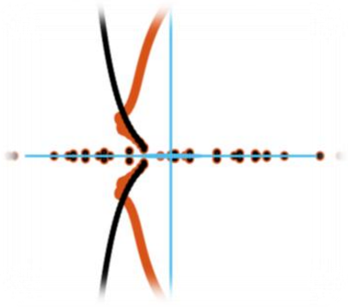
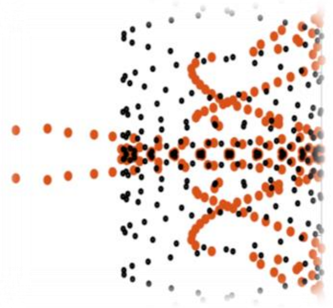
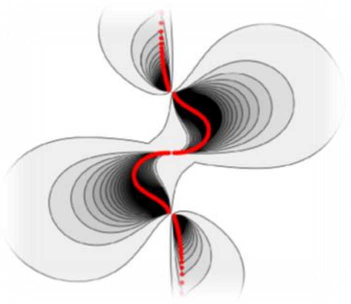
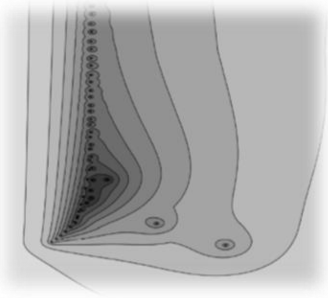








