Shouting at a Staircase:
Acoustic Scattering off a Plane with a Step

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Introduction

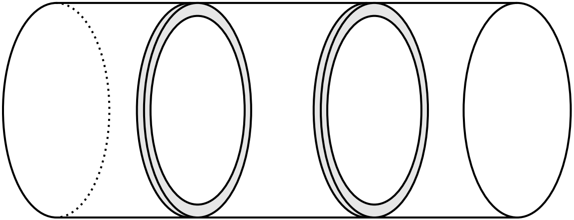
- 1st year PhD student, supervised by David Abrahams and Nigel Peake
- PhD broadly in area of Structural Acoustics
  - What is the internal and external response of a structure to a fluid scattering off it?
- Currently leaning towards solid mechanics more than fluid mechanics
- Techniques deployed include:
  - Matched Asymptotic Expansions
  - Complex-variable methods such as Schwarz–Christoffel mapping for fluids and solids
  - Not yet used Wiener–Hopf method but might do so in future

Motivation

- My supervisors told me to do it!
  (arguably no further motivation is needed)
- To optimise design of submarine hulls to reduce sound emitted by internal sources (*e.g.* vibrations from engine)
  - Defense purposes
  - Ecological purposes
  - Environmental purposes
- Also interesting from mathematical point of view — essentially singular perturbation problem in acoustics
- Analytical approach is useful as problem is difficult for computational approach — need very fine-grained mesh to simulate at sufficient detail but this requires heavy computational resources

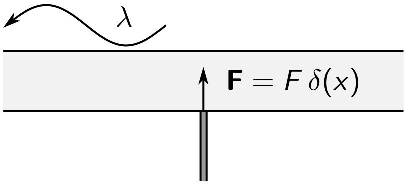
Motivation

- Question: What is the optimal design and placement of the bulkheads to minimise vibrations and hence noise?
- Initially model hull of submarine as a cylinder (neglecting complex end effects) with bulkheads as reinforcing ribs

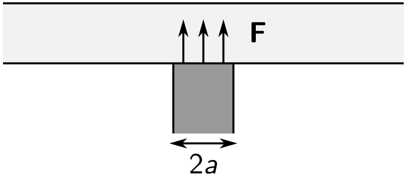


Motivation

- Further simplify by assuming wavelength of wave is very large compared to width of constraint ($a/\lambda \ll 1 \implies ka \ll 1$)



- Simplify even further by considering the ribs to be sufficiently far from each other so that we can consider each rib separately — this is a classical elastostatic 'punch' problem

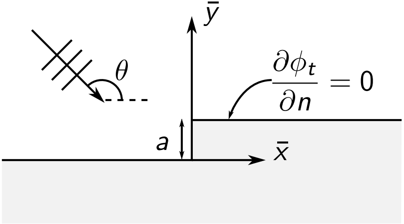


Motivation

- These define an 'outer' and 'inner' problem respectively, necessitating use of Matched Asymptotic Expansions:
  - Outer problem is on the scale of the wave
  - Inner problem is on the scale of the rib
- Solve each problem separately and then determine unknown constants in each problem by matching
- Hence form a composite solution that is valid for both the inner and outer problem
- A problem of this type was tackled by Abrahams *et al.*[1]
- Aim of PhD is to extend results to different geometries and in-plane vibrations
- Started by considering scattering off plane with step — topic of this talk!

[1] P. A. Cotterill *et al. The time-harmonic antiplane elastic response of a constrained layer*, Journal of Sound and Vibration, **348**, pp. 167-184 (2015)

Problem



- Given $\phi_t = \phi_i + \phi$, where ϕ_i is known time-harmonic incident wave plus a reflected time-harmonic wave so that ϕ is then purely scattering potential
- Wave equation for ϕ therefore simplifies to Helmholtz equation:

$$(\nabla^2 + k^2)\phi = 0$$

- Boundary conditions: Sommerfeld radiation condition and

$$\frac{\partial \phi_t}{\partial n} = 0 \implies \frac{\partial \phi}{\partial n} = -\frac{\partial \phi_i}{\partial n}$$

Strategy

- Key assumption: $ka \equiv \epsilon \ll 1$; *i.e.* small step
- Presence of two distinct length scales (a and $1/k$) implies singular perturbation problem
- Hence need to solve two problems: ‘outer’ and ‘inner problem’ — one for each length scale
- This is where method of Matched Asymptotic Expansions comes into use
- Will have undetermined constants in solutions to both problems due to impossibility of application of condition
  - $\partial\phi/\partial x = 0$ for outer problem (cannot meaningfully apply boundary condition to vanishingly thin edge)
  - Sommerfeld radiation condition for inner problem (cannot go to ∞)

