

Trustworthy Koopmanism:

Spectra, Forecasting, and Dictionary Learning

(managing non-closure in finite-dimensional approximations)

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Papers and talks:

[http://www.damtp.cam.ac.uk/
user/mjc249/home.html](http://www.damtp.cam.ac.uk/user/mjc249/home.html)

What is a Koopman operator?

- \mathcal{X} – *the state space*
- $\mathcal{X} \ni x$ – *the state*

cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – *the dynamics*: $x_{n+1} = F(x_n)$

Henri Poincaré
(Sorbonne)



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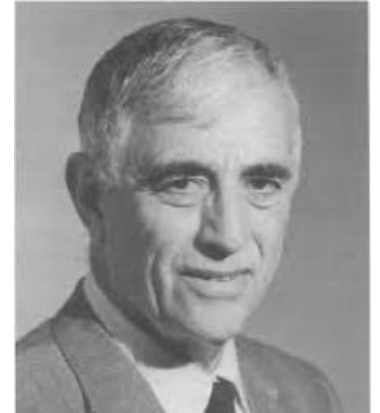
cts $F: \mathcal{X} \rightarrow \mathcal{X}$ – the dynamics: $x_{n+1} = F(x_n)$

- Functions $g: \mathcal{X} \rightarrow \mathbb{C}$ a.k.a “observables”, $g \in L^2(\mathcal{X}, \omega)$
- Koopman operator $\mathcal{K}_F: [\mathcal{K}_F g](x) = g(F(x))$

LINEAR!

Observe g one time step forward

Bernard Koopman
(Columbia)



John von Neumann
(IAS)



- Koopman, “Hamiltonian systems and transformation in Hilbert space,” *Proc. Natl. Acad. Sci. USA*, 1931.
- Koopman, v. Neumann, “Dynamical systems of continuous spectra,” *Proc. Natl. Acad. Sci. USA*, 1932.

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- Koopman operator $\mathcal{K}_F: [\mathcal{K}_F g](x) = g(F(x))$ **LINEAR!**
- Available snapshot data: $\left\{ \left(x^{(m)}, y^{(m)} = F(x^{(m)}) \right) : m = 1, \dots, M \right\}$

Can we compute spectral properties from trajectory data?

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

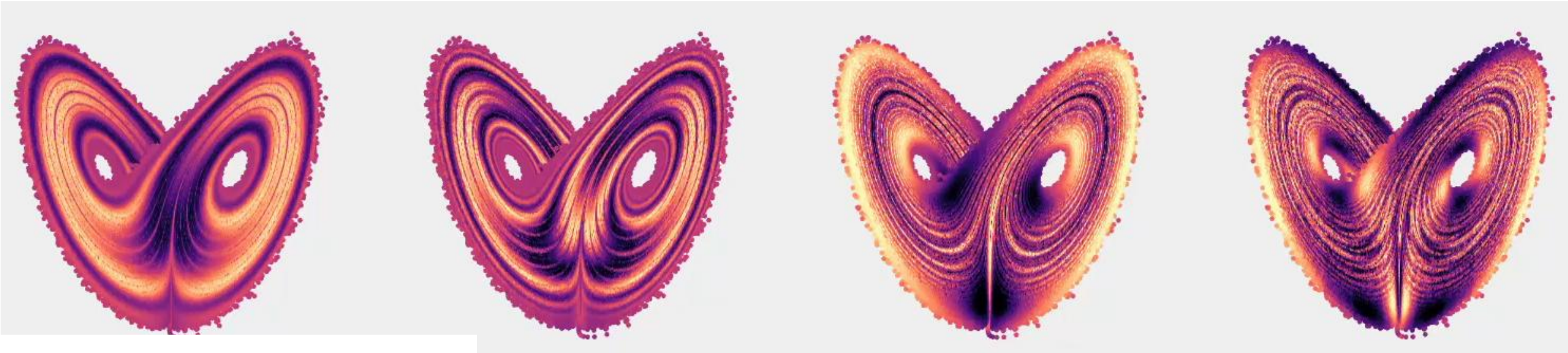
Why?

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$

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Coherent features!

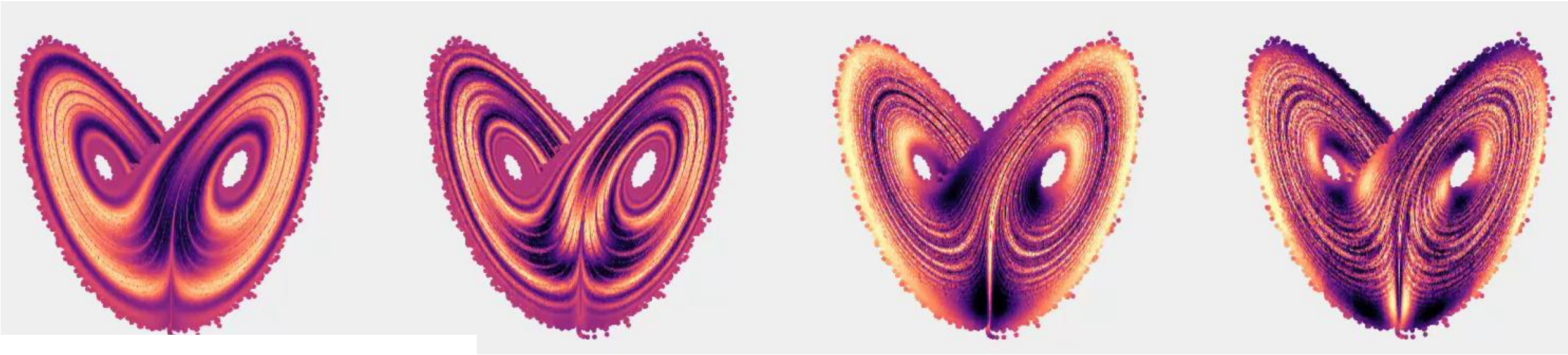
Lorenz attractor

- C., Drysdale, Horning, “Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators,” *SIAM Journal on Applied Dynamical Systems*, 2025. Code: <https://github.com/MColbrook/Rigged-Dynamic-Mode-Decomposition>

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Coherent features!

$$\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C} : \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

Trades: Nonlinear, finite-dimensional \Rightarrow Linear, infinite-dimensional.

Koopman Mode Decomposition

- Find (g_j, λ_j) with $\|\mathcal{K}g_j - \lambda_j g_j\| \leq \varepsilon$
- Expand observable:

$$h(x) \approx \sum_j c_j g_j(x)$$

Verified Eigenfunctions

coefficients, called
"Koopman modes"

- Forecasts:

$$h(x_n) = \sum_j \lambda_j^n c_j g_j(x) + \mathcal{O}(n\varepsilon)$$

Intuition: A nonlinear separation of variables through a linear operator!

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$

$$\begin{pmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ & & & & & & 0 & \ddots \\ & & & & & & & \ddots \end{pmatrix} \xrightarrow{\text{Two-way infinite}} \begin{pmatrix} 0 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{N \times N}$$

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Explicit example: Matrix approximation of \mathcal{K} (EDMD)

Observables $\psi_j: \mathcal{X} \rightarrow \mathbb{C}, j = 1, \dots, N$

$$\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^M$$

quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]^* \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]^* \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \dots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \dots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \right]_{jk}$$

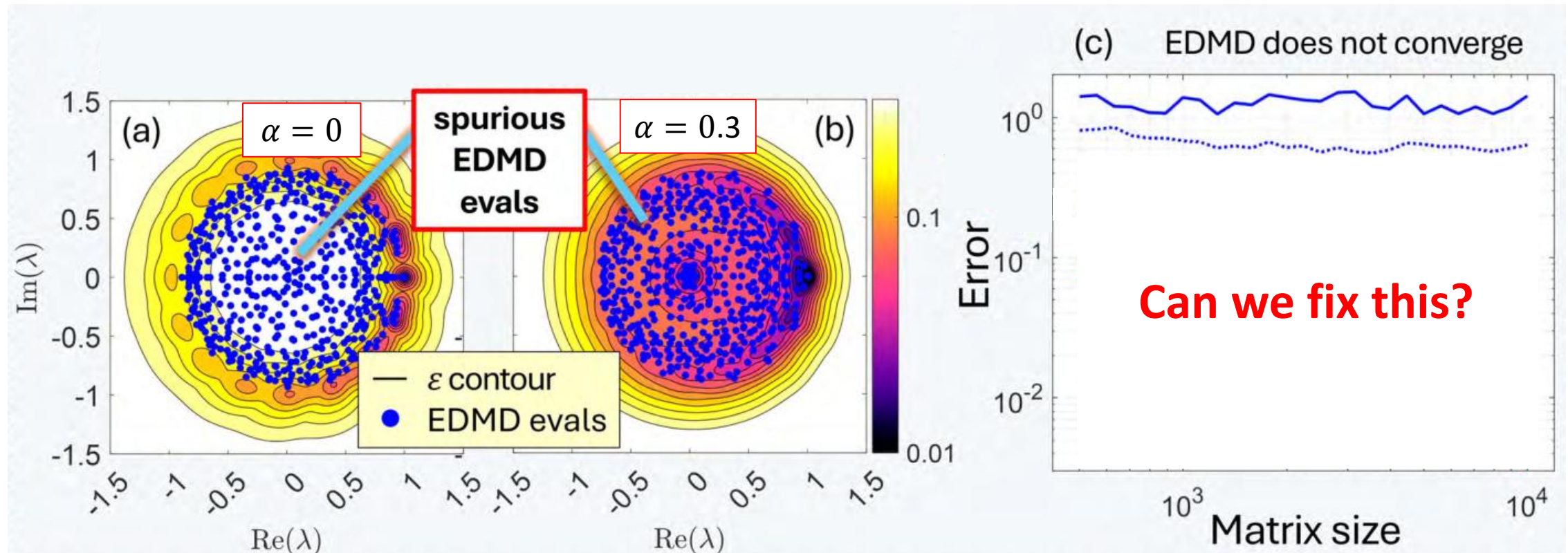
Galerkin
Approximation

$$\mathcal{K} \rightarrow (\Psi_X^* W \Psi_X)^{-1} \Psi_X^* W \Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, "Dynamic mode decomposition of numerical and experimental data," **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, "Spectral analysis of nonlinear flows," **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley "A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition," **J. Nonlinear Sci.**, 2015.

EDMD doesn't converge!

- Duffing oscillator: $\dot{x} = y$, $\dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)



The fix: Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\Psi_X^* W \Psi_X}_G \right]_{jk}$$

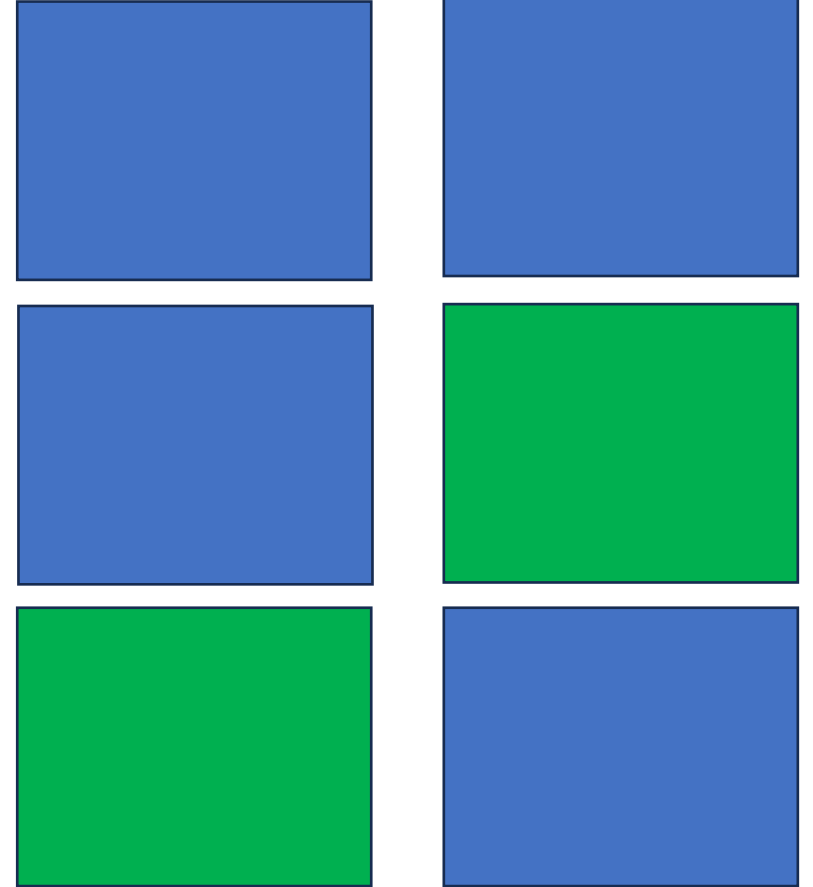
$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\Psi_X^* W \Psi_Y}_A \right]_{jk}$$