Four examples of discretization issues for Nonlinear Eigenvalue Problems

Matthew Colbrook

University of Cambridge

16/08/2023

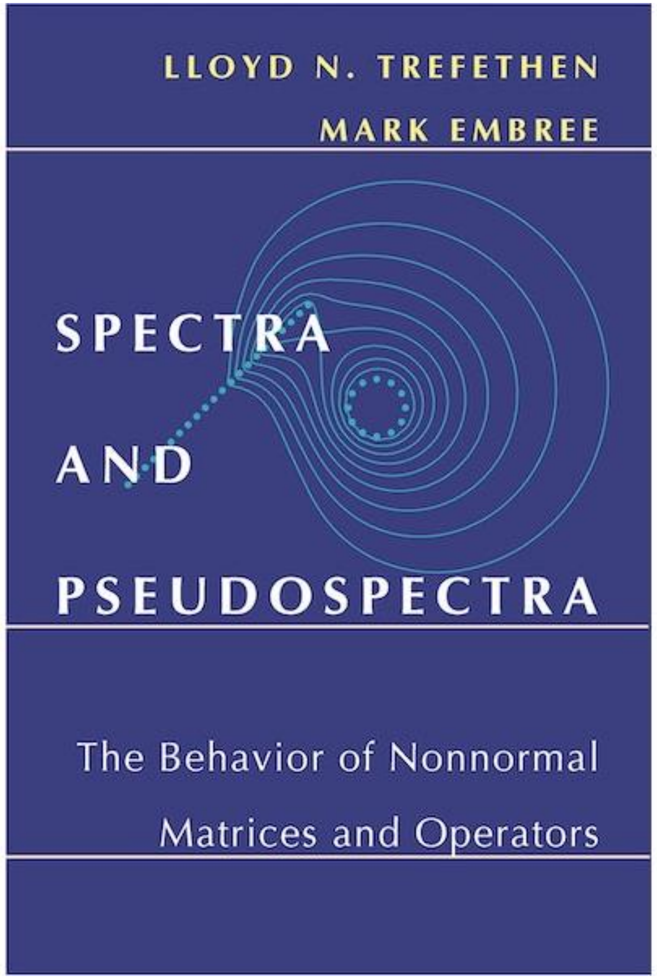
Joint work with

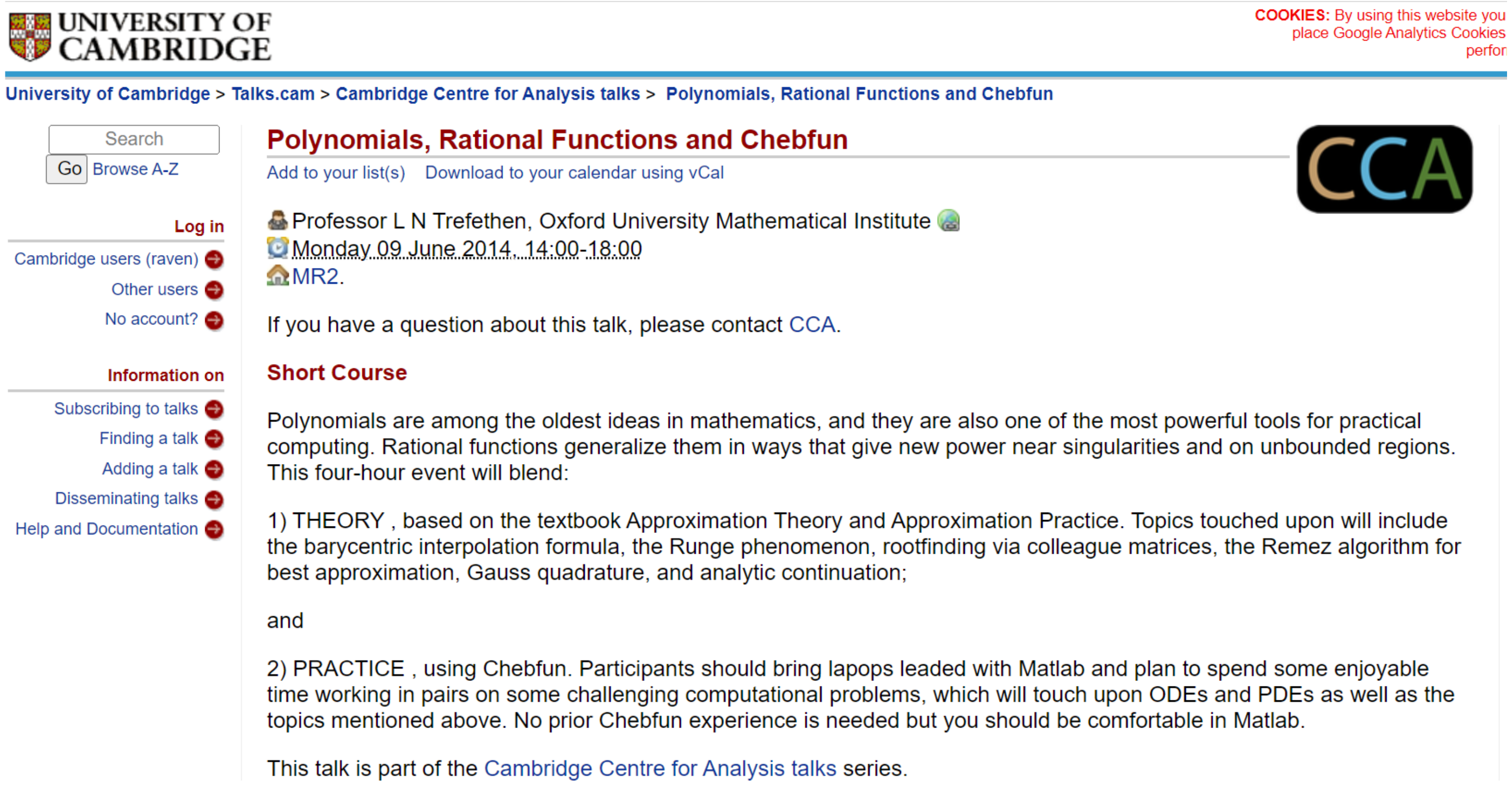
Alex Townsend

(Cornell)

C., Townsend, *"Avoiding discretization issues for nonlinear eigenvalue problem"*

THANKS NICK!

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Professor L N Trefethen, Oxford University Mathematical Institute
Monday 09 June 2014, 14:00-18:00
MR2.

If you have a question about this talk, please contact CCA.

Short Course

Polynomials are among the oldest ideas in mathematics, and they are also one of the most powerful tools for practical computing. Rational functions generalize them in ways that give new power near singularities and on unbounded regions. This four-hour event will blend:

1) THEORY , based on the textbook Approximation Theory and Approximation Practice. Topics touched upon will include the barycentric interpolation formula, the Runge phenomenon, rootfinding via colleague matrices, the Remez algorithm for best approximation, Gauss quadrature, and analytic continuation;

and

2) PRACTICE , using Chebfun. Participants should bring lapops leaded with Matlab and plan to spend some enjoyable time working in pairs on some challenging computational problems, which will touch upon ODEs and PDEs as well as the topics mentioned above. No prior Chebfun experience is needed but you should be comfortable in Matlab.

This talk is part of the Cambridge Centre for Analysis talks series.

Discretize-then-solve paradigm

Nonlinear eigenvalue problem

$$\lambda \in \Omega \subset \mathbb{C}, \quad T(\lambda): \mathcal{D}(T) \mapsto \mathcal{H} = \text{Hilbert space}$$

$\lambda \longrightarrow T(\lambda)u$ *holomorphic* for all $u \in \mathcal{D}(T)$

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$$T(\lambda)u = 0$$

discretize →

$$F(\hat{\lambda})\hat{u} = 0$$
$$F(\lambda) \in \mathbb{C}^{n\times n}$$

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discretize →

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$$F(\lambda) \in \mathbb{C}^{n\times n}$$

