

Aperiodic maximisers of Lyapunov dimension: A counterexample to the unrestricted form of Eden’s conjecture

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Abstract

We disprove the unrestricted form of Eden’s conjecture, which asserts that the maximal local Lyapunov dimension on a compact global attractor is attained at an equilibrium or a periodic orbit. For each $c > 4$, we construct a polynomial differential equation on $\mathbb{R}^5$ with uniformly negative divergence. Its compact global attractor has Lyapunov dimension $4 + 4/c$, attained precisely on an invariant two-torus carrying an irrational linear flow. The maximum over all equilibria and periodic orbits is 3. These conclusions hold both for the local dimension defined by singular values at every positive time and for the asymptotic Kaplan–Yorke dimension. The proof uses an exact singular-value identity on the whole attractor. The results of this paper are also formally verified in Lean.

1 Introduction

Lyapunov dimension relates the growth of infinitesimal perturbations to estimates for the dimension of invariant sets. The relation between local exponents, global exponents, and dimension estimates was developed by Eden [4] and Eden, Foias, and Temam [5]. In its unrestricted formulation for compact global attractors, Eden’s conjecture asserts that the maximal local Lyapunov dimension is attained at an equilibrium or a periodic orbit; see [10, following (50)] and [13, §2.1]. Such a reduction would replace a maximisation over the whole attractor by a calculation along distinguished trajectories.

Under explicit parameter conditions, Leonov, Kuznetsov, Korzhemanova, and Kusakin [12] prove an exact formula for the Lyapunov dimension of the Lorenz global attractor, attained at the origin. For Lur’e systems, Kato and Ishii [8, Theorem 2 and Corollary 1] give sufficient conditions under which the origin attains the Lyapunov dimension of every compact invariant set containing it. Recent computational approaches include polynomial optimisation for maximal sums of Lyapunov exponents [13] and optimisation over adapted metrics [1]. The latter gives numerical bounds close to proposed equilibrium and periodic maximisers in examples based on the Hénon and Rabinovich systems.

Aperiodic maximisation also occurs for products of matrices. Bousch and Mairesse [3] disproved the finiteness conjecture for the joint spectral radius. Hare, Morris, Sidorov, and Theys [6] subsequently gave an explicit pair of 2×2 matrices for which no periodic product attains the maximal asymptotic growth rate. Bochi [2] surveys the connection with ergodic optimisation of Lyapunov exponents.

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We disprove the unrestricted form of Eden's conjecture by constructing a forward-complete polynomial differential equation on $\mathbb{R}^5$. All pointwise Lyapunov exponents exist, but the maximum of the associated Kaplan–Yorke dimension is attained only on an invariant two-torus with no periodic orbits. The same maximising set occurs for the finite-time local dimension at every positive time. The counterexample also applies to the common-index definition in [13]; see Section 3. The attractor in this construction is nontransitive, and the result concerns the unrestricted statement. Formulations restricted to strange attractors or typical systems are discussed in [11, §§5.2–5.3].

1.1 Lyapunov dimensions

To state the result, we first recall some definitions. Let ϕ_t be the evolution of a forward-complete C^1 vector field on $\mathbb{R}^n$. All derivatives and singular values are taken in the ambient Euclidean space. The variational equation ensures that $D\phi_t(p)$ is invertible for every finite $t \geq 0$. For an invertible matrix $M \in \mathbb{R}^{n \times n}$, let $\sigma_1(M) \geq \dots \geq \sigma_n(M) > 0$ denote its singular values. The singular value function is defined by $\omega_n(M) = |\det M|$ and

$$\omega_{k+\alpha}(M) = \sigma_1(M) \cdots \sigma_k(M) \sigma_{k+1}(M)^\alpha, \quad k \in \{0, \dots, n-1\}, \quad 0 \leq \alpha < 1, \quad (1.1)$$

where the empty product is one. In particular, $\omega_0(M) = 1$.

The finite-time local Lyapunov dimension [9, §2] is

$$d_L(t, p) = \max\{d \in [0, n] : \omega_d(D\phi_t(p)) \geq 1\}, \quad t > 0, \quad (1.2)$$

and, for a compact invariant set K , define

$$\dim_L K = \inf_{t > 0} \sup_{p \in K} d_L(t, p). \quad (1.3)$$

The maximum in (1.2) exists because $d \mapsto \omega_d(M)$ is continuous and $d = 0$ is admissible.

