Data-Driven Koopman Methods:
Spectra, Forecasting, and Open Challenges

Matthew Colbrook
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UNIVERSITY OF CAMBRIDGE

Papers and talks:
http://www.damtp.cam.ac.uk/user/mjc249/home.html

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*

cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$

Henri Poincaré
(Sorbonne)

- Poincaré, ''*Les méthodes nouvelles de la mécanique céleste,*'' **Vol. 2. Gauthier-Villars et fils, imprimeurs-libraires**, 1893.

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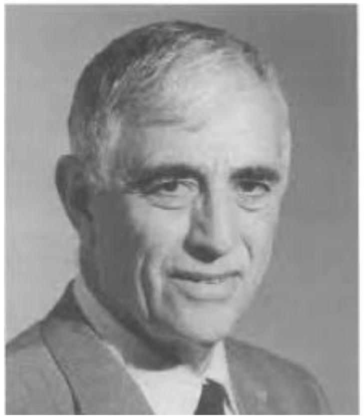
cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$

- Functions $g: \mathcal{X} \to \mathbb{C}$ a.k.a "observables", $g \in L^2(\mathcal{X}, \omega)$
- Koopman operator $\mathcal{K}_F$: $[\mathcal{K}_F g](x) = g(F(x))$ LINEAR!

Observe g one time step forward

Bernard Koopman
(Columbia)



John von Neumann
(IAS)

- Koopman, "*Hamiltonian systems and transformation in Hilbert space,*" **Proc. Natl. Acad. Sci. USA**, 1931.
- Koopman, v. Neumann, "*Dynamical systems of continuous spectra,*" **Proc. Natl. Acad. Sci. USA**, 1932.

What is a Koopman operator?

- $\mathcal{X}$ – *the state space*
- $\mathcal{X} \ni x$ – *the state*
- ***Unknown*** cts $F: \mathcal{X} \to \mathcal{X}$ – *the dynamics:* $x_{n+1} = F(x_n)$
- Functions $g: \mathcal{X} \to \mathbb{C}$ a.k.a "observables", $g \in L^2(\mathcal{X}, \omega)$
- Koopman operator $\mathcal{K}_F$: $[\mathcal{K}_F g](x) = g(F(x))$ LINEAR!
- ***Available*** snapshot data: $\left\{\left(x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right) : m = 1, \dots, M\right\}$

Can we compute spectral properties from trajectory data?

Why?

$\boxed{g(x_n) = [\mathcal{K}^n g](x_0)}$

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

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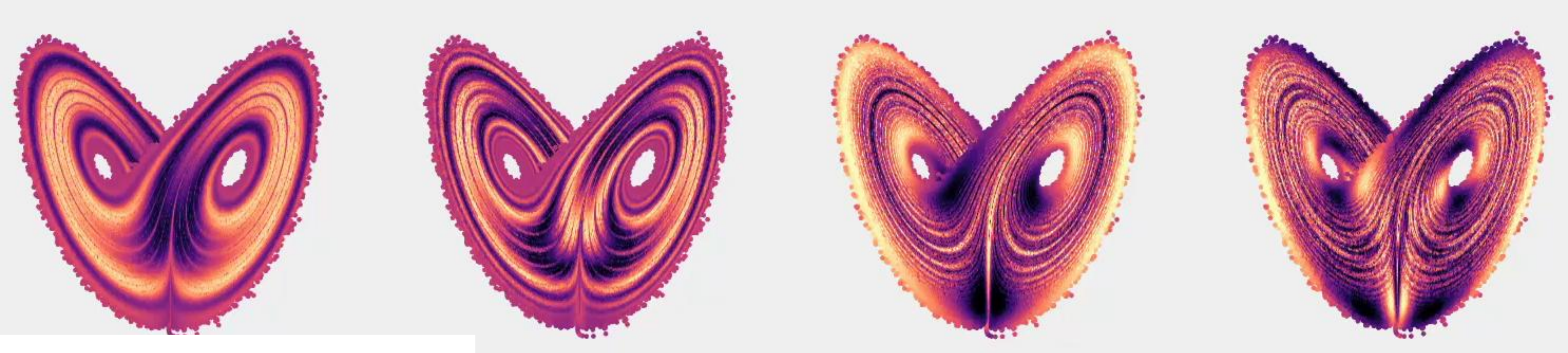


Coherent features!

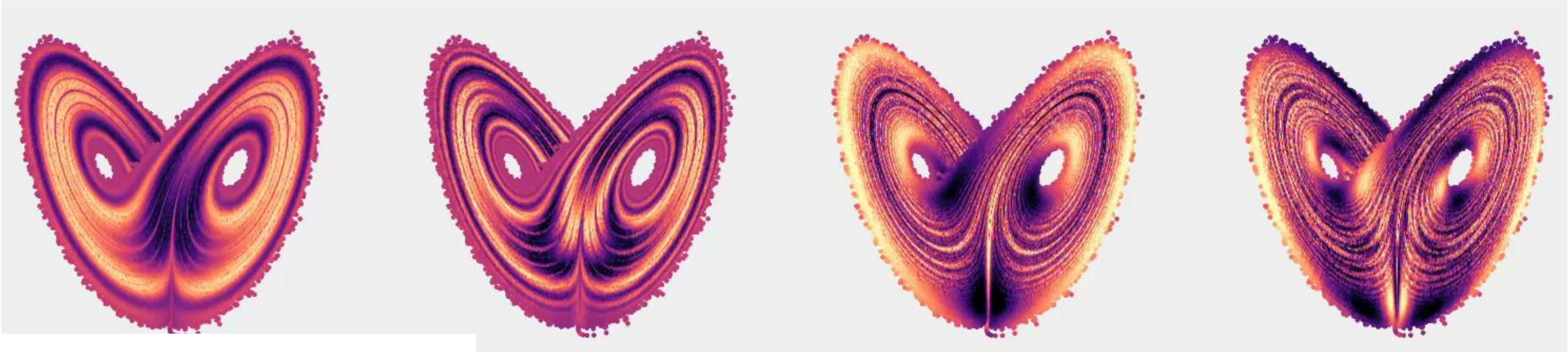
Lorenz attractor

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

$\boxed{g(x_n) = [\mathcal{K}^n g](x_0)}$

Why?

If $\|\mathcal{K}g - \lambda g\| \leq \varepsilon$, then $g(x_n) = [\mathcal{K}^n g](x_0) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$



Coherent features!

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}) = \{z \in \mathbb{C}: \exists g, \|g\| = 1, \|\mathcal{K}g - zg\| \leq \varepsilon\}$$

Trades: Nonlinear, finite-dimensional $\Longrightarrow$ Linear, infinite-dimensional.

Koopman Mode Decomposition

Verified Eigenfunctions

- Find (g_j, λ_j) with $\|\mathcal{K}g_j - \lambda_j g_j\| \leq \varepsilon$
- Expand state:

coefficients, called "Koopman modes"

$$x \approx \sum_j c_j g_j(x)$$

- Forecasts:

$$\boxed{x_n = \sum_j \lambda_j^n c_j g_j(x) + \mathcal{O}(n\varepsilon)}$$

Intuition: A nonlinear separation of variables through a linear operator!

Perils of discretization: Warmup on $\ell^2(\mathbb{Z})$

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & 1 & \\ & & & & 0 & \ddots \\ & & & & & \ddots \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{N \times N}$$

Two-way infinite

- Spectrum is unit circle.
- Spectrum is stable.
- Continuous spectra.
- Unitary evolution.

- Spectrum is $\{0\}$.
- Spectrum is unstable.
- Discrete spectra.
- Nilpotent evolution.

Lots of Koopman operators are built up from operators like these!

Matrix approximation of $\mathcal{K}$ (EDMD)

Observables $\psi_j : \mathcal{X} \to \mathbb{C}, j = 1, \dots, N$

$$\boxed{\left\{x^{(m)}, y^{(m)} = F\left(x^{(m)}\right)\right\}_{m=1}^{M}}$$

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} \underbrace{w_m}_{\text{quadrature weights}} \overline{\psi_j\left(x^{(m)}\right)} \psi_k\Big(\overbrace{x^{(m)}}^{\text{quadrature points}}\Big) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j\left(x^{(m)}\right)} \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \cdots & \psi_N(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \cdots & \psi_N(x^{(M)}) \end{pmatrix}^{*}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \cdots & \psi_N(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \cdots & \psi_N(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \right]_{jk}$$

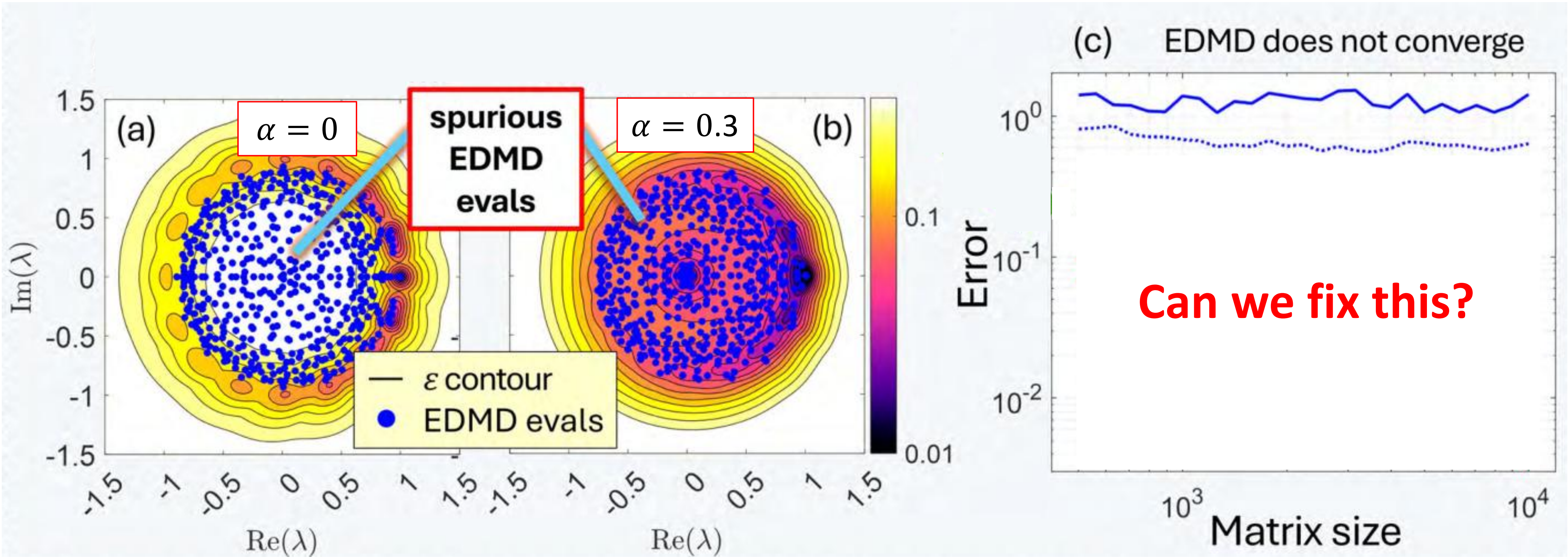
Galerkin Approximation

$$\mathcal{K} \longrightarrow \left({\Psi_X}^{*} W \Psi_X\right)^{-1} {\Psi_X}^{*} W \Psi_Y \in \mathbb{C}^{N \times N}$$

- Schmid, "*Dynamic mode decomposition of numerical and experimental data,*" **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, "*Spectral analysis of nonlinear flows,*" **J. Fluid Mech.**, 2009.
- Williams, Kevrekidis, Rowley "*A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition,*" **J. Nonlinear Sci.**, 2015.

EDMD doesn't converge!