solve →

$$\hat{\lambda}_j \in \mathbb{C},$$
$$\hat{u}_j \in \mathbb{C}^n$$

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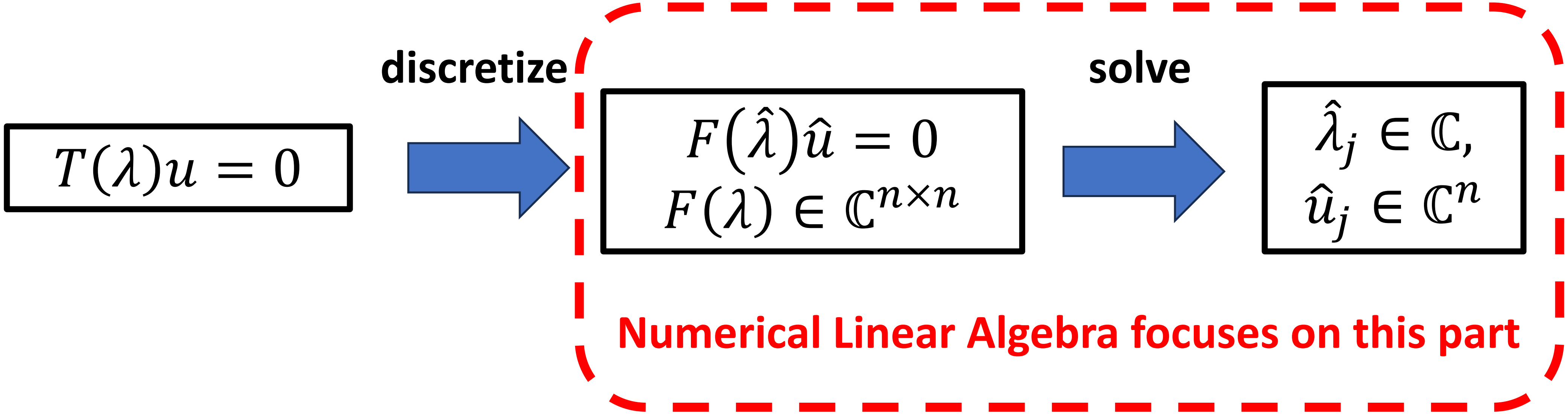
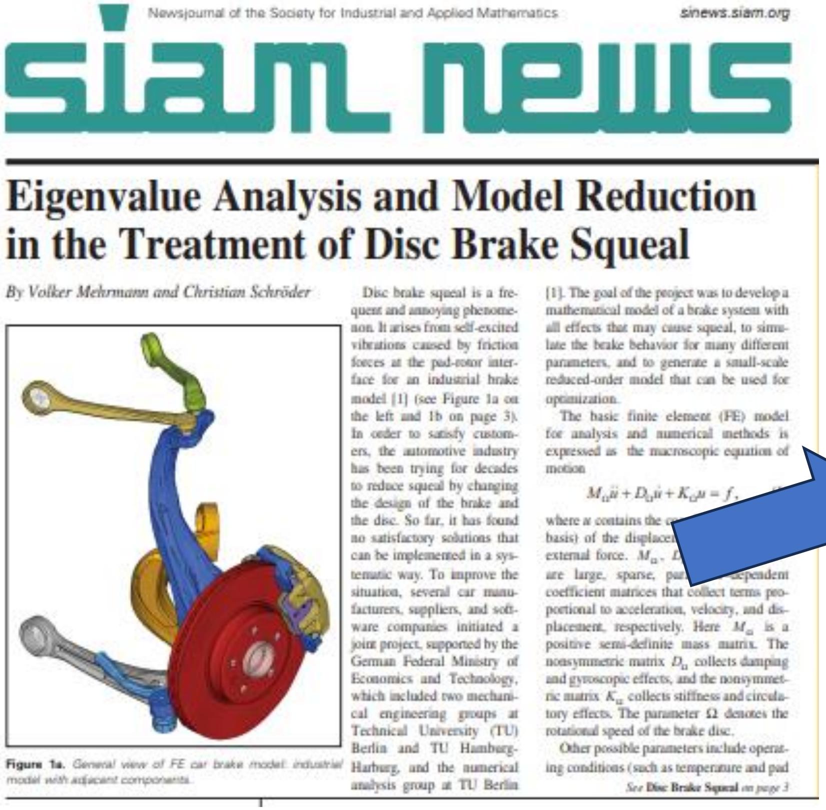


But are we solving the right problems?



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siam news

Eigenvalue Analysis and Model Reduction in the Treatment of Disc Brake Squeal

By Volker Mehrmann and Christian Schröder

Figure 1a. *General view of FE car brake model: industrial model with adjacent components.*

Disc brake squeal is a frequent and annoying phenomenon. It arises from self-excited vibrations caused by friction forces at the pad-rotor interface for an industrial brake model [1] (see Figure 1a on the left and 1b on page 3). In order to satisfy customers, the automotive industry has been trying for decades to reduce squeal by changing the design of the brake and the disc. So far, it has found no satisfactory solutions that can be implemented in a systematic way. To improve the situation, several car manufacturers, suppliers, and software companies initiated a joint project, supported by the German Federal Ministry of Economics and Technology, which included two mechanical engineering groups at Technical University (TU) Berlin and TU Hamburg-Harburg, and the numerical analysis group at TU Berlin [1]. The goal of the project was to develop a mathematical model of a brake system with all effects that may cause squeal, to simulate the brake behavior for many different parameters, and to generate a small-scale reduced-order model that can be used for optimization.

The basic finite element (FE) model for analysis and numerical methods is expressed as the macroscopic equation of motion

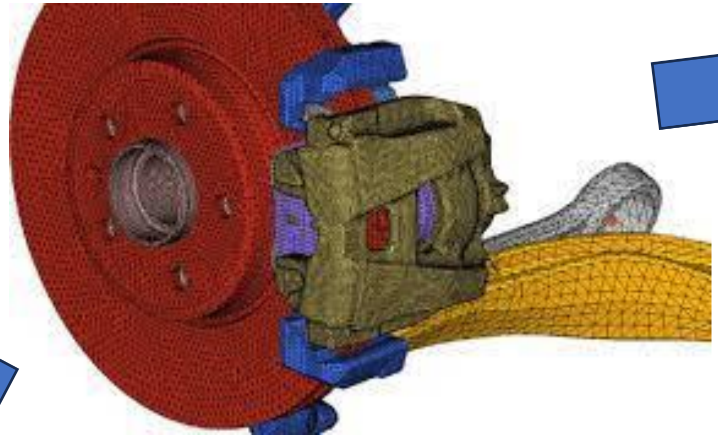
$$M_\Omega \ddot{u} + D_\Omega \dot{u} + K_\Omega u = f,$$

where *u* contains the co… basis) of the displace… external force. M_Ω, D… are large, sparse, par…-dependent coefficient matrices that collect terms proportional to acceleration, velocity, and displacement, respectively. Here M_Ω is a positive semi-definite mass matrix. The nonsymmetric matrix D_Ω collects damping and gyroscopic effects, and the nonsymmetric matrix K_Ω collects stiffness and circulatory effects. The parameter Ω denotes the rotational speed of the brake disc.

