Instanton effects in string/M-theory from 3d superconformal field theories

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References:

[M.H.-Okuyama, 1405.3653] [M.H.-Moriyama, 1404.0676]
[Hatsuda-M.H.-Moriyama-Okuyama, 1306.4297]

+ recent papers by
    Calvo, Codesido, Grassi, Hatsuda, Kallen, Marino, Matsumoto, Moriyama,
    Nosaka, Okuyama, Putrov, Yamazaki and Zakany

19th, Dec.    Indian Strings Meeting 2014
Non-perturbative effects in string/M-theory

⊃Worldsheet, D-brane and membrane instantons
Non-perturbative effects in string/M-theory

Worldsheet, D-brane and membrane instantons

In this talk, I will report

low-energy effective theories of M2-branes provide good laboratory to probe these effects via AdS/CFT.
M2-branes w/ fractional M2-branes in certain space

3d necklace quiver Chern-Simons matter theory
(N M2-branes) + (M fractional M2-branes) on $R^8/Z_k$

(=M5-branes wrapped on $S^3/Z_k \subset R^8/Z_k$)

[Aharony-Bergman-Jafferis-Maldacena ’08, Aharony-Bergman-Jafferis ’08]
(N M2-branes) + (M fractional M2-branes) on $\mathbb{R}^8/\mathbb{Z}_k$

(=M5-branes wrapped on $S^3/\mathbb{Z}_k \subset \mathbb{R}^8/\mathbb{Z}_k$)

Effective theory = ABJ(M) theory:

$$3d \, \mathcal{N} = 6 \, U(N)_k \times U(N+M)_{-k}$$

superconformal Chern-Simons theory
\((N \text{ M2-branes}) + (M \text{ fractional M2-branes})\) on \(R^8/Z_k\)

\((=\text{M5-branes wrapped on } S^3/Z_k \subset R^8/Z_k)\)

Effective theory = ABJ(M) theory:

\[ 3\text{d } \mathcal{N} = 6 \ U(N)_k \times U(N+M)_{-k} \]  

\((k: \text{CS level})\)

superconformal Chern-Simons theory

\[
\begin{align*}
\begin{cases}
\text{• Vector multiplet} \\
\text{• 2 bi-fundamental chiral multiplets} \\
\text{• 2 anti-bi-fundamental chiral multiplets}
\end{cases}
\end{align*}
\]

(in 3d \(\mathcal{N} = 2\) language)
CFT$_3$ / AdS$_4$

$U(N)_k \times U(N+M)_{-k}$

ABJ theory
$\text{CFT}_3 \quad / \quad \text{AdS}_4$

$U(N)_k \times U(N+M)_{-k}$

ABJ theory

$M$-theory on $\text{AdS}_4 \times S^7 / Z_k$

$k \ll N^{1/5}$

with

$$\frac{1}{2\pi} \int_{S^3 / Z_k} C_3 = \frac{1}{2} - \frac{M}{k}$$
\textbf{CFT}_3 \quad / \quad \textbf{AdS}_4

\textbf{U}(N)_k \times \textbf{U}(N+M)_{-k}

\textbf{ABJ theory}

\begin{align*}
\lambda & = \frac{N}{k} = \text{fixed, } N \gg 1 \\
k & \ll N^{1/5}
\end{align*}

M-theory

\text{on AdS}_4 \times S^7/Z_k

with \( \frac{1}{2\pi} \int_{S^3/Z_k} C_3 = \frac{1}{2} - \frac{M}{k} \)

Type IIA superstring

\text{on AdS}_4 \times \text{CP}^3

with \( \frac{1}{2\pi} \int_{\text{CP}^1} B_2 = \frac{1}{2} - \frac{M}{k} \)
CFT\textsubscript{3} \quad / \quad \text{AdS}_4

U(N)_k \times U(N+M)_-k

ABJ theory

M-theory
on AdS\textsubscript{4} \times S^7/Z\_k
with \( \frac{1}{2\pi} \int_{S^3/Z\_k} C_3 = \frac{1}{2} - \frac{M}{k} \)

Type IIA superstring
on AdS\textsubscript{4} \times \mathbb{CP}^3
with \( \frac{1}{2\pi} \int_{\mathbb{CP}^1} B_2 = \frac{1}{2} - \frac{M}{k} \)

Another limit I don’t consider here:
\( \frac{M}{k} = \text{fixed, } M \gg 1 \)

\( \leftrightarrow \quad \mathcal{N} = 6 \text{ Vasiliev theory on } \text{AdS}_4 \)


[Hirano-M.H.-Okuyama-Shigemori, to appear]
The figure is borrowed from Hatsuda-Marino-Moriyama-Okuyama [cf. Cagnazzo-Sorokin-Wulff ’09, Drukker-Marino-Putrov ’11].
$\text{AdS}_4 \times \mathcal{M} \times \mathcal{W}$

D2-brane instanton:
AdS$_4$ x 

**D2-brane instanton:**

**Worldsheet instanton:**

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Worldsheet instanton:

Membranane instantons

AdS$_4 \times$ D2-brane instanton:  

