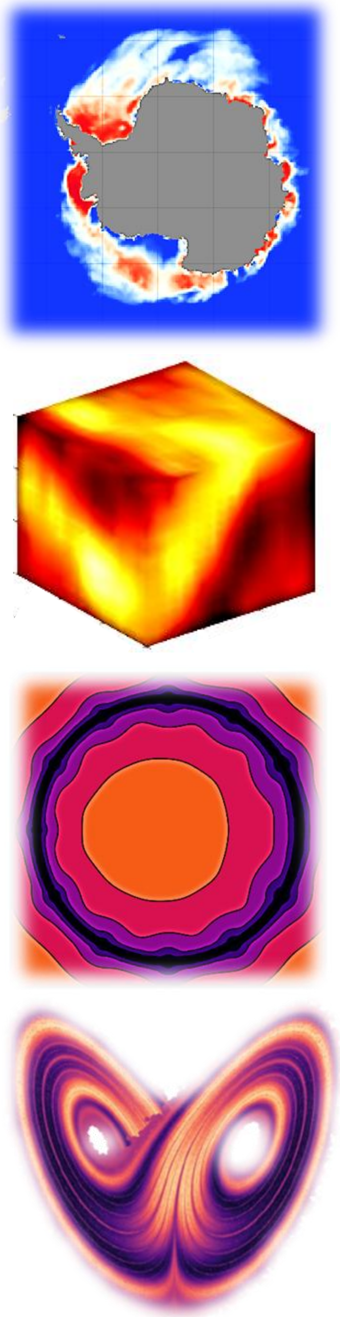

Rigorous Data-Driven Spectral Analysis of Nonlinear Dynamics

Matthew Colbrook

17th March 2026

UNIVERSITY OF CAMBRIDGE

“One of the great joys of doing mathematics is working with inspiring and brilliant people!” --- Arieh Iserles

Alex Townsend
(Cornell)

Igor Mezić
(UC Santa Barbara)

Alexei Stepanenko
(Cam. -> Industry)

Nicolas Boullé
(Imperial)

Gustav Conradie
(Cambridge)

- C., Townsend. *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems."* **Communications on Pure and Applied Mathematics,** 2024.
- C., Mezić, Stepanenko*, “Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning.”* (under revision at **Nature Communications**).
- Boullé, C., Conradie, *“Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces.”* (***SpecRKHS*** - **hot off the press: https://arxiv.org/abs/2506.15782**)

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*

cts $F\colon \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$

Henri Poincaré
(Sorbonne)

- Poincaré, ''*Les méthodes nouvelles de la mécanique céleste,*'' **Vol. 2. Gauthier-Villars et fils, imprimeurs-libraires**, 1893.

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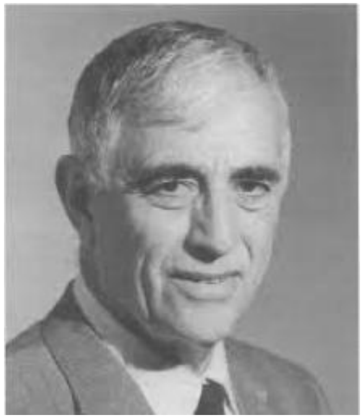
cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$

- Functions $g: \mathcal{X} \to \mathbb{C}$ a.k.a "observables", $g \in L^2(\mathcal{X}, \omega)$
- Koopman operator $\mathcal{K}_F$: $[\mathcal{K}_F g](x) = g(F(x))$

LINEAR!

Observe g one time step forward

Bernard Koopman
(Columbia)



John von Neumann
(IAS)

- Koopman, "*Hamiltonian systems and transformation in Hilbert space,*" **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, "*Dynamical systems of continuous spectra,*" **Proc. Natl. Acad. Sci. USA**, 1932.

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*
- ***Unknown*** cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$
- Functions $g: \mathcal{X} \to \mathbb{C}$ a.k.a "observables", $g \in L^2(\mathcal{X}, \omega)$
- Koopman operator $\mathcal{K}_F$: $[\mathcal{K}_F g](x) = g(F(x))$ LINEAR!
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right) : m = 1, \ldots, M\right\}$

Can we compute spectral properties from trajectory data?

$\boxed{g(x_n) = [\mathcal{K}^n g](x_0)}$

Why?

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Why?

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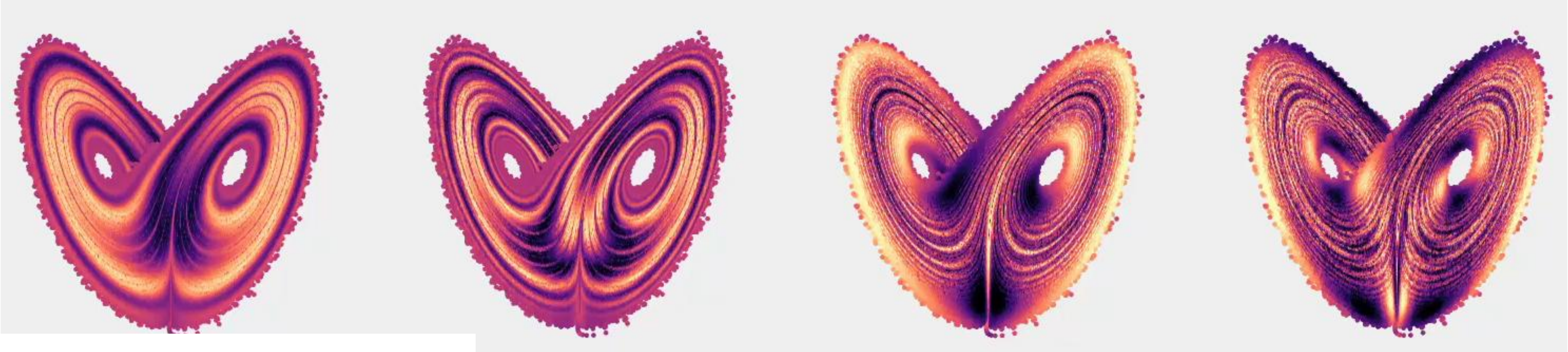