For the pointwise asymptotic dimension, suppose that the limits

$$\lambda_i(p) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \sigma_i(D\phi_t(p)), \quad 1 \leq i \leq n, \quad (1.4)$$

exist. They are in nonincreasing order. Define $S_0(p) = 0$, $S_k(p) = \sum_{i=1}^k \lambda_i(p)$ for $1 \leq k \leq n$, and $j(p) = \max\{k \in \{0, \dots, n\} : S_k(p) \geq 0\}$. The Kaplan–Yorke dimension [7] is

$$d_{KY}(p) = \begin{cases} 0, & j(p) = 0, \\ j(p) + \frac{S_{j(p)}(p)}{|\lambda_{j(p)+1}(p)|}, & 0 < j(p) < n, \\ n, & j(p) = n. \end{cases} \quad (1.5)$$

In the middle case, maximality of $j(p)$ implies $\lambda_{j(p)+1}(p) < 0$. The convention $S_k(p) \geq 0$ includes neutral directions when a partial sum vanishes. Equivalently, (1.5) is the largest point at which the piecewise linear interpolant of the partial sums is nonnegative. The same description applies to (1.2) with $\lambda_i(p)$ replaced by $t^{-1} \log \sigma_i(D\phi_t(p))$.

1.2 Main result

Our construction combines two planar systems with an independent contracting coordinate. Identify $\mathbb{R}^5$ with $\mathbb{C}^2 \times \mathbb{R}$, equipped with its Euclidean norm. For $c > 4$ and $q(s) = -(s-1)(s-2)$, consider the following real polynomial vector field of degree five:

$$\dot{z}_1 = (q(|z_1|^2) + i)z_1, \quad \dot{z}_2 = (q(|z_2|^2) + i\sqrt{2})z_2, \quad \dot{w} = -cw. \quad (1.6)$$

Let ϕ_t denote its forward evolution, and define

$$A = \{(z_1, z_2, 0) : |z_1| \leq \sqrt{2}, |z_2| \leq \sqrt{2}\}, \quad \mathcal{T} = \{(z_1, z_2, 0) : |z_1| = |z_2| = 1\}. \quad (1.7)$$

The following theorem shows that this construction is a counterexample.

Theorem 1.1. *For every $c > 4$, system (1.6) is forward complete and has compact global attractor A . Its vector field F satisfies $\operatorname{div} F \leq 4 - c < 0$ on $\mathbb{R}^5$. For every $t > 0$,*

$$\max_{p \in A} d_L(t, p) = 4 + \frac{4}{c}, \quad \left\{ p \in A : d_L(t, p) = 4 + \frac{4}{c} \right\} = \mathcal{T}. \quad (1.8)$$

All singular-value Lyapunov exponents exist at every point of A , and

$$\dim_L A = \max_{p \in A} d_{KY}(p) = 4 + \frac{4}{c}, \quad \left\{ p \in A : d_{KY}(p) = 4 + \frac{4}{c} \right\} = \mathcal{T}. \quad (1.9)$$

If $\mathcal{P}(A)$ denotes the set of equilibria and points on periodic orbits in A , then

$$\max_{p \in \mathcal{P}(A)} d_{KY}(p) = \max_{p \in \mathcal{P}(A)} d_L(t, p) = 3 \quad (t > 0). \quad (1.10)$$

The flow on $\mathcal{T}$ has no periodic orbit. For each $p \in A$, $d_L(t, p) \rightarrow d_{KY}(p)$ as $t \rightarrow \infty$.

For example, $c = 8$ gives $\dim_L A = 9/2$, whereas the maximum in (1.10) is 3. Here $\dim_H A = 4$ and $\dim_H \mathcal{T} = 2$, while the Lyapunov dimensions concern the derivative in the ambient space $\mathbb{R}^5$. In Section 3, we show that two further conventions for Lyapunov dimension also give the value $4 + 4/c$ for A .

Lean verification for the results of this paper can be found here:

https://github.com/MColbrook/Eden_conjecture.

The formal verification is also registered on Palomar:

<https://palomar-registry.org/entry.html?id=PALOMAR-2026-09-22-000001&version=1>.

