Fast and Spectrally Accurate Numerical Methods for Perforated Screens (with Applications to Robin Boundary Conditions)

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Abstract

This paper considers the use of compliant or Robin boundary conditions to provide a homogenised model of a finite array of collinear plates, modelling a perforated screen or grating. This geometry forms a canonical model in scattering theory, with applications from electromagnetism to aeroacoustics. Interest in perforated media incorporated within larger structures motivates interrogating the appropriateness of homogenised boundary conditions in this case, especially as the homogenised model changes the junction behaviour considered at the extreme edges of the screen. To facilitate effective investigation we consider three numerical methods: the unified transform and an iterative Wiener–Hopf approach for the exact problem of a set of collinear rigid plates (the difficult geometry of the problem means that such methods, which converge exponentially, are crucial), and a novel Mathieu function collocation approach to consider a variable compliance applied along the length of a single plate. We detail the relative performance and practical considerations for applying each method, of broader interest to those considering applying the methods. We verify the appropriateness of the constant compliance given in previous theoretical research to gain a good estimate of the solution even for a modest number of plates, provided we are sufficiently far into the asymptotic regime in which the homogenisation is valid, which we describe. We further investigate tapering the compliance near the extreme endpoints of the screen, and find that tapering with tanh functions reduces the error in the approximation of the far-field (if we are sufficiently far into the asymptotic regime). We also find that the number of plates and wavenumber have significant effects, even far into the asymptotic regime. These last two points indicate the importance of modelling end effects to achieve highly accurate results.

Keywords: spectral methods, acoustic scattering, perforated screens, Robin boundary conditions

1 Introduction

The scattering of waves by sets of collinear plates or gratings forms a canonical scattering problem of interest in a range of applications including optics [49], electromagnetism [8, 16, 20, 39, 42] and acoustics [25]. While diffraction gratings exist in a plethora of designs, the most simplistic transmission grating is that consisting of periodically spaced plates as illustrated in Figure 1. Despite its simplicity, there has been continued interest in the problem since the 1950’s [1, 17, 22], particularly for acoustic wave scattering. We remark that whilst many acoustic problems for a single finite plate can be solved in closed form with the use of special functions (for example via separation of variables in elliptic coordinates), there is no such solution for multiple plates. This has spawned a variety
of numerical solution approaches, including several recently developed spectrally accurate numerical schemes [11, 43, 47].

Previous work most commonly considers infinite gratings. However, with recent renewed interest in the silent flight of owls [19, 24, 32], there have been studies into the effects of noise reduction due to truncated perforated plates (also referred to as porous plates) both experimentally and theoretically [18, 27, 44]. A natural model for such physical designs is a finite grating, but since it is difficult to model the precise setup of a porous plate, the typical approach in these finite grating investigations is to homogenise the boundary condition, finding an effective compliance of the plate and applying this over the whole grating’s length. This provides a Robin boundary condition constructed to capture the macroscopic phenomena of interest, allowing a more straightforward analysis of structures involving the perforated material. It is desirable to understand the appropriateness of this approximation to inform the development of physical designs, especially as the precise nature of material junctions is known to be significant in scattering problems and aerodynamic considerations encourage the use of small porous elements.

Early work by Lamb [28] calculated the effective compliance of a two-dimensional infinite grating comprised of thin plates and slits, and obtained a constant compliance in this case, dependent on the plate and slit lengths. Much later, Howe [23] calculated the Rayleigh conductivity of a circular aperture, and thus the effective compliance of an infinite screen perforated with circular holes. This once again gave rise to a constant effective compliance which has been used to classify the porosity of plates in recent applications relating to quiet flight [7, 24, 27]. The effective compliance of more complex pore geometries has been recently considered theoretically in [29], and it may also be determined experimentally by measuring acoustic impedance [15]. We emphasise that for aeroacoustic investigations such a uniform effective compliance associated with an infinite medium has been repeatedly applied on truncated sections [7, 24, 27].

However, Leppington’s [30] analysis of the edge effects of semi-infinite perforated screens indicates that the effective compliance ought not to be constant, and the constant approximation holds true only in the infinite grating limit. Despite this finding by Leppington, to the best of the authors’ knowledge, no one has taken account of this variable compliance when considering acoustic scattering by perforated plates when finite edges are present. Previous studies have considered both scattering by finite diffraction gratings [20] and explicit consideration of truncation effects on electromagnetic scattering by a semi-infinite array of dipoles [5, 6], but this is typically isolated from the notion of an effective boundary condition.

This paper, therefore, investigates the effective compliance of finite gratings and the influence of edge effects on the far-field scattered noise due to an incident plane wave. To do so, we employ numerical methods to accurately solve for the scattered field accounting for the precise grating geometry and compare to the scattered field when a constant or variable homogenised compliance is imposed. As Leppington’s results only provide asymptotic limits for the compliance at the edge and far from the edge, we further investigate the influence on the scattered field of how one transitions from these two values.

The ubiquity of singular integral equations and singular edge behaviour in these problems means care is required in all numerical implementations. Whilst standard boundary element techniques could, in principle, be employed, this would be difficult for the appropriate asymptotic regime considered in this paper. We therefore take the opportunity to compare two recently developed approaches based upon a spectral formulation. These two methods, the extension of the unified transform method to scattering problems [11], and an iterative Wiener–Hopf method [43] have both individually been seen to be spectrally convergent for acoustic scattering problems consisting of finite flat plates. We also remark that the unified transform approach can treat more general geometries. As well as using these two methods to consider the impact of homogenised boundary conditions on perforated plates and gratings, we also discuss their relative performance for such scattering problems giving indications as to which method is best suited for grating-related problems, particularly as parameters vary; for instance the number of plates and characteristic Helmholtz numbers associated with the problem. We
also seek to note qualitative and practical differences of interest for those wishing to apply or extend our approaches.

In order to consider variable porosity, this paper also presents an accurate (and simple) collocation method based on Mathieu functions for solving scattering by a finite plate with a variable Robin boundary condition imposed along the chord. Such a variable Robin condition has been investigated for considering the aerodynamic problem (solving Laplace) of thin aerofoils with porosity gradients [3, 21]; this Mathieu collocation method may be used to solve the complementary aeroacoustic problem (solving Helmholtz), and so we believe it may be of wider interest in the community. Future work will seek to extend this approach to include other boundary conditions, such as those modelling elasticity.

The layout of this paper is as follows. We first describe the mathematical modelling of scattering by a set of collinear plates and review results on the use of homogenised boundary conditions to be interrogated. We then outline the unified transform and iterative Wiener–Hopf method applied to the problem of collinear finite plates, highlighting similarities and divergences between these approaches and compare their performance. To investigate a variable Robin condition imposed on a finite screen, we introduce a separation of variables boundary collocation method (which we believe may be of further interest in the acoustic community). Finally, we compare the use of the homogenised boundary condition to the exact case, indicating the appropriateness of its use in relevant parameter regimes.

2 The Mathematical Model

In this paper, we are primarily concerned with the scattering of acoustic sources by a finite collection of collinear plates as illustrated in Figure 2. We denote the set of plates by \( \gamma = \bigcup_{i=1}^{M} \gamma_i \), where each plate is located at \( y = 0 \) and \( x \in [x_{i,0}, x_{i,1}] = \gamma_i \) with \( x_{i,0} < x_{i,1} < x_{i+1,0} \) (i.e. the plates are non-touching). Thus our total scattering domain consists of \( \mathbb{R}^2 \setminus \gamma \), where the scattered field \( q \) must satisfy the Helmholtz equation

\[
\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + k_0^2 q = 0. \tag{2.1}
\]

An example of the set up for three plates with labelled endpoints is shown in Figure 2.

A canonical heterogeneous obstacle is a perforated screen or grating, that is a sound-hard wall with several small open apertures - see Figure 1. On the sound-hard plates we impose a Neumann boundary condition, and away from the plates a continuity condition, which here is equivalent to a sound-soft boundary:

\[
\frac{\partial q}{\partial y}(x, 0) + \frac{\partial q_I}{\partial y}(x, 0) = 0 \quad x \in \gamma, \quad [q](x, 0) := q(x, 0^+) - q(x, 0^-) = 0 \quad x \in \mathbb{R} \setminus \gamma, \tag{2.2}
\]

where \( q_I \) denotes the incident field. The typical choice in this paper is

\[
q_I(x, y) = e^{-ik_0 x \cos(\theta) - ik_0 y \sin(\theta)} \tag{2.3}
\]

corresponding to a wave incident at angle \( \theta \) measured in the anti-clockwise direction from the positive \( x \)-axis. Throughout, we will also impose the Sommerfeld radiation condition on the scattered field \( q \).