- Duffing oscillator: $\dot{x} = y$, $\dot{y} = -\alpha y + x(1 - x^2)$, sampled $\Delta t = 0.3$.
- Gaussian radial basis functions, Monte Carlo integration ($M = 50000$)



The fix: Residual DMD (ResDMD)

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \psi_k\left(x^{(m)}\right) = \Big[\underbrace{\Psi_X{}^* W \Psi_X}_{G}\Big]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^{M} w_m \overline{\psi_j(x^{(m)})}\, \underbrace{\psi_k\left(y^{(m)}\right)}_{[\mathcal{K}\psi_k]\left(x^{(m)}\right)} = \Big[\underbrace{\Psi_X{}^* W \Psi_Y}_{K_1}\Big]_{jk}$$

- C., Townsend, "*Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,*" **Commun. Pure Appl. Math.**, 2023.
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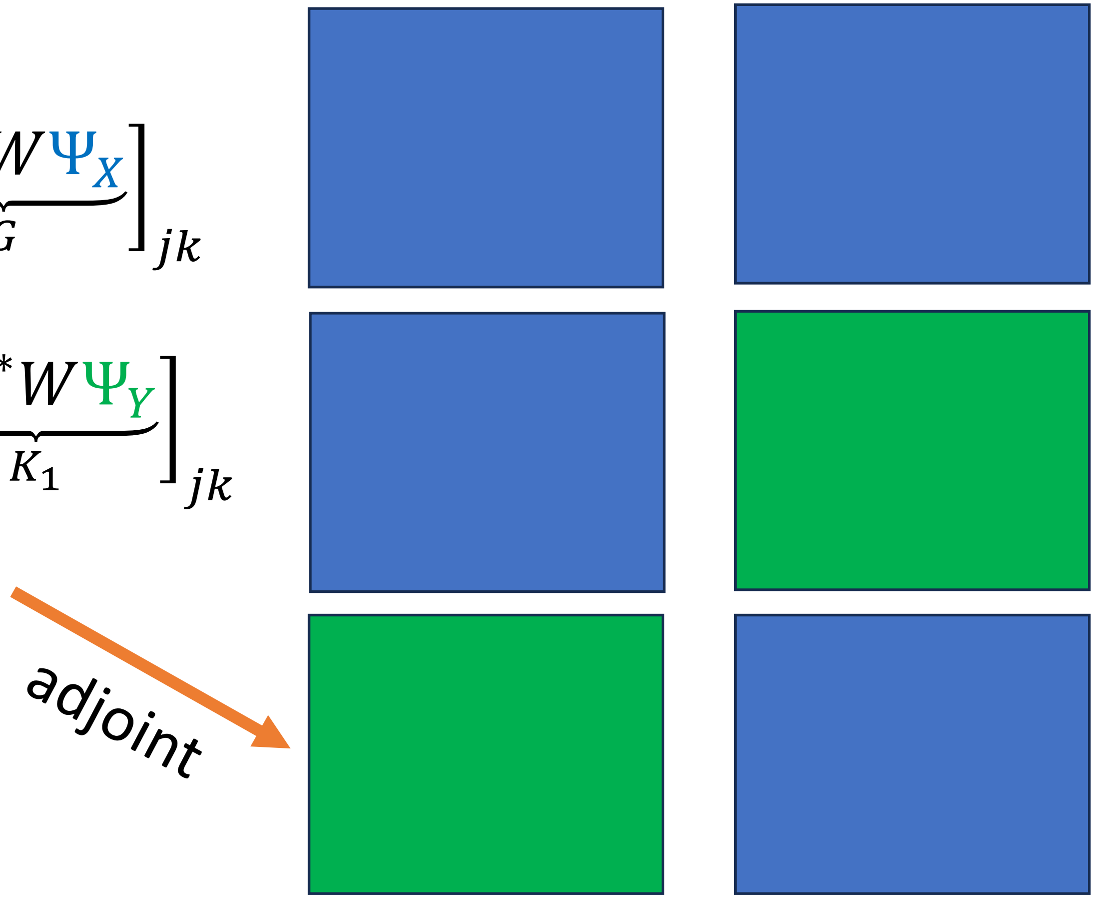

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Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \langle \mathcal{K}g - \lambda g, \mathcal{K}g - \lambda g \rangle$

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<u>Residuals:</u> $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j,\ \|\mathcal{K}g - \lambda g\|^2 = \sum_{k,j=1}^{N} \mathbf{g}_k \overline{\mathbf{g}_j}\, \langle \mathcal{K}\psi_k - \lambda\psi_k, \mathcal{K}\psi_j - \lambda\psi_j \rangle$

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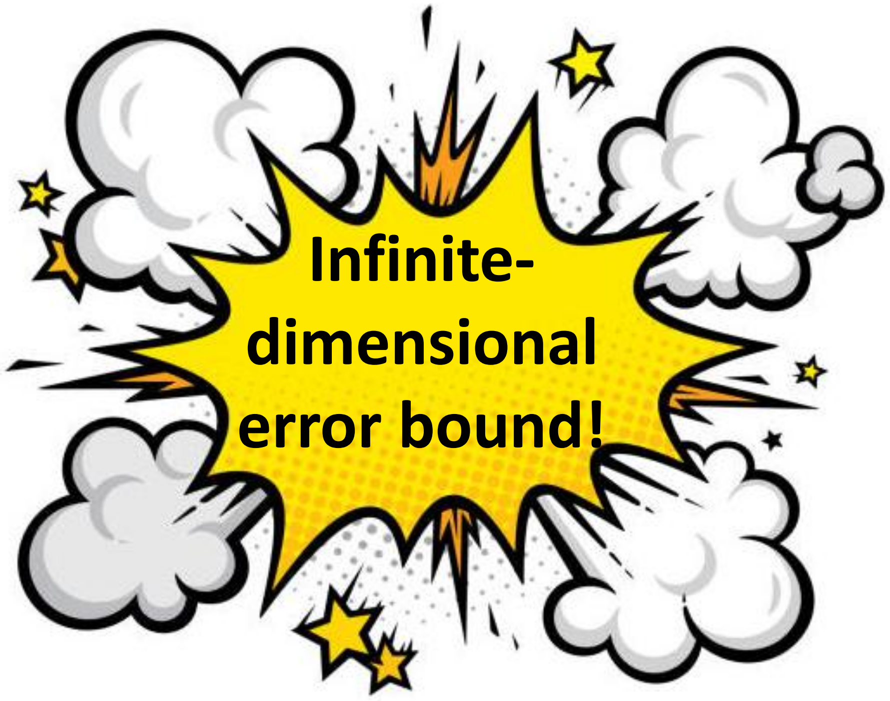

Residuals: $g = \sum_{j=1}^{N} \mathbf{g}_j \psi_j$, $\|\mathcal{K}g - \lambda g\|^2 = \lim_{M\to\infty} \mathbf{g}^* [K_2 - \lambda K_1{}^* - \bar{\lambda} K_1 + |\lambda|^2 G]\mathbf{g}$

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ResDMD does converge!

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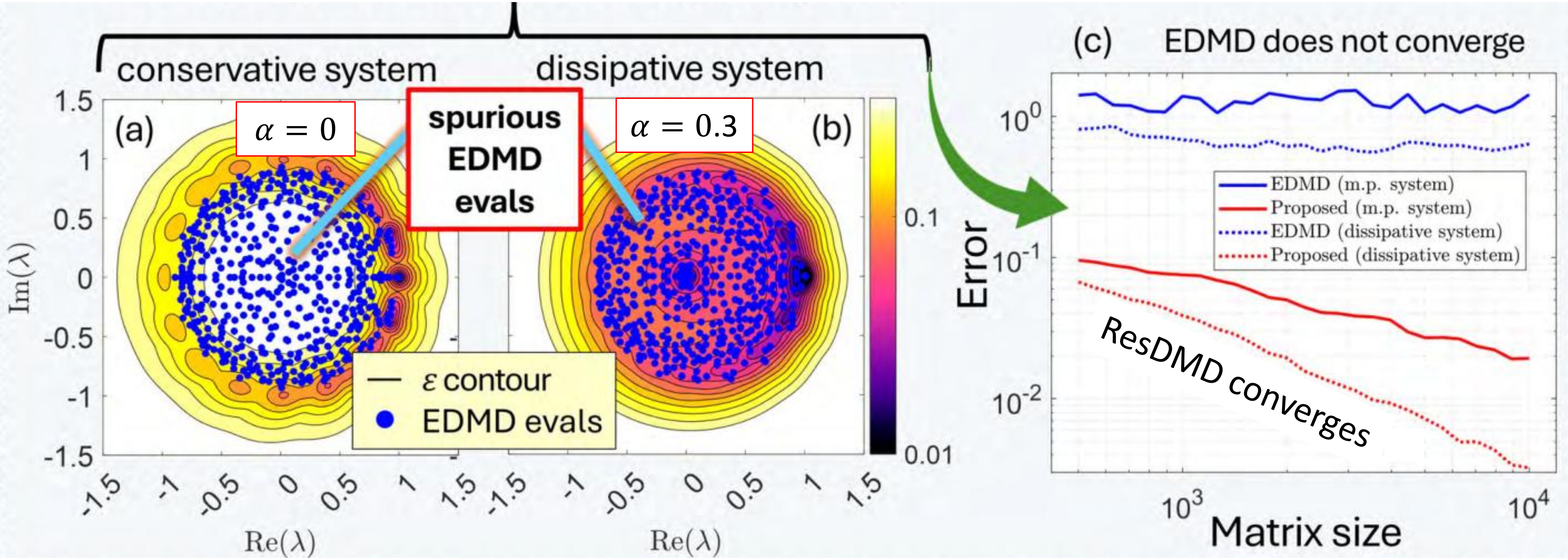
Compute $\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K})$, local adaptive control on $\varepsilon \downarrow 0$



What can we do?

Consider space of observables with finite energy: $L^2(\mathcal{X},\omega)$

Theorem: There **exists** algorithms $\Gamma_{N,M}$ using snapshots such that

$$\lim_{N\to\infty}\lim_{M\to\infty}\Gamma_{N,M}(F)=\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}_F)$$

for all systems.

N **= size of basis,** M **= amount of data (quadrature)**

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K})=\{z\in\mathbb{C}:\exists g,\|g\|=1,\|\mathcal{K}g-zg\|\leq\varepsilon\}$$

- C., Mezić, Stepanenko, *"Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning,"* **preprint**, 2025.

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Double limit $\lim_{N\to\infty}\lim_{M\to\infty}$ **Can we do better?**

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Adversaries: Double limit is necessary!

Implies $\mathcal{K}$ is unitary

Class of systems: $\Omega_{\mathbb{D}} = \{F: \overline{\mathbb{D}} \to \overline{\mathbb{D}} \mid F \text{ cts, measure preserving, invertible}\}.$

Data an algorithm can use: $\mathcal{T}_F = \{(x, y_m) \mid x \in \overline{\mathbb{D}}, \|F(x) - y_m\| \leq 2^{-m}\}.$

Theorem: There **does not exist** any sequence of deterministic algorithms $\{\Gamma_n\}$ using $\mathcal{T}_F$ such that $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}_F)\ \forall F \in \Omega_{\mathbb{D}}$.

NB:

- n can index anything.
- Universal - any type of algorithm or computational model.
- Similarly, no random algorithms converging with probability $> 1/2$.

(Ask me about the proof if interested.)

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Classifications: *Solvability Complexity Index (SCI)*

SCI: Fewest number of limits needed to solve a computational problem.

- Δ_1: One limit, full error control. E.g., $d(\Gamma_n(F), \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
- Δ_{m+1}: SCI $\leq m$.
- Σ_m: SCI $\leq m$, final limit from below.