1.3 Proof ideas

The proof exploits the independence of radial growth and angular motion, as illustrated in Figure 1. The scalar radial equations give A as the global attractor. Each planar subsystem has an unstable invariant circle of radius one, and their product $\mathcal{T}$ has two expanding normal directions and two neutral tangential directions. Together with the contracting w direction, these give the exponents $(2, 2, 0, 0, -c)$ and dimension $4 + 4/c$. Constant angular velocities contribute only orthogonal rotations to the derivative. The irrational frequency ratio excludes periodic orbits on $\mathcal{T}$. Replacing $\sqrt{2}$ in the angular frequency by a positive rational number makes every trajectory on $\mathcal{T}$ periodic, without changing any singular value. Thus arbitrarily small changes of the angular frequency restore periodic attainment of the maximum.

The main estimate proves that this torus is the entire maximising set. In rotating orthonormal frames, the planar variational equations are diagonal. The choice of q makes the sum of the radial and tangential growth rates equal to $2 - 6(|z|^2 - 1)^2$, which is maximal exactly on the unit circle. Each planar rate is at least -4 on A , so $c > 4$ makes e^{-ct} the

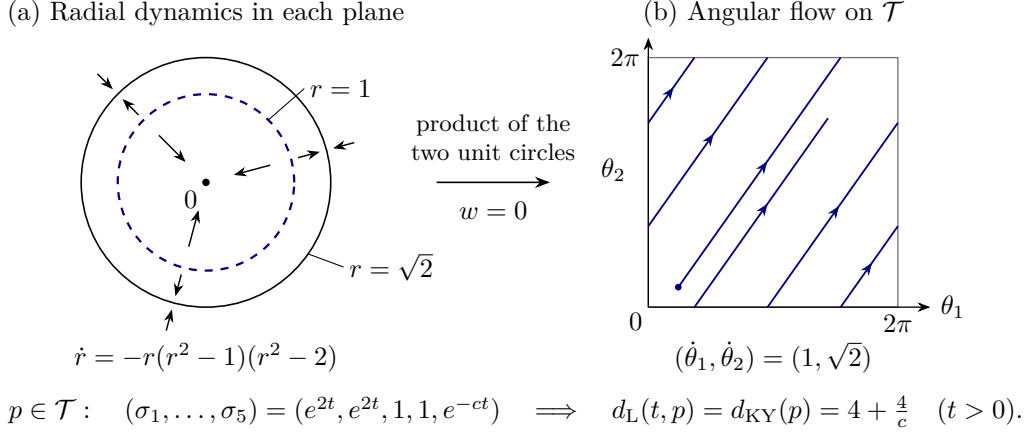


Figure 1: Mechanism of the construction, with $c > 4$. Left: the radial dynamics of either planar subsystem. The unit circle is unstable and the circle of radius $\sqrt{2}$ is attracting; the arrows show only radial motion. Right: a finite trajectory segment on the product torus $\mathcal{T}$, with opposite sides of the angular square identified. The irrational frequency ratio excludes periodic orbits. On $\mathcal{T}$, the two radial rates are 2, the two tangential rates are 0, and the w rate is $-c$.

smallest singular value. Thus, along the trajectory through $p \in A$ and for $t > 0$, the ordered singular values $\sigma_1 \geq \dots \geq \sigma_5$ of $D\phi_t(p)$ satisfy $\sigma_5 = e^{-ct}$ and

$$\left(\prod_{i=1}^4 \sigma_i \right) \sigma_5^{4/c} = \exp \left[-6 \int_0^t \sum_{j=1}^2 (|z_j(\tau)|^2 - 1)^2 d\tau \right] \leq 1.$$

Equality holds precisely when $p \in \mathcal{T}$. This identity gives the upper bound $4 + 4/c$ for the local dimension throughout A , with equality only on $\mathcal{T}$, at every positive time.

The radial limits determine the asymptotic dimensions. Each radius tends to 0, 1, or $\sqrt{2}$, and it tends to 1 only if it starts at 1. The resulting spectra give the same maximising set $\mathcal{T}$. A periodic solution must have constant radii; the irrational frequency ratio then forces at least one planar coordinate to vanish. The remaining periodic orbits have dimension at most 3.