Rather than solving for each disjoint boundary condition, the screen can be considered to have one homogenised Robin boundary condition in appropriate limits as we now discuss. In scattering problems, the fundamental quantities of interest are typically the far-field wave scattered by an obstacle, or if the obstacle divides two regions, the reflection and transmission coefficients across this border such as that calculated in [28]. If the obstacle is heterogeneous, but only due to defects that are small relative to all other length scales, then we might naturally seek a homogenised boundary condition that captures the important macroscopic effects on these physically relevant quantities.

If the wavelength \( 2\pi/k_0 \) of the incident disturbance is much larger than the length scales of the grating, that is the aperture width \( 2a \) and spacing \( d \), then the defects are compact, and we observe the
desired separation of scales to permit homogenisation of the boundary condition. Our focus concerns the use of effective compliance [30] as this homogenised boundary condition, that is approximating the boundary conditions on the plates by a boundary condition on the full screen of the form

\[ \frac{\partial q}{\partial y}(x, 0) + \frac{\partial q}{\partial y}(x, 0) = \mu(x)[q](x, 0), \quad x \in [x_{1,0}, x_{M,1}] \]  

in order to model a perforated screen of finite extent. Typically, we will take \([x_{1,0}, x_{M,1}] = [-1, 1]\) which is without loss of generality since we can always non-dimensionalise lengths by the semi-chord of the screen. We will refer to this boundary condition as a Robin boundary condition, consistent with the PDE literature.

Leppington [30] considered scattering by a perforated screen of semi-infinite extent \((x \in \mathbb{R}_{\geq 0})\) formed from a set of collinear plates and the homogenised boundary condition this required. By first constructing an integral equation associated with the scattering problem and restricting attention to the asymptotic regime

\[ a \ll d \ll k_0^{-1}, \]  

the effective compliance at \(x \gg d\) was found to be

\[ \mu = \mu_0 = \frac{\pi}{2a} \left\{ \log \left( \frac{d}{\pi a} \right) \right\}^{-1} \]  

consistent with results in [28] for an infinite screen. Leppington also analysed a formula for the effective compliance when \(x \sim d\) near the plate edge, however the analysis breaks down for smaller \(x\) and it is not clear how one should let \(\mu\) approach zero at the end of the plate. In fact the formula given in [30] becomes negative for small enough \(x \sim a\). A correction enforcing \(\mu \geq 0\) was also proposed but this leads to \(\mu(x) \uparrow \infty\) as \(x \downarrow 0\) as opposed to the physically correct value \(\mu(0) = 0\). In general, there is very little said in the literature about the values of \(\mu\) close to the plate ends. For our finite screen, we will therefore consider the constant compliance given by (2.6) as well as different approaches of tapering the effective compliance to zero at both plate edges (see §6.3).

The numerical methods considered for solving the non-homogenised problem (2.2) in this paper enable the direct interrogation of when these homogenised boundary conditions (2.4) may be appropriate: i.e. how “deep” within the asymptotic regimes we must be in order for a homogenised boundary condition to offer a good approximation. Three primary limits present themselves:

(a) How small must the open fraction (or void fraction or porosity in other subjects) \(2a/d\) be?

(b) How small must the grating Helmholtz number \(k_0d\) be?

(c) How large must the screen Helmholtz number \(2k_0\) or the number of plates \(M\) be for end effects to be negligible?

2.1 The numerical approaches

We consider two spectral methods to solve the problem of scattering by a set of collinear finite rigid plates. The differences between the two approaches are illustrated in Figure 3. Both approaches start with essentially the same equation relating integral transforms of the unknown boundary values (the global relation) though employ distinct solution methods. The unified transform simply views this

\[ y = 0 \]

Figure 1: Diagram of a perforated screen for aperture width \(2a\) and separation \(d\).
equation as providing a set of linear relations parametrised by a variable $\alpha$ (and hence is applicable to more general problems/geometries). This may be discretised by choosing a suitable basis for the unknown physical boundary values, and the resulting linear system is solved by collocation employing closed form expressions for the Fourier transforms of the basis functions. By contrast, the iterative Wiener–Hopf approach views the global relation as defining a jump problem between sectionally analytic functions: a matrix Wiener–Hopf problem which may be solved by considering a sequence of scalar Wiener–Hopf problems to give a fixed-point iteration scheme. For the problem considered, this iterative process has a physical interpretation as the analogue of Schwarzschild diffraction series [45] in the spectral domain. Practically this requires choosing a basis for the jumps of functions along branch cuts and employing closed form expressions for their Cauchy transforms. Whereas the unified transform recovers an expression for the physical boundary values and their Fourier transform, the iterative Wiener–Hopf approach only solves directly for their Fourier transform and the physical boundary values must be recovered by numerically inverting a Fourier transform. This makes the iterative Wiener–Hopf approach most suitable for recovering the far-field directivity using a saddle-point approximation. The entire solution may be recovered using integral representations of the boundary values or their Fourier transform.

The key differences to note are:

- **Nature of the functions to be approximated:** oscillatory boundary values vs Fourier transform of boundary values, evaluated on steepest descent contour. For the unified transform the boundary values along each segment of the domain must be represented accurately, typically requiring more degrees of freedom for large wavenumbers. In the iterative Wiener–Hopf method, the Fourier transforms of the boundary values are represented in terms of Cauchy transforms of smooth non-oscillating functions.

- **Solution method:** collocation of discretised global relation vs iteration of scalar Wiener–Hopf problems associated with scattering events. This means the iterative method is more naturally suited to problems involving large wavenumbers.

- **Solution quantity most easily recovered:** boundary values (jump over plate) vs Fourier transform of boundary values (and so far-field directivity) in the iterative Wiener–Hopf method.

We now outline each method and its implementation in more detail, before comparing the performance of each method for multiple plates in §5.

### 3 Unified Transform

In this section, we briefly discuss the unified transform and how it can be used as a numerical method for scattering problems [2, 11]. For a full discussion of the method applied to elastic plate geometries, we refer the reader to [10]. The first step is to obtain the so-called ‘global relation’. Once the global relation is obtained, we can expand all unknown boundary data in terms of carefully selected basis functions to obtain a linear system for the expansion coefficients. Finally, the linear system can be evaluated at collocation points to solve for the unknown expansion coefficients.
3.1 The global relation for collinear plates

The unified transform can be applied to arbitrary elliptic PDEs with constant coefficients [11, 12] and more general separable PDEs [9]. However, in this paper we are solely concerned with the Helmholtz scattering problem for the geometry outlined in §2. Let \( \Lambda = (-1, 0) \cup (1, \infty) \cup \{ e^{i\theta} : \pi < \theta < 2\pi \} \) and \( \beta = k_0/2 \), then the global relation for our problem (see [10] for a simple derivation using Green’s theorem) is

\[
\int_{\gamma} e^{-i\beta x(\lambda + \frac{1}{\lambda})} q_y(x,0) dx + \frac{i}{\sqrt{2}} \int_\gamma e^{-i\beta x(\lambda + \frac{1}{\lambda})} \left[ q_y(x,0) + \beta \left( \lambda - \frac{1}{\lambda} \right) [g](x,0) \right] dx = 0, \quad \lambda \in \Lambda. \tag{3.1}
\]

The idea is to expand the unknowns in this relation in suitable basis functions and evaluate at enough collocation points \( \lambda \) to set up a well-conditioned linear system for the unknown coefficients. In the special case of this collinear geometry, there is another interpretation of the global relation which we wish to highlight. Using the fact that the scattered field is anti-symmetric in the \( y \)-direction yields the boundary integral equations:

\[
\frac{1}{4} [g](x,0) + \int_{\mathbb{R}} G(x - x') q_y(x',0) dx' = 0, \tag{3.2}
\]

where \( G \) is the Green’s function

\[
G(x) = \frac{i}{4} H_0^{(1)} (k_0 |x|)
\]

and where \( H_0^{(1)} (\cdot) \) denotes the Hankel function of the first kind of order \( \alpha \). Taking the Fourier transform of (3.2) with frequency parameter \( w = \beta (\lambda + 1/\lambda) \) and using the convolution theorem yields

\[
\int_\gamma e^{-i\beta x(\lambda + \frac{1}{\lambda})} [g](x,0) dx + \frac{i}{\sqrt{2}} \int_\gamma e^{-i\beta x(\lambda + \frac{1}{\lambda})} q_y(x,0) dx \int_{\mathbb{R}} e^{-i\beta x(\lambda + \frac{1}{\lambda})} H_0^{(1)} (k_0 |x|) dx.
\]