Coherent features!

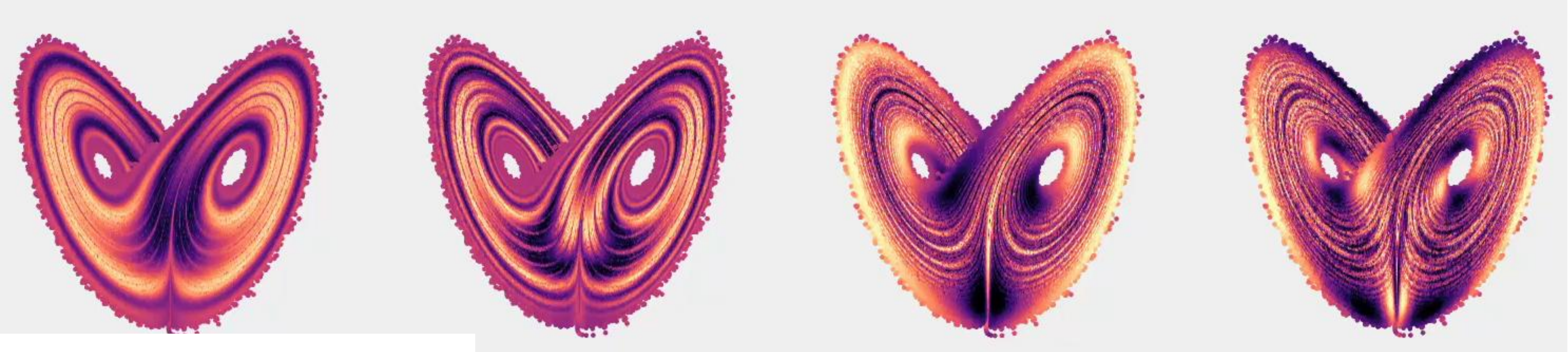
Lorenz attractor

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Why?

$$\boxed{g(x_n) = [\mathcal{K}^n g](x_0)}$$

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$



Coherent features!

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C}: \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Koopman Mode Decomposition

- Find (g_j, λ_j) with $\|\mathcal{K} g_j - \lambda_j g_j\| \leq \varepsilon$

Verified Eigenfunctions

- Expand state:

$$x \approx \sum_j c_j g_j(x)$$

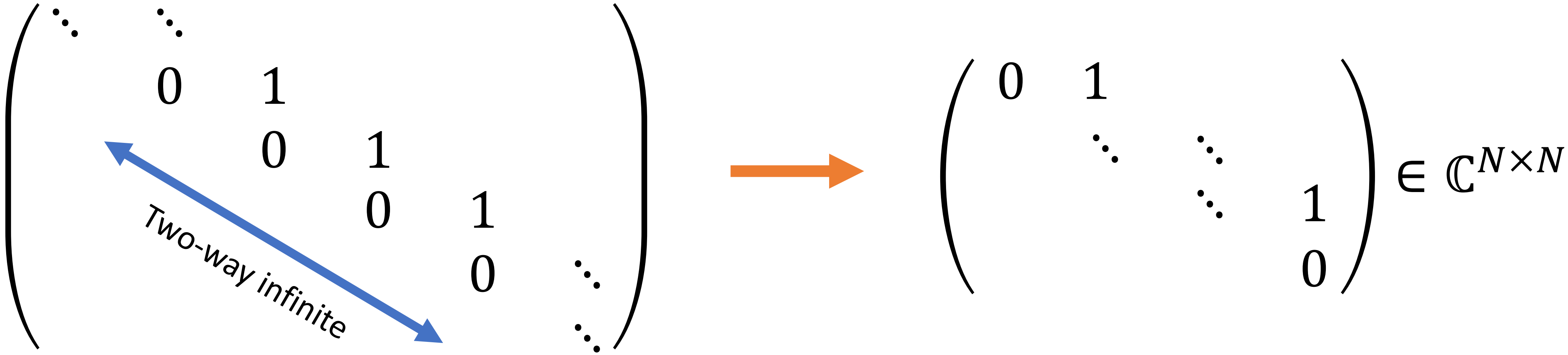
coefficients, called "Koopman modes"

- Forecasts:

$$\boxed{x_n = \sum_j \lambda_j^n c_j g_j(x) + \mathcal{O}(n\varepsilon)}$$

Intuition: A nonlinear separation of variables through a linear operator!