- C., Townsend, "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems," **Commun. Pure Appl. Math.**, 2023.
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Residuals: $g = \sum_{j=1}^N \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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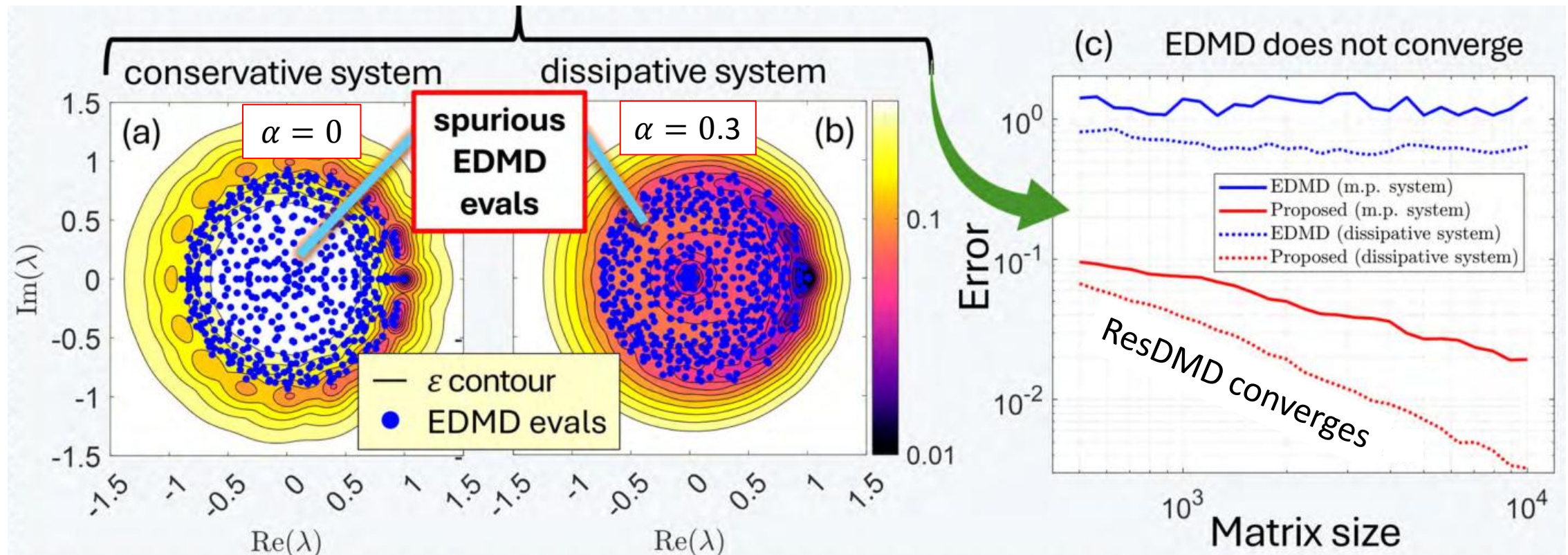
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- C., Ayton, Szóke, "Residual Dynamic Mode Decomposition," *J. Fluid Mech.*, 2023.
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ResDMD does converge!

- Duffing oscillator: $\dot{x} = y, \dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)

Compute $\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K})$, local adaptive control on $\varepsilon \downarrow 0$



What can we do?

Consider space of observables with finite energy: $L^2(\mathcal{X}, \omega)$

Theorem: There **exists** algorithms $\Gamma_{N,M}$ using snapshots such that

$$\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \Gamma_{N,M}(F) = \text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}_F)$$

for all systems.



N = size of basis, M = amount of data (quadrature)

$$\text{Sp}_{\text{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C} : \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

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Double limit $\lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty}$

Can we do better?

Adversaries: Double limit is necessary!

Implies \mathcal{K} is unitary

Class of systems: $\Omega_{\mathbb{D}} = \{F: \bar{\mathbb{D}} \rightarrow \bar{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}$.

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) \mid x \in \bar{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}$.

Theorem: There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using \mathcal{T}_F such that $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}_F) \forall F \in \Omega_{\mathbb{D}}$.

NB:

- n can index anything.
- Universal - any type of algorithm or computational model.
- Similarly, no random algorithms converging with probability $> 1/2$.

Proof idea: Constructing an adversary

$$F_0: \text{rotation by } \pi, \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$$

Phase transition lemma: Let $X = \{x_1, \dots, x_N\}, Y = \{y_1, \dots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} \mid 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism H such that H acts as the identity on $\mathbb{D} \setminus \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \dots, N$.

Conjugacy of data ($x_j \rightarrow y_j$) with F_0

Idea: Use lemma to trick any algorithm into oscillating between spectra.

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses \mathcal{J}_F , $\lim_{n \rightarrow \infty} \Gamma_n(F) = \text{Sp}(\mathcal{K}_F) \quad \forall F \in \Omega_{\mathbb{D}}$.

Build an **adversarial** F ...

$$\mathcal{J}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$$

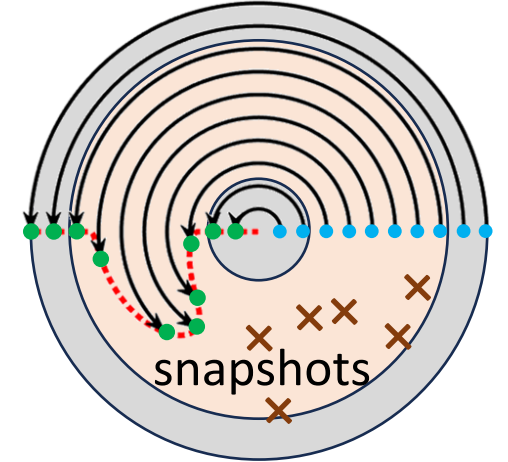
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Build an **adversarial** F ...

$$\widetilde{F}_1(r, \theta) = (r, \theta + \pi + \phi(r)), \text{supp}(\phi) \subset [1/4, 3/4]$$

$$\text{Sp}(\mathcal{K}_{\widetilde{F}_1}) = \mathbb{T} \text{ (unit circle).}$$



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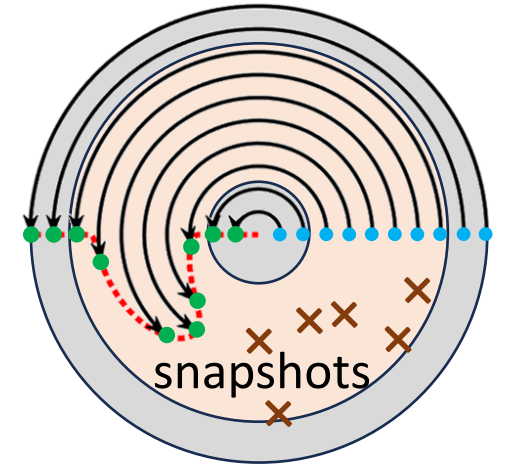
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$$\lim_{n \rightarrow \infty} \Gamma_n(\widetilde{F}_1) = \text{Sp}(\mathcal{K}_{\widetilde{F}_1}) \Rightarrow \exists n_1 \text{ s.t. } \text{dist}(i, \Gamma_{n_1}(\widetilde{F}_1)) \leq 1.$$

BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}(\widetilde{F}_1)$.

Let X, Y correspond to these snapshots.



$$\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$$

Proof idea: Constructing an adversary

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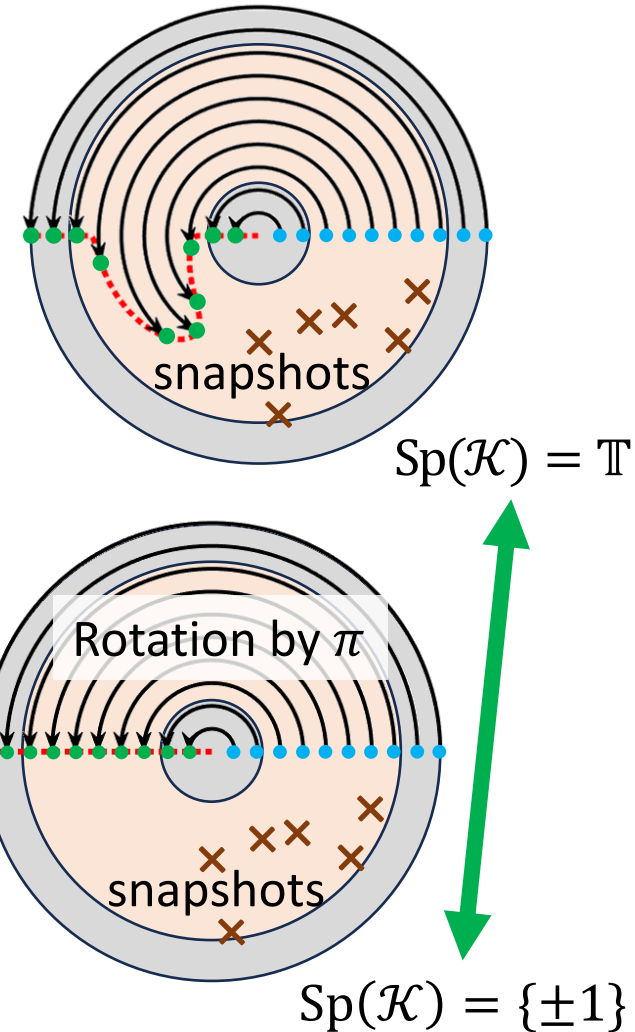
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Let X, Y correspond to these snapshots.

Lemma: $F_1 = H_1^{-1} \circ F_0 \circ H_1$ on annulus \mathcal{A}_1 .

Consistent data $\Rightarrow \Gamma_{n_1}(F_1) = \Gamma_{n_1}(\widetilde{F}_1)$, $\text{dist}(i, \Gamma_{n_1}(F_1)) \leq 1$

BUT $\text{Sp}(\mathcal{K}_{F_1}) = \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$



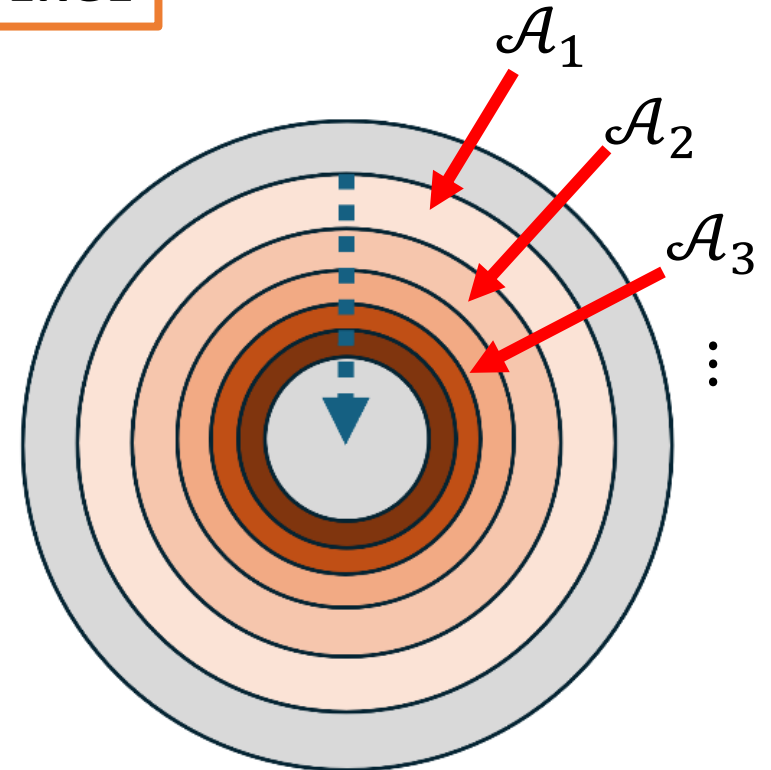
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on \mathcal{A}_k . $F = \lim_{k \rightarrow \infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F}_k)$, $\text{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \rightarrow \infty$

BUT $\text{Sp}(\mathcal{K}_F) = \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Cascade of disks