Note that the allowed \( \lambda \) values correspond exactly to real \( w \). The Fourier transform of the Hankel function is known [40] and yields

\[
i \int_{\mathbb{R}} e^{-i\beta x(\lambda + \frac{1}{\lambda})} H_0^{(1)} (k_0 |x|) dx = \frac{2}{\beta(\lambda - 1/\lambda)}. \]
It follows that the global relation (3.1) in this case is exactly the Fourier transform of the boundary integral equation (3.2). The convolution with the singular integral is transformed into a multiplication in Fourier space and hence the unified transform avoids the need for difficult quadratures. We should stress that this interpretation as collocating the Fourier transform of boundary integral equations does not hold in generality. However, it does serve as an intuition behind the method for more complicated geometries. Note also that we obtain the same equation when using the Wiener–Hopf method in (4.3) through the formal substitution $\beta(\lambda + \frac{1}{\lambda}) = -\alpha$ and 

$$K(\alpha) = \sqrt{\alpha^2 - k_0^2} = \beta \left( \lambda - \frac{1}{\lambda} \right).$$

In other words, we can view the unified transform, in particular the global relation (3.1), as a natural generalisation (when considering more complex domains) of Fourier transforms of the boundary integral equations. The method itself can be viewed as a numerical method for solving Wiener–Hopf type problems for arbitrary domains.

### 3.2 Basis functions and the approximate global relation

We split $\mathbb{R} \setminus \gamma$ into the following intervals. Let $I_1 = (-\infty, x_{1,0})$ and $I_2 = (x_{M,1}, \infty)$. Then $[x_{1,0}, x_{M,1}] \setminus \gamma$ can be split into disjoint open intervals $J_i$ for $i = 1, \ldots, M - 1$. For notational convenience, we introduce $L_i = |J_i| = x_{i,1} - x_{i,0}$ and $m_i = (x_{i,1} + x_{i,0})/2$ for $i = 1, \ldots, M$ as well as $L'_i = |J_i|$ and $m'_i$ equal to the midpoint of $J_i$ for $i = 1, \ldots, M - 1$. We suppose that along each plate $\gamma_i$ we are given a relation of the form

$$q_y(x, 0) - \mu_i[q](x, 0) = f_i(x),$$

for known functions $f_i$ and parameters $\mu_i$ (which when considering the unified transform, we assume to be constant along each plate). The unknowns in the global relation (3.1) are $[q]$ on each interval $\gamma_i$ and $q_y$ on $I_1, I_2$ and on each $J_i$. For the finite intervals $\gamma_i$, in order to capture the square-root type singularities near the edge tips we define

$$C_m(t) = \sqrt{1 - t^2} \cdot U_m(t),$$

where $U_m(\cdot)$ denote Chebyshev polynomials of the second kind. These have the following Fourier transform [40]:

$$\int_{-1}^{1} e^{i\lambda t} \sqrt{1 - t^2} \cdot U_m(t) dt = \frac{(m + 1)i^m \pi}{\lambda} J_{m+1}(\lambda),$$

(3.4)

where $J_{\alpha}(\cdot)$ denotes the Bessel function of the first kind of order $\alpha$. We expand $[q_i] = [q]$ on $\gamma_i$ as

$$[q_i](x) \approx \sum_{n=1}^{N_i} a_{i,n} C_{n-1} \left( \frac{2x - 2m_i}{L_i} \right).$$

Similarly, for the $M - 1$ intervals $J_i$ we expand $q_{y,i} = q_y$ as

$$q_{y,i}(x) \approx \sum_{n=1}^{N'_i} b_{i,n} S_{n-1} \left( \frac{2x - 2m'_i}{L'_i} \right),$$

where $S_m(t) = T_m(t)(\sqrt{1 - t^2})^{-1}$ and $T_m(\cdot)$ denote Chebyshev polynomials of the first kind. These are chosen to capture the derivatives of the relevant square root type singularity and have

$$\int_{-1}^{1} e^{i\lambda t} \frac{T_m(t)}{\sqrt{1 - t^2}} dt = i^m \pi J_m(\lambda).$$

(3.5)
For the semi-infinite intervals $I_1$ and $I_2$ we expand (after a relevant affine change of variables) in terms of the Bessel functions $\{J_{n+1/2}(k_0 x)/x\}_{n \geq 0}$. These have the advantage of capturing the correct singular behaviour near the plate edges when $n$ is even. They decay with the correct algebraic rate at infinity, and have easy to compute Fourier transforms \[40\]:

\[
\int_0^\infty e^{i t \lambda} \frac{J_\alpha(bt)}{t} \, dt = \begin{cases} 
\exp(i \alpha \arcsin(\lambda/b)), & \text{for } 0 \leq \lambda \leq b \\
\frac{\alpha}{b^\alpha \exp(\alpha \pi i/2)} \frac{\exp(\alpha \sqrt{\lambda^2 - b^2})}{\alpha (\lambda + \sqrt{\lambda^2 - b^2})}, & \text{for } 0 < b \leq \lambda.
\end{cases} \tag{3.6}
\]

Explicitly, we approximate $q_{y,0} = q_y$ on $I_1$ via

$q_{y,0} \approx \sum_{n=1}^{N_0'} b_{0,n} J_{\frac{1}{2}}(k_0(x_{1,0} - x)) / x_{1,0} - x$

and $q_{y,M} = q_y$ on $I_2$ via

$q_{y,0} \approx \sum_{n=1}^{N_M'} b_{M,n} J_{\frac{1}{2}}(k_0(x - x_{M,1})) / x - x_{M,1}.$

Using the formulae for the relevant Fourier transforms, we thus form an approximate global relation

$\sum_{i=1}^M \sum_{n=1}^{N_i} A_{i,n} \lambda a_{i,n} + \sum_{i=0}^M \sum_{n=1}^{N_i'} B_{i,n} \lambda b_{i,n} \approx -\sum_{i=1}^M \int_{\gamma_i} e^{-i \beta_2 (\lambda + \frac{1}{\lambda})} f_i(x) \, dx. \tag{3.7}$

The coefficients $A$ are given by

$A_{i,n}(\lambda) = \left( \frac{1}{\lambda} - \frac{2 \mu_i}{\beta} \right) e^{-i \beta n_{i,1}(\lambda + 1/\lambda)} \frac{n_{i,n-1} \pi}{2 (\lambda + \frac{1}{\lambda})} J_n \left( -\frac{\beta L_i}{2} \left( \lambda + \frac{1}{\lambda} \right) \right).$

We also have

$B_{0,n}(\lambda) = e^{-i \beta x_{1,0}(\lambda + 1/\lambda)} \int_0^\infty e^{i \beta t(\lambda + 1/\lambda)} \frac{J_{\frac{1}{2}}(k_0 t)}{t} \, dt,$

$B_{M,n}(\lambda) = e^{-i \beta x_{M,1}(\lambda + 1/\lambda)} \int_0^\infty e^{-i \beta t(\lambda + 1/\lambda)} \frac{J_{\frac{1}{2}}(k_0 t)}{t} \, dt.$

Finally, for $0 < i < M$ we have

$B_{i,n} = \frac{L_i^{n-1} \pi}{2} e^{-i \beta n_{i,1}(\lambda + 1/\lambda)} J_{n-1} \left( -\frac{\beta L_i}{2} \left( \lambda + \frac{1}{\lambda} \right) \right).$

We evaluate (3.7) at $C \geq \sum N_i + \sum N_i'$ collocation points $\lambda \in \Lambda$ to set up a linear system for the unknown coefficients (typically with $|C| \sim \sum N_i + \sum N_i'$). This is then inverted in the least-squares sense. Once the coefficients are computed, we can reconstruct approximations of the unknown functions.

**Remark 1** (Remark on convergence). As we will see later, when $\mu_i = 0$ the unified transform with the above basis choices converges at least exponentially. However, the introduction of Robin boundary conditions induces (poly-)logarithmic type singularities due to resonance phenomena in the poles of the Mellin symbol. This is a well-studied phenomenon in the PDE literature [33, 35]. The dominant singularities are still square-root and hence, the unified transform will converge in this case algebraically with a large order of convergence. See §6.3 for this effect and another interpretation of the above choice of basis functions in elliptic coordinates as a sine series.
3.3 Collocation points and obtaining the scattered field

Unfortunately, in contrast to standard spectral methods [4], there is no current theory describing the best choices for collocation points. For collocation points \( \lambda \in \Lambda \), we chose Halton nodes (scattered points with a lack of regularity use in quasi-Monte Carlo integration) in the interval \((-1, 0)\), minus their reciprocal values in \((1, \infty)\), and points in \(\{e^{-i\theta} : 0 < \theta < \pi\}\) with \(\theta\) corresponding to Halton nodes in \((0, \pi)\). This corresponds to sampling frequencies along the entire real line of the Fourier transforms of the relevant functions. The complex collocation points along the unit circle are allowed precisely because the solution satisfies the Sommerfeld radiation condition so that the contribution of Green’s identity along the relevant semi-circular arc vanishes in the infinite radius limit (see [48]). We found that to obtain accurate numerical solutions, we needed to sample these points, and hence, we considered the full complex solution. This corresponds to implementing the boundary conditions that make the boundary value problem well-posed.