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[cf. Cagnazzo-Sorokin-Wulff ’09, Drukker-Marino-Putrov ’11]
**Worldsheet instanton:**
\[
\exp \left[ -T_{D2} \text{Vol}(RP^3) \right] = \exp \left[ -\pi \sqrt{\frac{2N^2}{\lambda}} \right]
\]

**D2-brane instanton:**
\[
\exp \left[ -\frac{1}{2\pi \alpha'} \text{Area}(CP^1) \right] = \exp \left[ -2\pi \sqrt{2\lambda} \right]
\]
AdS$_4 \times S^7 / \mathbb{Z}_k$

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*non-perturbative in the sense of genus expansion!!*

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Here I will overview recent progress on probing \textit{instanton} effects in string/M-theory from \textit{M2-brane} theories.
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ABJ(M) partition function on sphere:
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- **Ideal Fermi gas formalism**  
  [Marino-Putrov, Okuyama, Awata-Hirano-Shigemori, M.H.]
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- **Exact** computation of the ABJ partition function for various \((k,M,N)\)  
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Ex.) For \((k,M)=(2,1)\) up to \(N=65\), etc...
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- **Non-perturbative structure from the refined topological string**  
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**Some generalizations:**

- **BPS Wilson loop**  

- **Less SUSY theories**  
  [M.H.-Moriyama, Grassi-Marino, Hatsuda-Okuyama, Moriyama-Nosaka]
Main result

\[ e^{J(\mu)} \sim \sum_N Z_{\text{ABJ}}(k, M; N) e^{\mu N}, \]

\[ J(\mu) = J_{\text{perturbative}}(\mu) + J_{\text{WS-inst}}(\mu) + J_{\text{D2-inst}}(\mu) + J_{\text{mixed}}(\mu) \]

standard topological string

refined topological string (in Nekrasov-Shatashvili limit)

\[ Z_{\text{D2,\ell-inst;WS,m-inst}} = g_{\ell,m} \left( k, M; \frac{\partial}{\partial N} \right) \text{Ai} \left[ C^{-\frac{1}{3}}(k) \left( N - B(k, M) + 2\ell + \frac{4m}{k} \right) \right] \]

\[
\left( \frac{Z_{\text{D2,\ell-inst;WS,m-inst}}}{Z_{\text{perturbative}}} \right) \sim e^{-\pi \ell \sqrt{2kN - 2\pi m \sqrt{\frac{2N}{k}}}}
\]
Instanton effects from \( \text{ABJ(M)} \) partition function
$$Z_{\text{ABJ}(M)} = \int [D\Phi] \ e^{-S_{\text{ABJ}(M)}[\Phi]}$$
In this representation,

analysis is basically limited to perturbative expansion of $\lambda=N/k$. (inconvenient to study the instantons)
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SUSY Localization

[Kapustin-Willett-Yaakov, Jafferis, Hama-Hosomichi-Lee]

\[ Z_{\text{ABJ}}(M) = (\text{Finite dimensional integral}) \]
In this representation,

analysis is basically limited to perturbative expansion of \( \lambda = N/k \).
(inconvenient to study the instantons)

Standard matrix model technique is available to study genus expansion, which is convenient to study worldsheet instanton \( \mathcal{O}(e^{-2\pi \sqrt{2\lambda}}) \), but not D2-instanton \( \mathcal{O}(e^{-\pi \sqrt{2N^2/\lambda}}) \),
ABJ(M) theory as a Fermi gas

Localization + some explicit calculations lead us to

\[
\hat{Z}^{(N,N+M)}(k) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int_{-\infty}^{\infty} \frac{d^N y}{(4\pi k)^N} \prod_{a=1}^{N} \rho(y_a, y_{\sigma(a)}),
\]

\[
\rho(x, y) = \frac{\sqrt{V(x)V(y)}}{\cosh \frac{x-y}{2k}}.
\]

\[
V(x) = \frac{1}{e^{x/2} + (-1)^M e^{-x/2}} \prod_{s=-\frac{M-1}{2}}^{\frac{M-1}{2}} \tanh \frac{x + 2\pi is}{2|k|}.
\]
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\]

Ideal Fermi gas!!

Switch to grand canonical formalism

\[
\Xi_k^{(M)}(\mu) = \sum_{N=0}^{\infty} e^{\mu N} \hat{Z}^{(N,N+M)}(k) = \text{Det} \left[ 1 + e^{\mu \rho} \right]
\]
ABJ(M) Fermi gas as QM

Quantum mechanical description:

\[ \rho(x, y) = \langle x | e^{-\hat{H}(\hat{q}, \hat{p})} | y \rangle, \quad e^{-\hat{H}(\hat{q}, \hat{p})} = \sqrt{V(\hat{q})} \frac{1}{2 \cosh \frac{\hat{p}}{2}} \sqrt{V(\hat{q})}, \quad [\hat{q}, \hat{p}] = 2\pi i k, \]
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CS level \( k \) can be regarded as Planck constant: \( \hbar = 2\pi k \)
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Semi-classical expansion

