R$ and $T$ from this.

- Inner region appears like step problem but now with additional boundary condition on top side of structure; need to analyse to match to outer problem for $R$ and $T$.
Summary

- PhD in Structural Acoustics with eventual goal of minimising sound from underwater vehicles, with work of interest to Thales
- Described how MAEs might be used for problems with different length scales
- Showed how to apply MAEs to simple problem of scattering off plane with step
- Future work: extend ideas of scattering off step to new and exciting geometries as well as in-plane vibrations for elastic structures
- Questions?
Recall that we needed to solve Laplace’s equation \((\nabla^2 \phi = 0)\) in the inner problem.

We can do this via Schwarz–Christoffel (SC) mappings which are a particular type of conformal mapping.

SC mappings allow us to map complicated polygonal geometries (and their boundary conditions) to nice geometries, (e.g. upper half plane or disc).

Can solve Laplace’s equation in nice geometry and then map solution back to original geometry via inverse mapping.
Bonus: Crash Course in Schwarz–Christoffel Mappings

In our case:

- **Step 1:** Switch to complex coordinates and then map plane with step (\(z\)-plane) to upper half plane (\(\zeta\)-plane)\(^2\)

\[
\nabla^2 \phi(z) = 0 \quad \rightarrow \quad \zeta(z) \quad \nabla^2 \phi(\zeta) = 0
\]

- **Step 2:** Solve \(\nabla^2 \phi(\zeta) = 0\) in upper half plane via Fourier transforms

- **Step 3:** Map solution \(\phi(\zeta)\) back to plane with step

\[
\phi(z) \quad \rightarrow \quad z(\zeta) \quad \phi(\zeta)
\]

\(^2\)We also choose \(\infty\) in the \(z\)-plane to be mapped to \(\infty\) in the \(\zeta\)-plane