In all of the examples encountered in this paper, the scattered field is an odd function in the \(y\) variable. Hence, by considering the reflected Green’s function

\[
G_R(x, y, x', y') = \frac{1}{4i} \left( H_0^{(1)}(k_0 \sqrt{(x-x')^2 + (y-y')^2}) - H_0^{(1)}(k_0 \sqrt{(x-x')^2 + (y+y')^2}) \right),
\]

and its normal derivative, we can write

\[
q(x, y) = \frac{ik_0y}{4} \int_{\gamma} \frac{H_1^{(1)}(k_0 \sqrt{(x-x')^2 + y'^2})}{\sqrt{(x-x')^2 + y'^2}} [q](x', 0) \, dx.
\]

For points off the union of plates \(\gamma\), this can be evaluated rapidly using standard Gaussian quadrature. Near the plates (where the integrand becomes singular) we can use first order approximations from the computed \(q\) and its normal derivative along the plates [13]. We can also evaluate the far-field using steepest descent.

4 An Iterative Wiener–Hopf Method

In this section, we discuss an alternative approach to scattering problems using an iterative Wiener–Hopf formulation. The method we use was introduced in [27] and implemented for a set of \(M\) collinear finite plates in [43]. A matrix Wiener–Hopf problem is formulated, viewed as a set of coupled scalar problems, and then solved by fixed-point iteration. We briefly discuss the key steps and refer the reader to [43] for details.

4.1 Matrix Wiener–Hopf equation for collinear plates

The fundamental equation to be solved is analogous to the global relation (3.1). However, this is typically derived in the context of Wiener–Hopf problems by exploiting the \(y\)-anti-symmetry of the scattered field \(q\) and Fourier transforming the boundary value problem in the \(x\) variable using the convention

\[
Q(\alpha, y) = \int_{-\infty}^{\infty} q(x, y) e^{i\alpha x} \, dx.
\]  

The \(x\)-Fourier transform \(Q(\alpha, y)\) of the general solution \(q(x, y)\) satisfying the governing Helmholtz equation and decaying as \(|y| \to \infty\) may then be represented as

\[
Q(\alpha, y) = \text{sgn}(y) A(\alpha) e^{-K(\alpha)|y|}
\]  

\(^1\)For a good choice for bounded convex polygons, see [13], which we cannot adopt here due to the restrictions on the values of \(\lambda\).
where \( K(\alpha) = \sqrt{\alpha^2 - K_0^2} \). Here the branch cuts are taken to be the rays \( \{\alpha = \pm k_0 \pm is : 0 < s < \infty\} \) parallel to the imaginary axis. For \( y = 0\pm \) this provides the relationship \( Q' \pm KQ = 0 \), where \( \partial Q / \partial y \equiv Q' \), and so

\[
\int_{\mathbb{R} \setminus \gamma} e^{ixy} q_y(x,0) \, dx + \int_{\gamma} e^{ixy} [q_y(x,0) + K(\alpha)q(x,0)] \, dx = 0 \tag{4.3}
\]

which may be recovered from (3.1) using the substitution \( \beta(\lambda + \frac{1}{\lambda}) = -\alpha \) and using the fact that \( q \) is an odd function in the \( y \) variable (as noted in [11]). We now restrict attention to \( y = 0^+ \). The normal derivative of the scattered field \( q \) on each plate is prescribed by the incident field \( q_I \), say

\[ Q'_{\gamma m} = F_{\gamma m} \tag{4.4} \]

where \( W_\gamma \) denotes the Fourier transform of a function \( w \) along the contour \( \gamma \). We rewrite equation (4.3) as

\[ 0 = \sum_{m=1}^{M} (KQ_{\gamma m} + F_{\gamma m}) + Q'_{I1} + Q'_{I2} + \sum_{m=1}^{M-1} Q'_{\gamma m} \tag{4.5} \]

This is the fundamental equation to be solved that relates Fourier transforms of unknown boundary values. In order to apply the iterative Wiener–Hopf technique we must identify the analyticity and growth at infinity of each term in the upper and lower half planes, annotating those analytic in the upper half plane by + and the lower half plane by −. We first denote the unknown functions by

\[ V_1 = Q'_{I1}, \quad V_{2m+1} = Q'_{\gamma m} : m = 1 : M - 1, \quad V_{2M+1} = Q'_{I2}, \quad V_{2m} = Q_{\gamma m} : m = 1 : M \tag{4.6} \]

and then to ensure + and − functions do not grow exponentially in their half-plane of analyticity we define the shifted functions

\[
\begin{align*}
\Psi^{(m)}_{-} & = e^{-i\alpha x_m} V_m \tag{4.7a} \\
\Psi^{(m)}_{+} & = e^{-i\alpha x_m} V_{m+1} \tag{4.7b}
\end{align*}
\]

for \( 1 \leq m \leq 2M \), each \( \Psi^{(m)}_{-} \) now having algebraic behaviour in the lower half plane, and \( \Psi^{(m)}_{+} \) in the upper half plane. We may now find an \( 2M \times 2M \) matrix Wiener–Hopf system suitable for the iterative scheme as follows. The \( m^{th} \) row may be obtained from equation (4.5) by rescaling by \( e^{-i\alpha x_m} \) and recasting in terms of \( \Psi_{\pm} \). We find

\[ H \Psi_- + G \Psi_+ = F \tag{4.8} \]

where \( H \) and \( G \) are triangular matrices with entries given by

\[
H_{lm} = \begin{cases} 
0 & l < m \\
E^{(l,m)} & m \text{ odd and } l \geq m \\
E^{(l,m)}K(\alpha) & m \text{ even and } l \geq m
\end{cases}, \quad
G_{lm} = \begin{cases} 
0 & l > m \\
E^{(l,m)}K(\alpha) & m \text{ odd and } l \leq m \\
E^{(l,m)} & m \text{ even and } l \leq m
\end{cases} \tag{4.9}
\]

where \( E^{(l,m)} = e^{i(x_m - x_l)\alpha} \). The forcing term \( F \) is given by

\[ F^{(m)} = -e^{-i\alpha x_m} \sum_{l=1}^{M} F_{\gamma l} \tag{4.10} \]

This formulates a matrix Wiener–Hopf equation where a partial matrix factorisation has been achieved to ensure all terms involving unknown functions are analytic with algebraic behaviour at infinity in the upper or lower half planes.
4.2 Solution by iteration

We now look to solve equation (4.8) by constructing a fixed-point iteration scheme. The \((2m + 1)\)th row of the matrix equation (4.8) is

\[
\sum_{l=1}^{m+1} E^{(2m+1,2l-1)} \Psi_{(2l-1)}^{(2l-1)} + \sum_{l=1}^{m} E^{(2m+1,2l)} K \Psi_{(2l)}^{(2l)} + \sum_{l=m+2}^{M+1} E^{(2m+1,2l-1)} \Psi_{(2l-1)}^{(2l-1)} + \sum_{l=m+1}^{M} E^{(2m+1,2l)} K \Psi_{(2l)}^{(2l)} = F^{(m)}
\]

which we solve at the \(r\)th iterative step by considering

\[
\Psi_{(2m+1)r}^{(2m+2)r} + K \Psi_{(2m+2r)}^{(2m+2r)} = -\sum_{l=1}^{m} E^{(2m+1,2l-1)} \Psi_{(2l-1)}^{(2l-1)r-1} - \sum_{l=1}^{m} E^{(2m+1,2l)} K \Psi_{(2l)r-1}^{(2l)r-1} - \sum_{l=m+2}^{M+1} E^{(2m+1,2l-1)} \Psi_{(2l-1)}^{(2l-1)r-1} - \sum_{l=m+1}^{M} E^{(2m+1,2l)} K \Psi_{(2l)r-1}^{(2l)r-1} + F^{(m)}
\]

where \(\Psi_{(m)r}^{(m)}\) denotes the estimate of \(\Psi_{(m)}^{(m)}\) at the \(r\)th solution step. An analogous equation may be found for even rows. Since the terms on the right-hand side are known, equation (4.12) may be solved by the standard scalar Wiener–Hopf technique. This solution may then be used to update the ‘forcing’ in the remaining rows of the matrix equation. We initialise the scheme by setting unknown terms on the left-hand side to vanish. The iteration sequence may be terminated when an appropriate error threshold between consecutive iterations is reached.