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$



$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N\times N}$$

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Explicit example: Matrix approximation of $\mathcal{K}$ (EDMD)

Observables $\psi_j : \mathcal{X} \to \mathbb{C},\ j = 1, \dots, N$

$$\boxed{\left\{x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right\}_{m=1}^{M}}$$

quadrature points

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)}\psi_k\left(x^{(m)}\right) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X}\right]_{jk}$$

quadrature weights

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)}\ \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y}\right]_{jk}$$

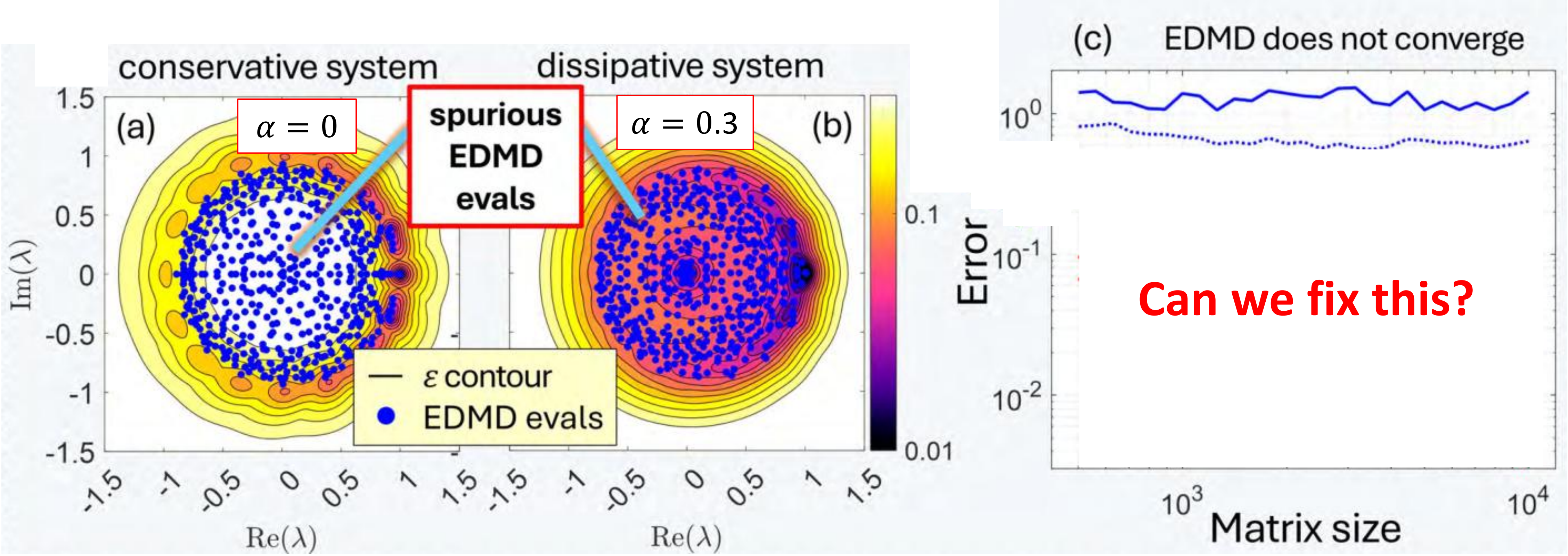
Galerkin
Approximation

$$\mathcal{K} \longrightarrow \left(\Psi_X{}^{*} W \Psi_X\right)^{-1} \Psi_X{}^{*} W \Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, “*Dynamic mode decomposition of numerical and experimental data,*” **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, “*Spectral analysis of nonlinear flows,*” **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley “*A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,*” **J. Nonlinear Sci.**, 2015.

EDMD doesn't converge!

- Duffing oscillator: $\dot{x} = y$, $\dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)



$\alpha = 0$

spurious EDMD evals

$\alpha = 0.3$

Can we fix this?

The fix: Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k(x^{(m)}) = \Big[\underbrace{\Psi_X{}^{*} W \Psi_X}_{G}\Big]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \Big[\underbrace{\Psi_X{}^{*} W \Psi_Y}_{K_1}\Big]_{jk}$$