2 Proof of the theorem

We first use the radial equations to establish forward completeness and identify the global attractor (Section 2.1). We then compute the singular values of the derivative in rotating orthonormal frames (Section 2.2). These formulas determine the maximum local dimension and all its maximisers at each positive time (Section 2.3). The radial limits give the limiting exponents, while the angular frequencies determine which trajectories are periodic (Section 2.4). Finally, we prove convergence of the local dimensions, treating separately the cases with vanishing leading exponents (Section 2.5).

2.1 The global attractor

In polar coordinates $z_j = r_j e^{i\theta_j}$, with $z_j \neq 0$ and $(\Omega_1, \Omega_2) = (1, \sqrt{2})$, system (1.6) becomes

$$\dot{r}_j = h(r_j) := -r_j(r_j^2 - 1)(r_j^2 - 2), \quad \dot{\theta}_j = \Omega_j, \quad \dot{w} = -cw. \quad (2.1)$$

A zero complex coordinate remains zero. The equilibrium radii are $0, 1, \sqrt{2}$. The sign of h gives

$$\rho(r_0) := \lim_{t \rightarrow \infty} r(t) = \begin{cases} 0, & 0 \leq r_0 < 1, \\ 1, & r_0 = 1, \\ \sqrt{2}, & r_0 > 1, \end{cases} \quad r(0) = r_0. \quad (2.2)$$

Indeed, the radius decreases on $(0, 1)$ and $(\sqrt{2}, \infty)$ and increases on $(1, \sqrt{2})$. Uniqueness prevents a solution from crossing an equilibrium radius.

Lemma 2.1. *System (1.6) is forward complete. The set A is compact, satisfies $\phi_t(A) = A$ for every $t \geq 0$, and attracts every bounded subset of $\mathbb{R}^5$ uniformly.*

Proof. Local existence and uniqueness follow from smoothness. The radial equation and the equation for w give

$$0 \leq r_j(t) \leq \max\{r_j(0), \sqrt{2}\}, \quad |w(t)| = e^{-ct}|w(0)|.$$

Thus solutions remain bounded on every finite forward interval and extend to all $t \geq 0$. The set A is compact and forward invariant. A solution in A also remains there backwards in time: each interval $[0, 1]$ and $[1, \sqrt{2}]$ is bounded by equilibrium radii, the angles are defined for all time, and $w = 0$. Compactness then gives backward continuation in A , and hence $\phi_t(A) = A$.

Let $B \subset \mathbb{R}^5$ be bounded. Choose $M > \sqrt{2}$ and $W \geq 0$ such that both initial radii are at most M and $|w| \leq W$ on B . Let R_M solve the scalar radial equation with $R_M(0) = M$. By order preservation,

$$(r_j(t) - \sqrt{2})_+ \leq R_M(t) - \sqrt{2}, \quad R_M(t) \downarrow \sqrt{2},$$

where $x_+ = \max\{x, 0\}$. Therefore

$$\sup_{p \in B} \text{dist}(\phi_t(p), A)^2 \leq 2(R_M(t) - \sqrt{2})^2 + W^2 e^{-2ct} \rightarrow 0.$$

This proves uniform attraction. $\square$

These are the defining properties of a compact global attractor. They also imply minimality: if a compact set C attracts every bounded set, apply attraction to A and use $\phi_t(A) = A$ to obtain $\sup_{p \in A} \text{dist}(p, C) = 0$. Hence $A \subseteq C$.

The sets $\{p \in A : |z_1| < 1\}$ and $\{p \in A : |z_1| > 1\}$ are nonempty, disjoint, relatively open in A , and invariant. Every orbit lies in one of these sets or in the level set $\{p \in A : |z_1| = 1\}$. Hence no orbit is dense in A , and A is not transitive.