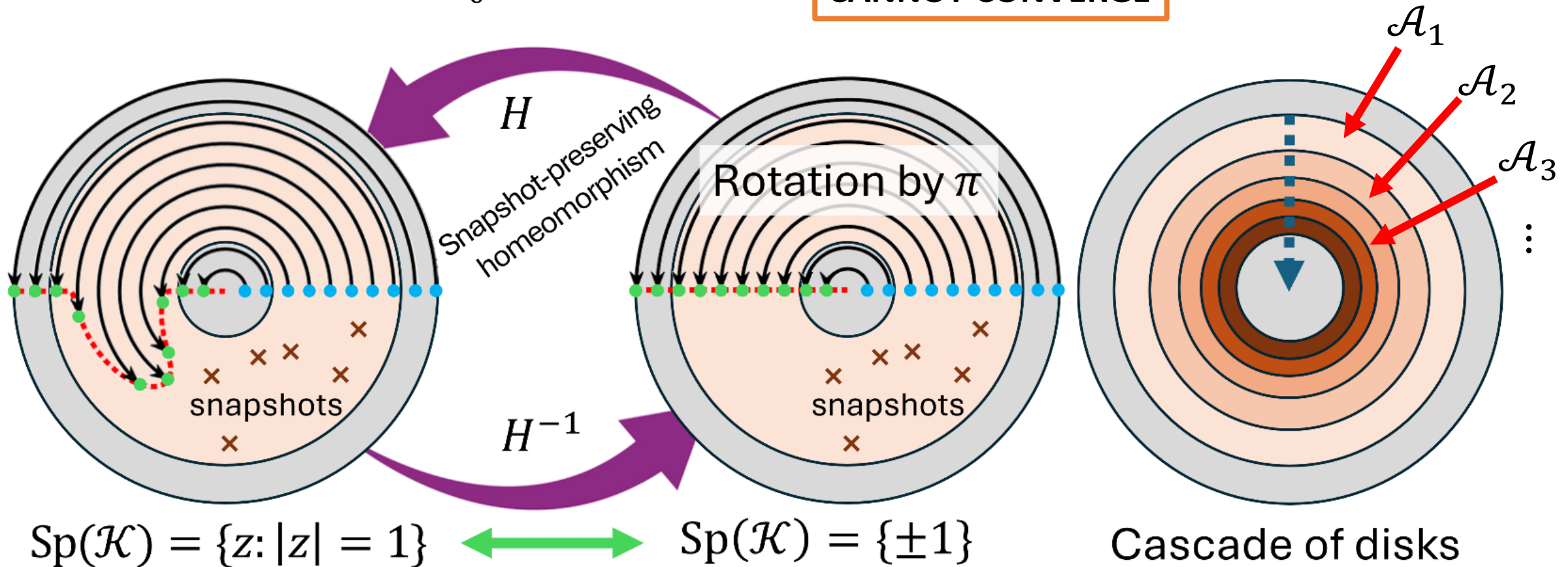
Proof idea: Constructing an adversary

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Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F}_k)$, $\text{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \rightarrow \infty$

BUT $\text{Sp}(\mathcal{K}_F) = \text{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1 : One limit, full error control. E.g., $d(\Gamma_n(F), \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.

- Δ_{m+1} : $\text{SCI} \leq m$.

- Σ_m : $\text{SCI} \leq m$, final limit from below.

$$\text{E.g., } \Sigma_1: \sup_{z \in \Gamma_n(F)} \text{dist}(z, \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}.$$

- Π_m : $\text{SCI} \leq m$, final limit from above.

$$\text{E.g., } \Pi_1: \sup_{z \in \text{Sp}(\mathcal{K}_F)} \text{dist}(z, \Gamma_n(F)) \leq 2^{-n}.$$

-
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 - C., "The foundations of infinite-dimensional spectral computations," **PhD diss.**, University of Cambridge, 2020.
 - C., Hansen, "The foundations of spectral computations via the solvability complexity index hierarchy," **J. Eur. Math. Soc.**, 2022.
 - C., Antun, Hansen, "The difficulty of computing stable and accurate neural networks," **Proc. Natl. Acad. Sci. USA**, 2022.
 - Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "On the solvability complexity index hierarchy and towers of algorithms," arXiv, 2020.

Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1 : One limit, full error control. E.g., $d(\Gamma_n(F), \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.

- Δ_{m+1} : $\text{SCI} \leq m$.

- Σ_m : $\text{SCI} \leq m$, final limit from below.

E.g., Σ_1 : $\sup_{z \in \Gamma_n(F)} \text{dist}(z, \text{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.

- Π_m : $\text{SCI} \leq m$, final limit from above.

E.g., Π_1 : $\sup_{z \in \text{Sp}(\mathcal{K}_F)} \text{dist}(z, \Gamma_n(F)) \leq 2^{-n}$.

trust output

verification

covers spectrum

- Hansen, "On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators." *J. Am. Math. Soc.*, 2011.
- C., "The foundations of infinite-dimensional spectral computations," **PhD diss.**, University of Cambridge, 2020.
- C., Hansen, "The foundations of spectral computations via the solvability complexity index hierarchy," *J. Eur. Math. Soc.*, 2022.
- C., Antun, Hansen, "The difficulty of computing stable and accurate neural networks," *Proc. Natl. Acad. Sci. USA*, 2022.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "On the solvability complexity index hierarchy and towers of algorithms," arXiv, 2020.

Lots of SCI upper bounds lurking in Koopman literature!

SCI: Fewest number of limits needed to solve a computational problem.

| Algorithm | Comments/Assumptions | Spectral Problem's Corresponding SCI Upper Bound | | | |
|-------------------------------------|---|--|-----------------------|--|---|
| | | <i>KMD</i> | <i>Spectrum</i> | <i>Spectral Measure (if m.p.)</i> | <i>Spectral Type (if m.p.)</i> |
| Extended DMD [47] | general L^2 spaces | $\text{SCI} \leq 2^*$ | N/C | N/C | n/a |
| Residual DMD [44] | general L^2 spaces | $\text{SCI} \leq 2^*$ | $\text{SCI} \leq 3^*$ | $\text{SCI} \leq 2^*$ | varies, see [84] e.g., a.c. density: $\text{SCI} \leq 2^*$ |
| Measure-preserving EDMD [45] | m.p. systems | $\text{SCI} \leq 1$ | N/C | $\text{SCI} \leq 2^*$ (general) $\text{SCI} \leq 1$ (delay-embedding) | n/a |
| Hankel DMD [85] | m.p. ergodic systems | $\text{SCI} \leq 2^*$ | N/C | N/C | n/a |
| Periodic approximations [86] | m.p. + ω a.c. | $\text{SCI} \leq 2$ | N/C | $\text{SCI} \leq 2$ (see [87]) | a.c. density: $\text{SCI} \leq 3$ |
| Christoffel–Darboux kernel [40] | m.p. ergodic systems | $\text{SCI} \leq 3$ | n/a | $\text{SCI} \leq 2$ | e.g., a.c. density: $\text{SCI} \leq 2$ |
| Generator EDMD [88] | cts.-time, samples ∇F (otherwise additional limit) | $\text{SCI} \leq 2$ | N/C | $\text{SCI} \leq 2$ (see [89]) | n/a |
| Compactification [42] | cts.-time, m.p. ergodic systems | $\text{SCI} \leq 4$ | N/C | $\text{SCI} \leq 4$ | n/a |
| Resolvent compactification [43] | cts.-time, m.p. ergodic systems | $\text{SCI} \leq 5$ | N/C | $\text{SCI} \leq 5$ | n/a |
| Diffusion maps [90] (see also [10]) | cts.-time, m.p. ergodic systems | $\text{SCI} \leq 3$ | n/a | n/a | |

Are these sharp?

Previous techniques prove upper bounds on SCI.

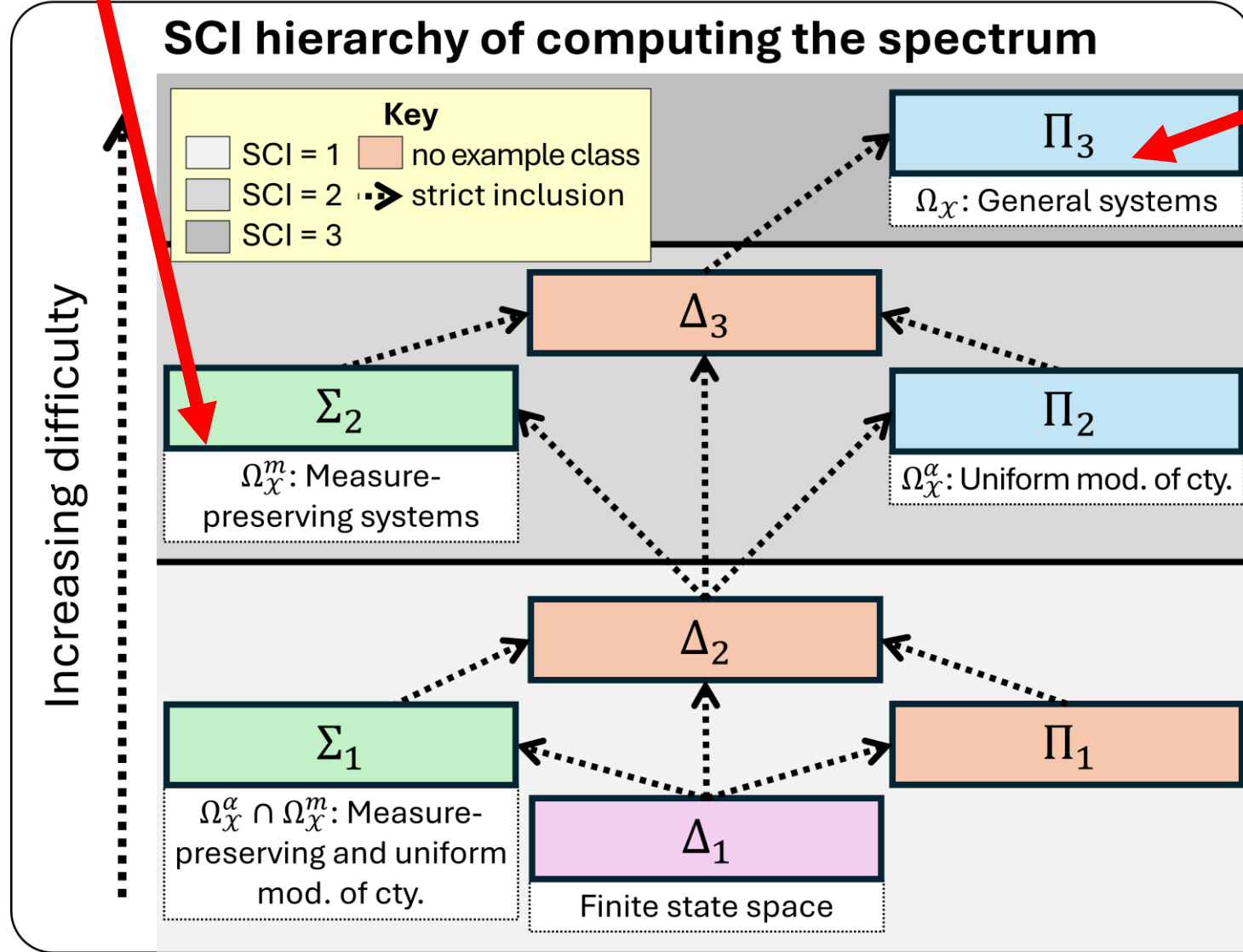
“N/C”: method need not converge. “n/a”: algorithm not applicable to problem.

Also in Ulam’s method for Markov processes, SRB measure computation, control,...

Classification for Koopman

3 limits needed in general!

SCI hierarchy of computing the spectrum



Different classes:

$$\Omega_{\mathcal{X}} = \{F: \mathcal{X} \rightarrow \mathcal{X} \mid F \text{ cts}\}$$

$$\Omega_{\mathcal{X}}^m = \{F: \mathcal{X} \rightarrow \mathcal{X} \mid F \text{ cts, m. p.}\}$$

$$\Omega_{\mathcal{X}}^\alpha = \{F: \mathcal{X} \rightarrow \mathcal{X} \mid F \text{ mod. cty. } \alpha\}$$

$$[d_{\mathcal{X}}(F(x), F(y)) \leq \alpha(d_{\mathcal{X}}(x, y))]$$

Optimal algorithms and classifications of dynamical systems.

Peter Lax:

“The trick of the successful mathematician is to turn the question being asked into one he knows how to answer.”

Johann Wolfgang von Goethe:

“Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different.”

Let's perform this trick by changing the space...