Outer Problem

- Select scalings of $\bar{x} = x/k$, $\bar{y} = y/k$ to obtain (also rewriting ϕ_i for convenience)

$$(\nabla^2 + 1)\phi = 0, \qquad \phi_i = 2\cos(y\sin\theta)\mathrm{e}^{-\mathrm{i}x\cos\theta}.$$

- Boundary conditions are then

$$\begin{aligned}
-\infty < x \le 0, \qquad & y = 0: \qquad \frac{\partial \phi}{\partial y} = 0, \\
x = 0, \quad 0 < y < \epsilon: \qquad & \frac{\partial \phi}{\partial x} = 2\mathrm{i}\cos\theta\cos(y\sin\theta), \\
0 \le x < \infty, \qquad & y = \epsilon: \qquad \frac{\partial \phi}{\partial y} = 2\sin\theta\sin(\epsilon\sin\theta)\mathrm{e}^{-\mathrm{i}x\cos\theta}.
\end{aligned}$$

- Pose asymptotic expansion for ϕ in outer variables of form

$$\phi(x,y) \sim \epsilon\phi_1(x,y) + \sum_{n=2} g_n(\epsilon)\phi_n(x,y); \qquad g_{n+1} = o(g_n)$$

- No ϕ_0 term since forcing in outer problem is of $\mathcal{O}(\epsilon)$

Outer Problem ($\mathcal{O}(\epsilon)$ Solution)

- For $\mathcal{O}(\epsilon)$ solution, plane appears uniformly flat so that only boundary condition is on $y = 0$:

$$-\infty < x \leq 0, \; y = 0 : \qquad \frac{\partial \phi_1}{\partial y} = 0,$$
$$0 \leq x < \infty, y = 0 : \qquad \frac{\partial \phi_1}{\partial y} = 2 \sin^2 \theta \mathrm{e}^{-\mathrm{i} x \cos \theta},$$

- Solve $(\nabla^2 + 1)\phi_1 = 0$, so by Fourier transforms:

$$\phi_1 = -\frac{\mathrm{i} \sin^2 \theta}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\alpha x - (\alpha^2 - 1)^{1/2} y}}{(\alpha^2 - 1)^{1/2}(\alpha + \cos\theta)} \, \mathrm{d}\alpha + a_1 H_0^{(1)}(r),$$

where we have added the Hankel function as a source term at the origin due to the singular behaviour at step

Inner Problem

- This time select scalings of $\bar{x} = aX$, $\bar{y} = aY$ so that

$$(\nabla^2 + \epsilon^2)\phi = 0, \qquad \phi_i = 2\cos(\epsilon Y \sin\theta)\mathrm{e}^{-\mathrm{i}\epsilon X \cos\theta}$$

- Corresponding boundary conditions are

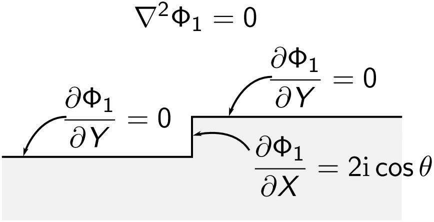
$$\begin{aligned}
-\infty < X \leq 0, \qquad Y = 0: \qquad & \frac{\partial \phi}{\partial Y} = 0, \\
X = 0, \quad 0 < Y < 1: \qquad & \frac{\partial \phi}{\partial X} = 2\mathrm{i}\epsilon\cos\theta\cos(\epsilon Y \sin\theta), \\
0 \leq X < \infty, \qquad Y = 1: \qquad & \frac{\partial \phi}{\partial Y} = 2\epsilon\sin\theta\sin(\epsilon\sin\theta)\mathrm{e}^{-\mathrm{i}\epsilon X\cos\theta}.
\end{aligned}$$

- Asymptotic expansion posed is now in terms of inner variables

$$\phi(X, Y) \sim \epsilon\Phi_1(X, Y) + \sum_{n=2} G_n(\epsilon)\Phi_n(X, Y); \qquad G_{n+1} = o(G_n)$$

- Forcing in inner region is also of $\mathcal{O}(\epsilon)$ so no Φ_0 term here

Inner Problem ($\mathcal{O}(\epsilon)$ Solution)



- Solve this via Schwarz–Christoffel mapping — type of conformal mapping that maps open/closed polygons to the Upper Half Plane
- Solve $\nabla^2\Phi_1 = 0$ in Upper Half Plane (ζ-plane) and then map solution back to original geometry