- C., Townsend, “*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems*,” **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szőke, “*Residual Dynamic Mode Decomposition*,” **J. Fluid Mech.**, 2023.
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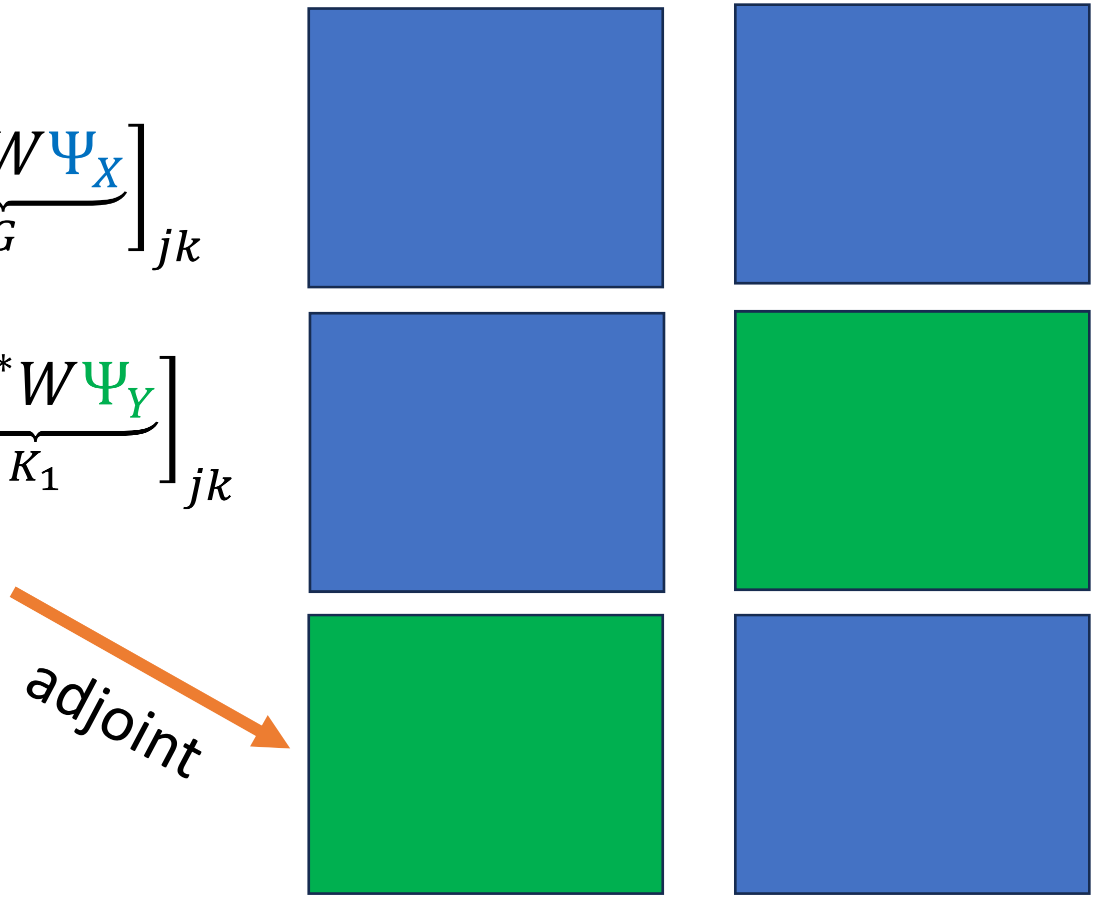

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- C., Townsend, “*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems*,” **Commun. Pure Appl. Math.**, 2023.
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Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j,\ \|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \sum_{k,j=1}^{N} \mathbf{g}_k \overline{\mathbf{g}_j} \langle \mathcal{K}\psi_k - \lambda\psi_k, \mathcal{K}\psi_j - \lambda\psi_j \rangle$

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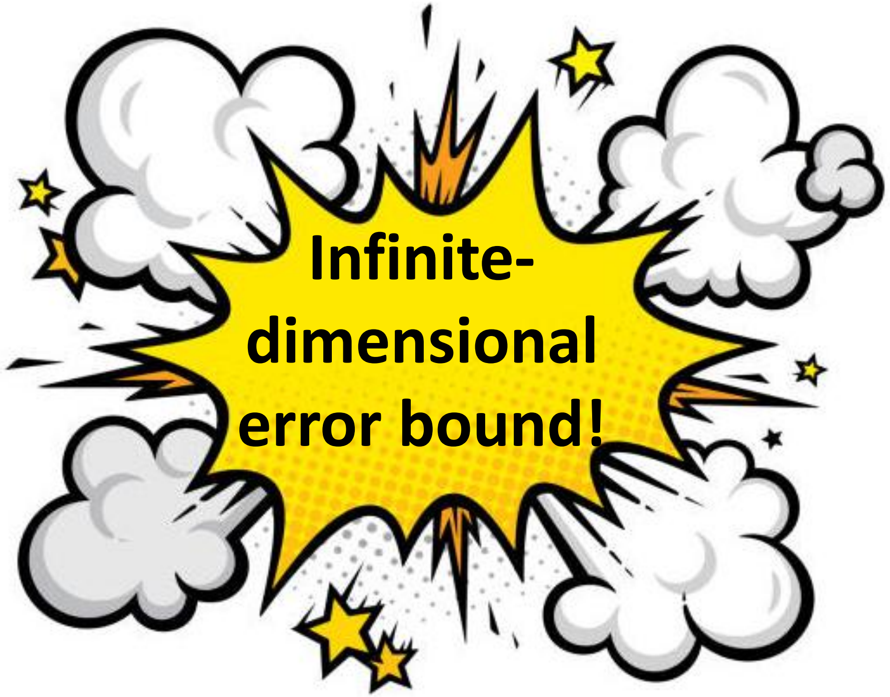

Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \lim_{M\to\infty} \mathbf{g}^{*}[K_2 - \lambda K_1{}^{*} - \bar{\lambda} K_1 + |\lambda|^2 G]\mathbf{g}$