2.2 The variational equation

The product structure reduces the singular-value calculation to the planar equations. Their radial and tangential growth rates are

$$a(s) = q(s) + 2sq'(s) = -5s^2 + 9s - 2, \quad b(s) = q(s) = -s^2 + 3s - 2. \quad (2.3)$$

Lemma 2.2. *For a fixed $\Omega \in \mathbb{R}$ and a planar solution of $\dot{z} = (q(|z|^2) + i\Omega)z$, the two singular values of the derivative of its evolution map at time $t > 0$ are, in unspecified order,*

$$\exp\left(\int_0^t a(s(\tau)) \, d\tau\right), \quad \exp\left(\int_0^t b(s(\tau)) \, d\tau\right), \quad s(t) = |z(t)|^2. \quad (2.4)$$

Proof. Let $v = (\Re z, \Im z)^\top$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The Cartesian Jacobian is

$$q(s)I + 2q'(s)vv^\top + \Omega J. \quad (2.5)$$

If $z(0) \neq 0$, then $z(t) \neq 0$ at every finite time. In the rotating orthonormal frame, a perturbation has the form $\delta z = e^{i\theta(t)}(\xi + i\eta)$. Since $\dot{\theta} = \Omega$, the derivative of this frame cancels the last term in (2.5), leaving

$$\dot{\xi} = a(s)\xi, \quad \dot{\eta} = b(s)\eta.$$

If $Q(\theta)$ denotes rotation through angle θ , the planar derivative consequently has the factorisation

$$Q(\theta(t)) \operatorname{diag}\left(e^{\int_0^t a(s(\tau)) d\tau}, e^{\int_0^t b(s(\tau)) d\tau}\right) Q(\theta(0))^\top. \quad (2.6)$$

The outer factors are orthogonal, which proves (2.4). At $z(0) = 0$, the derivative is $e^{-2t}Q(\Omega t)$, in agreement with the same formula since $a(0) = b(0) = -2$. $\square$

The full system is a direct product. Its five singular values are the four values from Lemma 2.2 and e^{-ct} . The planar growth rates satisfy

$$a(s) + 4 = (2 - s)(5s + 1), \quad b(s) + 2 = s(3 - s), \quad a(s) + b(s) = 2 - 6(s - 1)^2. \quad (2.7)$$

In particular,

$$\operatorname{div} F = 4 - c - 6(|z_1|^2 - 1)^2 - 6(|z_2|^2 - 1)^2 \leq 4 - c < 0 \quad (2.8)$$

throughout $\mathbb{R}^5$.

2.3 The maximum at each positive time

Fix $p \in A$ and $t > 0$. Along the trajectory $\phi_\tau(p) = (z_1(\tau), z_2(\tau), w(\tau))$, let $s_j(\tau) = |z_j(\tau)|^2$ for $j = 1, 2$. These values belong to $[0, 2]$ for all $\tau \geq 0$. The first two identities in (2.7) give $a(s_j(\tau)) \geq -4$ and $b(s_j(\tau)) \geq -2$. Every planar singular value is therefore at least e^{-4t} . Since $c > 4$,

$$\sigma_5(D\phi_t(p)) = e^{-ct}. \quad (2.9)$$

The last identity in (2.7) gives

$$\log \prod_{i=1}^4 \sigma_i(D\phi_t(p)) = 4t - 6I_t(p), \quad I_t(p) = \int_0^t [(s_1(\tau) - 1)^2 + (s_2(\tau) - 1)^2] d\tau. \quad (2.10)$$

Let $d_* = 4 + 4/c \in (4, 5)$. Combining (2.9) and (2.10) yields the exact identity

$$\omega_{d_*}(D\phi_t(p)) = e^{-6I_t(p)} \leq 1. \quad (2.11)$$

For $d_* < d \leq 5$, we also have

$$\omega_d(D\phi_t(p)) = e^{-6I_t(p) - ct(d - d_*)} < 1. \quad (2.12)$$

Thus $d_L(t, p) \leq d_*$ on A .

The integrand defining I_t is continuous and nonnegative. Its integral vanishes precisely when $s_1 = s_2 = 1$ on $[0, t]$, or equivalently when $p \in \mathcal{T}$. If $p \notin \mathcal{T}$, then (2.11) is strict, so (2.12) and the existence of the maximum in (1.2) give $d_L(t, p) < d_*$. On $\mathcal{T}$, the singular values are exactly

$$e^{2t}, \quad e^{2t}, \quad 1, \quad 1, \quad e^{-ct}. \quad (2.13)$$

Hence $d_L(t, p) = d_*$ there. This proves (1.8), and (1.3) gives $\dim_L A = d_*$.