Inner Problem ($\mathcal{O}(\epsilon)$ Solution)

- One such mapping is

$$Z(\zeta) = \frac{2}{\pi}\left[\sqrt{\zeta}\sqrt{\zeta - 1} - \log\left(\mathrm{e}^{-2\mathrm{i}/\pi}\left\{\sqrt{\zeta - 1} + \sqrt{\zeta}\right\}\right)\right]$$

- Solution is

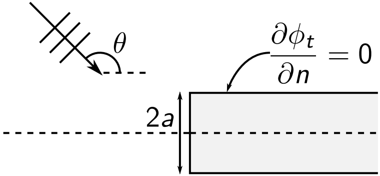
$$\Phi_1 = 2\mathrm{i}\cos X + A_1\,\mathrm{Re}(\zeta^2) + B_1$$

- Nasty but as we are interested in limit of inner solution for matching purposes, can just take leading order term:

$$Z(\zeta) \sim \frac{2}{\pi}\zeta + \dots \qquad |Z| \to \infty$$

Comparison with Crighton & Leppington (1973)

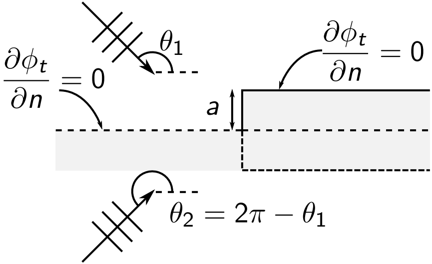
- All well and good, but is there an easier way of solving problem?
- Answer: Yes! Problem is really just disguised version of problem considered by Crighton and Leppington in 1973[2]
- They consider diffraction by a plate of finite thickness



[2] D.G. Crighton & F.G. Leppington *Singular perturbation methods in acoustics: diffraction by a plate of finite thickness*, Proc. R. Soc. Lond. A, **335**, pp. 313-339 (1973)

Comparison with Crighton & Leppington (1973)

- Our problem is their problem with two incident waves such that they cancel out along line of symmetry



- Should hence be able to obtain solution to our problem by discarding odd solutions (in y) and keeping even solutions from their paper

Crighton & Leppington (1973)

- The $\mathcal{O}(1)$ solution for their outer problem reduces to the classical Sommerfeld problem of diffraction by a thin plate
- Entails a rather nasty contour integral — needs to be evaluated in terms of Fresnel or error functions
- Not needed in my problem as this is an odd solution in y
- The next term in their solution is composed of Hankel functions, hence suggesting that we should expect Hankel functions in our solution too
- The third term requires the Wiener-Hopf method — not needed here again

The Putative Solution

- After matching, total acoustic potential ϕ_t is (in dimensional coordinates)

$$\phi_t(x,y) = \underbrace{\mathrm{e}^{-\mathrm{i}k(x\cos\theta + y\sin\theta)}}_{\text{incident wave}} + \underbrace{\mathrm{e}^{-\mathrm{i}k(x\cos\theta - y\sin\theta)}}_{\text{reflected wave}}$$

$$+\epsilon\left[-\frac{\mathrm{i}\sin^2\theta}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx-(\alpha^2-1)^{1/2}ky}}{(\alpha^2-1)^{1/2}(\alpha+\cos\theta)}\,\mathrm{d}\alpha + \underbrace{\frac{\cos\theta}{2}H_0^{(1)}(kr)}_{\text{like a monopole source}}\right]$$

$$+o(\epsilon)$$

- Somewhat surprising that the scattering potential is like a monopole and not a dipole

Historical Note

- Paper was important due to being the first to apply MAEs to acoustic problems and one of first discussions of new matching principle
- Standard matching principle at time was due to Van Dyke, in which terms such as $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon \log \epsilon)$ were regarded as 'asymptotically distinct'
- Crighton and Leppington argued that such terms should instead be treated as being of same order — group by algebraic powers of ϵ, regardless of any $\log \epsilon$ terms
- Failure to do so led to inconsistencies in higher-order terms

Summary

- PhD in Structural Acoustics with eventual goal of minimising sound from underwater vehicles, with work of interest to Thales
- Read papers by Abrahams *et al.* and Crighton & Leppington
- Studied starter problem of scattering off step
- Learning how to use techniques such as Schwarz–Christoffel and Matched Asymptotic Expansions
- Future work: extend ideas of scattering off step to new and exciting geometries as well as in-plane vibrations for elastic structures
- Questions?