Other possible parameters include operating conditions (such as temperature and pad

See **Disc Brake Squeal** on page 3

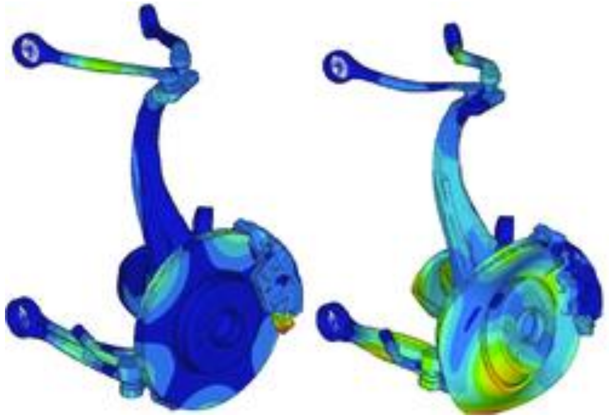
Continuous physics



Discretize to
nonlinear eigenproblem

$$M\ddot{q} + C(\omega)\dot{q} + K(\omega)q :=$$

$$M\ddot{q} + (C_1 + \frac{\omega_r}{\omega}C_R + \frac{\omega}{\omega_r}C_G)\dot{q} + (K_1 + K_R + (\frac{\omega}{\omega_r})^2 K_G)q = f,$$



Eigenfunctions

Volker Mehrmann

- Gräbner, Mehrmann, Quraishi, Schröder, von Wagner, *Numerical methods for parametric model reduction in the simulation of disk brake squeal.* **ZAMM**, 20016.

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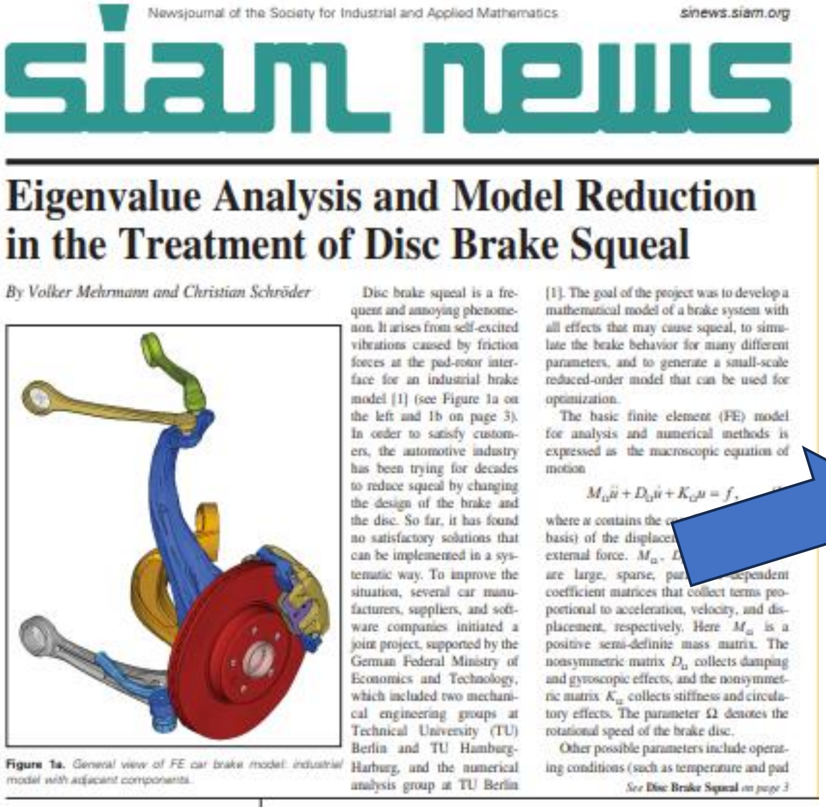
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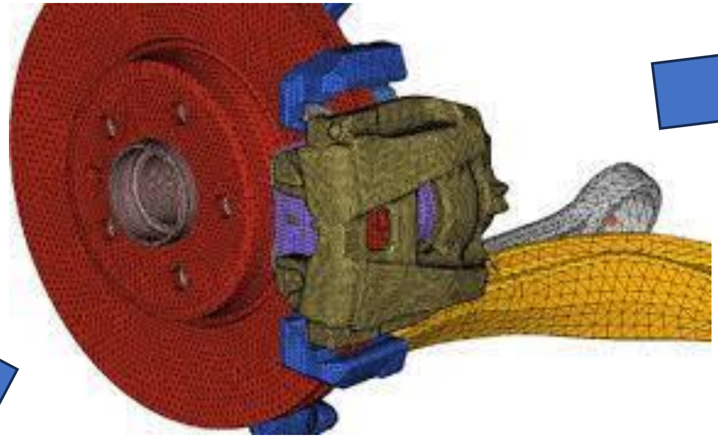

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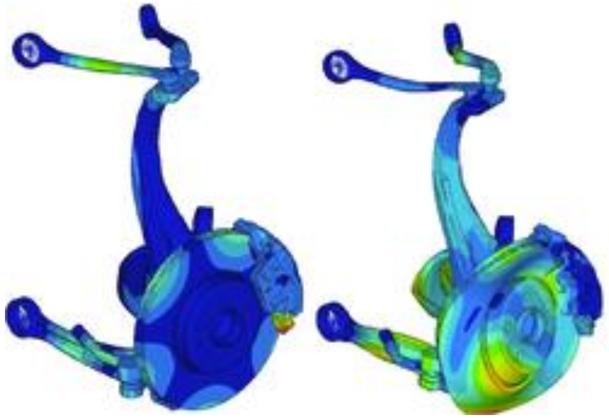

Eigenfunctions

Volker Mehrmann

Do we lose anything when we discretize?