Expansion in M-theory regime
ABJ(M) Fermi gas as QM

Quantum mechanical description:

\[ \rho(x, y) = \langle x | e^{-\hat{H}(\hat{q}, \hat{p})} | y \rangle, \quad e^{-\hat{H}(\hat{q}, \hat{p})} = \sqrt{V(\hat{q})} \frac{1}{2 \cosh \frac{\hat{p}}{2}} \sqrt{V(\hat{q})}, \quad [\hat{q}, \hat{p}] = 2\pi i k, \]

CS level \( k \) can be regarded as Planck constant: \( \hbar = 2\pi k \)

Semi-classical expansion

Expansion in M-theory regime

In this expansion,

D2-instanton: \( \mathcal{O}(e^{-\pi \sqrt{2kN}}) \) appears perturbatively

but not for worldsheet instanton: \( \mathcal{O}(e^{-2\pi \sqrt{2N/k}}) \)
Simple derivation of $N^{3/2}$ law

$$Z_{ABJ}^{(N,N+M)}(k) = \int d\mu \ e^{J_k^{(M)}(\mu) - N\mu}$$
Simple derivation of $N^{3/2}$ law

\[ Z_{\text{ABJ}}^{(N,N+M)}(k) = \int d\mu \, e^{J_k^{(M)}(\mu) - N\mu} \]

\[ N \to \infty \]

\[ \log \hat{Z}^{(N,N+M)}(k) \simeq J_k^{(M)}(\mu_*) - \mu_* N, \quad \text{with} \quad \left. \frac{\partial J_k^{(M)}(\mu)}{\partial \mu} \right|_{\mu=\mu_*} = N. \]
Simple derivation of $N^{3/2}$ law

$$Z^{(N,N+M)}_{\text{ABJ}}(k) = \int d\mu \ e^{J^{(M)}_{k}(\mu)} - N\mu$$

$N \to \infty$

$$\log \hat{Z}^{(N,N+M)}(k) \simeq J^{(M)}_{k}(\mu_\ast) - \mu_\ast N, \quad \text{with} \quad \frac{\partial J^{(M)}_{k}(\mu)}{\partial \mu} \bigg|_{\mu=\mu_\ast} = N.$$  

Classical Hamiltonian:

$$H_{\text{cl}}(q, p) = \log \left( 2 \cosh \frac{q}{2} \right) + \log \left( 2 \cosh \frac{p}{2} \right) \sim \frac{|q| + |p|}{2}$$
Simple derivation of $N^{3/2}$ law

$$Z_{ABJ}^{(N,N+M)}(k) = \int d\mu \ e^{J_k^{(M)}(\mu)} - N\mu$$

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with $$\frac{\partial J_k^{(M)}(\mu)}{\partial \mu} \bigg|_{\mu=\mu_*} = N.$$

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Classical grand potential:

$$J_k^{(M)}(\mu) \sim \int dE \ \frac{\text{Vol}(H_{cl} \leq E)}{1 + ze^{-E}} \sim \frac{2}{3\pi^2 k} \mu^3,$$

$$\mu_* = \pi \sqrt{\frac{kN}{2}}$$

$H(q,p) = E = 4$
Simple derivation of $N^{3/2}$ law

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$N \to \infty$

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Classical grand potential:

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$$\log Z_{ABJ}^{(N,N+M)}(k) \sim -\frac{\pi \sqrt{2k}}{3} N^{3/2}$$
Semi-classical analysis shows

\[ J(\mu) = \frac{C}{3} \mu^3 + B\mu + A + \text{(instantons)} \]

\( (C,B,A: \text{independent of } \mu) \)

need information only on leading and sub-leading

need full order information

[ Marino-Putrov]

Perturbative part

Semi-classical analysis shows

\[ J(\mu) = \frac{C}{3} \mu^3 + B\mu + A + \text{(instantons)} \]

(C, B, A: independent of \( \mu \))

Need information only on leading and sub-leading

Need full order information

This is true also for general \( \mathcal{N} \geq 3 \) necklace quiver.
One-loop test of AdS/CFT

\[ \hat{Z}_{\text{pert}}^{(N,N+M)}(k) = C^{-1/3} e^A \text{Ai}[C^{-1/3}(N - B)]. \]
One-loop test of AdS/CFT

\[ \hat{Z}_{\text{pert}}^{(N,N+M)}(k) = C^{-1/3} e^\Lambda \text{Ai}[C^{-1/3}(N - B)]. \]

\[ N \gg 1 \]

\[ \log \hat{Z}_{\text{pert}}^{(N,N+M)}(k) = -\frac{2}{3} C^{-1/2} N^{3/2} + C^{-1/2} B N^{1/2} - \frac{1}{4} \log N + \mathcal{O}(1). \]
One-loop test of AdS/CFT

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(classical SUGRA)
One-loop test of AdS/CFT

\[ \hat{Z}_{\text{pert}}^{(N,N+M)}(k) = C^{-1/3}e^{A} \text{Ai}\left[C^{-1/3}(N - B)\right]. \]

\[ \text{log} \hat{Z}_{\text{pert}}^{(N,N+M)}(k) = \frac{-2}{3}C^{-1/2}N^{3/2} + C^{-1/2}BN^{1/2} - \frac{1}{4}\log N + \mathcal{O}(1). \]

$N \gg 1$

classical SUGRA

universal term coming from Airy
One-loop test of AdS/CFT

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The logarithmic term appears in 11d SUGRA on AdS$_4 \times X_7$ at 1-loop.