Reproducing kernel Hilbert space (RKHS)

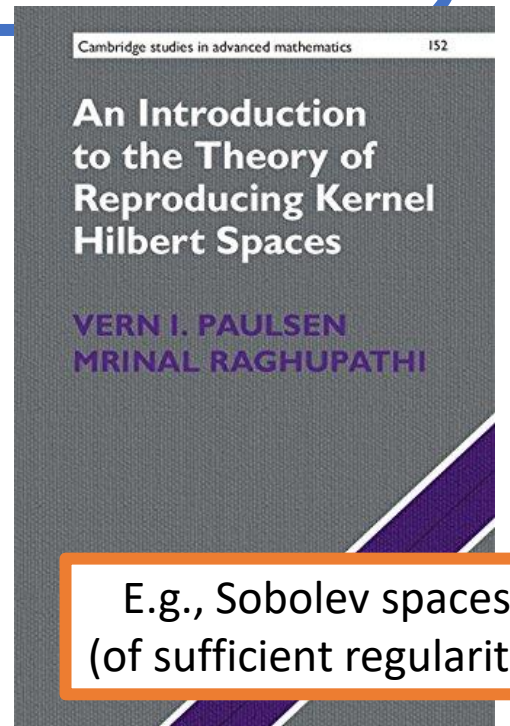
Hilbert space of functions on \mathcal{X} s.t. $g \mapsto g(x)$ bounded $\forall x \in \mathcal{X}$.

Generated by a kernel $\mathfrak{K}: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$

$$g(x) = \langle g, \mathfrak{K}_x \rangle, \quad \mathfrak{K}(x, y) = \langle \mathfrak{K}_x, \mathfrak{K}_y \rangle = \mathfrak{K}_x(y)$$

Advantages over $L^2(\mathcal{X}, \omega)$:

- Forecasts: space bounds \Rightarrow pointwise bounds.
- High-dimensional systems practical through kernel trick.
- Fast methods for evaluating \mathfrak{K} .
- Different $\mathfrak{K} \Rightarrow$ different \mathcal{K} ! Can be tailored to application. (This is where the community is currently heading.)
- Leads to fundamental “possibility” gains...



Reproducing kernel Hilbert space (RKHS)

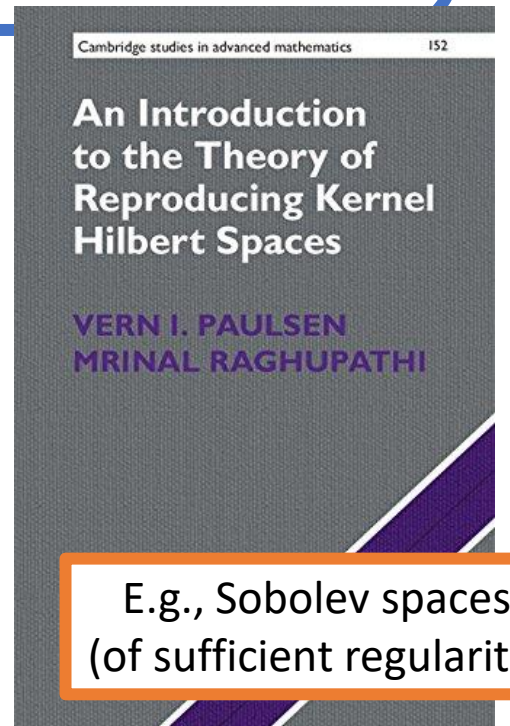
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SpecRKHS: Avoiding large data limit $M \rightarrow \infty$

Look at “Left eigenpairs” through \mathcal{K}^* :

$$\mathcal{K}^* \mathfrak{K}_x = \mathfrak{K}_{F(x)}$$

Evolution of functionals.
 $g(x) = \langle g, \mathfrak{K}_x \rangle_{\mathcal{H}}$

No quadrature needed:

$$G_{jk} = \langle \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}(x^{(k)}, x^{(j)})$$

$$A_{jk} = \langle \mathcal{K}^* \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}(y^{(k)}, x^{(j)})$$

$$L_{jk} = \langle \mathcal{K}^* \mathfrak{K}_{x^{(k)}}, \mathcal{K}^* \mathfrak{K}_{x^{(j)}} \rangle = \langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{y^{(j)}} \rangle = \mathfrak{K}(y^{(k)}, y^{(j)})$$

$$g = \sum_{m=1}^M \mathbf{g}_m \mathfrak{K}_{x^{(m)}}, \quad \|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2 = \mathbf{g}^* (L - \lambda A^* - \bar{\lambda} A + G) \mathbf{g}$$

SpecRKHS: Example algorithm

$$\text{res}^*(\lambda, \mathbf{g})^2 = \frac{\|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2}{\|g\|_{\mathcal{H}}^2} = \frac{\mathbf{g}^* [L - \lambda A^* - \bar{\lambda} A + |\lambda|^2 G] \mathbf{g}}{\mathbf{g}^* G \mathbf{g}}$$

1. Compute $G, A, L \in \mathbb{C}^{N \times N}$ ($N = M$)
2. For z_k in grid, compute $\tau_k = \min_{g = \sum_{m=1}^N \mathbf{g}_m \mathfrak{K}_x(m)}$ $\text{res}^*(z_k, \mathbf{g})$, corresponding g_k (gen. SVD).
3. **Output:** $\{z_k: \tau_k < \varepsilon\}, \{g_k: \tau_k < \varepsilon\}$ (ε -pseudoeigenfunctions).

Theorem:

- **Error control:** $\{z_k: \tau_k < \varepsilon\} \subseteq \text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}^*)$
- **Convergence:** Converges locally uniformly to $\text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}^*)$ (as $N \rightarrow \infty$)

$$\text{Sp}_{\text{ap}, \varepsilon}(\mathcal{K}^*) = \{z \in \mathbb{C}: \exists g, \|g\|_{\mathcal{H}} = 1, \|\mathcal{K}^* g - z g\|_{\mathcal{H}} \leq \varepsilon\}$$

Practical gains: Sea ice forecasting



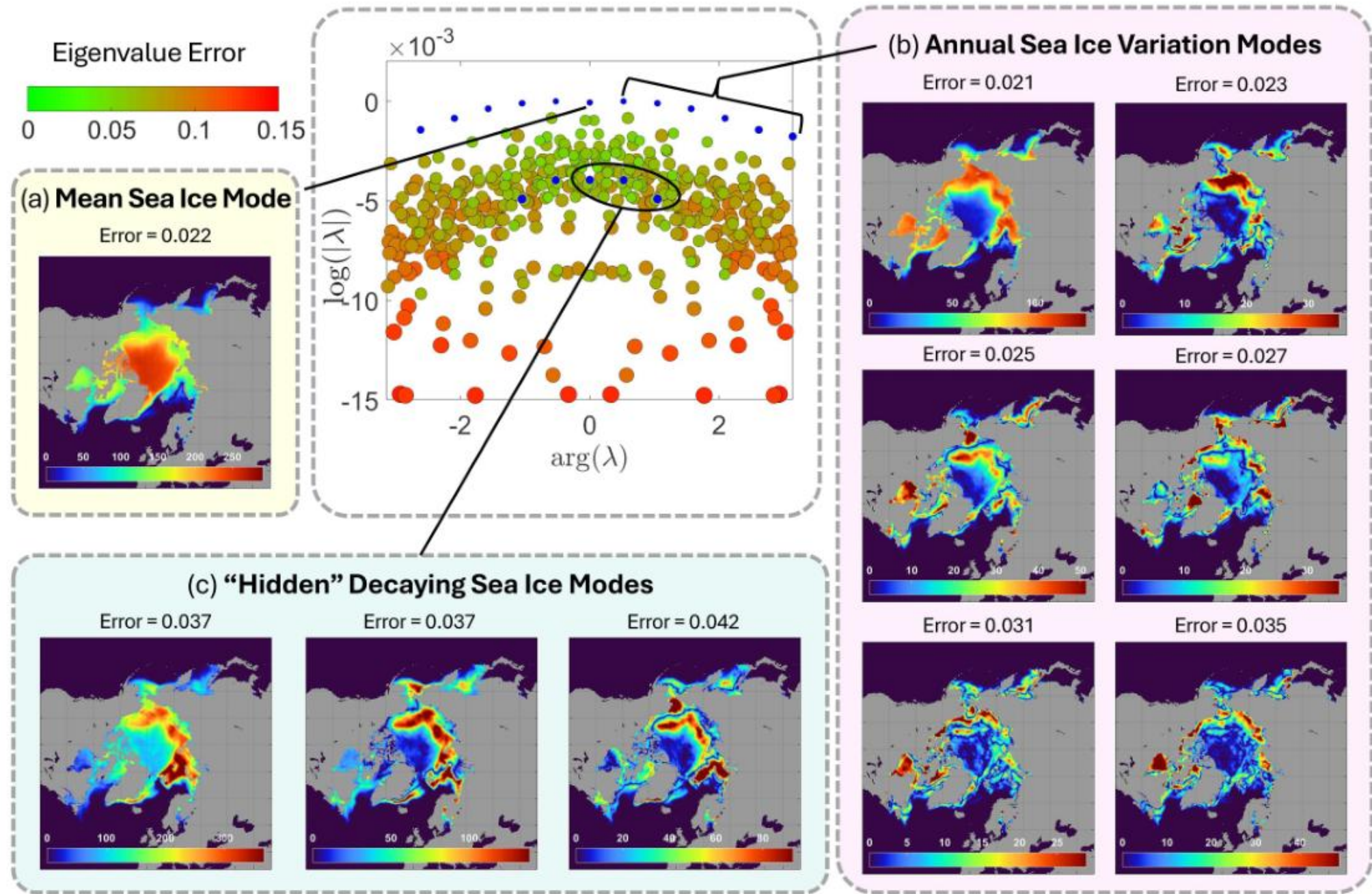
Satellite data



Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,...

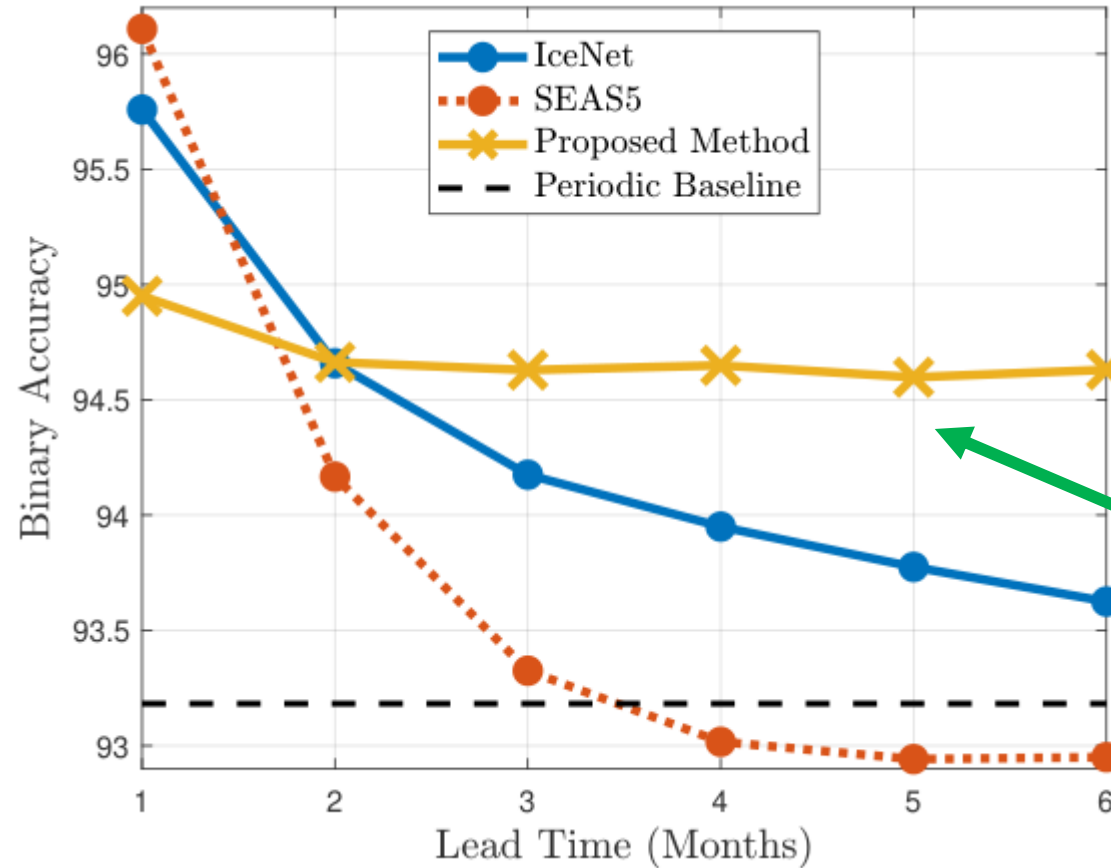
Problems:

1. Very hard to locate geographical significant regions.
2. Very hard to predict more than two months in advance.



- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," preprint, 2025.