- Gräbner, Mehrmann, Quraishi, Schröder, von Wagner, *Numerical methods for parametric model reduction in the simulation of disk brake squeal.* **ZAMM**, 20016.

A landmark benchmark

NLEVP: A collection of nonlinear eigenvalue problems

T Betcke, NJ Higham, V Mehrmann… - ACM Transactions on …, 2013 - dl.acm.org

… **collection** of 52 **nonlinear eigenvalue problems** in the form of a MATLAB toolbox. The **collection** contains **problems** … A classification is given of polynomial **eigenvalue problems** according …

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Most of the problems in NLEVP come from inf-dim problems.

Some discretization issues.

Example	Observed discretization woes
acoustic_wave_1d	spurious eigenvalues slow convergence
acoustic_wave_2d	spurious eigenvalues wrong multiplicity
butterfly	spectral pollution missed spectra wrong pseudospectra
damped_beam	slow convergence resolved eigenfunctions with inaccurate eigenvalues
loaded_string	ill-conditioning from discretization
planar_waveguide	collapse onto ghost essential spectrum failure for accumulating eigenvalues spectral pollution

- Betcke, Higham, Mehrmann, Schröder, Tisseur, *"NLEVP: A collection of nonlinear eigenvalue problems,"* **ACM Trans. Math. Soft.**, 2013.

Example: Pollution, invisibility…

$$\mathcal{A}(\varepsilon) = \left\{E: \Omega \to \mathcal{B}(\mathcal{H}): \sup_{\lambda\in\Omega} \|E(\lambda)\| < \varepsilon\right\}$$

$$\mathrm{Sp}_{\varepsilon}(T) = \bigcup_{E\in\mathcal{A}(\varepsilon)} \mathrm{Sp}(T+E) = \{\lambda \in \Omega: \|T(\lambda)^{-1}\|^{-1} < \varepsilon\}$$

Stability of spectrum

Characterization through resolvent

Example: Pollution, invisibility…

butterfly from NLEVP, $T(\lambda) = F(\lambda, S)$
S bilateral shift on $l^2(\mathbb{Z})$, F a rational function

$$\mathcal{A}(\varepsilon) = \left\{E: \Omega \to \mathcal{B}(\mathcal{H}): \sup_{\lambda\in\Omega} \|E(\lambda)\| < \varepsilon\right\}$$