[ Bhattacharyya –Grassi-Marino-Sen ’12]
One-loop test of AdS/CFT

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The logarithmic term appears in 11d SUGRA on AdS\(_4\) x X\(_7\) at 1-loop.  

[Dabholkar-Drukker-Gomes]

Airy function behavior also appears from localization of the SUGRA.  

[Bhattacharyya–Grassi-Marino-Sen ‘12]
We can also obtain exact values for various \((k,M,N)\) by applying integrability-like technique to the ideal Fermi gas.

Ex.) For \((k,M)=(2,1)\) up to \(N=65\) and for \((k,M)=(4,1)\) up to \(N=64\), etc...
We can also obtain exact values for various \((k, M, N)\) by applying integrability-like technique to the ideal Fermi gas

Ex.) For \((k, M) = (2, 1)\) up to \(N = 65\) and for \((k, M) = (4, 1)\) up to \(N = 64\), etc...

**Exact values for \((k, M) = (2, 1)\)**

\[
\hat{Z}(1, 2) (2) = \frac{1}{4\pi}, \quad \hat{Z}(2, 3) (2) = \frac{1}{128} - \frac{1}{16\pi^2}, \quad \hat{Z}(3, 4) (2) = \frac{5\pi^2 - 48}{4608\pi^3},
\]

\[
\hat{Z}(4, 5) (2) = \frac{9}{32768} + \frac{5}{3072\pi^4} - \frac{53}{18432\pi^2}, \quad \hat{Z}(5, 6) (2) = \frac{6240 - 800\pi^2 + 17\pi^4}{29491200\pi^5},
\]

\[
\hat{Z}(6, 7) (2) = \frac{-218880 + 1413600\pi^2 - 1160264\pi^4 + 103275\pi^6}{8493465600\pi^6},
\]

\[
\hat{Z}(7, 8) (2) = \frac{-4677120 - 8631840\pi^2 + 14206864\pi^4 - 1345977\pi^6}{1664719257600\pi^7},
\]

\[
\hat{Z}(8, 9) (2) = \frac{61608960 - 1051438080\pi^2 + 2363612608\pi^4 - 1477376224\pi^6 + 126511875\pi^8}{213084064972800\pi^8},
\]

\[
\hat{Z}(9, 10) (2) = \frac{633830400 + 6140897280\pi^2 - 22473501120\pi^4 + 16465544384\pi^6 - 1444050207\pi^8}{23013079017062400\pi^9},
\]
Exact values for \((k,M) = (2,1)\)
Exact values for \((k,M) = (3,1)\)

\[ z^{(1,2)}(3) = \frac{1}{17} \left( 2\sqrt{3} - 3 \right), \quad z^{(2,3)}(3) = \frac{1}{34} \left( -27 + 14\sqrt{3} + 9 \right), \quad z^{(3,4)}(3) = -\frac{45 + 18\sqrt{3} - 14\sqrt{3}}{1728\pi}, \quad z^{(4,5)}(3) = \frac{702 + 84(27 + 2\sqrt{3})\pi + (1152\sqrt{3} - 2881)\pi^2}{248832\pi^4}, \]

\[ z^{(5,6)}(3) = \frac{54(14\sqrt{3} - 37) + 840\sqrt{3}\pi + (5797 - 3574\sqrt{3})\pi^2}{993528\pi^3}, \quad z^{(6,7)}(3) = \frac{17882 + 162(182\sqrt{3} - 1647)\pi + 27(2394\sqrt{3} - 5905)\pi^2 - 7(42110\sqrt{3} - 78327)\pi^3}{322486272\pi^3}, \]

\[ z^{(7,8)}(3) = -2430(61 + 18\sqrt{3}) + 83916\sqrt{3}\pi + 27(28553 + 16770\sqrt{3})\pi^2 + (78732 - 335594\sqrt{3})\pi^3, \]

\[ z^{(7,8)}(3) = \frac{1472580 + 9072(2295 + 74\sqrt{3})\pi + 324(9108\sqrt{3} - 105709)\pi^2 - 168(793233 + 60254\sqrt{3})\pi^3 + (178071703 - 76499136\sqrt{3})\pi^4}{1289945088\pi^4}, \]

\[ z^{(9,10)}(3) = \frac{2916(774\sqrt{3} - 2857) + 5533920\sqrt{3}\pi - 324(152814\sqrt{3} - 521209)\pi^2 - 48(1483907\sqrt{3} - 511758)\pi^3 + (22686378\sqrt{3} - 370195279)\pi^4}{371504185344\pi^4}, \]

\[ z^{(9,10)}(3) = \frac{299318280 + 72900(7070\sqrt{3} - 171747)\pi + 24300(882464\sqrt{3} - 411157)\pi^2 - 2700(6698762\sqrt{3} - 62488989)\pi^3 - 9(5065099200\sqrt{3} - 9212744479)\pi^4 + 25(6475592722\sqrt{3} - 11826421389)\pi^5}{4012245210175200\pi^5}, \]