Avoid spurious evals \Rightarrow State-of-the-art forecasts



$$g(x_n) = [\mathcal{K}^n g](x_0)$$

$$\|\mathcal{K}g - \lambda g\| \leq \varepsilon$$

$$\Rightarrow g(x_n) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$$

Use ε to filter evals!

Figure: Mean binary accuracy over test years 2012-2020.

(IceNet: Andersson et al, "Seasonal Arctic sea ice forecasting with probabilistic deep learning." Nature Communications, 2021.)

Optimal algorithms and classifications of systems

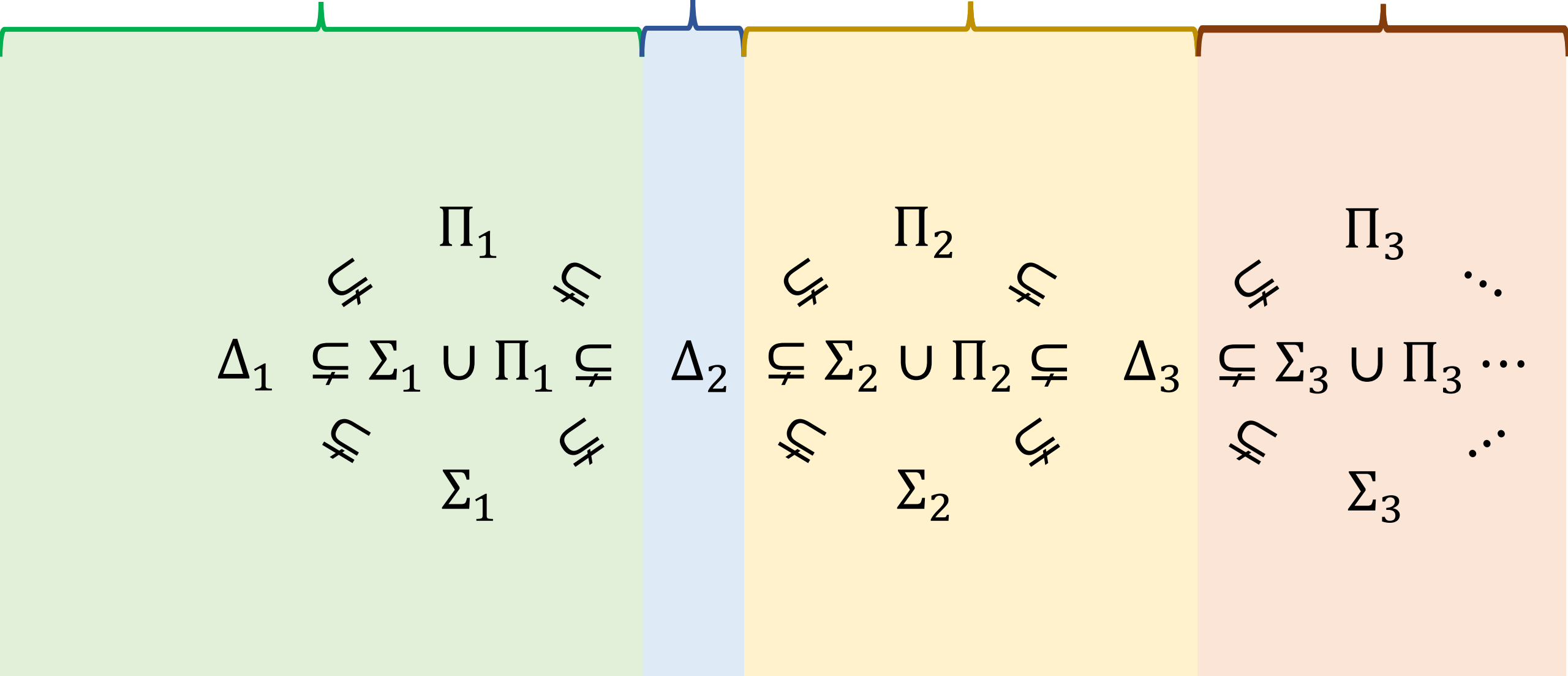


Error control


1 limit

2 limits

3 limits



Optimal algorithms and classifications of systems

$\varepsilon = 0$ 

increasing difficulty 

Error control

1 limit

2 limits

3 limits

$L_2(x, \omega)$

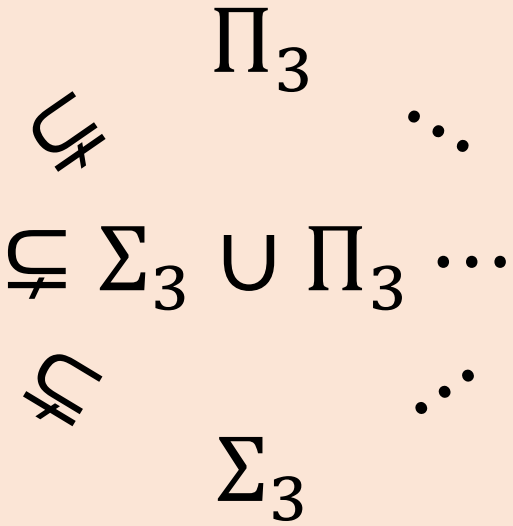
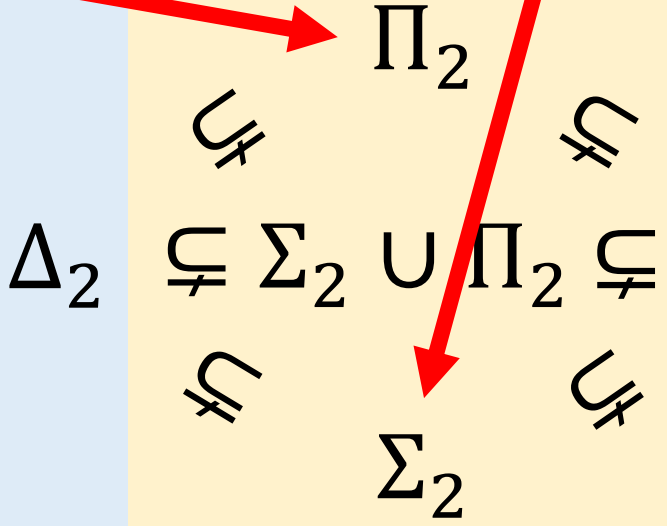
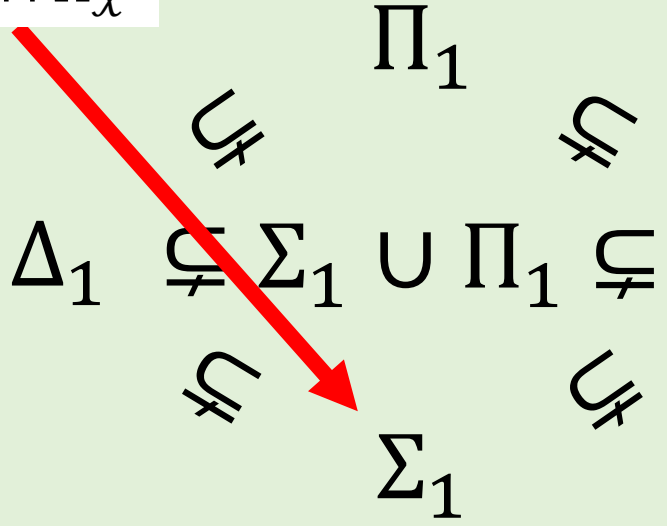
$$\Omega_x^\alpha = \{F: \text{mod. cty. } \alpha\}$$

$$d(F(x), F(y)) \leq \alpha(d(x, y))$$


$$\Omega_x^m \cap \Omega_x^\alpha$$

$$\Omega_x^m = \{F : F \text{ cts, m. p.}\}$$

$$\Omega_x = \{F : F \text{ cts}\}$$



Optimal algorithms and classifications of systems

$\varepsilon = 0$ 

increasing difficulty 

Error control

1 limit

2 limits

3 limits

$L^2(\mathcal{X}, \omega)$

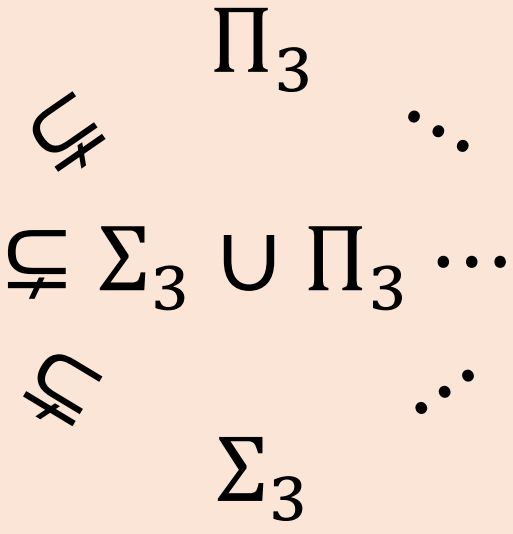
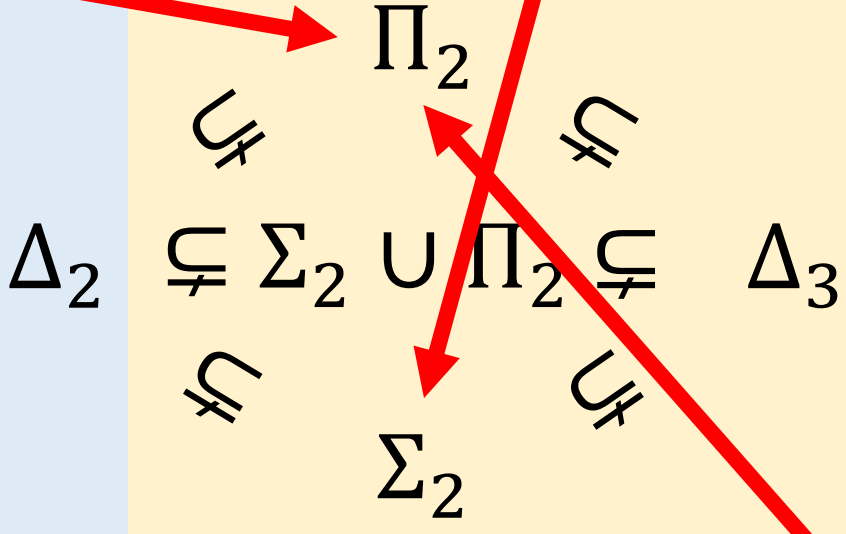
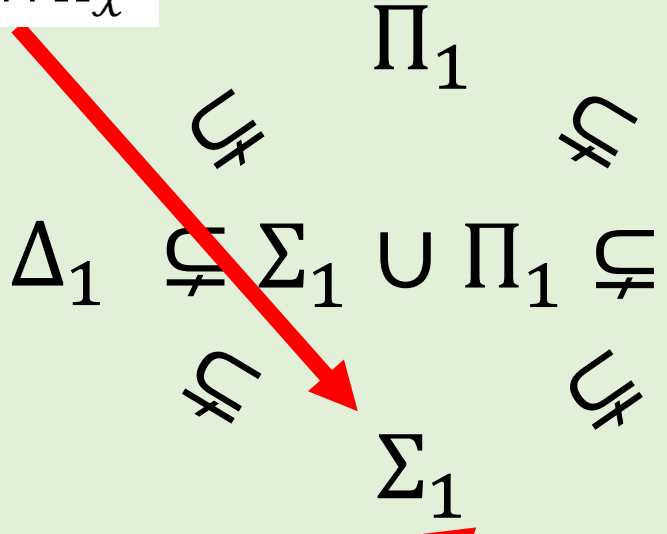
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$$\Omega_{\mathcal{X}} = \{F : F \text{ cts}\}$$