4.3 Numerical implementation

The problem has been reduced to solving a sequence of scalar Wiener–Hopf problems of the form

\[
K \Psi_{+} + \Psi_{-} = F
\]

which are well understood. Using exact scalar multiplicative factorisations of \(K = K^+ K^+\) and additive Wiener–Hopf factorisations, we may represent the solution in terms of Cauchy transforms along appropriately chosen contours [38]. We define the branch cuts of \(K(\alpha)\) to be parallel to the imaginary axis in order to be paths of steepest descent of the exponential factors \(e^{i\alpha x_m}\). By then deforming the Cauchy integration contours onto these branch cuts we induce square root endpoint singularities at both ends of the branch cut in the integrand. All the Cauchy transforms that must be computed are then of the form

\[
\int_{0}^{\infty} f(z) \frac{1}{z^{1/2}(z - \alpha)} dz
\]

where \(f\) is a smooth, non-oscillatory function on \((0, \infty)\) decaying as \(z \to \infty\). To perform such singular integrals numerically, we first relate the Cauchy transform on the half-line to one on the finite interval \((-1, 1)\) with similar endpoint and non-oscillatory behaviour using a Möbius map [50]. We then employ the spectral approach to Cauchy transforms introduced in [41] and implemented in Julia in [46], encoding the square root endpoint singularities through the weight \(1/(\sqrt{1 - \sqrt{z}} + 1)\) of “modified” Chebyshev polynomials \(T_n^+\) on \((-1, 1)\) introduced in [50]. Specifically, we define \(T_n^+\) by

\[
T_0^+ = 1, \quad T_1^+ = z, \quad T_n^+ = T_n - T_{n-2}, \quad n \geq 2
\]
where the $T_n$ are Chebyshev polynomials of the first kind. The associated moments of the Cauchy transform on $(-1, 1)$ have simple expressions in terms of elementary functions. The application of the iterative scheme may then be cast in terms of fast transforms between function values and coefficients on the branch cuts, and the evaluation of Cauchy transforms by contracting matrices of precomputed Cauchy transform moments with vectors of coefficients. In [47], where the Wiener–Hopf problems for scattering by a half-plane is considered numerically, the square root endpoint singularities in the spectral function $Q (\Phi$ in [47]) on a doubly infinite interval are encoded through the use of rational mappings which have multiple inverses. Since we only consider semi-infinite rays (having deformed a doubly infinite interval onto each side of a semi-infinite branch cut), we avoid this complication.

Whilst a small number of degrees of freedom are typically required to approximate a function with given exponential or algebraic decay by tuning the mapping, for ease of implementation we consider a single mapping for all cases. Therefore the present implementation requires more degrees of freedom for problems involving high and low wavenumbers, and those involving larger numbers of plates; this is associated with approximating a range of different exponential decay rates.

To recover the spatial field $q(x, y)$ we must invert the $x$-Fourier transform:

$$q(x, y) = \frac{\text{sgn}(y)}{2\pi} \int_{-\infty}^{\infty} Q(\alpha, 0+) e^{-i\alpha x - K(\alpha)|y|} d\alpha$$

Again, the asymptotic far-field may be readily recovered by the method of steepest descent. For evaluation of the far-field in or near the direction of the reflected wave one must take care of the removable singularity in the spectral forcing term $F$. One approach is to compute $Q$ using the Cauchy integral formula.

5 Analysis of Methods for Multiple Plates

We now compare the two spectral methods for scattering by a set of collinear finite sound-hard plates. All experiments in this section were performed with an incident field given by (2.3) with $\theta = \pi/4$. It is of primary interest to quantify performance at different wavenumbers and plate numbers in order to identify which method should be preferred. It is also pertinent to highlight a number of qualitative points regarding performance differences for obtaining different aspects of the solution, ease of implementation and adaptability.

We start with the case of two equally spaced plates with $d = 2a = 2/3$ in the interval $[-1, 1]$. Figure 4 (left) shows the total computed field for $k_0 = 50$. To measure error, we will consider both near-field and far-field. For the near-field, we compute an approximation $[\tilde{q}]$ to $[q]$, the jump in $q$ across the plates, at equally spaced points $\{x_j\} \subset \gamma$ and define

$$E_{\text{near}} = \frac{\sum_j ||\tilde{q}(x_j, 0) - [q](x_j, 0)||}{\sum_j ||[q](x_j, 0)||}.$$  

The unified transform computes $[\tilde{q}]$ via a series expansion whereas the iterative Wiener–Hopf method obtains $[\tilde{q}]$ from the inverse Fourier transform of $\tilde{Q}$, which we compute using Gaussian quadrature. For the far-field, the asymptotic form of the solution can be computed via steepest-descent applied to the Fourier inversion integral to obtain

$$q(r, \theta) \sim e^{ik_0 r} \sin \theta \sqrt{\frac{k_0}{2\pi r}} \int_{\gamma} e^{-ik_0 x \cos \theta} q(x, 0+) dx, \quad r \to \infty.$$  

We therefore set

$$D(\theta) = \sin(\theta) \int_{\gamma} e^{-ik_0 x \cos \theta} q(x, 0+) dx.$$
Figure 4: Left: Example computed total field for $k_0 = 50$. Right: Exponential convergence of the unified transform (near-field and far-field) for different $k_0$.

Note that the finite extent of the plates means that the Fourier transform of $q$ is pole free; poles would give rise to Fresnel regions and require special treatment to achieve a uniformly valid expression (in our case the far-field is beyond the Rayleigh distance). We compute an approximation $\tilde{D}$ to $D$ at equally spaced angles $\{\theta_j\} \subset (0, \pi)$ and define

$$E_{\text{far}} = \frac{\sum_j |\tilde{D}(\theta_j) - D(\theta_j)|}{\sum_j |D(\theta_j)|}. $$

Note that $\tilde{D}$ can be computed using the unified transform and the relevant Fourier transforms of the functions $\{C_m\}$ given by (3.4), whereas the iterative Wiener–Hopf method directly computes these Fourier transforms at points of interest. In reality, we do not have access to the true scattered field $q$ to compute the above errors. Hence we will use a “converged” solution (computed using a larger number of degrees of freedom). In what follows, $E_{\text{near}}$ was computed using 201 evenly-spaced points on each plate and $E_{\text{far}}$ was computed using 99 evenly-spaced angles.

Figure 4 (right) shows the exponential convergence of the unified transform for different wavenumbers $k_0$ and $N_i = N'_i = N$. As discussed in [10], we found that for larger $k_0$, larger $N$ is needed before we see exponential convergence. This is due to the more oscillatory solution and typically the $N$ needed to gain a given accuracy scales linearly with $k_0$. However, we also see another effect. For small $k_0 = 1$, the convergence is slower, plateauing at a larger relative error $\approx 10^{-9}$. This can be overcome by increasing the number of collocation points and, for the examples in this paper, was not an issue in practice. There does seem to be an inherent ill-conditioning of (3.1) and equivalently (4.3) for smaller $k_0$ which can be understood as a consequence of the branch points of $K(\alpha)$ coalescing. Another interpretation is the slower decay properties of the solution at infinity - the unified transform expands the $y$-derivative of the scattered field along the infinite portions of $\mathbb{R}\setminus\gamma$, whereas the iterative Wiener–Hopf method involves integrands that decay at a slower rate for smaller $k_0$. Also, in the case of small $k_0$, it is well-known that the large incompressible region around a screen causes numerical difficulties. There is a large amount of energy in this region compared to the scattered field and hence standard numerical methods must resolve both regimes. This highlights the importance of boundary approaches such as the ones we present, which can capture such behaviour.

Figure 5 shows the exponential convergence of the iterative Wiener–Hopf method. There are two key parameters in the numerical implementation of the iterative stage of the method: the number of degrees of freedom used to compute the Cauchy transforms using the spectral approach described in §4.3, and the number of iterations undertaken before the solution is extracted. The left-hand panel of Figure 5 demonstrates that the method converges exponentially in the number of degrees of freedom,
Figure 5: Left: Exponential convergence in degrees of freedom used for computing factorisations using Cauchy transforms of the iterative Wiener–Hopf method (near-field and far-field) for different $k_0$. Right: Exponential convergence in iteration of the iterative Wiener–Hopf method (near-field and far-field) for different $k_0$.