  E.g., Σ_1: $\sup_{z\in\Gamma_n(F)} \mathrm{dist}(z, \mathrm{Sp}(\mathcal{K}_F)) \leq 2^{-n}$.
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verification

trust output

covers spectrum

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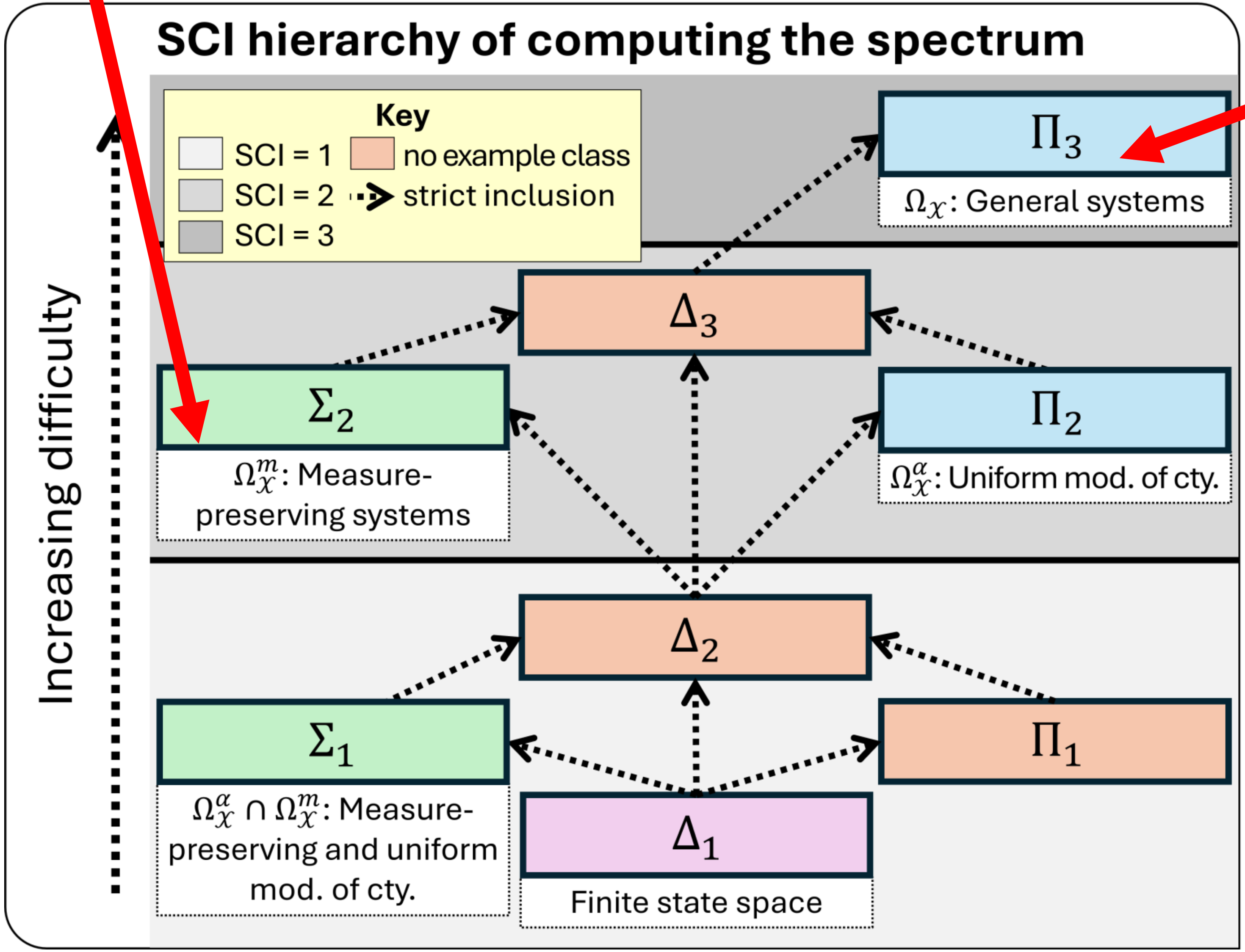
Classification for Koopman

Lower + upper bounds

3 limits needed in general!



Different classes:

$\Omega_{\mathcal{X}} = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts}\}$

$\Omega_{\mathcal{X}}^m = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ cts, m. p.}\}$

$\Omega_{\mathcal{X}}^\alpha = \{F: \mathcal{X} \to \mathcal{X} \mid F \text{ mod. cty. } \alpha\}$

$[d_{\mathcal{X}}(F(x), F(y)) \leq \alpha(d_{\mathcal{X}}(x,y))]$

Optimal algorithms and classifications of dynamical systems.

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Peter Lax:

"The trick of the successful mathematician is to turn the question being asked into one he knows how to answer."

Let's perform this trick by **changing the space of observables**…

Reproducing kernel Hilbert space (RKHS)

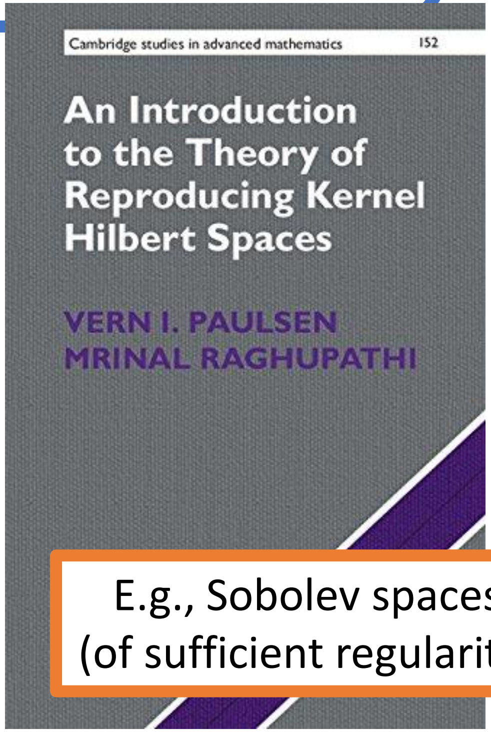
Hilbert space of functions on $\mathcal{X}$ s.t. $g \mapsto g(x)$ bounded $\forall x \in \mathcal{X}$.

Generated by a kernel $\mathfrak{K}: \mathcal{X} \times \mathcal{X} \to \mathbb{C}$

$$g(x) = \langle g, \mathfrak{K}_x \rangle, \qquad \mathfrak{K}(x,y) = \langle \mathfrak{K}_x, \mathfrak{K}_y \rangle = \mathfrak{K}_x(y)$$

Advantages over $L^2(\mathcal{X}, \omega)$:

- Forecasts: space bounds ⇒ pointwise bounds.
- High-dimensional systems practical through kernel trick.
- Fast methods for evaluating $\mathfrak{K}$.
- Different $\mathfrak{K}$ ⇒ different $\mathcal{K}$! Can be tailored to application.
  (This is where the community is currently heading.)
- Leads to fundamental "possibility" gains…



E.g., Sobolev spaces
(of sufficient regularity)

SpecRKHS: Avoiding large data limit $M \to \infty$

Look at "Left eigenpairs" through $\mathcal{K}^*$: $\boxed{\mathcal{K}^*\mathfrak{K}_x = \mathfrak{K}_{F(x)}}$

$\boxed{\text{Evolution of functionals.}}$

$\boxed{g(x) = \langle g, \mathfrak{K}_x \rangle_{\mathcal{H}}}$

<u>*No quadrature needed:*</u>

$$G_{jk} = \langle \mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \mathfrak{K}\left(x^{(k)}, x^{(j)}\right)$$

$$A_{jk} = \langle \mathcal{K}^*\mathfrak{K}_{x^{(k)}}, \mathfrak{K}_{x^{(j)}} \rangle = \left\langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{x^{(j)}} \right\rangle = \mathfrak{K}\left(y^{(k)}, x^{(j)}\right)$$

$$R_{jk} = \langle \mathcal{K}^*\mathfrak{K}_{x^{(k)}}, \mathcal{K}^*\mathfrak{K}_{x^{(j)}} \rangle = \left\langle \mathfrak{K}_{y^{(k)}}, \mathfrak{K}_{y^{(j)}} \right\rangle = \mathfrak{K}\left(y^{(k)}, y^{(j)}\right)$$

$$g = \sum_{m=1}^{M} \mathbf{g}_m \mathfrak{K}_{x^{(m)}}, \qquad \|\mathcal{K}^* g - \lambda g\|_{\mathcal{H}}^2 = \mathbf{g}^*(R - \lambda A^* - \bar{\lambda} A + G)\mathbf{g}$$

- Boullé, C., Conradie, *"Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces,"* **preprint**, 2025.

SpecRKHS: Example algorithm

$$\mathrm{res}^*(\lambda,\mathbf{g})^2=\frac{\|\mathcal{K}^*g-\lambda g\|_{\mathcal{H}}^2}{\|g\|_{\mathcal{H}}^2}=\frac{\mathbf{g}^*\left[R-\lambda A^*-\bar{\lambda}A+|\lambda|^2G\right]\mathbf{g}}{\mathbf{g}^*G\mathbf{g}}$$

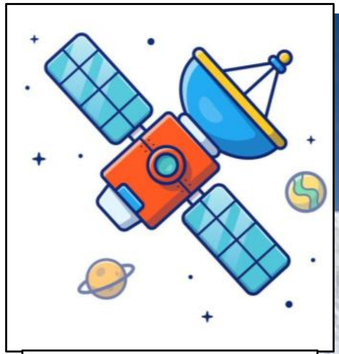
1. Compute $G, A, R \in \mathbb{C}^{N\times N}$ $(N=M)$
2. For z_k in grid, compute $\tau_k = \min_{g=\sum_{m=1}^{N}\mathbf{g}_m\mathfrak{K}_{x^{(m)}}} \mathrm{res}^*(z_k,\mathbf{g})$, corresponding g_k (gen. SVD).
3. **Output:** $\{z_k:\tau_k<\varepsilon\}$, $\{g_k:\tau_k<\varepsilon\}$ (ε-pseudoeigenfunctions).

Theorem:
- **Error control:** $\{z_k:\tau_k<\varepsilon\}\subseteq \mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)$
- **Convergence:** Converges locally uniformly to $\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)$ (as $N\to\infty$)

$$\mathrm{Sp}_{\mathrm{ap},\varepsilon}(\mathcal{K}^*)=\{z\in\mathbb{C}:\exists g,\|g\|_{\mathcal{H}}=1,\|\mathcal{K}^*g-zg\|_{\mathcal{H}}\le\varepsilon\}$$

- Boullé, C., Conradie, *"Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces,"* **preprint**, 2025.

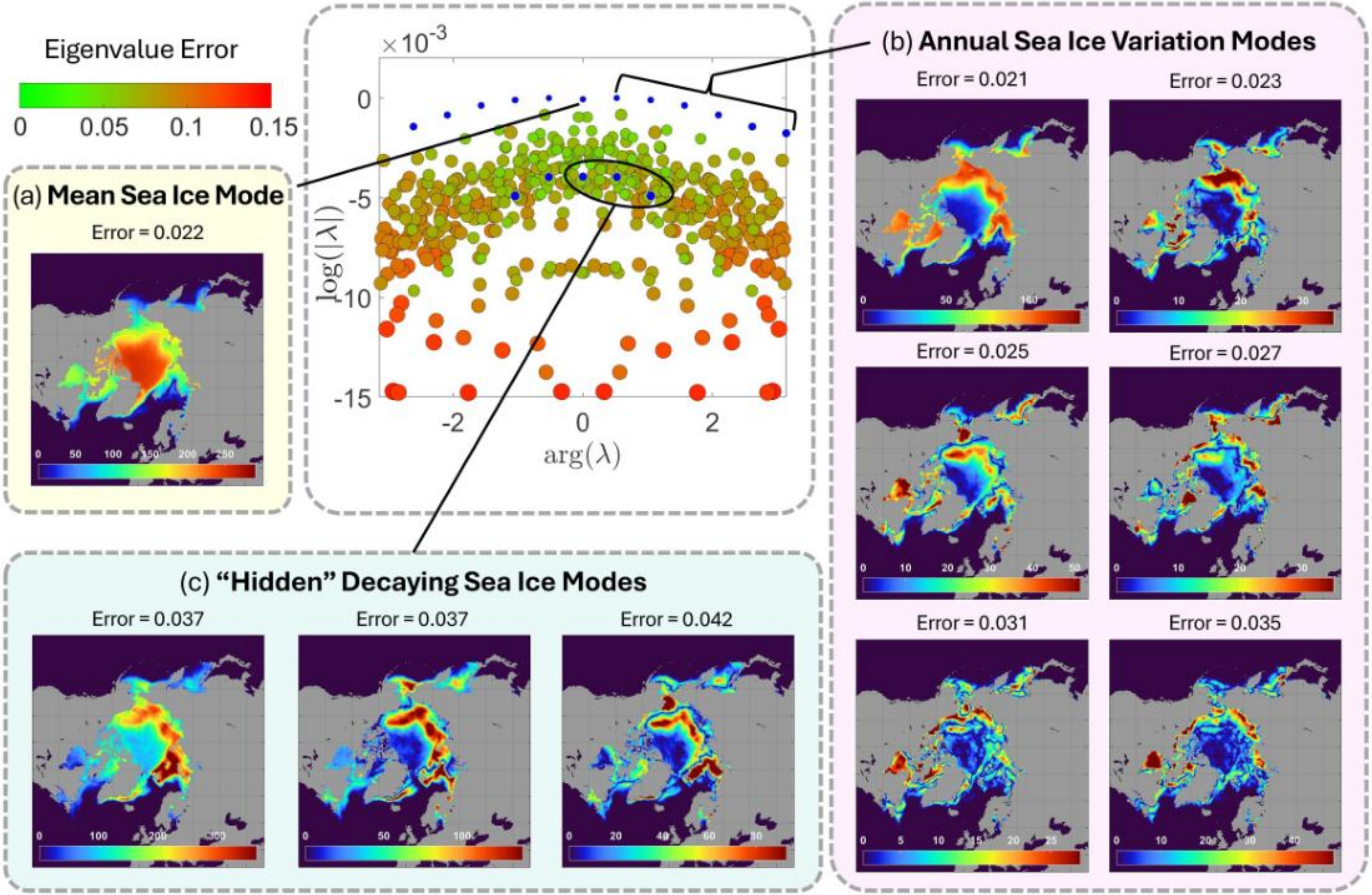
Practical gains: Sea ice forecasting



Satellite data

Motivation: Arctic amplification, polar bears, local communities, effect on extreme weather in Northern hemisphere,…

Problems: 1. Very hard to locate geographical significant regions.
2. Very hard to predict more than two months in advance.