2.4 Limiting exponents and periodic orbits

The radial limits determine the maximisers of the asymptotic dimension. By Lemma 2.2, the logarithm of each planar singular value, divided by t , is a time average of $a(s(t))$ or $b(s(t))$. Equation (2.2) and convergence of Cesàro averages give the limiting pairs

$\rho(r_0)$	0	1	$\sqrt{2}$
planar exponents	$(-2, -2)$	$(2, 0)$	$(0, -4)$.

The planar values need not remain in a fixed order, but arranging a finite family in decreasing order preserves convergence. Thus the two planar pairs, together with $-c$, give the limits of all five ordered exponents. For $p \in A$, let $\rho_j = \rho(|z_j(0)|)$, $j = 1, 2$. These limits give Table 1.

(ρ_1, ρ_2)	Ordered exponents	$d_{KY}(p)$
$(0, 0)$	$(-2, -2, -2, -2, -c)$	0
$(0, 1)$	$(2, 0, -2, -2, -c)$	3
$(0, \sqrt{2})$	$(0, -2, -2, -4, -c)$	1
$(1, 1)$	$(2, 2, 0, 0, -c)$	$4 + 4/c$
$(1, \sqrt{2})$	$(2, 0, 0, -4, -c)$	$7/2$
$(\sqrt{2}, \sqrt{2})$	$(0, 0, -4, -4, -c)$	2

Table 1: Exponents and Kaplan–Yorke dimensions on A , up to interchange of the two planar coordinates. The numbers ρ_1, ρ_2 are the limiting radii.

The largest value is d_* , and it occurs precisely when $\rho_1 = \rho_2 = 1$, which by (2.2) is equivalent to $p \in \mathcal{T}$. This proves (1.9).

The radial equations and angular frequencies also determine which of these spectra occur on equilibria and periodic orbits. Since $q(s) + i\Omega_j \neq 0$, the origin is the only equilibrium. The exponents at the origin are $(-2, -2, -2, -2, -c)$, so both local dimensions are zero. A nonconstant periodic solution must have $w = 0$. Each radius must be constant, since a nonconstant solution of a scalar autonomous equation cannot be periodic. Thus each radius belongs to $\{0, 1, \sqrt{2}\}$. If both were nonzero, a period $T > 0$ would satisfy

$$T = 2\pi m, \quad \sqrt{2}T = 2\pi n \quad (2.14)$$

for positive integers m, n , which is impossible. There are consequently four geometric periodic orbits: either coordinate has radius 1 or $\sqrt{2}$, and the other coordinate is zero. On the radius-one orbits, the exponents are $(2, 0, -2, -2, -c)$ and the dimension is 3. On the radius- $\sqrt{2}$ orbits, they are $(0, -2, -2, -4, -c)$ and the dimension is 1. All radii are constant on these orbits, so the logarithms of the singular values equal t times the listed exponents for every $t > 0$. This proves (1.10). Finally, (2.14) excludes every periodic orbit on $\mathcal{T}$.

2.5 Convergence of the local dimensions

Convergence of the exponents alone need not imply convergence of the local dimensions when leading exponents vanish. For the spectra in Table 1, the signs of the tangential rates resolve these cases. For fixed $p \in A$, let $\ell_i(t, p) = t^{-1} \log \sigma_i(D\phi_t(p))$, $1 \leq i \leq 5$. The functions $d \mapsto t^{-1} \log \omega_d(D\phi_t(p))$ converge uniformly on $[0, 5]$ to the piecewise linear interpolant of the partial sums of $\lambda_i(p)$. This convergence is uniform because the interpolants are linear on each interval between consecutive integers, and their values converge at all six endpoints.

These interpolants are concave, since their slopes are nonincreasing. For limiting radii $(0, 1)$ and $(1, \sqrt{2})$, this interpolant is strictly positive immediately to the left of its largest zero and strictly negative to the right. Uniform convergence therefore gives convergence of the largest zeros, to 3 and $7/2$, respectively. For $(1, 1)$ the dimension is constant in time. For $(0, 0)$ all $\ell_i(t, p)$ are eventually negative, so the dimension is eventually zero.