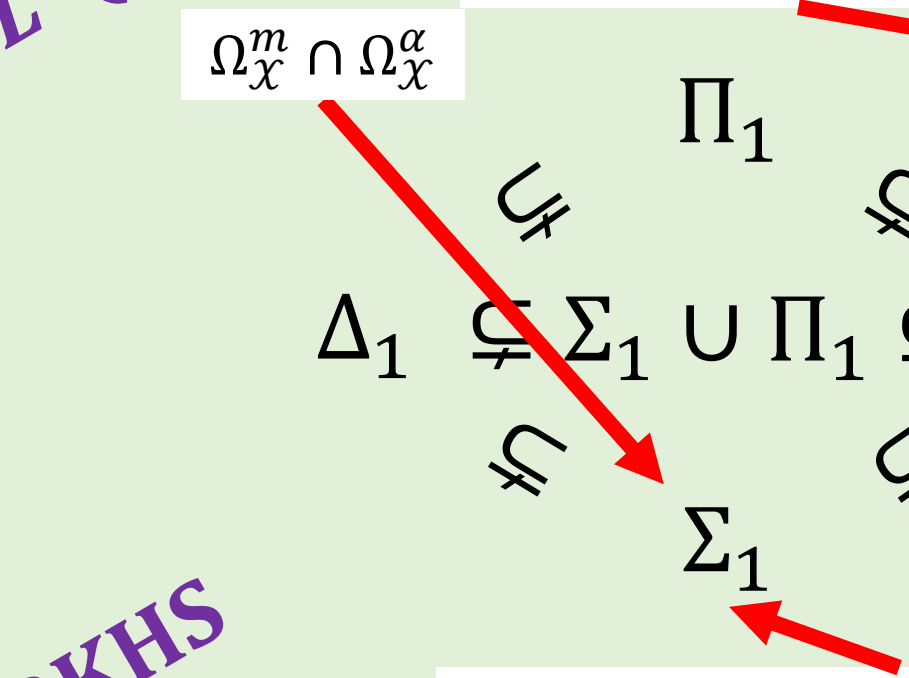


$$\Omega_r^{\mathfrak{K}} = \{F : \mathcal{K}_F \text{ res. control}\}$$

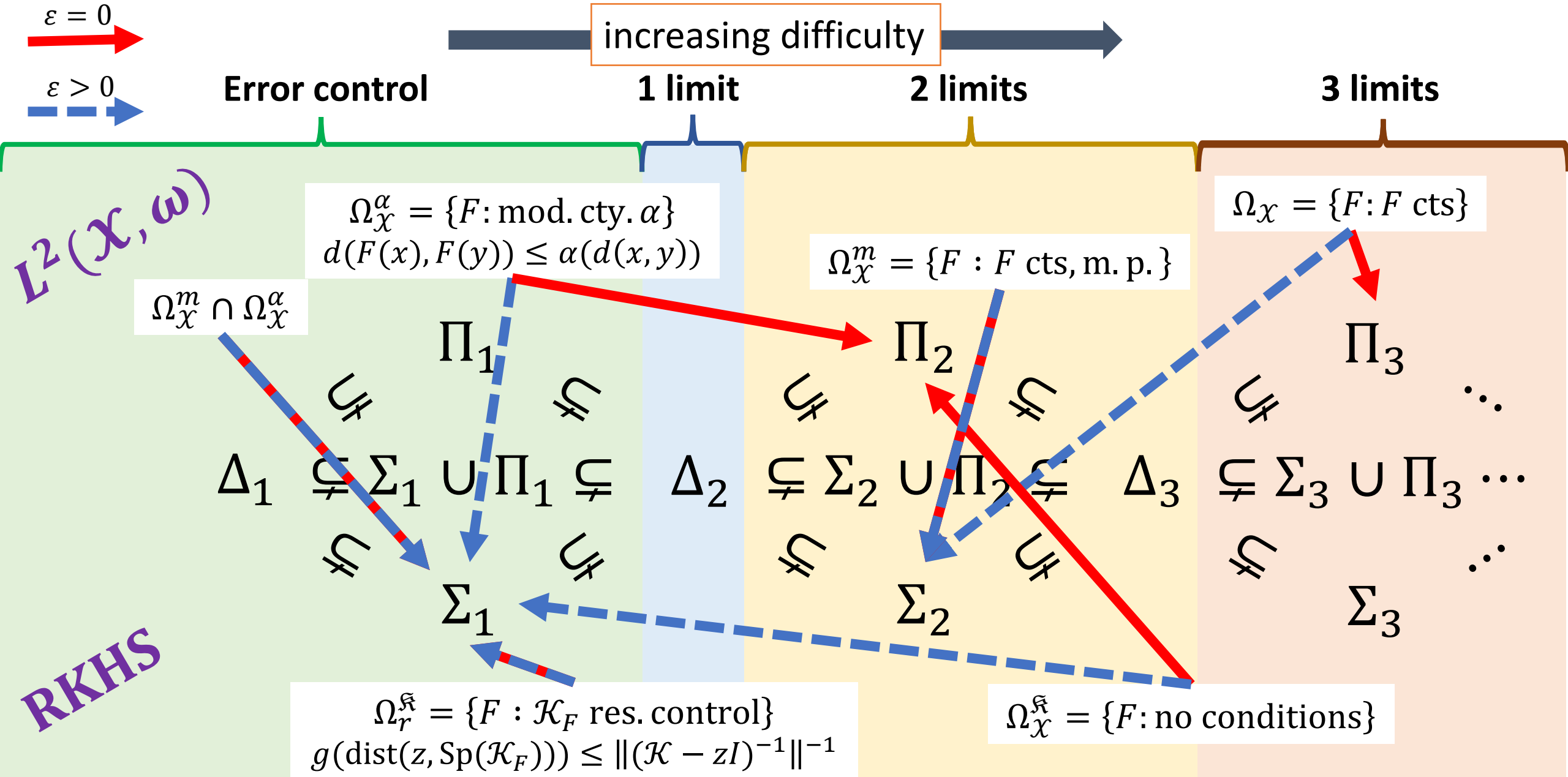
$$g(\text{dist}(z, \text{Sp}(\mathcal{K}_F))) \leq \|(\mathcal{K} - zI)^{-1}\|^{-1}$$

$$\Omega_{\mathcal{X}}^{\mathfrak{K}} = \{F : \text{no conditions}\}$$

$RKHS$



Optimal algorithms and classifications of systems



What about forecast bounds and dictionary learning?

Principle Angle Decomposition (PAD)

Algorithm 3.1 Principal angles and observables between \mathcal{V} and \mathcal{KV} .

Input: Matrices \mathbf{G} , \mathbf{A} and \mathbf{L} in (2.3) and (2.9) for dictionary $\{\psi_j\}_{j=1}^N$, subspace selection matrix $\mathbf{B} \in \mathbb{C}^{N \times n}$, cut-off tolerance $\epsilon_c \geq 0$. For RKHS, use instead the matrices \mathbf{G} , \mathbf{A} and \mathbf{R} (instead of \mathbf{L}) from (2.6) and (2.10).

- 1: Compress the matrices to form $\mathbf{G}_{\mathcal{V}} = \mathbf{B}^* \mathbf{G} \mathbf{B}$, $\mathbf{A}_{\mathcal{V}} = \mathbf{B}^* \mathbf{A} \mathbf{B}$, and $\mathbf{L}_{\mathcal{V}} = \mathbf{B}^* \mathbf{L} \mathbf{B}$.
- 2: Compute the matrix

$$\mathbf{J}_{\mathcal{V}} = \begin{pmatrix} \mathbf{G}_{\mathcal{V}} & \mathbf{A}_{\mathcal{V}} \\ \mathbf{A}_{\mathcal{V}}^* & \mathbf{L}_{\mathcal{V}} \end{pmatrix} \in \mathbb{C}^{2n \times 2n},$$

and its eigendecomposition $\mathbf{J}_{\mathcal{V}} = \mathbf{U} \mathbf{D} \mathbf{U}^*$.

- 3: Set $\mathcal{I} = \{j : \mathbf{D}_{jj} > \epsilon_c, j = 1, \dots, 2n\}$, and compute

$$\mathbf{C}_1 = \sqrt{\mathbf{D}(\mathcal{I}, \mathcal{I})} (\mathbf{U}(1:n, \mathcal{I}))^*, \quad \mathbf{C}_2 = \sqrt{\mathbf{D}(\mathcal{I}, \mathcal{I})} (\mathbf{U}(n+1:2n, \mathcal{I}))^*.$$

- 4: Compute the principal angles and observables

$$\{[\theta_j]_{j=1}^q, \mathbf{U}_1, \mathbf{U}_2\} = \text{subspacea}(\mathbf{C}_1, \mathbf{C}_2), \quad q \leq n,$$

and convert back to the original dictionary via (3.1) and (3.2).

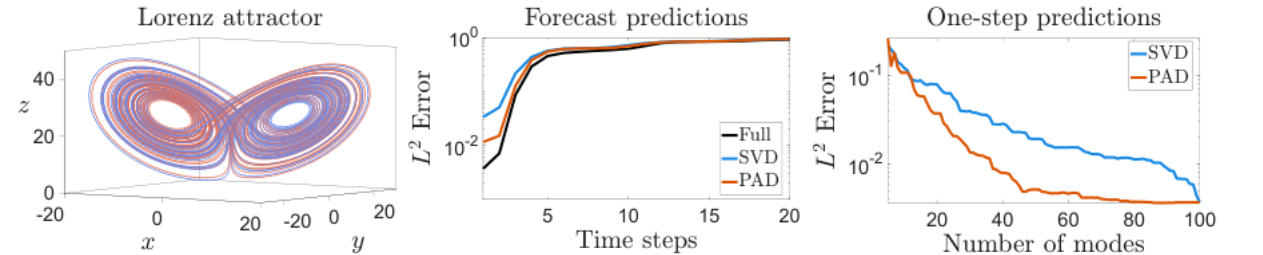
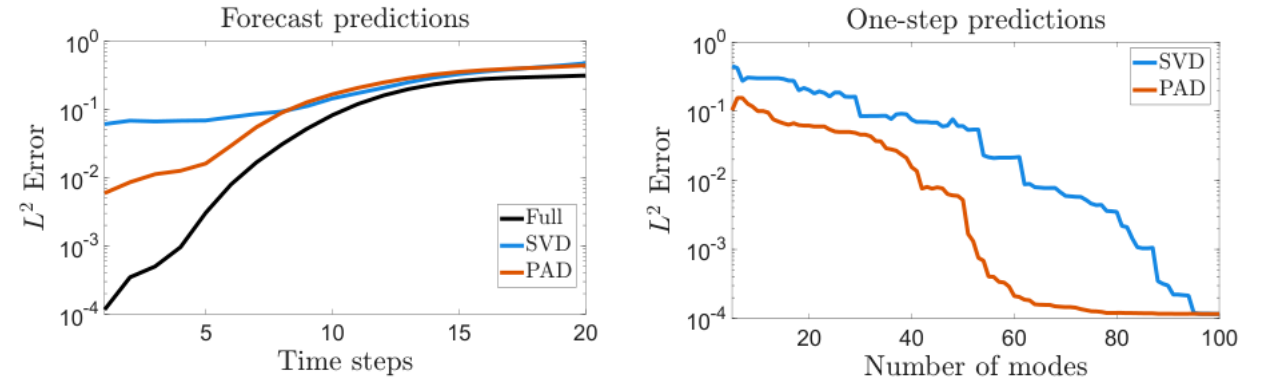
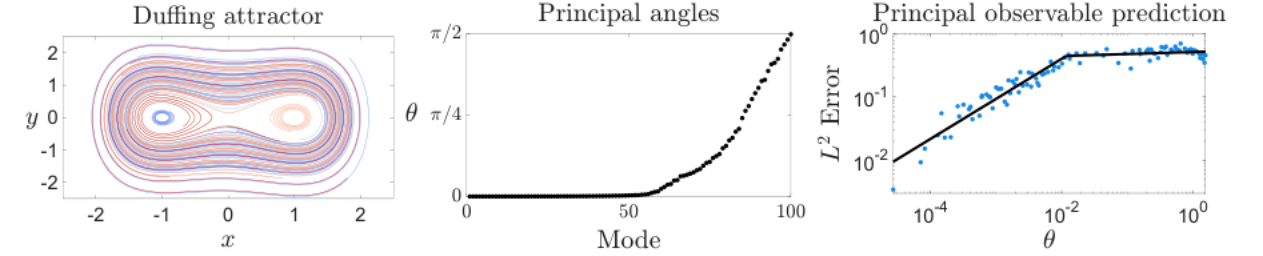
Output: Principal angles $\{\theta_j\}_{j=1}^q$ and principal observables $\{(u_j, v_j)\}_{j=1}^q$.

Algorithm 3.2 Principal Angle Decomposition (PAD).

Input: Matrices \mathbf{G} , \mathbf{A} , \mathbf{K} and \mathbf{L} in (2.3), (2.4), and (2.9) for dictionary $\{\psi_j\}_{j=1}^N$, cut-off tolerance $\epsilon_c \geq 0$, number of post-compression modes $r > 0$.

- 1: Compute principal angles $\{\theta_j\}_{j=1}^q$ and observables \mathbf{U}_1 via Algorithm 3.1 ($\mathbf{B} = \mathbf{I}_N$).
- 2: Define $\mathbf{U}'_1 = \mathbf{U}_1(1:q, :) + \mathbf{K} \mathbf{U}_1(q+1:2q, :)$, and truncate to $\mathbf{U} = \mathbf{U}'_1(:, 1:r)$.
- 3: Construct the matrices $\mathbf{A}_{\text{pad}} = \mathbf{U}^* \mathbf{A} \mathbf{U} = \mathbf{K}_{\text{pad}}$ and $\mathbf{L}_{\text{pad}} = \mathbf{U}^* \mathbf{L} \mathbf{U}$.