- C., Mezić, Stepanenko, *"Adversarial Dynamical Systems Reveal Limits and Rules for Trustworthy Data-Driven Learning,"* **preprint**, 2025.

Avoid spurious evals ⇒ State-of-the-art forecasts

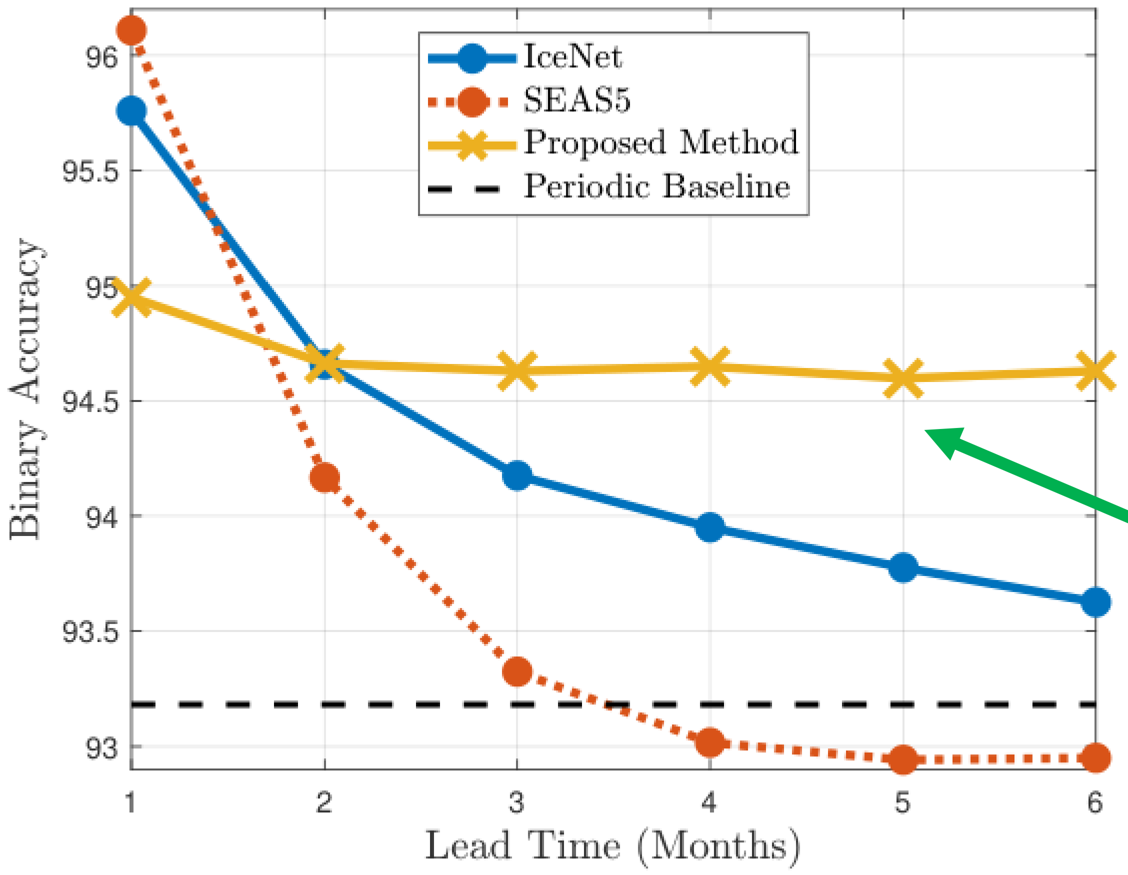


Figure: Mean binary accuracy over test years 2012-2020.
(IceNet: Andersson et al, "Seasonal Arctic sea ice forecasting with probabilistic deep learning." Nature Communications, 2021.)

$$g(x_n) = [\mathcal{K}^n g](x_0)$$

$$\|\mathcal{K}g - \lambda g\| \leq \varepsilon$$
$$\Rightarrow \; g(x_n) = \lambda^n g(x_0) + \mathcal{O}(n\varepsilon)$$

Use ε to filter evals!

Four Major Problems Driving Current Research

- Implementing new tools for continuous spectra in applications.
- Dictionary learning with guarantees.
- Control: Residual and subspace invariance (i.e., closure errors).
- Koopman frameworks for non-autonomous systems that avoid state augmentation blow-up.

Common thread: closure in infinite dimensions.

Tools for continuous spectra and beyond

- Fourier filtering.

  *Korda, Putinar, Mezic, **ACHA,** 2020,   C., Townsend, **CPAM,** 2024.*

- Resolvent based methods and generalized eigenfunctions.

  *C. Drysdale, Horning, **SIADS,** 2025.*

Smoothed spectral measures

- Eigenvalue-based methods.

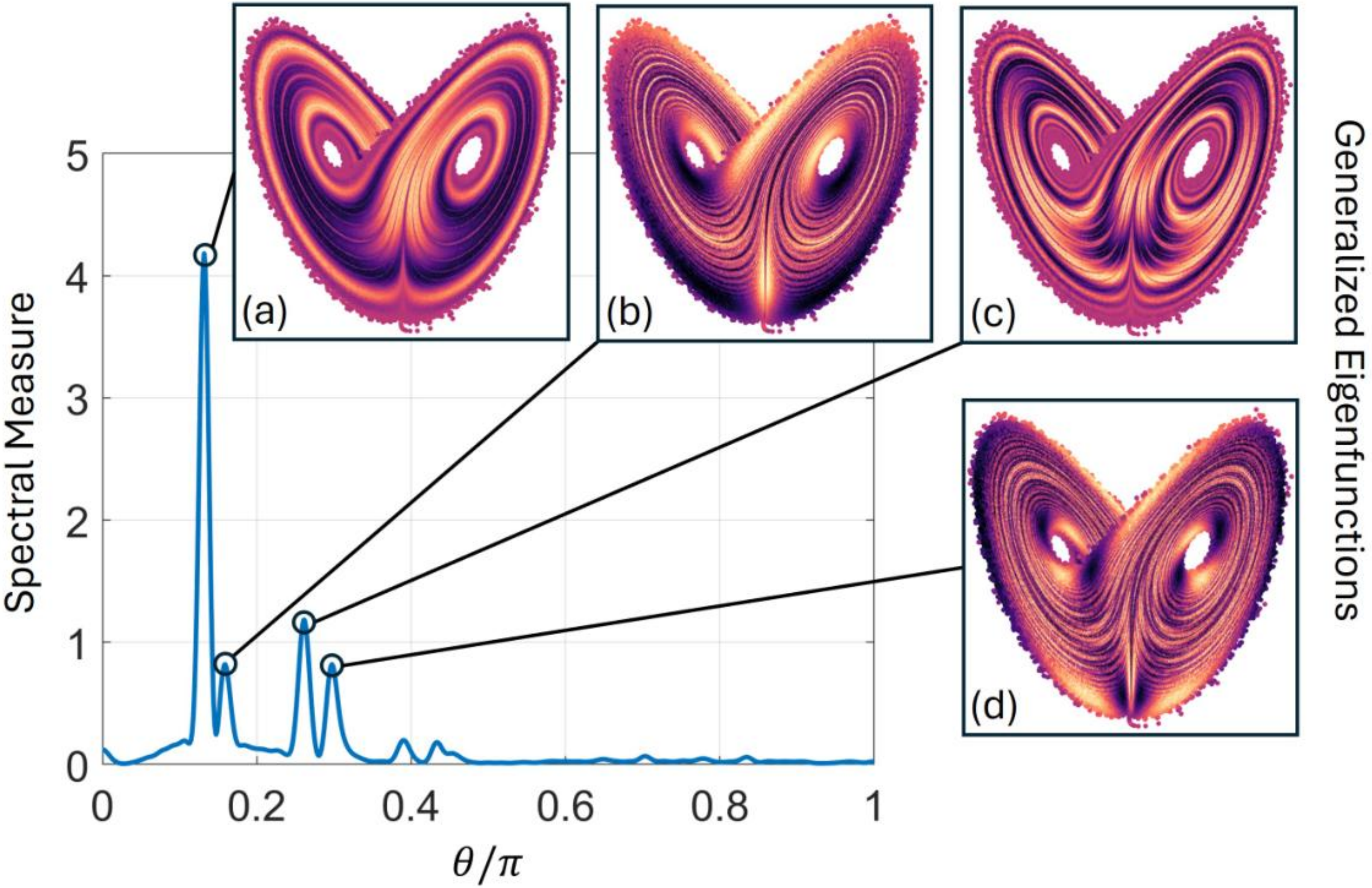
  *Govindarajan, Mohr, Chandrasekaran, Mezic, **SIADS**, 2019,   C., **SINUM**, 2023.*

- Methods based on compactification.

  *Das, Giannakis, Slawinska, **ACHA**, 2021, Valva, Giannakis, **Nonlinearity**, 2023.*

- Resonances (and connections with EDMD).

  *Slipantschuk, Bandtlow, Just, **CNSNS**, 2020,   Wormell, **SINUM**, 2025.*



Spectral Measure
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1
θ/π
(a)
(b)
(c)
(d)
Generalized Eigenfunctions

Tools for continuous spectra and beyond

- Fourier filtering.