For an initial radius in $(1, \sqrt{2}]$, we have $b(s(t)) \geq 0$ for all $t \geq 0$, so its tangential finite-time exponent is nonnegative. Let $m = 1$ for limiting radii $(0, \sqrt{2})$ and $m = 2$ for $(\sqrt{2}, \sqrt{2})$. Each planar component whose limiting radius is $\sqrt{2}$ therefore contributes a nonnegative tangential exponent at every positive time. All remaining exponents have strictly negative limits. For large t , the first m exponents $\ell_i(t, p)$ are nonnegative and tend to zero, whereas $\ell_{m+1}(t, p)$ tends to a strictly negative limit. Their first $m + 1$ partial sum is therefore eventually negative, and

$$d_L(t, p) = m + \frac{\sum_{i=1}^m \ell_i(t, p)}{-\ell_{m+1}(t, p)} \rightarrow m.$$

Thus $d_L(t, p) \rightarrow d_{KY}(p)$ for every $p \in A$, completing the proof of Theorem 1.1.

3 Other conventions for Lyapunov dimension

We compare the dimension in Theorem 1.1 with two further conventions, both of which give the same value $4 + 4/c$ for A .

The index $j(p)$ in (1.5) is chosen separately at each point. Some definitions instead fix one index for the whole invariant set. Parker and Goluskin [13, Definitions 2.2–2.3] use maximal asymptotic growth rates of exterior products to define the exponent sums. For (1.6), these equal $S_k(p)$ at each $p \in A$: the factorisation (2.6), including its zero-radius case, provides an orthonormal basis whose images remain orthogonal, and each exterior basis vector has growth rate equal to the corresponding sum of limiting planar exponents and, if present, $-c$. The largest such rate is $S_k(p)$. A fixed exterior vector is a finite linear combination of these basis vectors and cannot have a larger rate.

Let j be the smallest index in $\{0, \dots, 4\}$ with $\sup_{p \in A} S_{j+1}(p) < 0$. The dimension in [13, Definition 2.3] is

$$\widehat{d}_L(A) = j + \sup_{p \in A} \frac{S_j(p)}{-\lambda_{j+1}(p)}. \quad (3.1)$$

By Table 1,

$$j = 4, \quad \lambda_5(p) = -c, \quad \max_{p \in A} S_4(p) = 4, \quad \max_{p \in A} S_5(p) = 4 - c < 0.$$

Hence (3.1) also equals $4 + 4/c$ and its supremum is attained precisely on $\mathcal{T}$. Over equilibria and periodic points, the largest value of $4 + S_4(p)/c$ is $4 - 2/c$, which is strictly smaller. This value differs from the pointwise dimension 3 because the index remains fixed at 4.

Another convention maximises singular-value growth over A before taking the time limit [1, §2.2]. For $1 \leq k \leq 5$, define

$$G_k(t) = \log \sup_{p \in A} \omega_k(D\phi_t(p)), \quad t \geq 0.$$

In particular, $G_k(0) = 0$. Since $\omega_k(M) = \|\bigwedge^k M\|$, the cocycle identity and invariance of A imply $G_k(t + u) \leq G_k(t) + G_k(u)$. Boundedness of the Jacobian on A gives $|G_k(t)| \leq$

$C_k t$. For fixed $T > 0$, subadditivity with $t = mT + r$, $m \in \mathbb{N}_0$, and $0 \leq r < T$ gives $\limsup_{t \rightarrow \infty} G_k(t)/t \leq G_k(T)/T$. Consequently

$$\lim_{t \rightarrow \infty} \frac{G_k(t)}{t} = \inf_{T > 0} \frac{G_k(T)}{T}.$$

Equations (2.9), (2.10), and (2.13) give $G_4(t) = 4t$ and $G_5(t) = (4 - c)t$. The values on $\mathcal{T}$ also show that the first four global sums are nonnegative. The associated Kaplan–Yorke interpolation therefore has index 4 and value $4 + 4/c$. Restricting the supremum to $\mathcal{P}(A)$ gives the limiting sums $2, 2, 0, -2, -2 - c$, by the periodic spectra in Section 2.4. The resulting dimension is 3.

AI declaration. This counterexample arose from discussions between the author and ChatGPT 6. I used Codex first to attempt to prove the conjecture and then to obtain the counterexample presented here. The formal Lean verification was also carried out with assistance from ChatGPT 6. The resulting Lean proof and its problem formulation were thoroughly checked against the theorems stated in this paper, both by the author and using the Lean 4 Comparator. These checks included verifying that no new axioms or unproved results were introduced. All content in the paper was reviewed and adopted by the author, who takes full responsibility for it.

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