\[ z^{(11,12)}(3) = -131220(28925 + 664\sqrt{3}) + 2915854200\sqrt{3}\pi + 24300(2404421 + 1236762\sqrt{3})\pi^2 - 5400(879505\sqrt{3} - 7125246)\pi^3 - 9(4022657395 + 2716080898\sqrt{3})\pi^4 + 50(3671699105\sqrt{3} - 1298211948)\pi^5 \]

\[ + 160489808068608000000, \]

\[ z^{(12,13)}(3) = \frac{25651672920 + 3674160(281907 + 4562\sqrt{3})\pi + 218700(11516544\sqrt{3} - 6754681)\pi^2 - 26800(106545861 + 3385340\sqrt{3})\pi^3 - 162(232246756800\sqrt{3} - 175868541043)\pi^4}{160489808068608000000}, \]

\[ + 36(3302763448131 + 214002197506\sqrt{3})\pi^2 + 6(2873091396912\sqrt{3} - 647990832207)\pi^3, \]

\[ z^{(13,14)}(3) = \frac{1}{27732638834255462400\pi^6} \left[ 787320(66366\sqrt{3} - 316045) + 212550156000\sqrt{3}\pi - 218700(12640806\sqrt{3} - 69160621)\pi^2 - 64800(14749559\sqrt{3} - 104910390)\pi^3 \right. \]

\[ + 162(296655013938\sqrt{3} - 1076243018035)\pi^4 + 360(205884495833\sqrt{3} - 99859335845)\pi^5 - 175(1188902574054\sqrt{3} - 1946635604621)\pi^6, \]

\[ z^{(14,15)}(3) = \frac{1}{14676112471087990702080\pi^7} \left[ 9762153103980 + 38578680(456134\sqrt{3} - 24118263)\pi + 19289340(70712064\sqrt{3} - 37224121)\pi^2 - 3572100(37149442\sqrt{3} - 941265173)\pi^3 \right. \]

\[ - 23814(11647409241600\sqrt{3} - 760466097211)\pi^4 + 7938(492094175420\sqrt{3} - 4908526126129)\pi^5 + 9(10577413403217152\sqrt{3} - 16694829966566529)\pi^6 - 1225(2411451380806186\sqrt{3} - 436239059157621)\pi^7 \]

\[ - 23814(2803076569895 + 2326367522214\sqrt{3})\pi^4 + 7938[(9929504595129\sqrt{3} - 1601735224140)\pi^5 + 9(65702735219040679 + 4951016488530726\sqrt{3})\pi^6 - 2450(134862880599065\sqrt{3} - 58222941612138)\pi^7, \]

\[ z^{(15,16)}(3) = \frac{1}{58704498843519628083200\pi^8} \left[ -49601160(3662825 + 677322\sqrt{3}) + 170696384888400\sqrt{3}\pi + 19289340(210402593 + 126703602\sqrt{3})\pi^2 - 7144200(729601789\sqrt{3} - 1620343926)\pi^3 \right. \]

\[ - 23814(2803076569895 + 2326367522214\sqrt{3})\pi^4 + 7938[(9929504595129\sqrt{3} - 1601735224140)\pi^5 + 9(65702735219040679 + 4951016488530726\sqrt{3})\pi^6 - 2450(134862880599065\sqrt{3} - 58222941612138)\pi^7, \]

\[ + 360(205884495833\sqrt{3} - 99859335845)\pi^5 - 175(1188902574054\sqrt{3} - 1946635604621)\pi^6, \]

\[ \right] \]
Exact values for \((k,M) = (4,1)\)