- C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,"* **Commun. Pure Appl. Math.**, 2023.
- C., Ayton, Szőke, *"Residual Dynamic Mode Decomposition,"* **J. Fluid Mech.**, 2023.
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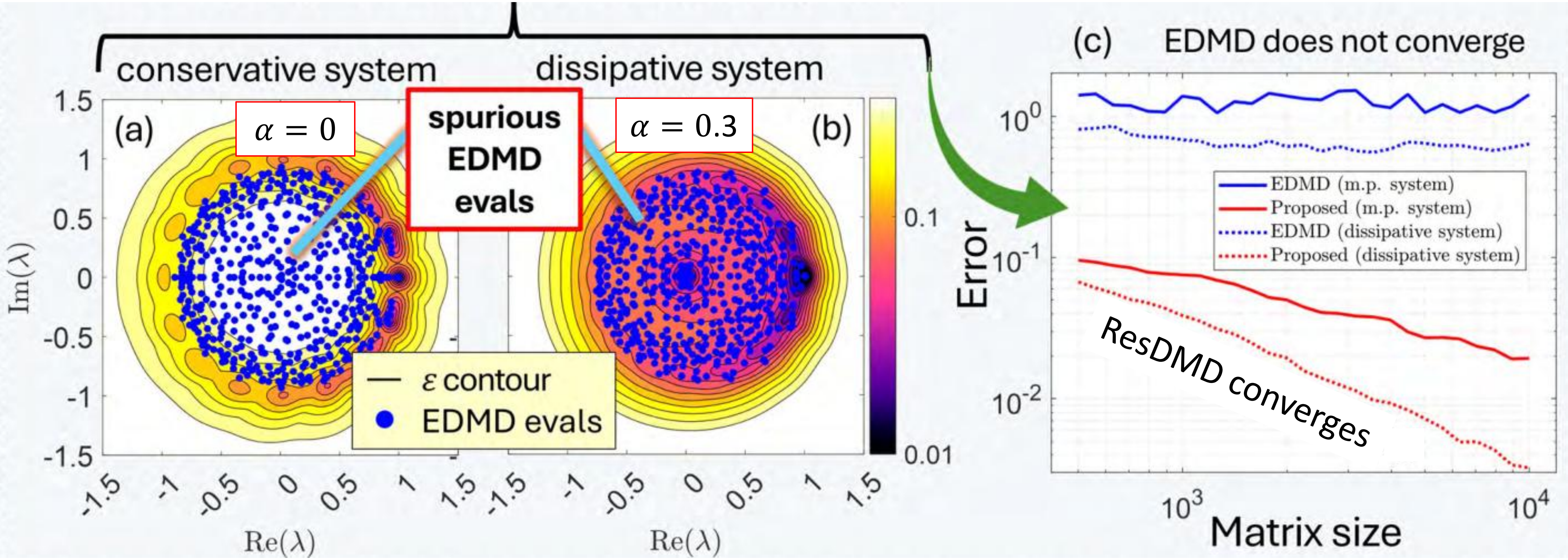
ResDMD does converge!

- Duffing oscillator: $\dot{x} = y, \dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)

Compute $\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K})$, local adaptive control on $\varepsilon \downarrow 0$



Reproducing kernel Hilbert space (RKHS)

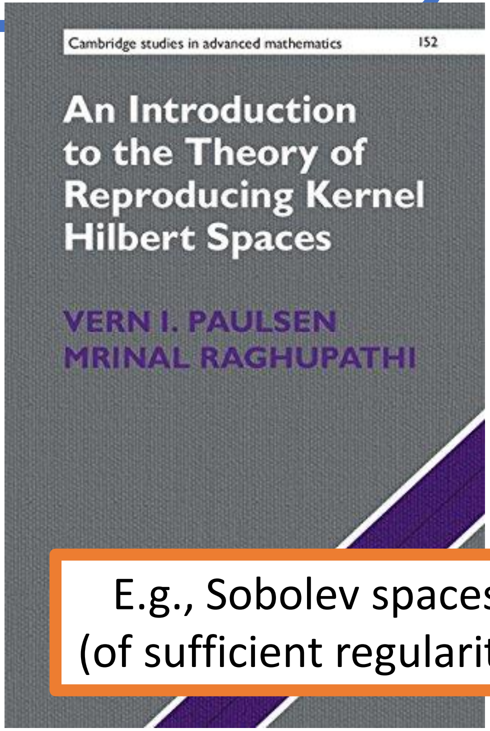
Hilbert space of functions on $\mathcal{X}$ s.t. $g \mapsto g(x)$ bounded $\forall x \in \mathcal{X}$.

Generated by a kernel $\mathfrak{K}: \mathcal{X} \times \mathcal{X} \to \mathbb{C}$

$$g(x) = \langle g, \mathfrak{K}_x \rangle, \qquad \mathfrak{K}(x, y) = \left\langle \mathfrak{K}_x, \mathfrak{K}_y \right\rangle = \mathfrak{K}_x(y)$$

Advantages over $L^2(\mathcal{X}, \omega)$:

- Forecasts: space bounds ⇒ pointwise bounds.
- High-dimensional systems practical through kernel trick.
- Fast methods for evaluating $\mathfrak{K}$.
- Different $\mathfrak{K}$ ⇒ different $\mathcal{K}$! Can be tailored to application. (This is where the community is currently heading.)