  *Korda, Putinar, Mezic, **ACHA,** 2020,   C., Townsend, **CPAM,** 2024.*

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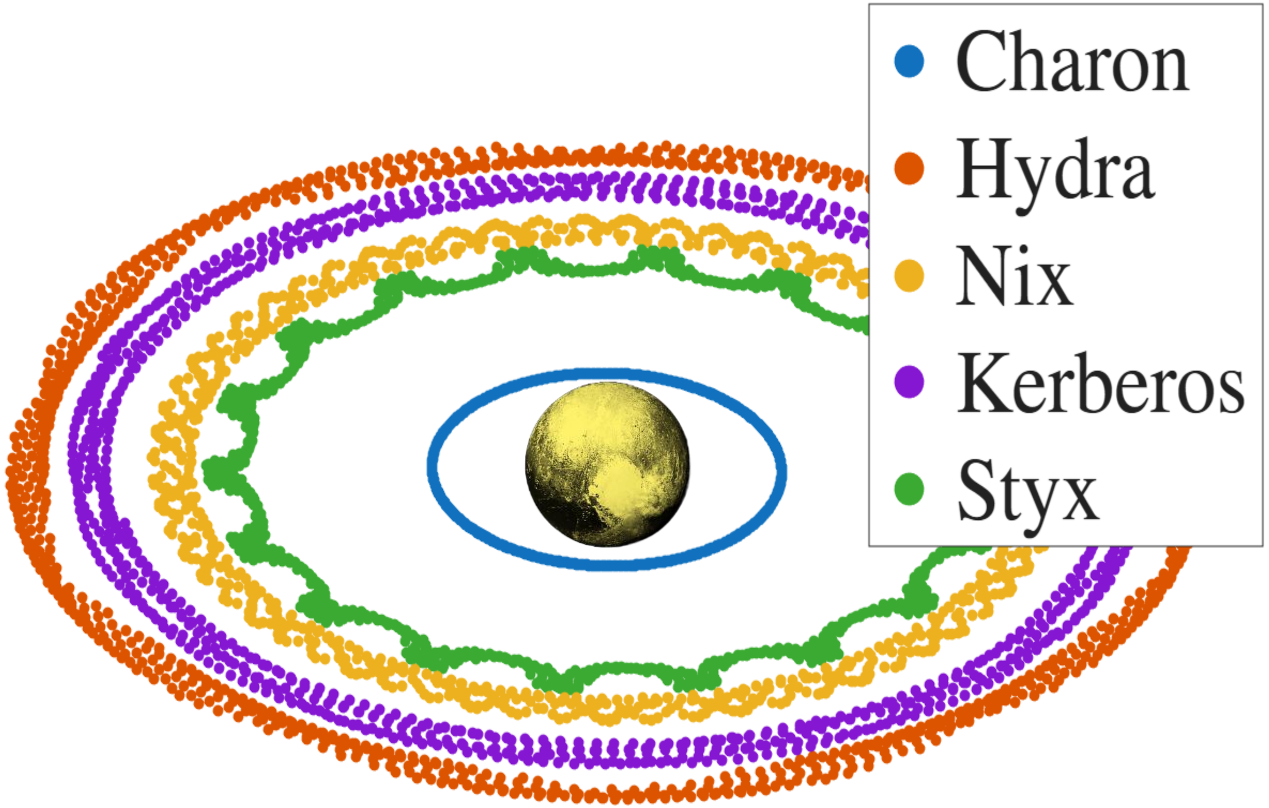
How do we integrate these new tools into very applied domains?

Dictionary learning and forecast bounds

- Pluto and its 5 satellites
- Comparable mass of Pluto and Charon → **complex dynamics**
- Obtain error bounds for both L^2 and RKHS cases

apply to dictionary refinement

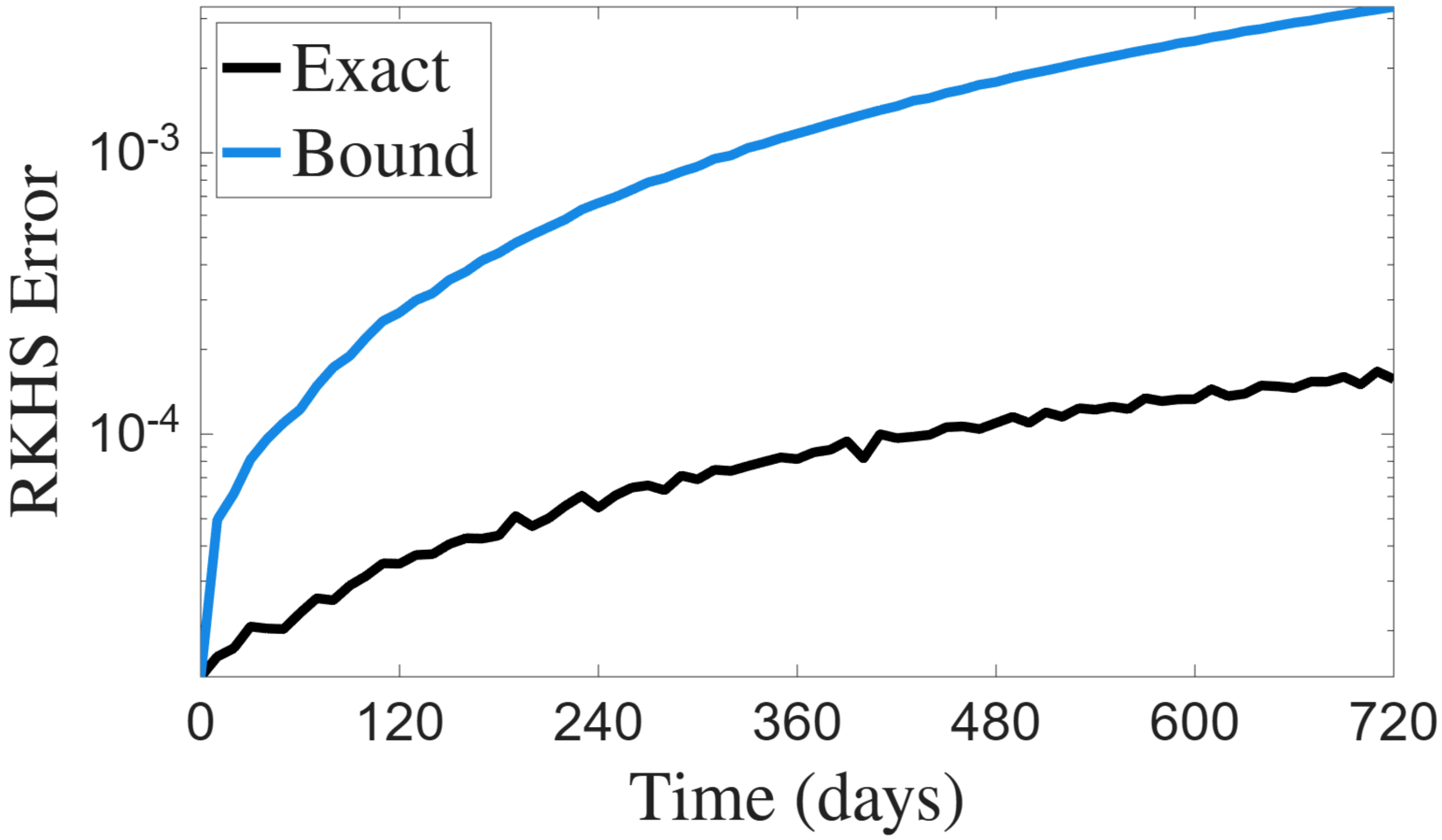
obtain pointwise error bounds, expected error surrogates



- Conradie, Boullé, Loiseau, Brunton, C., *"Trustworthy Koopman Operator Learning: Invariance Diagnostics and Error Bounds,"* **preprint**, 2026.

Dictionary learning and forecast bounds

- Apply algorithms to RKHS setting using Matérn kernel



- Conradie, Boullé, Loiseau, Brunton, C., *"Trustworthy Koopman Operator Learning: Invariance Diagnostics and Error Bounds,"* **preprint**, 2026.

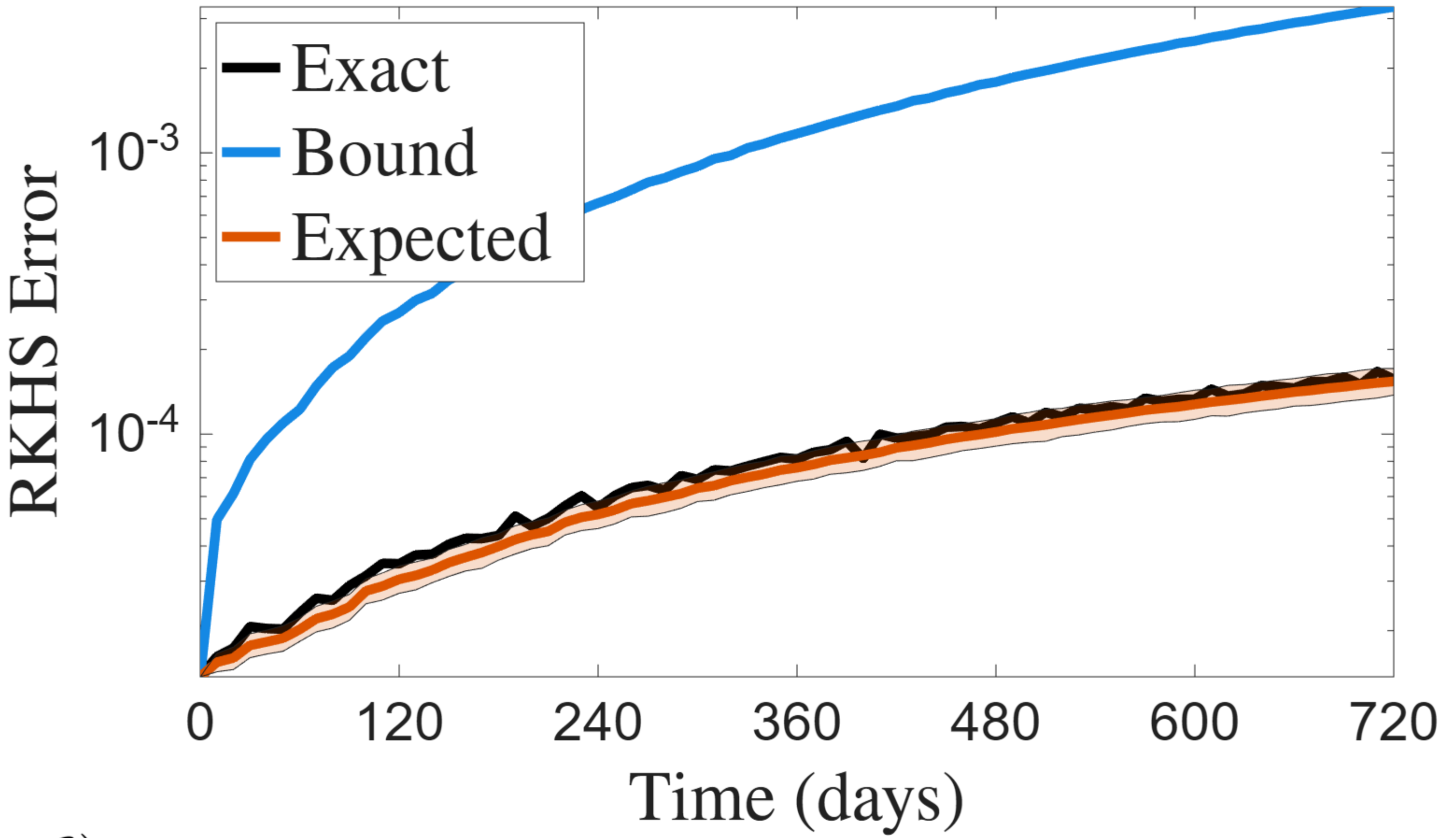
Dictionary learning and forecast bounds

- Apply algorithms to RKHS setting using Matérn kernel

- Expected errors using **Gaussian processes** avoid overestimation

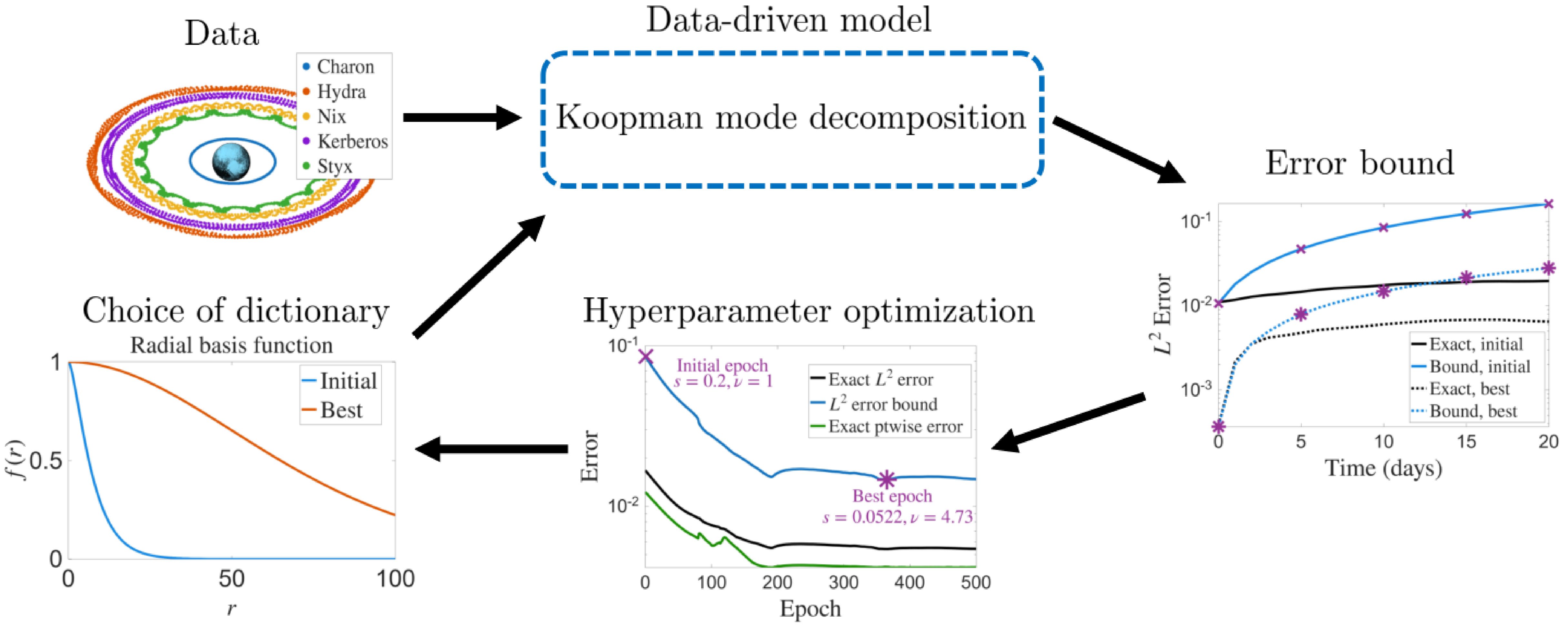
- E.g., replace $\|Ax\| \leq \|A\|\|x\|$ with $\|Ax\| \approx \mathbb{E}_{\mathcal{C}}[A]\|x\|$ where