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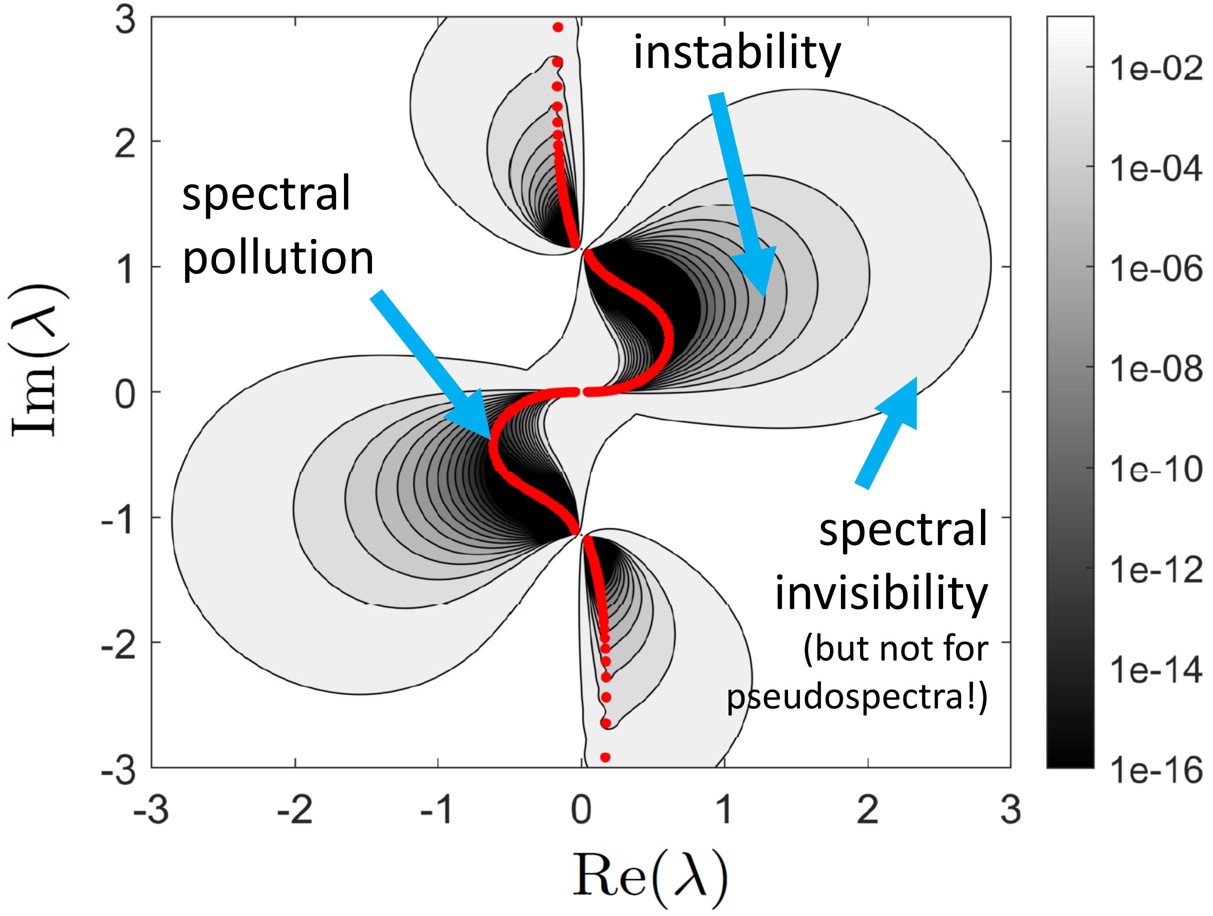
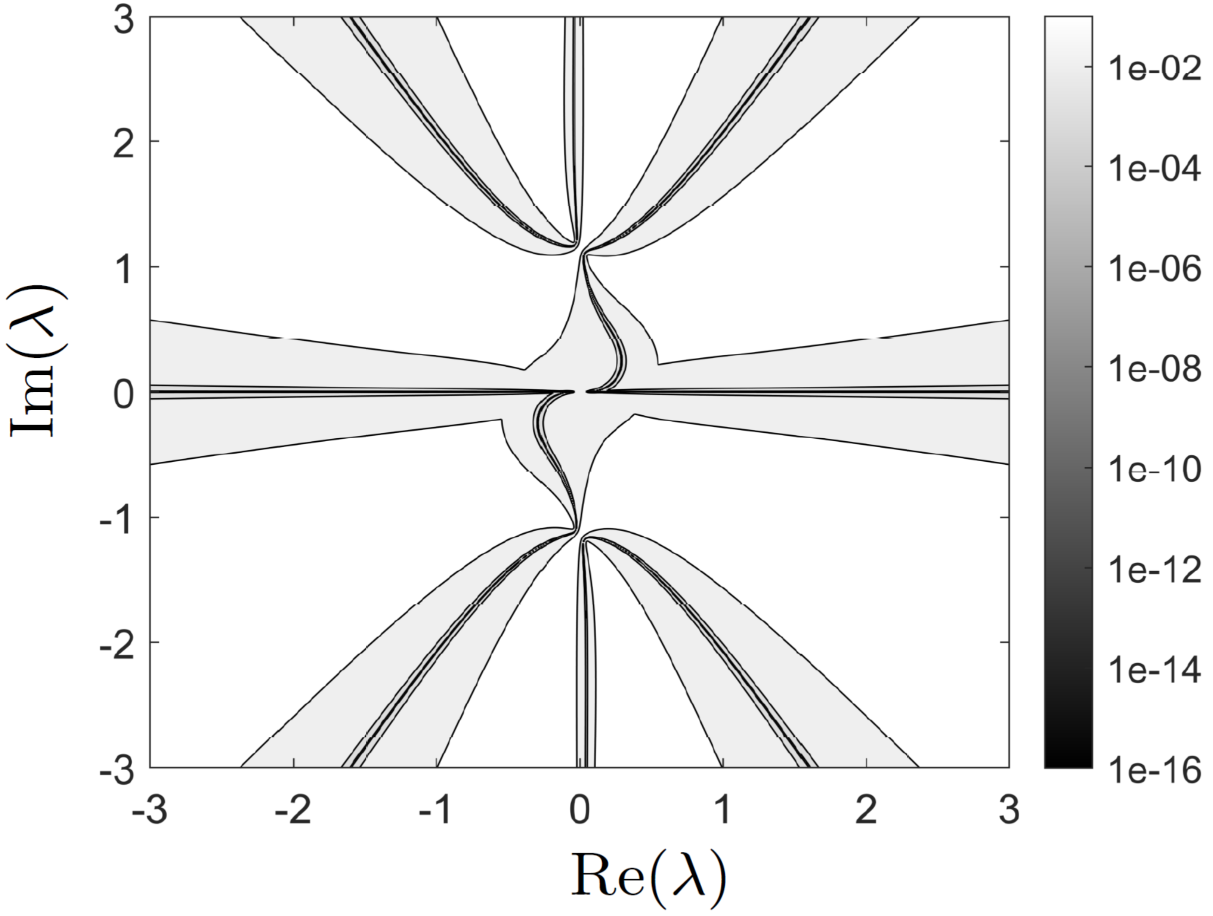
Discretized $\mathcal{P}_n T(\lambda) \mathcal{P}_n^* \ (n = 500)$



Verified method $\subset \mathrm{Sp}_\varepsilon(T)$



Computational tool: Contour methods

Technical point: $\mathrm{Sp_d}(T) = \{\lambda \in \mathrm{Sp}(T) : \lambda \text{ isolated}, T(\lambda) \text{ Fredholm}\}$, $\mathrm{Sp_{ess}}(T) = \mathrm{Sp}(T)\backslash\mathrm{Sp_d}(T)$

KELDYSH's THEOREM: Suppose $\mathrm{Sp_{ess}}(T) \cap \Omega = \emptyset$. Then for $z \in \Omega\backslash\mathrm{Sp}(T)$

assume finite

$$T(z)^{-1} = V(z-J)^{-1}W^* + R(z)$$

- m: sum of all algebraic multiplicities of eigenvalues inside Ω.
- V & W: quasimatrices with m cols of right & left gen. eigenvectors.
- J: Jordan blocks.
- $R(z)$: bounded holomorphic remainder.

$\Longrightarrow$ use contour integration to convert to a linear pencil...

- Keldysh, *"On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations,"* **Dokl. Akad. Nauk**, 1951.
- Keldysh, *"On the completeness of the eigenfunctions of some classes of non-self-adjoint linear operators,"* **UMN**, 1971.

Why is this so useful?

(1) Nonlinear $\longrightarrow$ linear.

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Why is this so useful?

(1) Nonlinear $\longrightarrow$ linear.

(2) Infinite-dimensional $\longrightarrow$ finite-dimensional.

(3) Computing spectra $\longrightarrow$ solving linear systems.