Output: The compressed matrices \mathbf{A}_{pad} , \mathbf{K}_{pad} and \mathbf{L}_{pad} .



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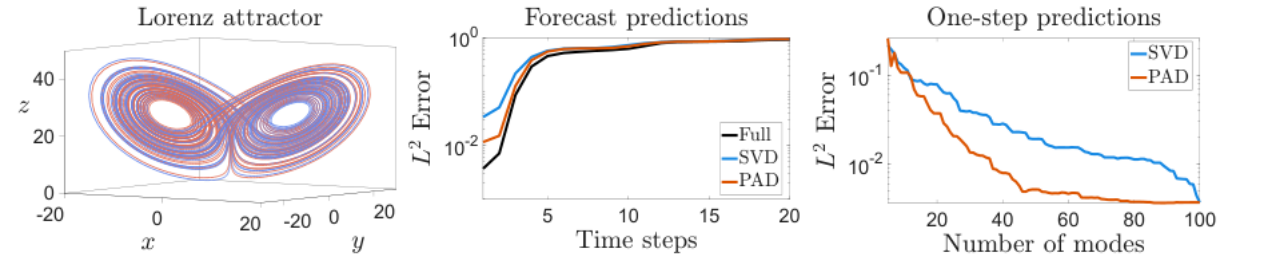
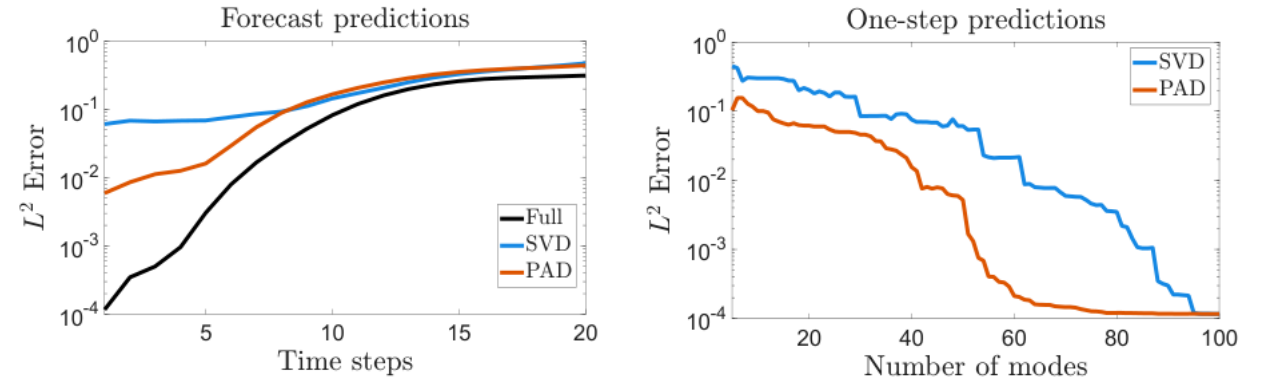
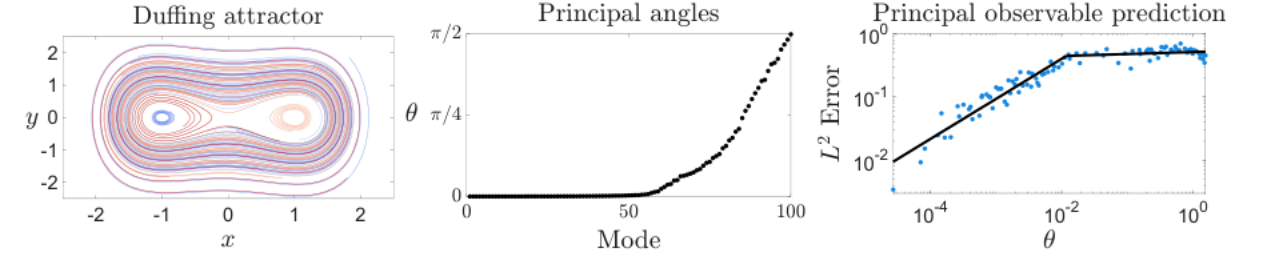
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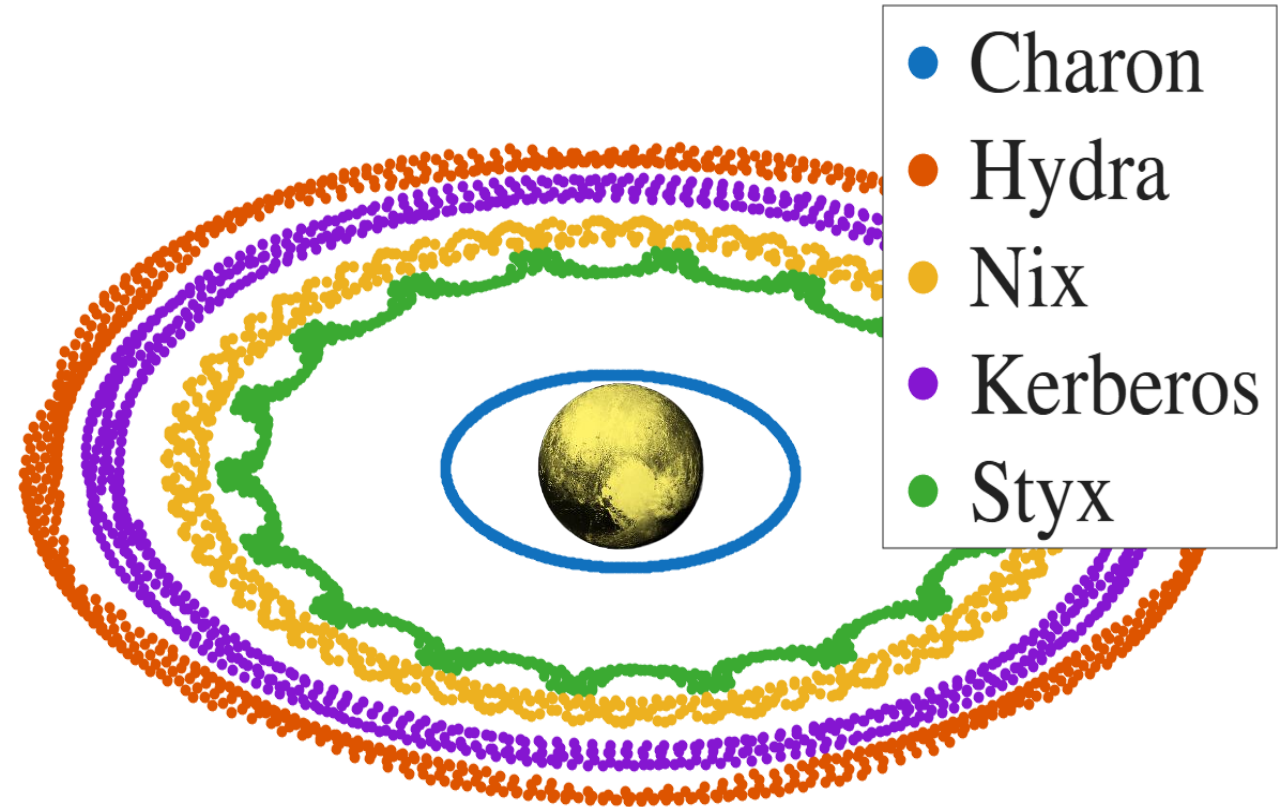


Dictionary learning and forecast multistep bounds

- Pluto and its 5 satellites
- Comparable mass of Pluto and Charon → **complex dynamics**
- Obtain error bounds for both L^2 and RKHS cases