$$\mathbb{E}_{\mathcal{C}}[A] = \mathbb{E}\left[\frac{\|Ax\|}{\|x\|}\right], \quad x \sim \mathcal{GP}(0, \mathcal{C})$$

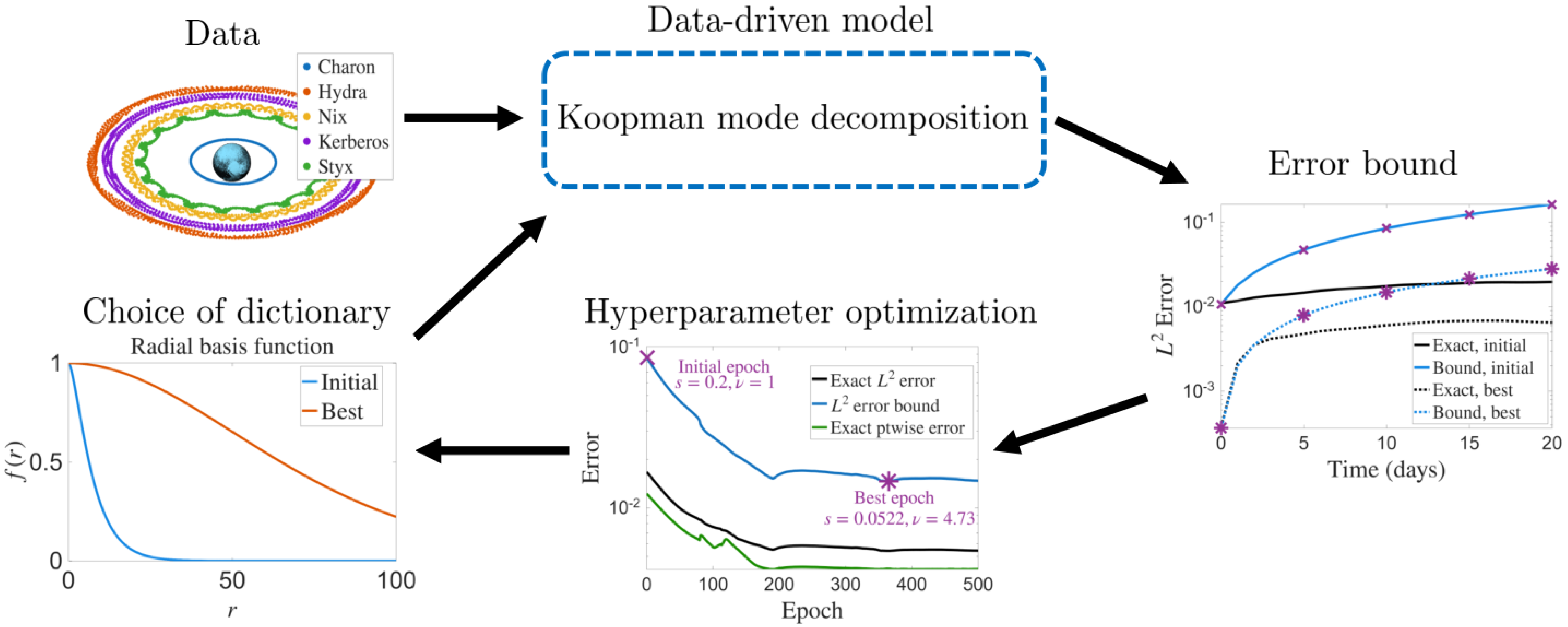


- Conradie, Boullé, Loiseau, Brunton, C., *"Trustworthy Koopman Operator Learning: Invariance Diagnostics and Error Bounds,"* **preprint**, 2026.

Dictionary learning and forecast bounds

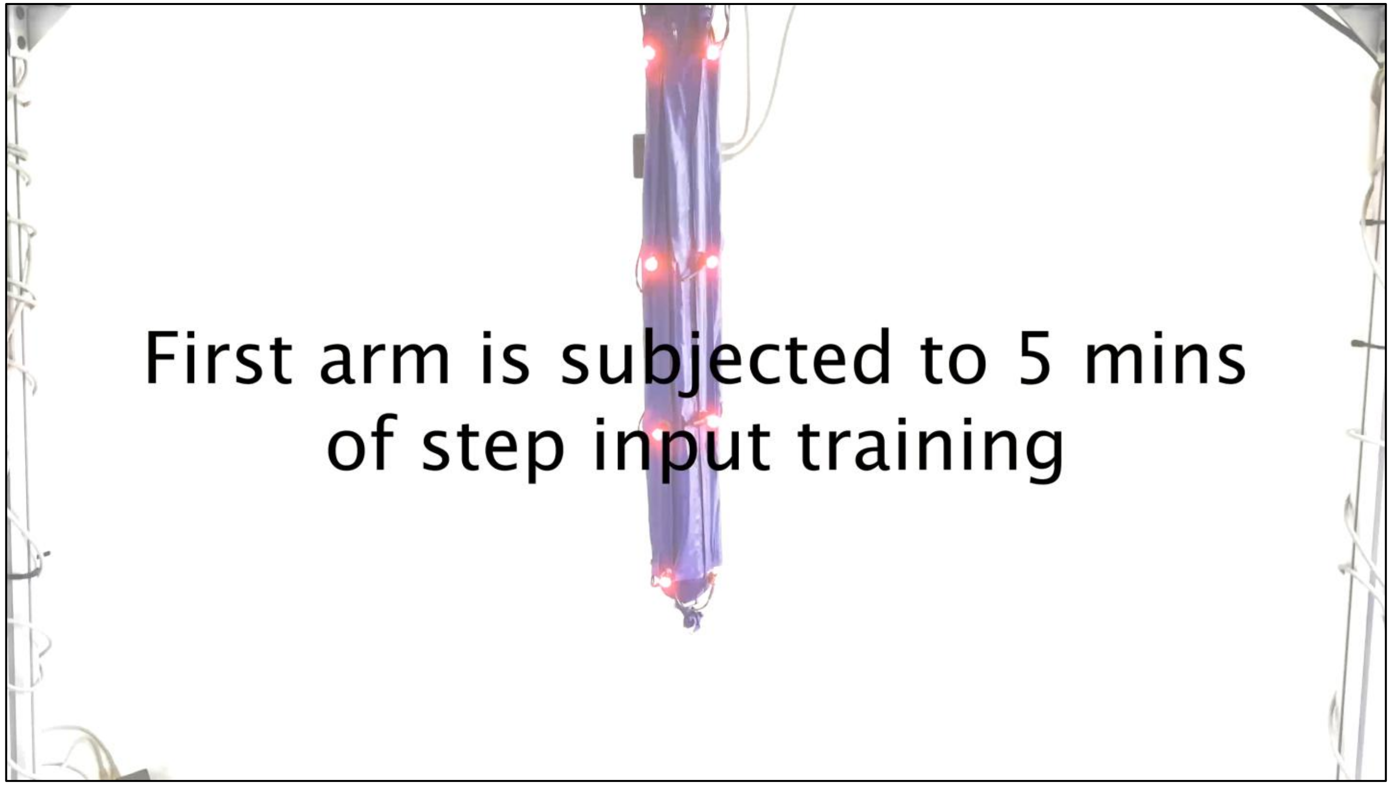


Dictionary learning and forecast bounds



Next steps?

Control



Haggerty, Banks, Kamenar, Cao, Curtis, Mezić, Hawkes, *“Control of soft robots with inertial dynamics,”* **Science robotics**, 2023.

How do we control design under non-invariant lifted subspaces?

Non-autonomous systems

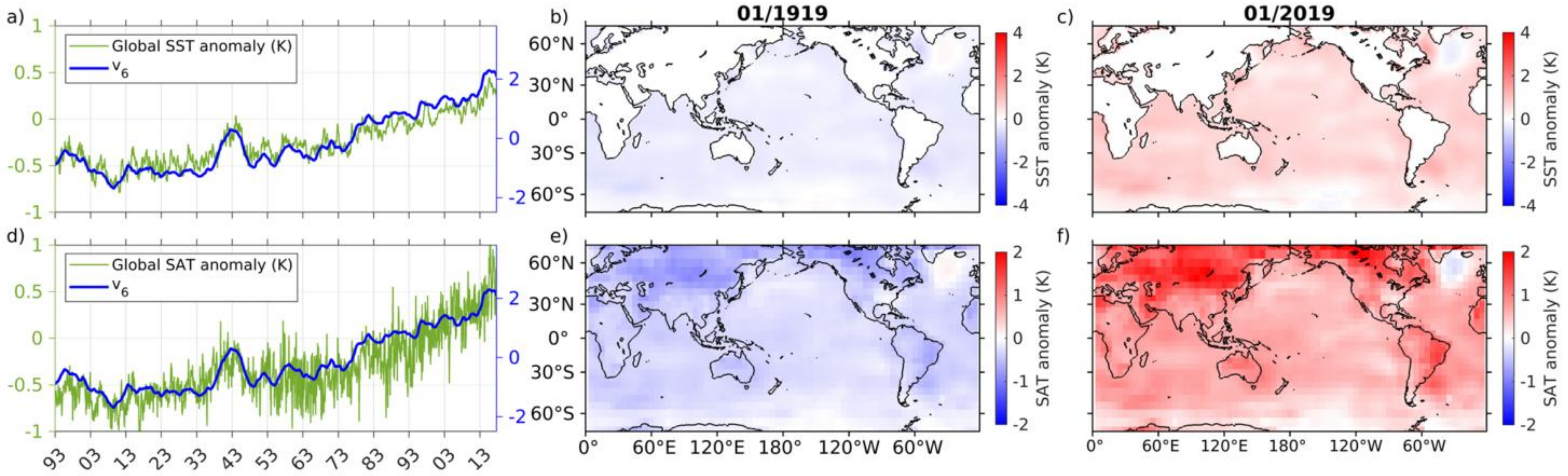
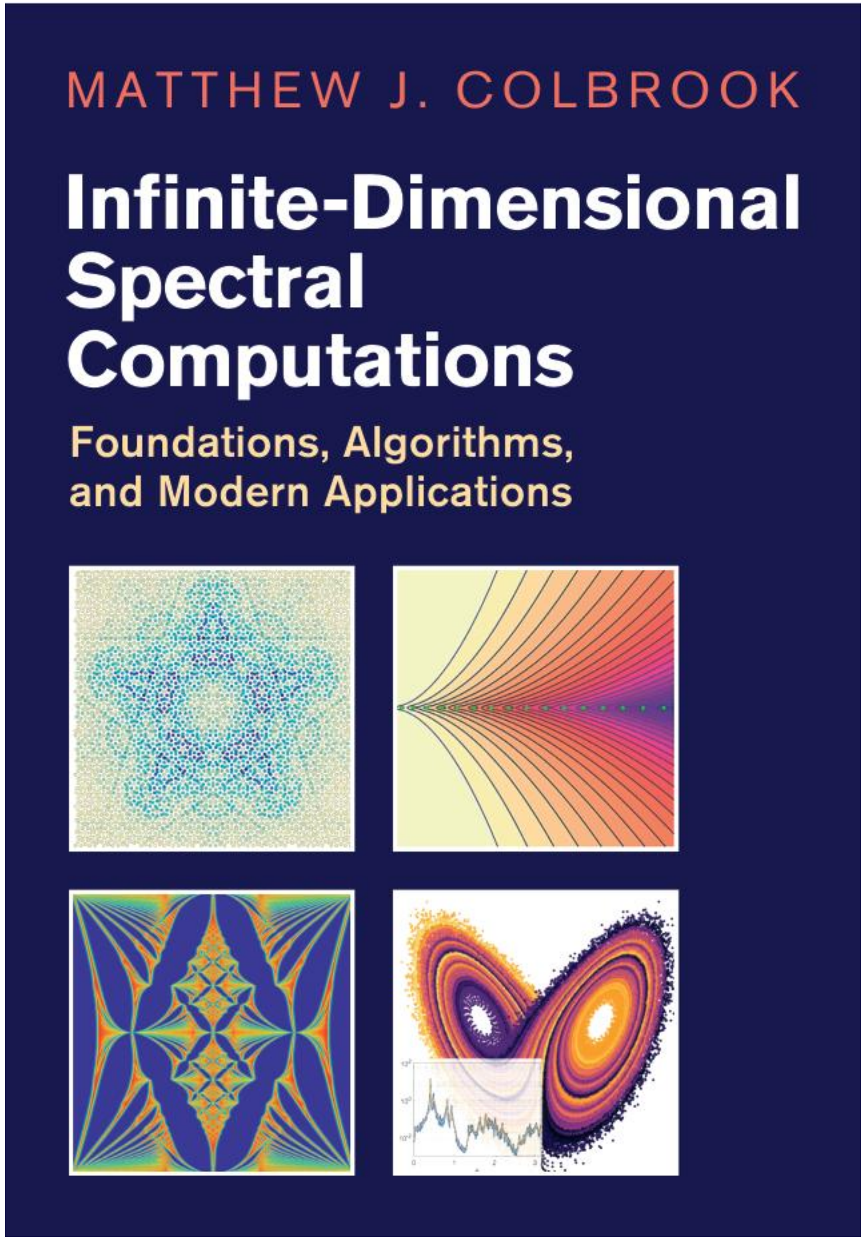