CANNOT OVERSTATE: Much easier for discretization to converge for solving linear systems than computing spectra!
This holds in theory and in practice.

InfBeyn algorithm

$\Gamma \subset \Omega$ contour enclosing m eigenvalues (not touching $\mathrm{Sp}(T)$).

$$A_0 = \frac{1}{2\pi i}\int_\Gamma T(z)^{-1}\mathcal{V}\,\mathrm{d}z, \qquad A_1 = \frac{1}{2\pi i}\int_\Gamma zT(z)^{-1}\mathcal{V}\,\mathrm{d}z$$

Random vectors drawn form a Gaussian process

Computed with adaptive discretization sizes (e.g., ultraspherical spectral method)

Approximate via quadrature: $\tilde{A}_0, \tilde{A}_1$.

Truncated SVD: $\tilde{A}_0 \approx \tilde{\mathcal{U}}\Sigma_0\tilde{V}_0^*$.

Eigenpairs (λ_j, x_j)
The eigenvectors of original problem are $\approx \mathcal{U}\Sigma_0 x_j$

Linear pencil: $\tilde{F}(z) = \tilde{\mathcal{U}}^*\tilde{A}_1\tilde{V}_0 - z\tilde{\mathcal{U}}^*\tilde{A}_0\tilde{V}_0 \in \mathbb{C}^{m\times m}$.

NB: $m = \mathrm{Trace}\left(\frac{1}{2\pi i}\int_\Gamma T'(z)T(z)^{-1}\,\mathrm{d}z\right)$ can compute this (another story).

- Beyn, *"An integral method for solving nonlinear eigenvalue problems,"* **Linear Algebra Appl.**, 2012.
- C., Townsend, *"Avoiding discretization issues for nonlinear eigenvalue problem"*, preprint.

Stability and convergence result

Keldysh: $T(z)^{-1} = V(z-J)^{-1}W^* + R(z)$, let $M = \sup_{z\in\Omega}\|R(z)\|$.

Suppose that $\|\tilde{A}_j - A_j\| \leq \varepsilon$.

THEOREM: For sufficiently oversampled $\mathcal{V}$, with overwhelming probability,

$$\left|\,\|F(z)^{-1}\|^{-1} - \left\|\tilde{F}(z)^{-1}\right\|^{-1}\right| \leq 2(\varepsilon + \|VJW^*\|\varepsilon/\sigma_m(VW^*) + |z|\varepsilon) \quad \text{(quad. err.)}$$

$$\mathrm{Sp}_{\frac{\varepsilon}{\|VW^*\|\|VW^*\mathcal{V}\|+M\varepsilon}}(T) \subset \mathrm{Sp}_\varepsilon(F) \subset \mathrm{Sp}_{\frac{\varepsilon}{\sigma_m(VW^*)\sigma_m(VW^*\mathcal{V})-M\varepsilon}}(T).$$

$\Longrightarrow$ **converges**
no spectral pollution
no spectral invisibility
method is stable

- C., Townsend, *"Avoiding discretization issues for nonlinear eigenvalue problems"*, preprint. ← Stability bound
- Horning, Townsend, *"FEAST for differential eigenvalue problems,"* **SIAM J. Math. Anal.**, 2020. ← How to control quad error
- C., *"Computing semigroups with error control,"* **SIAM J. Math. Anal.**, 2022. ← How to control quad error

Proof sketch (probably no time)

Keldysh: $T(z)^{-1} = V(z-J)^{-1}W^* + R(z)$, let $M = \sup_{z\in\Omega}\|R(z)\|$.
Introduce: $L_1 = (VW^*)^\dagger, \;\; L_2 = (VW^*\mathcal{V}V_0)^\dagger$.

$$T(z)^{-1}L_1F(z) = -VW^*\mathcal{V}V_0 + R(z)L_1F(z)$$

$$\sigma_{\inf}\big(F(z)\big) < \varepsilon \Longrightarrow \|T(z)^{-1}\| > \frac{\sigma_m(VW^*)\sigma_m(VW^*\mathcal{V})}{\varepsilon} - M$$