Figure 6: Average time taken to achieve 4 digits of accuracy in the far-field directivity $\tilde{D}$ using unified transform (solid blue) and iterative Wiener–Hopf method (dashed red). Left: Single plate with endpoints at $[-1, 1]$ for different wavenumbers $k_0$. Right: $M$ plates with plate length and spacing all equal, with extreme endpoints at $\{-1, 1\}$ for wavenumber $k_0 = 2$.

and the right-hand panel that the method converges exponentially in the number of iterations. For the present implementation, the number of degrees of freedom required to obtain a given accuracy increases outside an optimal regime centred near and around $k_0 = 10$. This is associated with the need to approximate smooth functions with a range of decay rates; for ease of implementation we consider a single quadrature scheme that can still provide a high degree of accuracy for a range of wavenumbers with around 100 degrees of freedom. Convergence with iteration is fastest for high wavenumbers. As mentioned previously, obtaining the near-field requires more care for large wavenumbers using the iterative approach due to issues around inverting the $x$-Fourier transform of functions with strong exponential behaviour, whilst also choosing a contour to avoid the branch points at $\pm k_0$. This difficulty is evidenced for the case of $k_0 = 100$ in each panel where the method struggles to achieve more than seven digits of accuracy (compared to about fourteen digits obtained with the unified transform). Should particular applications require higher accuracy, specialisation might mitigate these issues. The inversion method employed uses a single contour for each spatial location, which means errors are largest at angles $\pm \pi/2$ and $0, \pi$ from each plate edge.
Figure 7: Average time taken to achieve 2 digits of accuracy in the far-field directivity $\tilde{D}$ using unified transform (solid blue) and iterative Wiener–Hopf method (dashed red) for twenty plates ($M = 20$) and $k_0 = 2$. We have plotted against $d/a$ as opposed to the open fractions $2a/d$ since this best shows the growth in time taken.

Though each method converges exponentially, we found that when computing the far-field, the iterative Wiener–Hopf method scales better for large frequencies. On the other hand, both methods perform similarly for smaller $k_0$. This is shown in Figure 6 (left) where we have shown the time taken$^2$ by each method to reach $E_{\text{far}} < 10^{-4}$ for a single plate between $[−1, 1]$. The unified transform is more appropriate for computing the near-field (as shown in Figure 5), and so we have measured the error via $E_{\text{far}}$. We have also compared the time taken to reach $E_{\text{far}} < 10^{-4}$ for different values of $M$ (number of plates) with $d = 2a$ in Figure 6 (right) for $k_0 = 2$. In this case, we see that the unified transform is slightly faster, though both methods seem to scale with the same algebraic rate (roughly cubically). Finally, we compare the computation time to gain $E_{\text{far}} \leq 10^{-2}$ for twenty plates with $k_0 = 2$ and different open fractions $2a/d$ in Figure 7. The unified transform fares much better than the iterative Wiener–Hopf method when $2a/d$ is small, with a much slower increase in computation time as $2a/d$ decreases (the step shape for the unified transform is due to the increasing number of basis functions needed). This agrees with [43] where it was noted that the number of iterations required to achieve a given degree of accuracy is strongly correlated with the smallest Helmholtz number (wavenumber $\times$ lengthscale) present in the problem.

In summary:

- Both methods converge exponentially in the number of degrees of freedom used.
- Both methods offer similar scaling (computation time) as the number of plates is increased.
- The unified transform is better suited for computing the spatial field, especially on or near the domain boundary, since the solution is represented in physical space.
- The iterative Wiener–Hopf method is better suited to large wavenumbers, and generally appears to be less sensitive to changes in the wavenumber.
- The time taken for the iterative Wiener–Hopf method depends strongly on the smallest Helmholtz number (and so lengthscale) present and the method is therefore much slower than the unified transform when the open fraction $2a/d$ is small.
- The unified transform requires care in choosing the basis functions for the unknowns (in our case the functions in (3.3) to capture endpoint singularities) and also the collocation points.

$^2$All timings were performed using the BenchmarkTools.jl package on a 2018 MacBook Pro, a 2.3Ghz processor and 8GB memory, with the figures quoting the mean time from each trial.
• The iterative Wiener–Hopf method requires care in order to cope with singular features, such as singularities of the solution (captured by the orthogonal weights of the Chebyshev polynomials), evaluation near removable singularities and conducting effective spatial inversion. There is also a need to balance the number of iterations with the degrees of freedom. For the problems considered, 400 degrees of freedom are typically sufficient, and we iterate until a prescribed relative error between iterates is reached.

We also remark that, in general, the unified transform is more versatile and able to cope with more complicated geometries. We will use the unified transform to compute the near-field in §7.1 and both methods for the far-field (in order to verify computations) in §7.2.

6 A Collocation Method for a Single Compliant Plate

In this section, we discuss the numerical solution of the scattering by a single plate \( x \in [-1, 1] \) with Robin boundary conditions. Our solution will be obtained via a mixture of separation of variables and collocation. The geometry of a single plate can be transformed into a separable PDE on a rectangular domain. Essentially, a flat plate can be considered as a degenerate ellipse of zero thickness [36]. The solution can be written down as an infinite series of Mathieu functions, the appropriate eigenfunctions, the theory of which can be found in [34]. Separation of variables only works for the problem of a single plate, however, it does yield the solution everywhere in the domain and not just the unknown boundary values, bypassing the need for Green’s representation theorem or steepest descent to evaluate the solution in the exterior domain. Of course, we must still numerically sum a series of special functions (which can be represented efficiently using sine series - see below).

6.1 Separation of Variables

The first few steps, which we recall for the benefit of the reader, are standard for the case of \( \mu \equiv 0 \) and can be found, for example, in [11, 26, 37]. First, we introduce elliptic coordinates via \( x = \cosh(\nu) \cos(\eta), y = \sinh(\nu) \sin(\eta) \), where, with an abuse of notation, we write \( q(\nu, \eta), \mu(\eta) \) and \( f(\eta) \). The appropriate domain then becomes \( \nu \geq 0 \) and \( \eta \in [0, \pi] \) and the PDE becomes

\[
\frac{\partial^2 q}{\partial \eta^2} + \frac{\partial^2 q}{\partial \nu^2} + \frac{\cosh(2\nu) - \cos(2\eta)}{2} k_0^2 q = 0,
\]

\[
q|_{\eta=0} = q|_{\eta=\pi} = 0,
\]

\[
\frac{1}{\sin(\eta)} \frac{\partial q}{\partial \nu}(0, \eta) - 2\mu(\eta)q(0, \eta) = f(\eta),
\]

\[
\lim_{\nu \to \infty} \nu^{1/2} \left( \frac{\partial}{\partial \nu} - i k_0 \right) q(\nu, \eta) = 0.
\]

To simplify the formulae, we let \( Q = k_0^2/4 \). Separation of variables for solutions of the form \( V(\nu)W(\eta) \) leads to the regular Sturm–Liouville eigenvalue problem

\[
\begin{cases}
W''(\eta) + (\lambda - 2Q \cos(2\eta)) W(\eta) = 0, \\
W(0) = W(\pi) = 0.
\end{cases}
\]

The solutions of this are sine-elliptic functions, denoted by \( \text{se}_m \) with eigenvalue \( \lambda_m \), which we expand as

\[
\text{se}_m(\lambda_m, \eta) = \sum_{l=1}^{\infty} B_l^{(m)} \sin(l\eta).
\]

The eigenfunctions are real and orthogonal, and we choose the normalisation

\[
\int_0^\pi \text{se}_m(\eta)\text{se}_n(\eta) d\eta = \frac{\pi}{2} \delta_{mn}.
\]
We also split the solutions further by symmetry or antisymmetry about \( \eta = \pi / 2 \) and write
\[
\begin{align*}
se_{2m}(\eta) &= \sum_{l=1}^{\infty} B_{2l}^{(2m)} \sin(2l\eta), \\
se_{2m+1}(\eta) &= \sum_{l=0}^{\infty} B_{2l+1}^{(2m+1)} \sin((2l + 1)\eta).
\end{align*}
\]
(6.5)

For the even order solutions, the eigenvalue problem then becomes the tridiagonal system
\[
\begin{pmatrix}
2^2 - \lambda_{2m} & Q & 0 & 0 \\
Q & 4^2 - \lambda_{2m} & Q & 0 \\
0 & Q & 6^2 - \lambda_{2m} & Q \\
0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 2^2 - \lambda_{2m+1} \\
\end{pmatrix}
\begin{pmatrix}
B_2^{(2m)} \\
B_4^{(2m)} \\
B_6^{(2m)} \\
\vdots \\
B_{2N}^{(2m+1)}
\end{pmatrix} = 0. 
\]
(6.7)

A similar system holds for the odd order solutions:
\[
\begin{pmatrix}
1^2 - \lambda_{2m+1} - Q & Q \\
Q & 3^2 - \lambda_{2m+1} & Q \\
0 & Q & 5^2 - \lambda_{2m+1} & Q \\
0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 1^2 - \lambda_{2m+1} + Q \\
\end{pmatrix}
\begin{pmatrix}
B_1^{(2m+1)} \\
B_3^{(2m+1)} \\
B_5^{(2m+1)} \\
\vdots \\
B_{2N}^{(2m+1)}
\end{pmatrix} = 0. 
\]
(6.8)

These are solved using square \( n \times n \) truncations of the infinite matrix (also known as the finite section method or Galerkin method). Since the spectrum of the associated (self-adjoint) linear operator is discrete, we do not have to worry about issues such as spectral pollution [14, 31].