Fig. 7 | Reconstruction of SST and SAT trends over the industrial era using eigenfunction υ_6 of the generator. Panels (**a**, **d**) show time series of globally averaged SST and SAT anomalies, respectively, along with time series of υ_6. Panels (**b**, **c**) show global spatial maps of SST anomalies for January 1919 and 2019, respectively. Panels (**e**, **f**) show global SAT anomalies for the same dates.

Froyland, Giannakis, Luna, Slawinska, *"Revealing trends and persistent cycles of non-autonomous systems with autonomous operator-theoretic techniques,"* **Nature Communications,** 2024.

How do we build methods that don't augment with time and can rely on a single trajectory of data?

Shameless plug 1: CUP book out August 2026…



Contents

100s of:

- Classifications
- Algorithms
- Examples (full code)
- Exercises (full solutions)

If something of interest – speak to me!

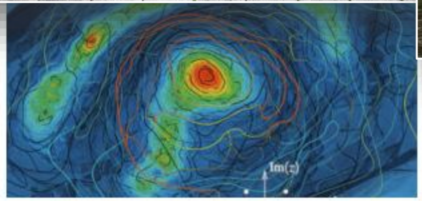
This talk

Shameless plug 2: Issac Newton Institute Programme

Operator Methods for Dynamical Systems

OMD

3 August 2026 to 28 August 2026



Programme theme

Dynamical systems lie at the heart of our understanding of complex phenomena, whether in modelling weather and ocean currents, molecular dynamics, population growth or stock market fluctuations. However, the nonlinear nature of systems poses a challenge to describing their behaviour. Operator methods offer a powerful way to tackle this challenge. By representing a finite-dimensional nonlinear system's evolution as a linear operator acting on an infinite-dimensional space of functions, tools from linear algebra and spectral analysis can be used to gain insights into the system's long-term behaviour. This operator-based perspective has roots in 20th-century mathematics. It has proven fruitful in classical settings like statistical mechanics, where it helped connect chaotic systems with well-understood linear techniques.

In recent years, interest in operator methods for dynamical systems has surged, driven by data-driven techniques to approximate and analyse these operators from real-world or simulated data. A wide range of frameworks and algorithms have emerged, creating an exciting opportunity and need to develop a unifying foundation for these approaches. This programme aims to bring together experts and young researchers from various communities who use operator-theoretic perspectives (for example, the Koopman and transfer operator frameworks) to study dynamics. By uniting participants from pure theory to practical applications, the programme will spark new collaborations and jointly tackle key open questions, building a more cohesive research community. In particular, a significant focus will be on exploring the spectral properties of these operators—essentially, understanding their eigenvalues, modes, and related features—as these provide crucial insights into a system's long-term behaviour and how such behaviour can be effectively analysed and computed.

Organisers

- Matthew Colbrook *University of Cambridge*
- Gary Froyland *University of New South Wales*
- Nathan Kutz *University of Washington*
- Julia Slipantschuk *University of Warwick; Universität Bayreuth*
- Caroline Wormell *University of Sydney*

Participants

- Wael Bahsoun *Loughborough University*
- Steve Brunton *University of Washington*
- Christopher Budd *University of Bath; Institute of Mathematics and its Applications*

Visit to find out more about operator theory and machine learning in dynamical systems!

(and to enjoy Cambridge in the summer!)

Pointers

1. Data-driven spectral problems for Koopman operators are hugely popular.
   BUT: Standard truncation methods can fail – NEED TO GO INF-DIM!

2. **General methods with convergence for spectral properties**
   (spectra, pseudospectra, spectral measures, etc.) of K. operators!
   E.g., Verification of approximate eigenfunctions leads to practical gains.

3. **SCI hierarchy** classifies computational problems:

   Lower bounds through method of *adversarial dynamics*.

   Upper bounds $\Longrightarrow$ new "inf.-dim." algorithms. *Rigorous, optimal, practical.*

$\longrightarrow$ We now have a near complete spectral picture for K. on $L^2(\mathcal{X},\omega)$ and RKHS!

4. **Related challenges** are shaping current research on **closure issues.**

Lots of SCI upper bounds lurking in Koopman literature!

SCI: Fewest number of limits needed to solve a computational problem.

Algorithm	Comments/Assumptions	Spectral Problem's Corresponding SCI Upper Bound			
		KMD	*Spectrum*	*Spectral Measure (if m.p.)*	*Spectral Type (if m.p.)*
Extended DMD [47]	general L^2 spaces	$\mathrm{SCI} \leq 2^*$	N/C	N/C	n/a
Residual DMD [44]	general L^2 spaces	$\mathrm{SCI} \leq 2^*$	$\mathrm{SCI} \leq 3^*$	$\mathrm{SCI} \leq 2^*$	varies, see [84] e.g., a.c. density: $\mathrm{SCI} \leq 2^*$
Measure-preserving EDMD [45]	m.p. systems	$\mathrm{SCI} \leq 1$	N/C	$\mathrm{SCI} \leq 2^*$ (general) $\mathrm{SCI} \leq 1$ (delay-embedding)	n/a
Hankel DMD [85]	m.p. ergodic systems	$\mathrm{SCI} \leq 2^*$	N/C	N/C	n/a
Periodic approximations [86]	m.p. $+\ \omega$ a.c.	$\mathrm{SCI} \leq 2$	N/C	$\mathrm{SCI} \leq 2$ (see [87])	a.c. density: $\mathrm{SCI} \leq 3$
Christoffel–Darboux kernel [40]	m.p. ergodic systems	$\mathrm{SCI} \leq 3$	n/a	$\mathrm{SCI} \leq 2$	e.g., a.c. density: $\mathrm{SCI} \leq 2$
Generator EDMD [88]	cts.-time, samples ∇F (otherwise additional limit)	$\mathrm{SCI} \leq 2$	N/C	$\mathrm{SCI} \leq 2$ (see [89])	n/a
Compactification [42]	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 4$	N/C	$\mathrm{SCI} \leq 4$	n/a
Resolvent compactification [43]	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 5$	N/C	$\mathrm{SCI} \leq 5$	n/a
Diffusion maps [90] (see also [10])	cts.-time, m.p. ergodic systems	$\mathrm{SCI} \leq 3$	n/a	n/a	n/a

Are these sharp?

Previous techniques prove upper bounds on SCI.

"N/C": method need not converge. "n/a": algorithm not applicable to problem.

Also in Ulam's method for Markov processes, SRB measure computation, control,…

Proof idea: Constructing an adversary

F_0: rotation by π, $\mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

Phase transition lemma: Let $X = \{x_1, \ldots, x_N\}$,$Y = \{y_1, \ldots, y_N\}$ be distinct points in annulus $\mathcal{A} = \{x \in \mathbb{D} | 0 < R < \|x\| < r < 1\}$ with $X \cap Y = \emptyset$. There exists a measure-preserving homeomorphism H such that H acts as the identity on $\mathbb{D} \backslash \mathcal{A}$ and $H(y_j) = F_0(H(x_j)), j = 1, \ldots, N$.

Conjugacy of <u>data</u> ($x_j \rightarrow y_j$) with F_0

Idea: Use lemma to trick any algorithm into oscillating between spectra.

- Brown and Halperin. "*On certain area-preserving maps.*" **Annals of Mathematics**, 1935.

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty}\Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

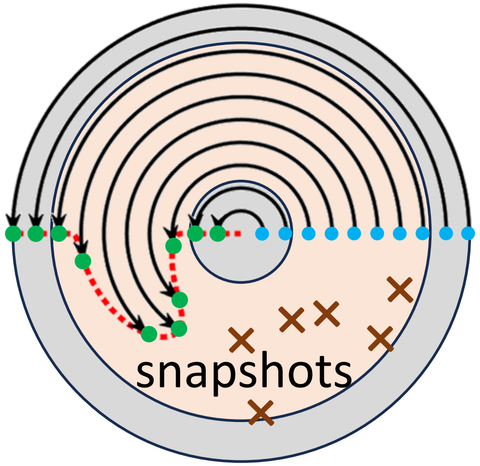
Build an adversarial F…

$\boxed{\mathcal{T}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{J}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F...



$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset$ [1/4, 3/4]

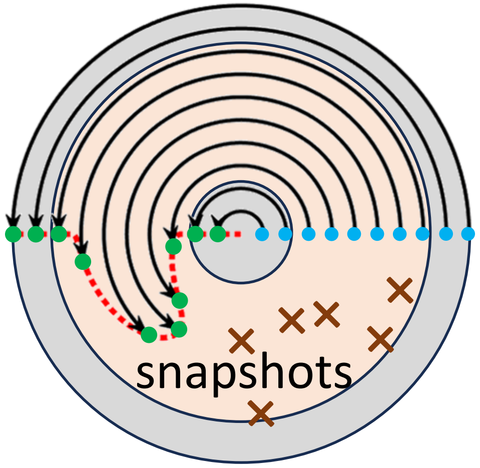
$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\mathcal{J}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{J}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F…



$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, $\mathrm{supp}(\phi) \subset [1/4, 3/4]$
$\mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) = \mathbb{T}$ (unit circle).

$\lim_{n\to\infty} \Gamma_n\big(\widetilde{F_1}\big) = \mathrm{Sp}(\mathcal{K}_{\widetilde{F_1}}) \Rightarrow \exists n_1$ s.t. $\mathrm{dist}\big(i, \Gamma_{n_1}\big(\widetilde{F_1}\big)\big) \leq 1.$

BUT Γ_{n_1} uses finite amount of info to output $\Gamma_{n_1}\big(\widetilde{F_1}\big)$.