\[
\begin{align*}
\tilde{z}^{(1,2)}(4) &= \frac{\pi}{10} - \frac{5\pi^2}{512}, \\
\tilde{z}^{(2,3)}(4) &= 12 + \frac{12 - 5\pi^2}{512^2}, \\
\tilde{z}^{(3,4)}(4) &= -168 + \frac{396\pi^2 + 20\pi^4 - 99\pi^3}{37378}, \\
\tilde{z}^{(4,5)}(4) &= \frac{1200 + 4320\pi - 3512\pi^2 - 4378\pi^3 + 1755\pi^4}{4718592}, \\
\tilde{z}^{(5,6)}(4) &= -\frac{38880 + 241200\pi + 18600\pi^2 - 40000\pi^3 - 203494\pi^4 + 96975\pi^5}{1887436800\pi^5}, \\
\tilde{z}^{(6,7)}(4) &= \frac{953200 + 8320320\pi - 737880\pi^2 - 9678480\pi^3 + 17373764\pi^4 + 27667476\pi^5 - 9333225\pi^6}{545581798400\pi^6}, \\
\tilde{z}^{(7,8)}(4) &= -\frac{52536960 + 69134680\pi + 56647920\pi^2 - 2914304100\pi^3 - 2041346488\pi^4 + 3962357364\pi^5 + 2156964930\pi^6 - 995722875\pi^7}{426168129945600\pi^7}, \\
\tilde{z}^{(8,9)}(4) &= \frac{478759680 + 8468167680\pi - 7157041920\pi^2 - 89293397760\pi^3 + 38691966624\pi^4 + 232256453184\pi^5 - 82822457776\pi^6 - 145218219408\pi^7 + 47021834475\pi^8}{54549520833936800\pi^8}, \\
\tilde{z}^{(9,10)}(4) &= -\frac{1}{2356539291347189760\pi^9} \left[ -12959654400 + 320811921600\pi + 24916749360\pi^2 - 2406136078080\pi^3 - 181379433120\pi^4 + 7622732486880\pi^5 \\
+ 5866548067808\pi^6 - 10929554789424\pi^7 - 6075970569810\pi^8 + 2721498152625\pi^9 \right], \\
\tilde{z}^{(10,11)}(4) &= \frac{1}{1885231439307751808000\pi^{10}} \left[ 646656988400 + 2085511936000\pi - 1590844752000\pi^2 - 423480742270000\pi^3 + 15545887162240\pi^4 + 2407085602588800\pi^5 \\
- 690712514324000\pi^6 - 485810278788960\pi^7 + 1434686348402316\pi^8 + 2720310664056300\pi^9 - 85388089265625\pi^{10} \right], \\
\tilde{z}^{(11,12)}(4) &= \frac{1}{364980805443852750028800\pi^{11}} \left[ -71248933324800 + 306683696309760\pi + 2147272497216000\pi^2 - 32994976801248000\pi^3 - 26307684678401280\pi^4 + 16982434233648576\pi^5 \\
+ 173665340769940800\pi^6 - 543644538181282640\pi^7 - 506552450721933352\pi^8 + 79131277810194444\pi^9 + 502106696970976050\pi^{10} - 218816278991454375\pi^{11} \right], \\
\tilde{z}^{(12,13)}(4) &= \frac{1}{210228943935691840165888000\pi^{12}} \left[ 2305385523302400 + 126291787634073600\pi - 84666760738560000\pi^2 - 4394402461709568000\pi^3 + 129932865729107840\pi^4 \\
+ 446228425933199385600\pi^5 - 9109124891322297600\pi^6 - 18733352163573261440\pi^7 + 38674114099980946736\pi^8 + 324147992295235894\pi^9 - 8269388051377884280\pi^{10} \\
- 168381450188362233000\pi^{11} + 517590537129378125\pi^{12} \right], \\
\tilde{z}^{(13,14)}(4) &= \frac{1}{5684590640202243358085611520000\pi^{13}} \left[ -326696029973913600 + 231110746777403084800\pi + 14222217559476326400\pi^2 - 296594132415241236000\pi^3 - 262735740032464258560\pi^4 \\
+ 17245225545827426956800\pi^5 + 5140617642131751546240\pi^6 - 934641469706594236160\pi^7 - 176787028975945443094\pi^8 + 38402692345719161274672\pi^9 + 45493756685677679170896\pi^{10} \\
- 63043272699716161765224\pi^{11} - 42976871049629192344650\pi^{12} + 18272369792404283180625\pi^{13} \right],
\end{align*}
\]
Comparison with classical SUGRA

\[ F_{\text{SUGRA}} = -\frac{\pi \sqrt{2k}}{3} N^{3/2} . \]
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Now we have information on

- genus expansion
- small-k expansion
- exact values for various specific \((k,M,N)\)
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However,

we have **not** obtained exact results for arbitrary \((k,M,N)\)
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However, we have not obtained exact results for arbitrary \((k, M, N)\)

To determine structures of non-perturbative effects completely,

we will "guess" the form of the grand potential and test this "guess" by using the above information.
Basic idea

ABJ(M) matrix model
Basic idea

ABJ(M) matrix model

Analytic continuation

Pure CS theory on $S^3/Z_2$
(Lens space matrix model)

[cf. Marino-Putrov]
Basic idea

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Pure CS theory on $S^3/Z_2$
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Geometric transition
[cf. Gopakumar-Vafa '98]

Topological string on certain space (local $\mathbb{P}^1 \times \mathbb{P}^1$)
Perturbative + Worldsheet instanton part
Perturbative + *Worldsheet instanton* part

|   |

Perturbative in the sense of genus expansion
Perturbative + \textit{Worldsheet instanton} part

\textmid\textmid

Perturbative \textit{in the sense of genus expansion}

This part is described by the \textit{standard} topological string.
Perturbative + Worldsheet instanton part

Perturbative in the sense of genus expansion

This part is described by the standard topological string.