The convergence to the eigenvalues and eigenfunctions depends on the parameter \( Q \), in general being slower for larger \( Q \). However, the convergence is exponential, yielding machine precision for small truncation parameter \( n \), even for very large \( Q \). Figure 8 shows the convergence for \( Q = 250,000 \), corresponding to \( k_0 = 1000 \) - at least an order of magnitude larger than those considered in the rest of this paper. Typically for the parameter regimes discussed in this paper, a few dozen sine functions are enough to yield machine precision.

The corresponding \( V(\nu) \) with the appropriate radiation condition at infinity are given by the Mathieu-Hankel functions
\[
Hse_m(\nu) = Jse_m(\nu) + iYse_m(\nu).
\]
These can be expanded in a rapidly convergent series using Bessel functions (see [34]). We choose the normalisation $H_{se}^\prime_m(0) = 1$ and the full solution can then be written as

$$q(\nu, \eta) = \sum_{m=1}^{\infty} a_m se_m(\eta) H_{se}^m(\nu).$$

In order to determine the coefficients $\{a_m\}$, we need to use the Robin boundary conditions. This yields the relation

$$\sum_{m=1}^{\infty} a_m se_m(\eta) [1 - 2\mu(\eta)H_{se}^m(0) \sin(\eta)] = \sin(\eta) f(\eta). \quad (6.9)$$

### 6.2 Numerical Approach

For non-zero $\mu$, there are at least two natural ways to proceed. We can truncate the relation (6.9) to $N$ terms and collocate at $\eta \in [0, \pi]$. Another option is to multiply by $se_n(\eta)$ and integrate along the interval $\eta \in [0, \pi]$, using the orthogonality of the sine-elliptic functions. For each $n = 1, ..., N$, this second approach yields the approximate linear relation

$$a_n - \frac{4}{\pi} \sum_{m=1}^{N} a_m H_{se}^m(0) \int_0^\pi \mu(\eta) se_m(\eta) se_n(\eta) \sin(\eta) d\eta = \frac{2}{\pi} \int_0^\pi \sin(\eta) se_n(\eta) f(\eta) d\eta. \quad (6.10)$$

Both of these approaches lead to a dense linear system and this is the price we pay for the more complicated boundary conditions (the system becomes diagonal when $\mu = 0$). When $\mu$ is constant, we can easily evaluate the relevant integrals in (6.10) to machine precision. For example,

$$\int_0^\pi se_m(\eta) se_n(\eta) \sin(\eta) d\eta = \sum_{l,k=1}^{\infty} B_l^{(m)} B_k^{(n)} \int_0^\pi \sin(l\eta) \sin(k\eta) \sin(\eta) d\eta \quad (6.11)$$

$$= \sum_{l,k=1}^{\infty} B_l^{(m)} B_k^{(n)} \begin{cases} \frac{2kl}{4k^2l^2 - (k^2 + l^2 - 1)^2}, & \text{if } |k - l| \neq 1 \\ 0, & \text{otherwise} \end{cases} \quad (6.12)$$

The typical forcing term corresponding to (2.3) can be written in terms of Bessel functions as

$$\frac{2}{\pi} \int_0^\pi \sin(\eta) se_n(\eta) f(\eta) d\eta = \frac{2}{\pi} i k_0 \sin(\theta) \sum_{l=1}^{\infty} B_l^{(n)} \int_0^\pi \sin(l\eta) \sin(l\eta) e^{-i k_0 \cos(\theta) \cos(\eta)} d\eta \quad (6.13)$$

$$= -2 \tan(\theta) \sum_{l=1}^{\infty} i^{-l} l B_l^{(n)} J_l(k_0 \cos(\theta)). \quad (6.14)$$

However, unless $\mu$ lends itself to integration against a triple product of highly oscillatory sine functions, the integrals in the left-hand side of (6.10) are very hard to numerically evaluate for large $m, n$. One approach is to expand $\mu$ in a sine series using the FFT and use formulae for integrating quadruple products of sine functions. Instead, we shall take the simpler approach of truncating (6.9) to $N$ terms and collocating at $N$ equally spaced points in $[0, \pi]$. In physical space, this corresponds to collocation at Chebyshev points.

### 6.3 Verification of Methods

We now verify that our computational method converges. We will measure the error via the same discrete relative absolute error in (5.1) but now over 199 equally spaced points in the interval $[-1, 1]$.
Figure 9: Examples of $\mu$ for ten plates between $[-1, 1]$ with $2a/d = 0.2$.

Figure 10 (left) shows the convergence of separation of variables (both collocation and Galerkin matrix), as well as the unified transform for $\theta = \pi/4$, $k_0 = 10$, $M = 10$ and $2a/d = 0.2$ with the constant choice

$$
\mu_0 = \frac{\pi}{2d} \left\{ \log \left( \frac{d}{\pi a} \right) \right\}^{-1}.
$$

We see that all three methods are comparable in terms of convergence with $N$. As noted in Remark 1, we also see that the convergence is now algebraic owing to the Robin boundary conditions (and induced (poly-)logarithmic singularities). The rate of convergence is large since the dominant singularity of the solutions is square-root at the endpoints, which is still captured by our choice of basis. The agreement between the unified transform and separation of variables can be understood in terms of basis functions

$$
\sqrt{1-x^2} \cdot U_m(x) = \sin((m+1)\eta).
$$

These sine functions also appear in (6.3), however as an infinite expansion. Despite this, we see the same qualitative convergence. Out of all three methods, the collocation approach is the most accurate with very stable convergence rates.

In order to consider the effect of varying the compliance near the extreme endpoints, as speculatively suggested in [30], we will also use two ways of tapering $\mu$ to 0 at the endpoints, shown in Figure 9. The first, $\mu_1$, will decay to zero like $1/\log(1/(|x^2-1|))$ at the end points. Whereas the second, $\mu_2$, will decay to zero exponentially fast near the endpoints (specifically using a $\tanh$ function). For both of these, the envelope of decay was chosen to be of the order $d$, consistent with the physical picture (we found this to be more accurate than other decay length-scales such as $a$). Specifically, we take

$$
\tilde{\mu}_1^{-1}(x) = \log \left( \frac{d + (x+1)}{a\pi x+1} \right) + \log \left( \frac{d + (1-x)}{a\pi 1-x} \right) - \log \left( \frac{d}{a\pi} \right), \quad \mu_1(x) = \mu_0 \tilde{\mu}_1(x) / \tilde{\mu}_1(0), \quad (6.15)
$$

$$
\tilde{\mu}_2(x) = \tanh \left( 3 \frac{(x+1)-d}{d} \right) + \tanh \left( 3 \frac{(1-x)-d}{d} \right), \quad \mu_2(x) = \mu_0 \frac{\tilde{\mu}_2(x) - \tilde{\mu}_2(-1)}{\tilde{\mu}_2(0) - \tilde{\mu}_2(-1)}. \quad (6.16)
$$

The choice of $\mu_1$ is motivated by the analysis in [28], but now has the correct limiting value of $\mu = 0$ at the endpoints.

The convergence of the collocation method, in these cases, is shown in Figure 10 (right) where we see algebraic convergence for $\mu_1$ but exponential convergence for $\mu_2$ owing to the rapid decay of $\mu_2$ near the endpoints. For our comparison between Robin boundary conditions and the perforated screen, the several digits typically obtained when $N = 100$ are easily sufficient for our purposes. Smaller errors can be obtained for larger $N$ if desired.
7 Robin Boundary Condition vs Small Rigid Plates

Now that we have discussed our numerical methods, and shown them to be effective for the considered problem, we turn to the application that motivated this paper, namely, the investigation of (2.4). We will begin with the near-field, where we use the unified transform (which is easier to compute the near-field with than the iterative Wiener–Hopf method) and note that we do not expect small errors very near the plate. In fact, we expect large errors near the plates up to distances of approximately $O(d)$. We then move onto the far-field where we use both the unified transform and the iterative Wiener–Hopf method in order to verify our numerical computations (however, the computation when $2a/d$ is small is much faster for the unified transform as noted in §5).