Let X, Y correspond to these snapshots.

$\boxed{\mathcal{J}_F = \{(x, y_m) \mid \|F(x) - y_m\| \leq 2^{-m}\}}$

Proof idea: Constructing an adversary

Suppose (for contradiction) $\{\Gamma_n\}$ uses $\mathcal{T}_F$, $\lim_{n\to\infty} \Gamma_n(F) = \mathrm{Sp}(\mathcal{K}_F)\ \ \forall F \in \Omega_{\mathbb{D}}$.

Build an adversarial F…

$\widetilde{F_1}(r,\theta) = \big(r, \theta + \pi + \phi(r)\big)$, supp$(\phi) \subset [1/4, 3/4]$

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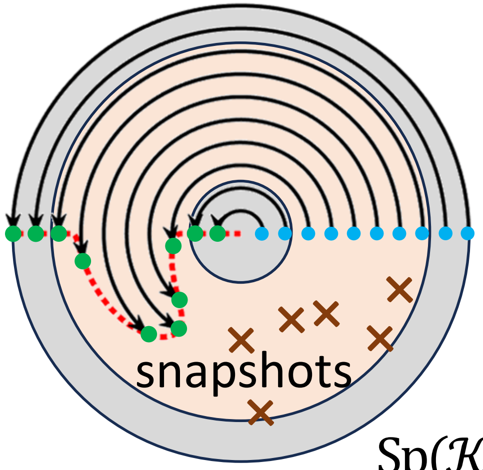

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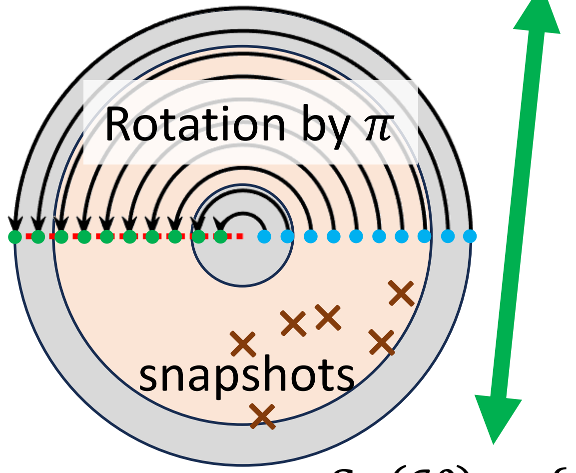


Lemma: $F_1 = H_1^{-1} \circ F_0 \circ H_1$ on annulus $\mathcal{A}_1$.

<u>Consistent data</u> $\Rightarrow \Gamma_{n_1}(F_1) = \Gamma_{n_1}\big(\widetilde{F_1}\big)$, dist$(i, \Gamma_{n_1}(F_1)) \leq 1$

BUT $\mathrm{Sp}(\mathcal{K}_{F_1}) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

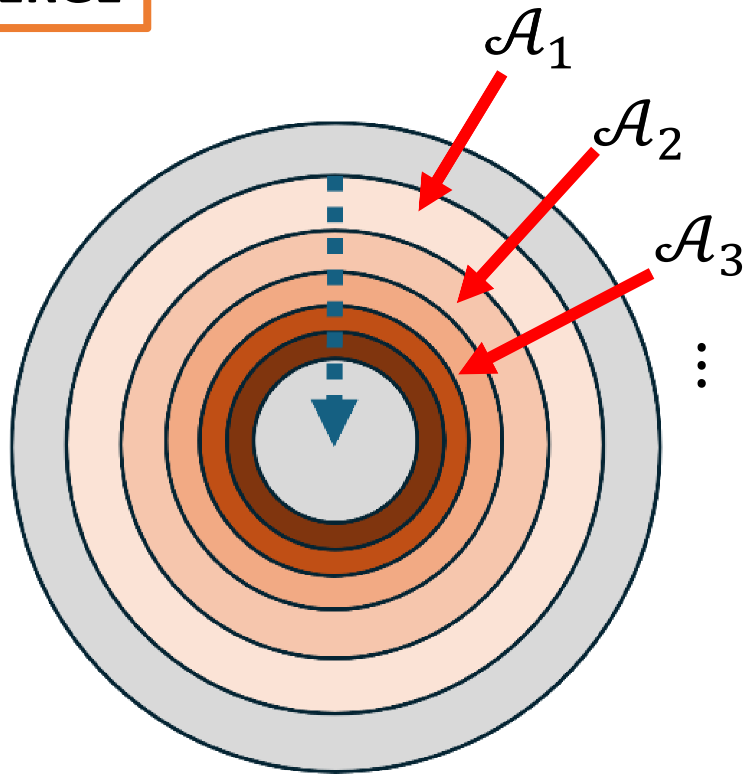
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

<u>Consistent data</u> $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

BUT $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



Cascade of disks

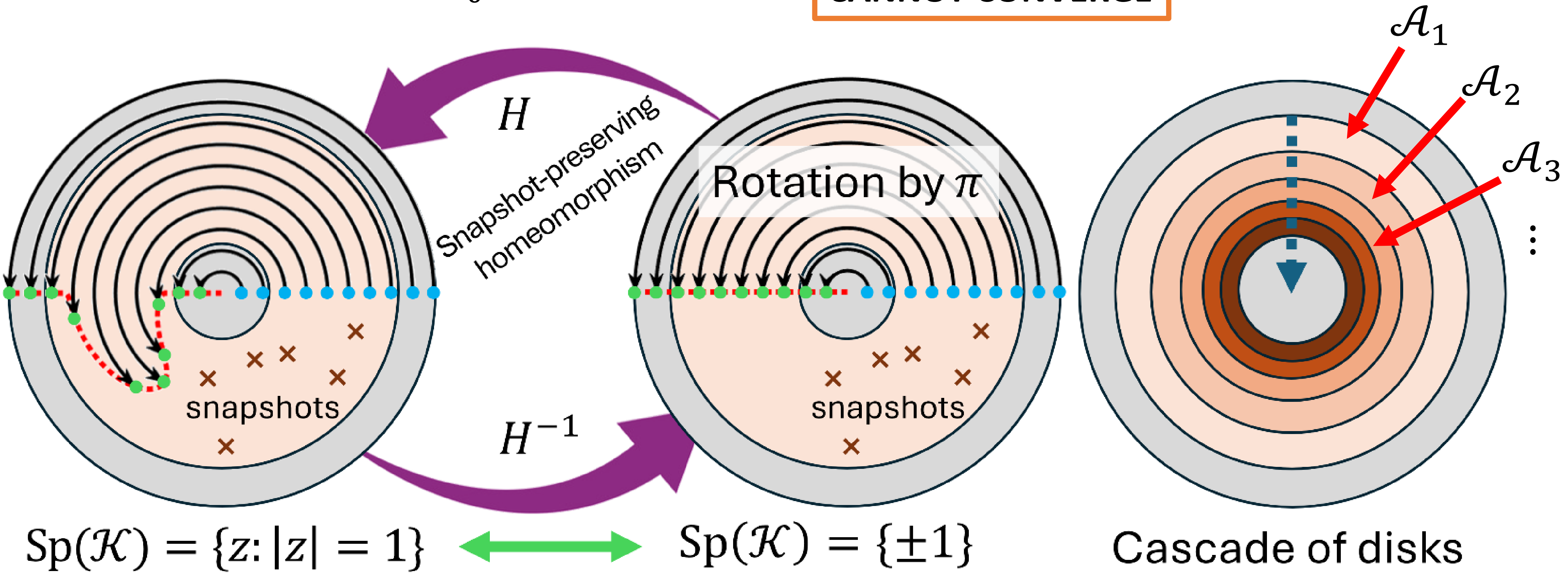
Proof idea: Constructing an adversary

Inductive step: Repeat on annuli, $F_k = H_k^{-1} \circ F_0 \circ H_k$ on $\mathcal{A}_k$. $F = \lim_{k\to\infty} F_k$

Consistent data $\Rightarrow \Gamma_{n_k}(F) = \Gamma_{n_k}(\widetilde{F_k})$, $\mathrm{dist}(i, \Gamma_{n_k}(F)) \leq 1$, $n_k \to \infty$

BUT $\mathrm{Sp}(\mathcal{K}_F) = \mathrm{Sp}(\mathcal{K}_{F_0}) = \{\pm 1\}$

CANNOT CONVERGE



References

[1] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." Communications on Pure and Applied Mathematics 77.1 (2024): 221-283.

[2] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[3] Colbrook, M. J., Li, Q., Raut, R. V., & Townsend, A. "Beyond expectations: residual dynamic mode decomposition and variance for stochastic dynamical systems." Nonlinear Dynamics 112.3 (2024): 2037-2061.

[4] Colbrook, Matthew J. "The Multiverse of Dynamic Mode Decomposition Algorithms." Handbook of Numerical Analysis, vol. 25, pp. 127-230. Elsevier, 2024..

[5] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." SIAM Journal on Numerical Analysis 61.3 (2023): 1585-1608.

[6] Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. "Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1150-1190.

[7] Boullé, Nicolas, and Matthew J. Colbrook. "Multiplicative Dynamic Mode Decomposition." SIAM Journal on Applied Dynamical Systems 24, no. 2 (2025): 1945-1968.

[8] Boullé, Nicolas and Matthew J. Colbrook, "On the Convergence of Hermitian Dynamic Mode Decomposition" Physica D: Nonlinear Phenomena, 472 (2025).

[9] Colbrook, Matthew J., Andrew Horning, and Tianyiwa Xie. "Computing Generalized Eigenfunctions in Rigged Hilbert Spaces." arXiv preprint arXiv:2410.08343 (2024).

[10] Zagli, Niccolò, et al. "Bridging the Gap between Koopmanism and Response Theory: Using Natural Variability to Predict Forced Response." arXiv preprint arXiv:2410.01622 (2024).

[11] Colbrook, Matthew J. "Another look at Residual Dynamic Mode Decomposition in the regime of fewer Snapshots than Dictionary Size." Physica D: Nonlinear Phenomena 469 (2024).

[12] Colbrook, Matthew. “The foundations of infinite-dimensional spectral computations.” Diss. University of Cambridge, 2020.

[13] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). “Computing Spectra--On the Solvability Complexity Index Hierarchy and Towers of Algorithms.” arXiv preprint arXiv:1508.03280.

[14] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale’s 18th problem." Proceedings of the National Academy of Sciences 119.12 (2022): e2107151119.

[15] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." SIAM review 63.3 (2021): 489-524.

[16] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.

[17] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society (2022).

[18] Colbrook, Matthew J. "Computing spectral measures and spectral types." Communications in Mathematical Physics 384 (2021): 433-501.

[19] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." Numerische Mathematik 143 (2019): 17-83.

[20] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." Foundations of Computational Mathematics (2022): 1-82.

[21] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective.“

[22] Colbrook, Matthew J., Igor Mezić, and Alexei Stepanenko. "Limits and Powers of Koopman Learning.“ arXiv preprint arxiv:2407.06312 (2024).

[23] Herwig, April, Matthew J. Colbrook, Oliver Junge, Péter Koltai, and Julia Slipantschuk. "Avoiding spectral pollution for transfer operators using residuals." arXiv preprint arXiv:2507.16915 (2025).

[24] Boullé, Nicolas, Matthew J. Colbrook, and Gustav Conradie. "Convergent Methods for Koopman Operators on Reproducing Kernel Hilbert Spaces." arXiv preprint arXiv:2506.15782 (2025).

[25] Drysdale, Catherine, Matthew Colbrook, and Michael TM Woodley. "Computation and Verification of Spectra for Non-Hermitian Systems." Physical review letters 135.17 (2025): 170202.

[26] Colbrook, Matthew J., and Catherine Drysdale. "Universal Methods for Nonlinear Spectral Problems." arXiv preprint arXiv:2504.17012 (2025).

[27] Colbrook, Matthew J., Zlatko Drmač, and Andrew Horning. "An Introductory Guide to Koopman Learning." arXiv preprint arXiv:2510.22002 (2025).