\[
\begin{align*}
Z_{\text{WS, m-inst}} &= d_m(k, M) \text{Ai} \left[ C^{-1/3} \left( N - B + \frac{4m}{k} \right) \right] \\
\frac{Z_{\text{WS, m-inst}}}{Z_{\text{pert}}} &\sim e^{-2\pi m \sqrt{\frac{2N}{k}}}
\end{align*}
\]
Test of WS 1-instanton

\[ Z^{(N,N+M)}_{\text{WS,1-instanton}}(k) = -2C^{-1/3} e^A \frac{\cos \pi \left( 1 - \frac{2M}{k} \right)}{\sin^2 \frac{2\pi}{k}} \text{Ai} \left[ C^{-1/3} \left( B - N - \frac{4}{k} \right) \right] \]
Test of WS 1-instanton

\[
Z_{WS,1\text{-inst}}(k) = -2C^{-1/3} e^{A \cos \pi \left( 1 - \frac{2M}{k} \right)} \frac{\sin^2 \frac{2\pi}{k}}{\text{Ai} \left[ C^{-1/3} \left( B - N - \frac{4}{k} \right) \right]}
\]

\[
e^{\frac{2\pi}{k} \sqrt{\frac{2N}{k}}} \left( Z_{ABJ}^{(N,N+M)}(k) - Z_{\text{pert}}^{(N,N+M)}(k) - Z_{WS,1\text{-inst}}^{(N,N+M)}(k) \right)
\]

\[(k, M) = (6, 1)\]
Problem on worldsheet instanton effect

[ Hatsuda-Moriyama-Okuyama, Matsumoto-Moriyama, M.H.-Okuyama]
The WS-instanton part is divergent for physical integer $k$. 
Problem on worldsheet instanton effect

The WS-instanton part is divergent for physical integer $k$.

For instance,

$$J_{\text{WS},1-\text{inst}} = \frac{\#}{\sin^2 \frac{2\pi}{k}}, \quad J_{\text{WS},2-\text{inst}} = \frac{\#}{\sin^2 \frac{4\pi}{k}} + \frac{\#}{\sin^2 \frac{2\pi}{k}}, \quad \text{etc.}$$
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However,

we know that the **exact result is finite**
Problem on worldsheet instanton effect

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However,

we know that the **exact result** is finite.

This divergence must be **apparent** and must **cancel out** if we include other sector: D2-instanton
D2-instanton + Mixture of D2- & WS-instanton part
D2-instanton + Mixture of D2- & WS-instanton part

|  |

Non-perturbative in the sense of genus expansion
D2-instanton + Mixture of D2- & WS-instanton part

Non-perturbative in the sense of genus expansion

This part is described by non-perturbative formulation of topological string: refined topological string in certain limit (Nekrasov-Shashvili limit)
D2-instanton + Mixture of D2- & WS-instanton part

Non-perturbative in the sense of genus expansion

This part is described by non-perturbative formulation of topological string: refined topological string in certain limit (Nekrasov-Shashvili limit)

\[
Z_{D2,\ell-\text{inst};\text{WS},m-\text{inst}} = g_{\ell,m} \left( k, M; \frac{\partial}{\partial N} \right) \text{Ai} \left[ C^{-1/3} \left( N - B + 2\ell + \frac{4m}{k} \right) \right]
\]

\[
\frac{Z_{D2,\ell-\text{inst};\text{WS},m-\text{inst}}}{Z_{\text{pert}}} \sim e^{-\pi \ell \sqrt{2kN} - 2\pi m \sqrt{\frac{2N}{k}}}
\]

[ Nekrasov-Shatashvili]
Test of our proposal

\[ 14 \pi \sqrt{\frac{2N}{k}} \frac{|\hat{Z} - \hat{Z}_{\text{inst}}|}{\hat{Z}_{\text{pert}}} \]

\( k = 4 \)

M=1

M=2
Drastic simplification for $\mathcal{N} = 8$ SUSY cases

Generally,

the ABJ(M) grand potential receives contributions from all-genus of topological string free energy.
Drastic simplification for $\mathcal{N} = 8$ SUSY cases

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However,

for $(k,M) = (1,0), (2,0)$ and $(2,1)$ (enhanced to $\mathcal{N} = 8$ SUSY),

the ABJ(M) grand potential after pole cancellation has contributions only from genus-0 and genus-1!!
Drastic simplification for $\mathcal{N} = 8$ SUSY cases

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However,

for $(k,M)=(1,0)$, $(2,0)$ and $(2,1)$ (enhanced to $\mathcal{N} = 8$ SUSY),

the ABJ(M) grand potential after pole cancellation has contributions only from genus-0 and genus-1 !!

\[
\Xi(\mu)|_{(k,M)=(1,0)} = \left(\vartheta_2(\xi/4, \bar{\tau}/4) + i\vartheta_1(\xi/4, \bar{\tau}/4)\right) \quad (\xi, \bar{\tau} : \text{determined by } F_0) \\
\times \exp\left[\frac{3\mu}{8} - \frac{3}{4} \log 2 + F_1 + F_{1\text{NS}} - \frac{1}{4\pi^2}\left(F_0 - \lambda \partial_\lambda F_0 + \frac{\lambda^2}{2} \partial_\lambda^2 F_0\right)\right]
\]

\[
\Xi(\mu)|_{(k,M)=(2,0)} = \vartheta_3(\xi, \bar{\tau}) \exp\left[\frac{\mu}{4} + F_1 + F_{1\text{NS}} - \frac{1}{\pi^2}\left(F_0 - \lambda \partial_\lambda F_0 + \frac{\lambda^2}{2} \partial_\lambda^2 F_0\right)\right]
\]

\[
\Xi(\mu)|_{(k,M)=(2,1)} = \vartheta_1(\xi+1/4, \bar{\tau}) \exp\left[\frac{\log 2}{2} + F_1 + F_{1\text{NS}} - \frac{1}{\pi^2}\left(F_0 - \lambda \partial_\lambda F_0 + \frac{\lambda^2}{2} \partial_\lambda^2 F_0\right)\right]
\]
Resumming the 1/N-expansion in ABJM

\[ F_{\text{ABJM}}|_{\text{genus-}g} \sim (2g)! \quad \text{asymptotic} \]
Resumming the $1/N$-expansion in ABJM