Cambridge studies in advanced mathematics 152

An Introduction to the Theory of Reproducing Kernel Hilbert Spaces

VERN I. PAULSEN
MRINAL RAGHUPATHI

E.g., Sobolev spaces (of sufficient regularity)

SpecRKHS: Avoiding large data limit $M \to \infty$

Look at "Left eigenpairs" through $\mathcal{K}^*$: $\boxed{\mathcal{K}^*\mathfrak{K}_x = \mathfrak{K}_{F(x)}}$

Evolution of functionals.

$g(x) = \langle g, \mathfrak{K}_x \rangle_{\mathcal{H}}$

No quadrature needed:

$$G_{jk} = \langle \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}\left(x^{(k)}, x^{(j)}\right)$$

$$A_{jk} = \langle \mathcal{K}^*\mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \left\langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{x^{(j)}} \right\rangle = \mathfrak{K}\left(y^{(k)}, x^{(j)}\right)$$

$$R_{jk} = \langle \mathcal{K}^*\mathfrak{K}_{x^{(k)}}, \mathcal{K}^*\mathfrak{K}_{x^{(j)}} \rangle = \left\langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{y^{(j)}} \right\rangle = \mathfrak{K}\left(y^{(k)}, y^{(j)}\right)$$

$$g = \sum_{m=1}^{M} \mathbf{g}_m \mathfrak{K}_{x^{(m)}}, \qquad \|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2 = \mathbf{g}^*(R - \lambda A^* - \bar{\lambda} A + G)\mathbf{g}$$

- Boullé, C., Conradie, *"Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces,"* **preprint**, 2025.

SpecRKHS: Example algorithm

$$\mathrm{res}^*(\lambda,\mathbf{g})^2=\frac{\|\mathcal{K}^*g-\lambda g\|_{\mathcal{H}}^2}{\|g\|_{\mathcal{H}}^2}=\frac{\mathbf{g}^*\left[R-\lambda A^*-\bar{\lambda}A+G\right]\mathbf{g}}{\mathbf{g}^*G\mathbf{g}}$$

1. Compute $G, A, R \in \mathbb{C}^{N\times N}$ $(N=M)$
2. For z_k in grid, compute $\tau_k = \min_{g=\sum_{m=1}^{N}\mathbf{g}_m\mathfrak{K}_{x^{(m)}},} \mathrm{res}^*(z_k,\mathbf{g})$, corresponding g_k (gen. SVD).
3. **Output:** $\{z_k:\tau_k<\varepsilon\}$, $\{g_k:\tau_k<\varepsilon\}$ (ε-pseudoeigenfunctions).

First convergent method on RKHS for general $\mathcal{K}$

Theorem:

- **Error control:** $\{z_k:\tau_k<\varepsilon\}\subseteq \mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)$
- **Convergence:** Converges locally uniformly to $\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)$ (as $N\to\infty$)

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)=\{z\in\mathbb{C}:\exists g,\|g\|_{\mathcal{H}}=1,\|\mathcal{K}^*g-zg\|_{\mathcal{H}}\leq\varepsilon\}$$

- Boullé, C., Conradie, *"Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces,"* **preprint**, 2025.

Practical gains: Sea ice forecasting

Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,…

Problems:
1. Very hard to locate geographical significant regions.
2. Very hard to predict more than two months in advance.

Koopman Mode Decomposition

Verified Eigenfunctions

- Find (g_j, λ_j) with $\|\mathcal{K} g_j - \lambda_j g_j\| \leq \varepsilon$
- Expand state:

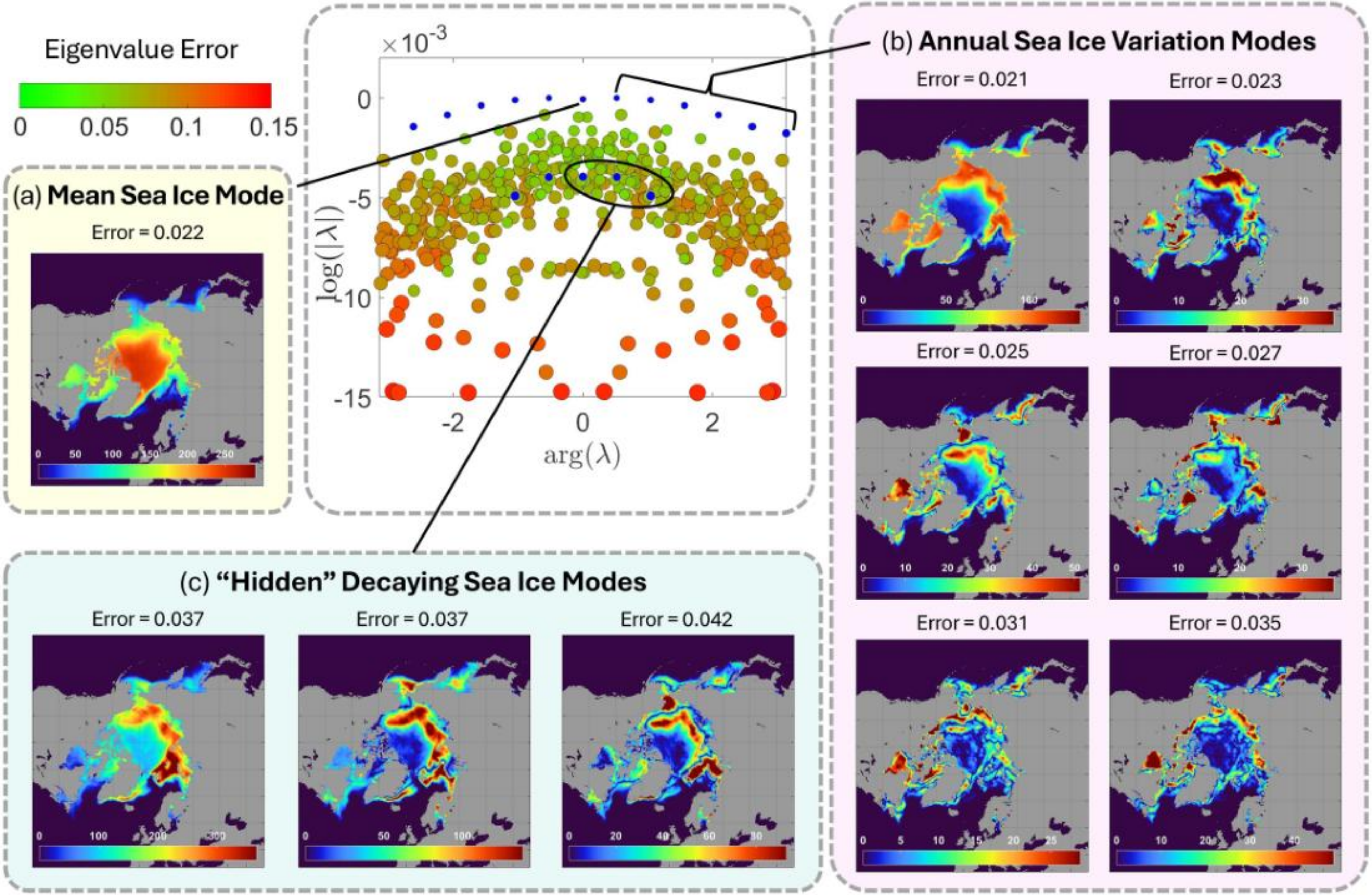
coefficients, called "Koopman modes"

$$x \approx \sum_j c_j g_j(x)$$

- Forecasts:

$$\boxed{x_n = \sum_j \lambda_j^n c_j g_j(x) + \mathcal{O}(n\varepsilon)}$$

Intuition: A nonlinear separation of variables through a linear operator!



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Avoid spurious evals ⇒ State-of-the-art forecasts

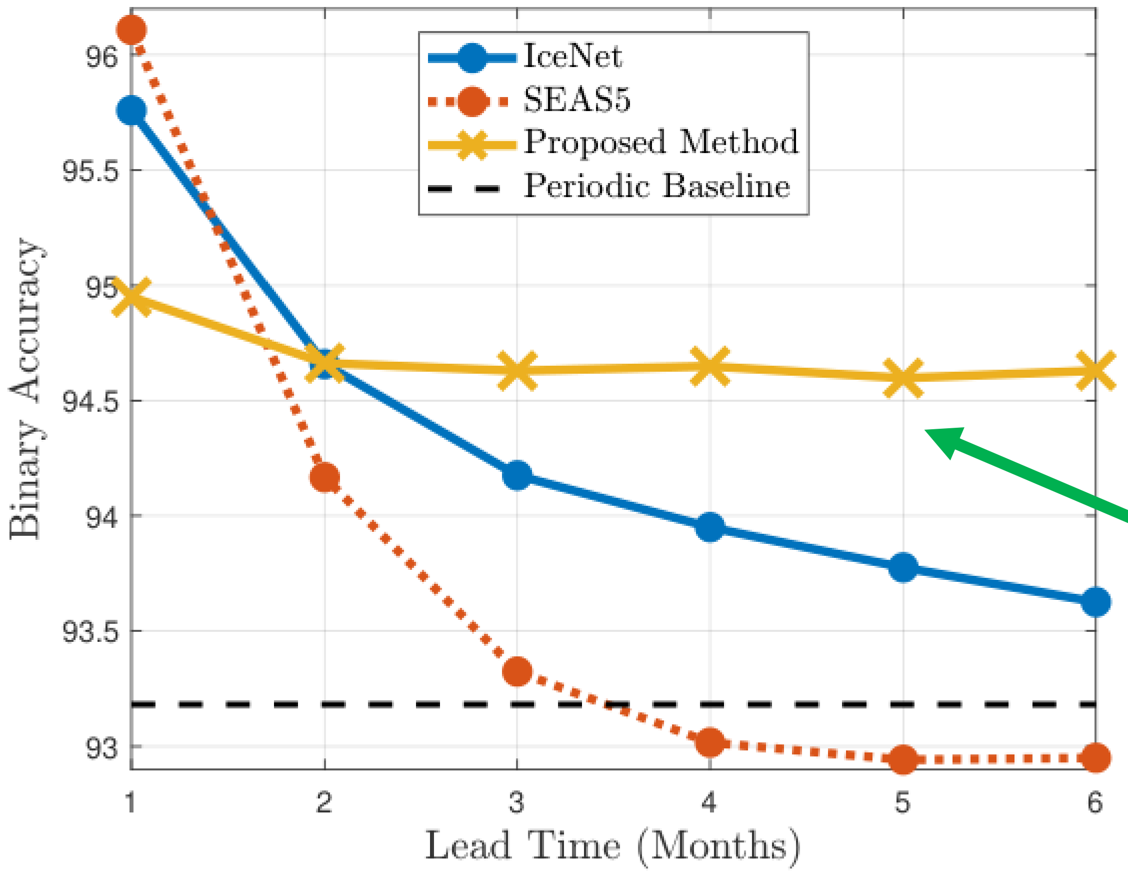

Figure: Mean binary accuracy over test years 2012-2020.
(IceNet: Andersson et al, "Seasonal Arctic sea ice forecasting with probabilistic deep learning." Nature Communications, 2021.)