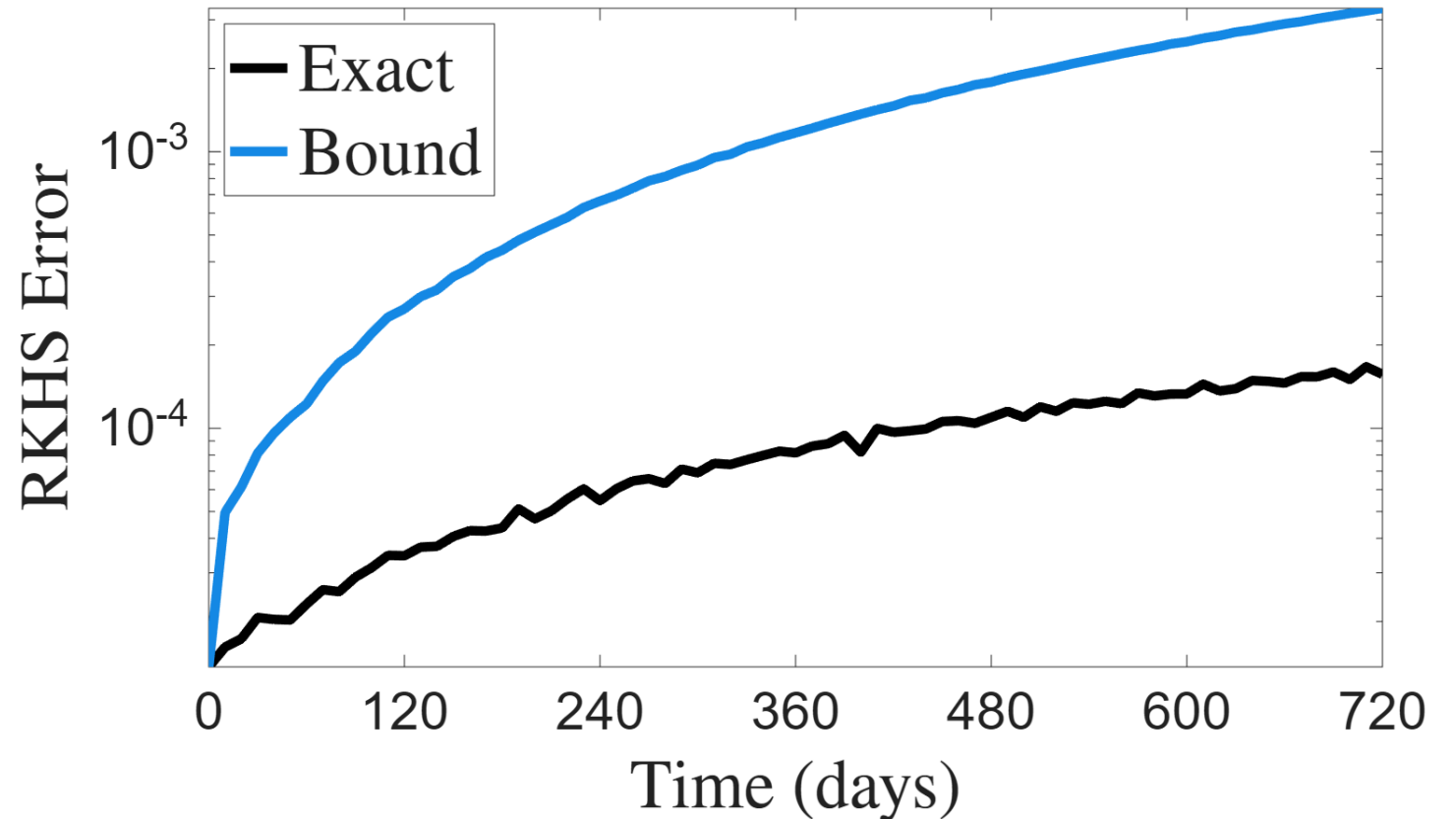
apply to dictionary refinement

obtain pointwise error bounds, expected error surrogates



Dictionary learning and forecast bounds

- Apply algorithms to RKHS setting using Matérn kernel

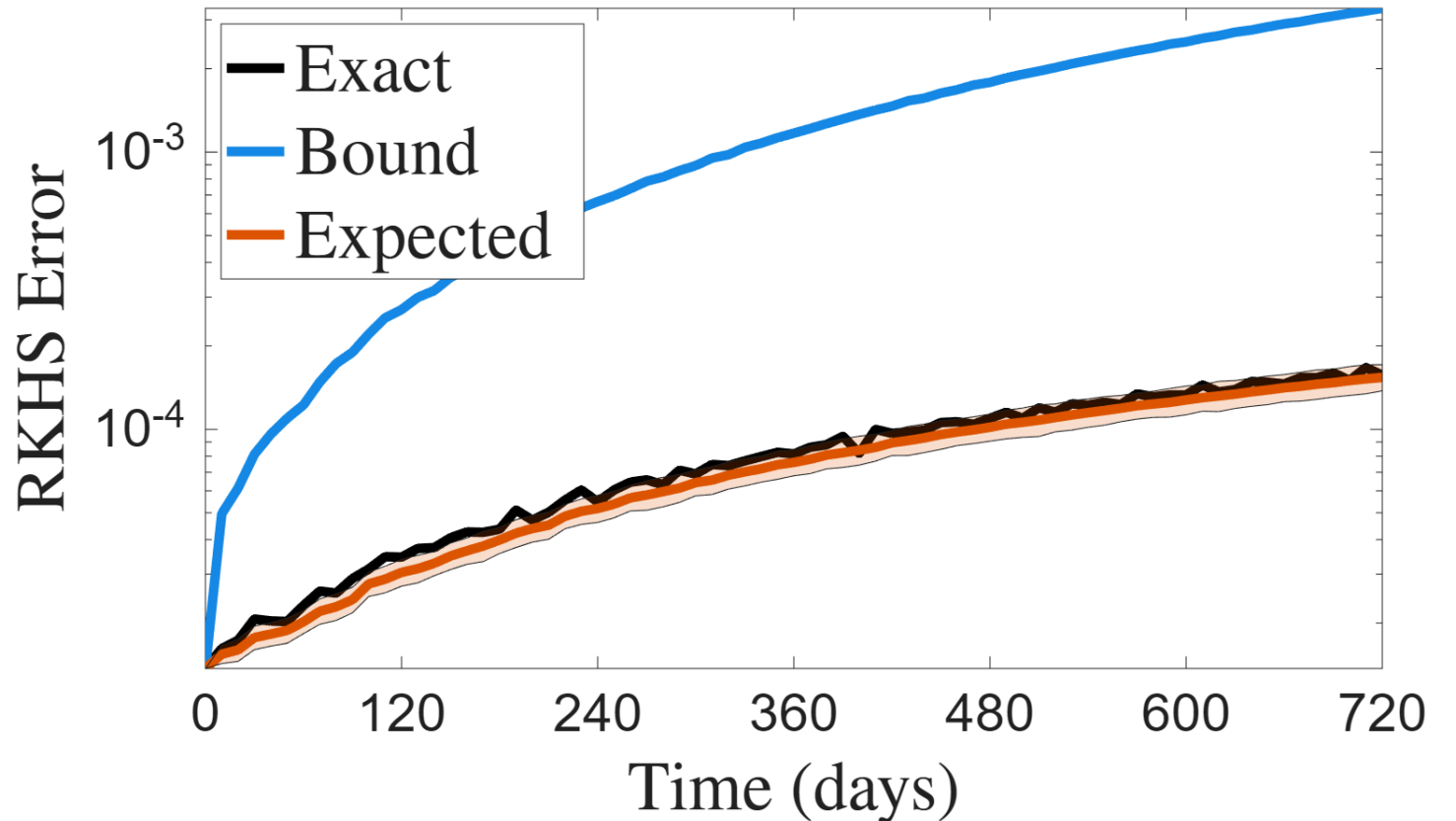


Dictionary learning and forecast bounds

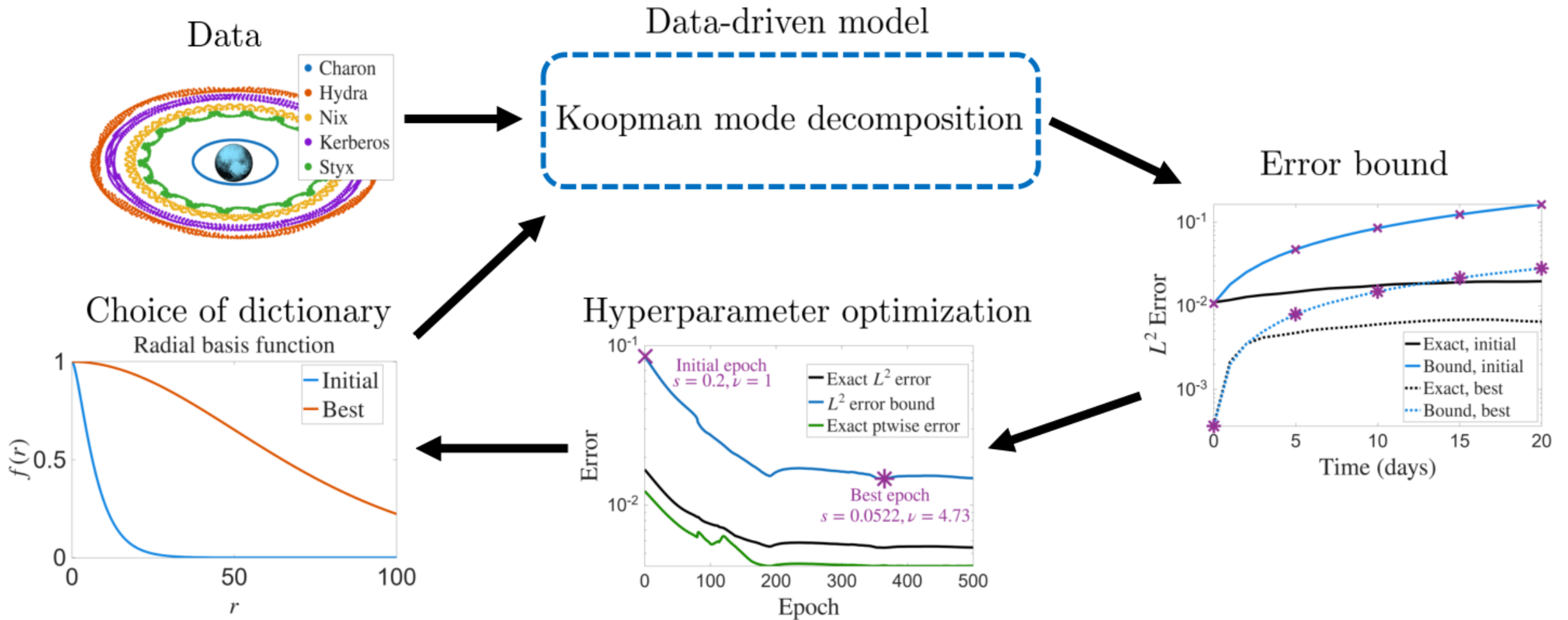
- Apply algorithms to RKHS setting using Matérn kernel
- Expected errors using **Gaussian processes** avoid overestimation

- E.g., replace $\|Ax\| \leq \|A\|\|x\|$ with $\|Ax\| \approx \mathbb{E}_{\mathcal{C}}[A]\|x\|$ where

$$\mathbb{E}_{\mathcal{C}}[A] = \mathbb{E} \left[\frac{\|Ax\|}{\|x\|} \right], \quad x \sim \mathcal{GP}(0, \mathcal{C})$$



Dictionary learning and forecast bounds

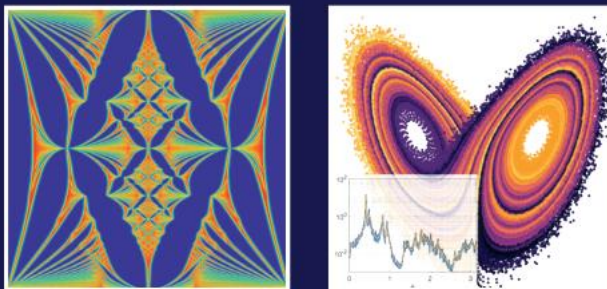
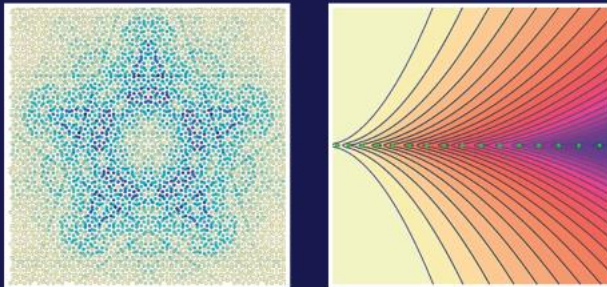


Shameless plug 1: CUP book out August 2026...

MATTHEW J. COLBROOK

Infinite-Dimensional Spectral Computations

Foundations, Algorithms, and Modern Applications



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100s of:

- Classifications
- Algorithms
- Examples (full code)
- Exercises (full solutions)

If something of interest – speak to me!

This talk

Shameless plug 2: Issac Newton Institute Programme

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Operator Methods for Dynamical Systems

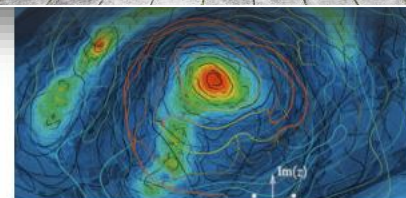
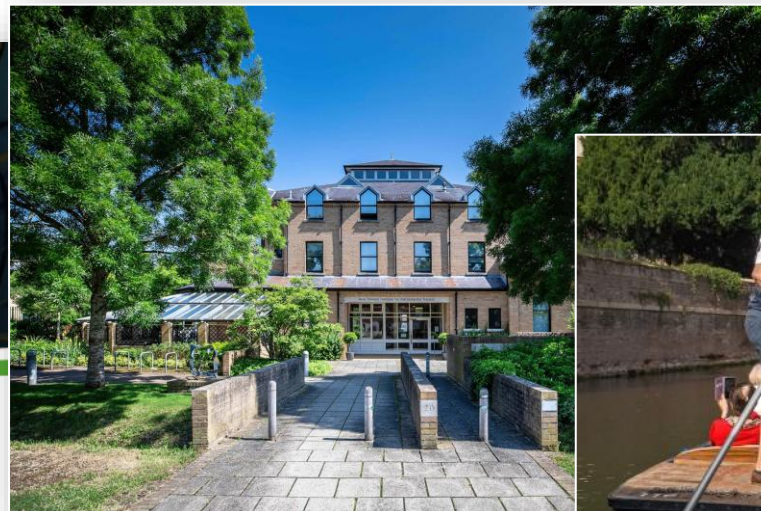
OMD

3 August 2026 to 28 August 2026

Programme theme

Dynamical systems lie at the heart of our understanding of complex phenomena, whether in modelling weather and ocean currents, molecular dynamics, population growth or stock market fluctuations. However, the nonlinear nature of systems poses a challenge to describing their behaviour. Operator methods offer a powerful way to tackle this challenge. By representing a finite-dimensional nonlinear system's evolution as a linear operator acting on an infinite-dimensional space of functions, tools from linear algebra and spectral analysis can be used to gain insights into the system's long-term behaviour. This operator-based perspective has roots in 20th-century mathematics. It has proven fruitful in classical settings like statistical mechanics, where it helped connect chaotic systems with well-understood linear techniques.

In recent years, interest in operator methods for dynamical systems has surged, driven by data-driven techniques to approximate and analyse these operators from real-world or simulated data. A wide range of frameworks and algorithms have emerged, creating an exciting opportunity and need to develop a unifying foundation for these approaches. This programme aims to bring together experts and young researchers from various communities who use operator-theoretic perspectives (for example, the Koopman and transfer operator frameworks) to study dynamics. By uniting participants from pure theory to practical applications, the programme will spark new collaborations and jointly tackle key open questions, building a more cohesive research community. In particular, a significant focus will be on exploring the spectral properties of these operators—essentially, understanding their eigenvalues, modes, and related features—as these provide crucial insights into a system's long-term behaviour and how such behaviour can be effectively analysed and computed.



Organisers

- Matthew Colbrook *University of Cambridge*
- Gary Froyland *University of New South Wales*
- Nathan Kutz *University of Washington*
- Julia Slipantschuk *University of Warwick; Universität Bayreuth*
- Caroline Wornell *University of Sydney*

Participants

- Wael Bahsoun *Loughborough University*
- Steve Brunton *University of Washington*
- Christopher Budd *University of Bath; Institute of Mathematics and its Applications*

Visit to find out more
about operator theory
and machine learning in
dynamical systems!

(and to enjoy Cambridge
in the summer!)

Pointers

1. Data-driven spectral problems for Koopman operators are hugely popular.
BUT: Standard truncation methods can fail – NEED TO GO INF-DIM!
2. **General methods with convergence for spectral properties**
 (spectra, pseudospectra, spectral measures, etc.) of K. operators!
E.g., Verification of approximate eigenfunctions leads to practical gains.
3. **SCI hierarchy** classifies computational problems:
Lower bounds through method of adversarial dynamics.
Upper bounds \Rightarrow new “inf.-dim.” algorithms. Rigorous, optimal, practical.
 \rightarrow We now have a near complete spectral picture for K. on $L^2(\mathcal{X}, \omega)$ and RKHS!
4. **Matrix L** leads to forecast bounds, ways to train dictionary, and PAD.

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