Remark 2. When we talk of errors in this section, we are referring to the difference between the scattered field produced by an array of sound-hard plates and the single plate with boundary condition (2.4), and not to numerical errors (all of the examples below were computed so that numerical errors are negligible compared to physical errors).

7.1 Near-field

To quantify the near-field error, we compute the pointwise relative error, $|\tilde{q}(x, y)/q(x, y)|$ for approximations $\tilde{q}$ of $q$ computed using a single plate with various $\mu$ (which were listed in §6.3). The converged reference solution $q$ for an array of sound-hard plates is computed using the unified transform. These errors are shown (in log scale) in Figure 11 for $k_0 = 0.2$ and Figure 12 for $k_0 = 2$, both for various values of $2a/d$ and $k_0 d$ (which must be small for the relevant asymptotic regime). We have not shown the errors for larger $k_0$ since they were very large. As expected, we see large relative errors for the rigid approximation $\mu = 0$. As $2a/d$ and $k_0 d$ decrease, the relative errors away from the plates for $\mu_0, \mu_1$ and $\mu_2$ all decrease. There is also an apparent oscillatory shape for the magnitude of the relative errors with height of order $d$, corresponding to the tips of each plate (hence 4 to 19 wave peaks depending on the parameters). It is also apparent that tapering can, in some cases, reduce the error, but can also sometimes increase the error. These comments underline the importance of edge effects in these types of physical models. It is not clear which choice of tapering is more effective for the smaller $k_0 = 0.2$ but the choice $\mu_1$ is more reliable and effective at reducing the error than $\mu_2$ when $k_0 = 2$. 

Figure 10: Left: Convergence of the collocation method, direct matrix method and unified transform for constant $\mu_0$. Right: Convergence of collocation method for $\mu_0$, $\mu_1$ and $\mu_2$. 

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Figure 11: Near-field relative errors for $k_0 = 0.2$ (screen Helmholtz number = 0.4) shown in $\log_{10}$ scale. The red lines show the positions of the plates.
Figure 12: Near-field relative errors for \( k_0 = 2 \) (screen Helmholtz number = 4) shown in \( \log_{10} \) scale. The red lines show the positions of the plates.
7.2 Far-field

Figures 13–15 show the relative far-field scattered noise $|D(\theta)|$ for $k_0 = 0.2, 0.2$ and 10 respectively, with the same parameter selections as before. We have plotted the results for an array of sound-hard plates (computed using the unified transform and iterative Wiener–Hopf method) and denoted by “True” in the figures. Immediately, we see that when both $2a/d$ and $k_0d$ are small, there is excellent agreement between the homogenised boundary conditions (with $\mu = \mu_0, \mu_1$ or $\mu_2$) and the sound-hard plates. This verifies the appropriateness of using a constant compliance (neglecting end effects) in this case to provide a good approximation to the physically relevant far-field directivity. However, we see that the number of plates and wavenumber also play important roles. For example, when comparing Figure 15 (d) where there is excellent agreement, to Figures 13 (b) and 14 (b) where there is less good agreement, despite the open fraction and grating Helmholtz numbers being smaller. This again highlights the importance of edge effects - for Figure 15 (d) there are more plates and a larger wavenumber so edge effects at $\pm 1$ are less important. The different choices also do not seem to affect the number of oscillations in the far-field, even for large wavenumber $k_0 = 10$.

To quantify these results further, we have plotted the relative power (the integral of $|D(\theta)|^2$ divided by the integral corresponding to the array of rigid plates) at infinity for the scattered field for different $k_0$ in Figure 16. The left plot shows the relative power as a function of the open fraction for twenty plates. We see that as $2a/d$ decreases, the curves approach 1 for each choice of $\mu$. However, for larger $k_0$, $\mu_2$ produces an overestimate of the power for small $2a/d$. We see however that if $k_0$ is small (so that $k_0d$ is small) then $\mu_2$ provides the best estimate for the power. Similarly, in the right
(a) $M = 5, 2a/d = 0.2, (k_0d)^{-1} = 0.69$

(b) $M = 5, 2a/d = 0.02, (k_0d)^{-1} = 0.79$

(c) $M = 20, 2a/d = 0.2, (k_0d)^{-1} = 0.17$

(d) $M = 20, 2a/d = 0.02, (k_0d)^{-1} = 0.20$

Figure 14: Relative absolute value of far-field for $k_0 = 2$ (screen Helmholtz number = 4).
Figure 15: Relative absolute value of far-field for $k_0 = 10$ (screen Helmholtz number = 20).
Figure 16: Relative power of far-field for different $k_0$. Left: For $M = 20$ and various $2a/d$. Right: For $2a/d = 0.02$ and various $M$.

plot of Figure 16, we have plotted the relative power for $2a/d = 0.02$ and various plate numbers $M$. We see that if $M$ is too small, $\mu_2$ can provide a severe overestimate (which seems to be worse for smaller $k_0$). For larger $k_0$, it is also possible for $\mu_0$ and $\mu_1$ to overestimate the power as well. Note also that in all cases, the power produced using $\mu_2$ is greater than when using $\mu_1$, which in turn is greater than $\mu_0$. This highlights the importance of edge effects when considering a finite screen. Moreover, whilst as we move deeper into the appropriate asymptotic regime (decreasing $2a/d$, $k_0d$ and increasing $M$), the agreement with the exact case improves. Finally, out of all the physical parameters, it appears that the most important for the homogeneous approximation to hold is the open fraction $2a/d$. This is consistent with the homogenisation approach in [30], which applies the asymptotic limit by considering each gap as a point.

We have also shown the corresponding errors in Figure 17. These highlight the above points - if $2a/d$ and $k_0d$ are small enough, then $\mu_2$ provides the best estimate of the far-field. However, for larger $k_0$ it can provide a worse estimate than even $\mu_0$ (no tapering). Overall we note that even the simple model of constant compliance can yield good agreement of utility for practical engineering studies, provided sufficiently

- large number of plates, $M > 10$
- small open fraction $2a/d < 0.04$
- small grating Helmholtz number $k_0d < 1$

However outside of these limits, one should take care in selecting a compliance parameter for finite sections of perforated materials.

8 Conclusion

This paper provides a contribution to understanding the applicability of a compliant or Robin boundary condition to modelling scattering by a truncated array of collinear rigid plates. In particular, we interrogate the role of end effects of such homogenised boundary conditions to finite perforated screens incorporated in larger structures of interest in applications, such as bio-inspired adaptations in aeroacoustics [27]. We solve the exact problem using two methods: the unified transform and an iterative Wiener–Hopf method. The former provides a versatile method, whilst the latter is more specialised, most appropriate for large wavenumbers. We detail the regimes in which each method is competitive and practical considerations which we hope to be of broader interest to those applying such methods. We note, however, that the unified transform was more appropriate for the asymptotic regimes studied...
in this paper. This paper also presents an effective collocation approach based on Mathieu functions for tackling the problem of scattering by a finite plate on which a variable Robin boundary condition is applied, which is likely to be of broader interest.

We verified the appropriateness of the homogenised constant Robin boundary condition given in [30] to achieve good agreement (< 10%) in the far-field directivity with the exact problem even for a modest number of plates (> 10) and sufficiently far into the asymptotic regime of validity for the homogenisation (2a/d < 0.04) and small k0. Investigating the role of end corrections to the compliance by taking μ ↓ 0 to include the sound-hard endpoint junctions can improve the approximation, but a generic manner tapering is unclear. Though a simple tanh-type tapering seemed the most effective choice in this paper for small 2a/d and k0d. This, and the slow convergence of the (physical) approximation as we proceed deeper into the appropriate asymptotic regime highlights the importance of junction effects in scattering problems, and may offer an interesting direction for future research should applications require more accurate models. Further, whilst we anticipate these findings to broadly carry across to alternative homogenisation models for structured media, their significance may warrant special interrogation. For each of these points, the extension of the unified transform to variable boundary conditions and the application of the Wiener–Hopf technique to semi-infinite truncated arrays [6] may be useful, in addition to the collocation Mathieu function method introduced here for the problem of a finite plate of variable compliance.

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