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

$$\|\mathcal{K}g - \lambda g\| \leq \varepsilon$$

$$\Rightarrow \; g(x_n) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$$

Use ε to filter evals!

- C., Mezić, Stepanenko, "Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning," **preprint**, 2025.

Pointers

1. Data-driven spectral problems for Koopman operators are hugely popular.
   BUT: Standard truncation methods often fail – NEED TO GO INF-DIM!

2. **General method with convergence for spectral properties**
   E.g., Verification of approximate eigenfunctions leads to practical gains.

NB: *Similar picture has emerged for spectral measures, dealing with continuous spectra (versus eigenvalues) and spectral type (different flavors of dynamics).*

NB: *Can also prove what is not possible!*

$\longrightarrow$ We now have a near complete picture for Koopman on $L^2(\mathcal{X},\omega)$ and RKHS!

Shameless final plug…

Upcoming book with CUP:

INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS

Foundations, Algorithms, and Modern Applications

Matthew J. Colbrook

100s of: classifications, algorithms, examples (webpage: full code), figures, exercises (webpage: full solutions).

*****Out in August 2026*****

Contents

References

[1] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." Communications on Pure and Applied Mathematics 77.1 (2024): 221-283.

[2] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[3] Colbrook, M. J., Li, Q., Raut, R. V., & Townsend, A. "Beyond expectations: residual dynamic mode decomposition and variance for stochastic dynamical systems." Nonlinear Dynamics 112.3 (2024): 2037-2061.

[4] Colbrook, Matthew J. "The Multiverse of Dynamic Mode Decomposition Algorithms." Handbook of Numerical Analysis, vol. 25, pp. 127-230. Elsevier, 2024..

[5] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." SIAM Journal on Numerical Analysis 61.3 (2023): 1585-1608.

[6] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1150-1190.

[7] Boullé, Nicolas, and Matthew J. Colbrook. "Multiplicative Dynamic Mode Decomposition." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1945-1968.

[8] Boullé, Nicolas and Matthew J. Colbrook, "On the Convergence of Hermitian Dynamic Mode Decomposition" Physica D: Nonlinear Phenomena, 472 (2025).

[9] Colbrook, Matthew J., Andrew Horning, and Tianyiwa Xie. "Computing Generalized Eigenfunctions in Rigged Hilbert Spaces." arXiv preprint arXiv:2410.08343 (2024).

[10] Zagli, Niccolò, et al. "Bridging the Gap between Koopmanism and Response Theory: Using Natural Variability to Predict Forced Response." arXiv preprint arXiv:2410.01622 (2024).

[11] Colbrook, Matthew J. "Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size." Physica D: Nonlinear Phenomena 469 (2024).

[12] Colbrook, Matthew. “The foundations of infinite-dimensional spectral computations.” Diss. University of Cambridge, 2020.

[13] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). “Computing Spectra--On the Solvability Complexity Index Hierarchy and Towers of Algorithms.” arXiv preprint arXiv:1508.03280.

[14] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale’s 18th problem." Proceedings of the National Academy of Sciences 119.12 (2022): e2107151119.

[15] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." SIAM review 63.3 (2021): 489-524.

[16] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.

[17] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society (2022).

[18] Colbrook, Matthew J. "Computing spectral measures and spectral types." Communications in Mathematical Physics 384 (2021): 433-501.

[19] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." Numerische Mathematik 143 (2019): 17-83.

[20] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." Foundations of Computational Mathematics (2022): 1-82.

[21] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective.“

[22] Colbrook, Matthew J., Igor Mezić, and Alexei Stepanenko. "Limits and Powers of Koopman Learning.“ arXiv preprint arxiv:2407.06312 (2024).

[23] Herwig, April, Matthew J. Colbrook, Oliver Junge, Péter Koltai, and Julia Slipantschuk. "Avoiding spectral pollution for transfer operators using residuals." arXiv preprint arXiv:2507.16915 (2025).

[24] Boullé, Nicolas, Matthew J. Colbrook, and Gustav Conradie. "Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces." arXiv preprint arXiv:2506.15782 (2025).

[25] Drysdale, Catherine, Matthew Colbrook, and Michael TM Woodley. "Computation and Verification of Spectra for Non-Hermitian Systems." Physical review letters 135.17 (2025): 170202.

[26] Colbrook, Matthew J., and Catherine Drysdale. "Universal Methods for Nonlinear Spectral Problems." arXiv preprint arXiv:2504.17012 (2025).

[27] Colbrook, Matthew J., Zlatko Drmač, and Andrew Horning. "An Introductory Guide to Koopman Learning." arXiv preprint arXiv:2510.22002 (2025).