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Can we resum the $1/N$-expansion (dual string perturbation series)?
Resumming the 1/N-expansion in ABJM

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Can we resum the 1/N-expansion (=dual string perturbation series)?

——— Yes, because this is Borel summable.
Resumming the $1/N$-expansion in ABJM

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Can we resum the $1/N$-expansion (=dual string perturbation series)?

—— Yes, because this is Borel summable.

Does the Borel resummation reproduce the exact results?

Does resummed string perturbation series describe D2-instanton?

—— No, Grassi-Marino-Zakany have found relevant differences.

We should resum each string perturbation series around each D2-instanton background (to get full result).
Some generalizations
Half-BPS Wilson loop in ABJM

[Hatsuda-M.H.-Moriyama-Okuyama, Grassi-Kallen-Marino]
Half-BPS Wilson loop in ABJM

By localization + some explicit calculations,

\[ \langle \text{Generating function} \rangle = (\text{Ideal Fermi gas}) \]
Half-BPS Wilson loop in ABJM

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The Wilson loop is described by the open topological string.
Half-BPS Wilson loop in ABJM

By localization + some explicit calculations,

\[
\langle \text{Generating function} \rangle = (\text{Ideal Fermi gas})
\]

The Wilson loop is described by the open topological string.

\[
Z_{\text{ABJM}} \langle W_R \rangle_{D_2, \ell-\text{inst}; WS, m-\text{inst}} = d_{\ell, m}(k) \text{Ai} \left[ C^{-\frac{1}{3}} \left( N - B + \frac{2|R|}{k} + 2\ell + \frac{4m}{k} \right) \right]
\]

\[
\langle W_R \rangle_{D_2, \ell-\text{inst}; WS, m-\text{inst}} \sim e^{\pi |R| \sqrt{\frac{2N}{k}} - \pi \ell \sqrt{2kN} - 2\pi m \sqrt{\frac{2N}{k}}}
\]
Less SUSY theories

[M.H.-Moriyama, Grassi-Marino, Hatsuda-Okuyama, Moriyama-Nosaka]
Less SUSY theories

Is the pole cancelation common in general M2-brane theories?
Is the pole cancelation common in general M2-brane theories?

Yes, probably.

Cancelation has been found also in some $\mathcal{N} = 4$ M2-brane theories (=special cases of Gaiotto-Witten theory).
Less SUSY theories

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——— Yes, probably.

Cancelation has been found also in some $\mathcal{N} = 4$ M2-brane theories (=special cases of Gaiotto-Witten theory).

Technical difficulties for less SUSY theories:

1. Corresponding topological string is unknown.

2. Except some special cases, density matrix of Fermi gas becomes complicated (given by integral)

3. For $\mathcal{N} = 2$, Fermi gas becomes interacting.
Summary & Outlook
Summary

ABJ(M) partition function on sphere:
Summary

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- Fermi gas formalism is powerful.
  
  Semi-classical expansion = M-theory expansion
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Summary

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  Semi-classical expansion = M-theory expansion

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Summary

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- Drastic simplification for \(\mathcal{N} = 8\) SUSY cases
- The 1/N-expansion in ABJM is Borel summable. But the resummation deviates from the exact values.
Summary

**ABJ(M) partition function on sphere:**

- Fermi gas formalism is powerful.
  Semi-classical expansion = M-theory expansion
- Exact computation of the ABJ partition function for various \((k,M,N)\)
- Non-perturbative structure from the refined topological string
- Drastic simplification for \(\mathcal{N} = 8\) SUSY cases
- The \(1/N\)-expansion in ABJM is **Borel summable.**
  But the resummation deviates from the exact values.

**Some generalizations:**

- Half-BPS Wilson loop in ABJM is described by open topological string.
- Pole cancelation occurs also in some less SUSY theories.
Outlook

- ABJ theory in higher spin limit [Hirano-M.H.-Okuyama-Shigemori, to appear]
- More general M2-brane theory [Hatsuda-M.H.-Okuyama, work in progress]
- Other quantities
  Ex.) Vortex loop, Energy-momentum tensor correlator, super-Renyi entropy
- Relation to Higgs branch localization formula [cf. Pasquetti, Fujitsuka-M.H.-Yoshida, Benini-Peelaers]
  Localization formula has another equivalent representation in terms of vortex partition functions for many 3d theories.
- Analysis on the gravity side
  Test many predictions.
  Probably, localization on the gravity side and string perturbation around instanton background would be useful.

Thanks!