$$F(z)L_2[T(z)^{-1} - R(z)] = -VW^*$$

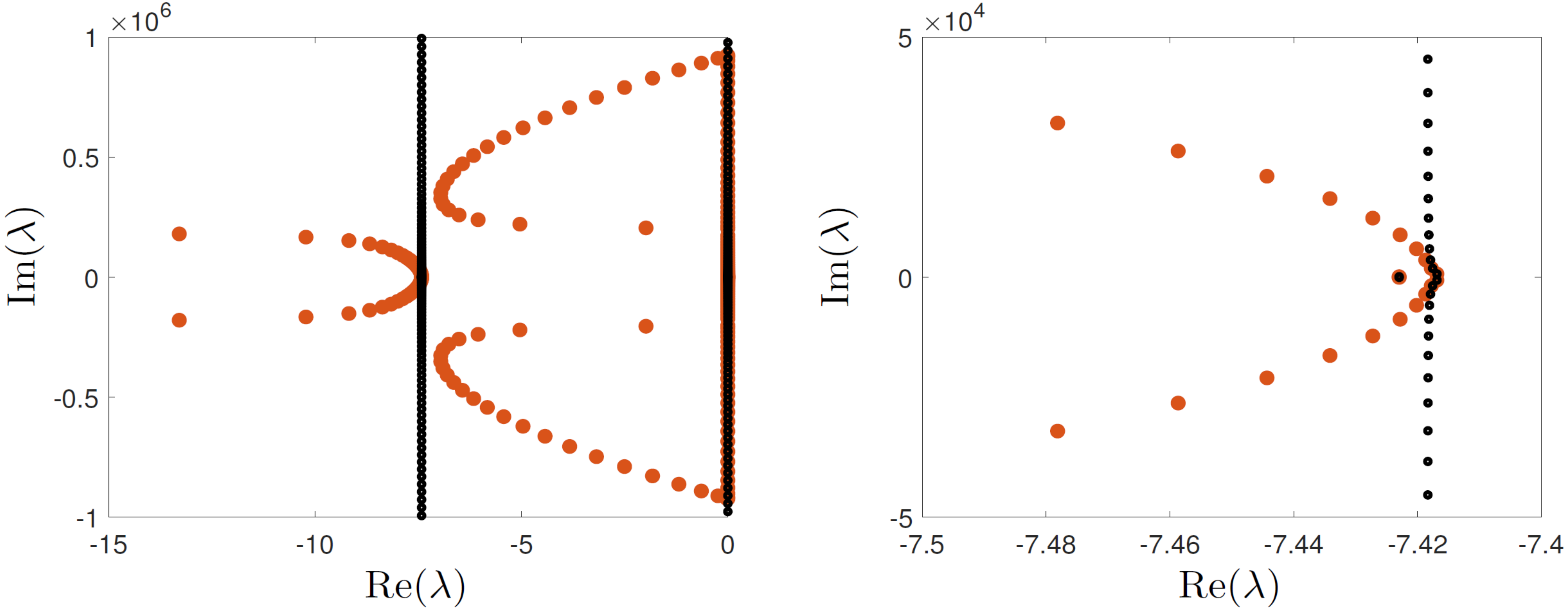
$$\|T(z)^{-1}\| > \varepsilon \Longrightarrow \sigma_{\inf}\big(F(z)\big) < \frac{\|VW^*\|\|VW^*\mathcal{V}\|}{1-M\varepsilon}\varepsilon$$

Use results from inf dim randomized NLA to bound terms with a $\mathcal{V}$.

- C., Townsend, *"Avoiding discretization issues for nonlinear eigenvalue problems"*, preprint.

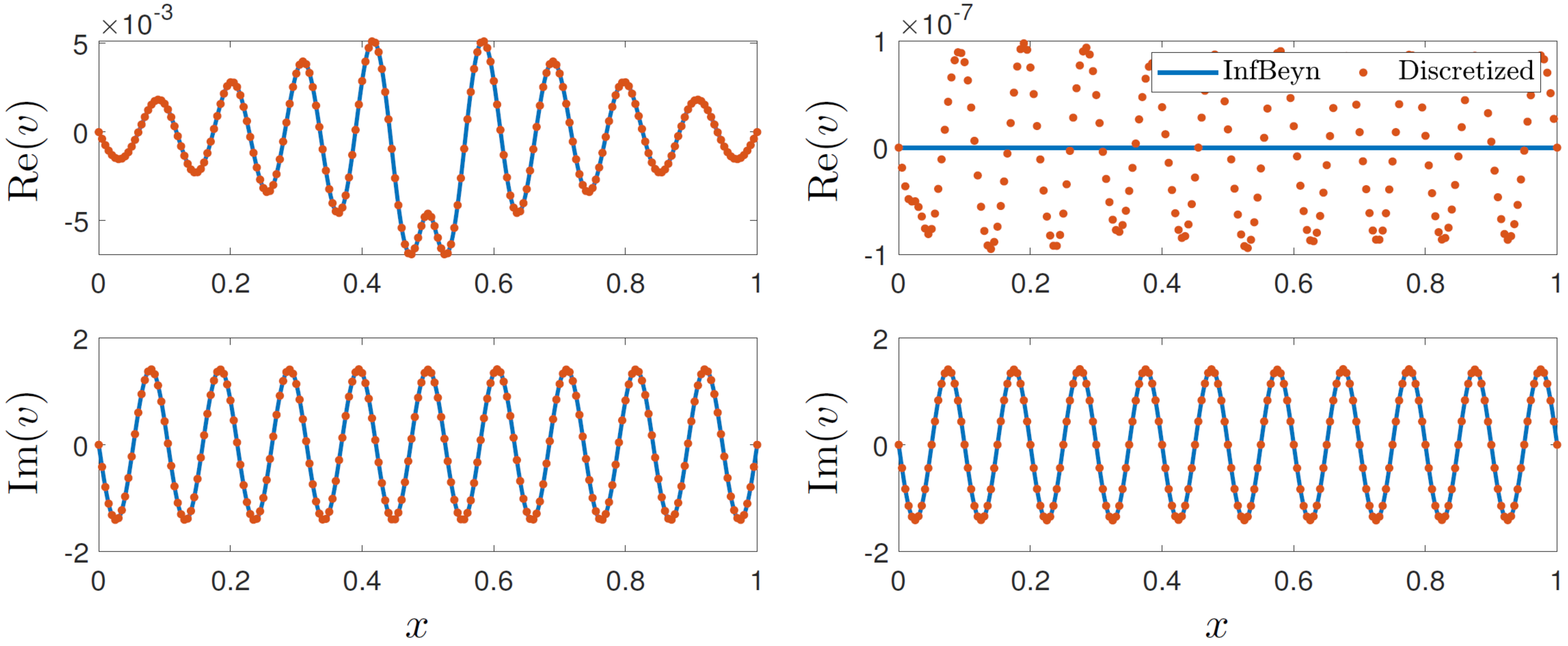
Example: E-functions resolved before e-values

$$\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} - \alpha\lambda^2 v = \beta\lambda\delta(x - 1/2)v, \qquad v(0) = v''(0) = v(1) = v''(1) = 0.$$



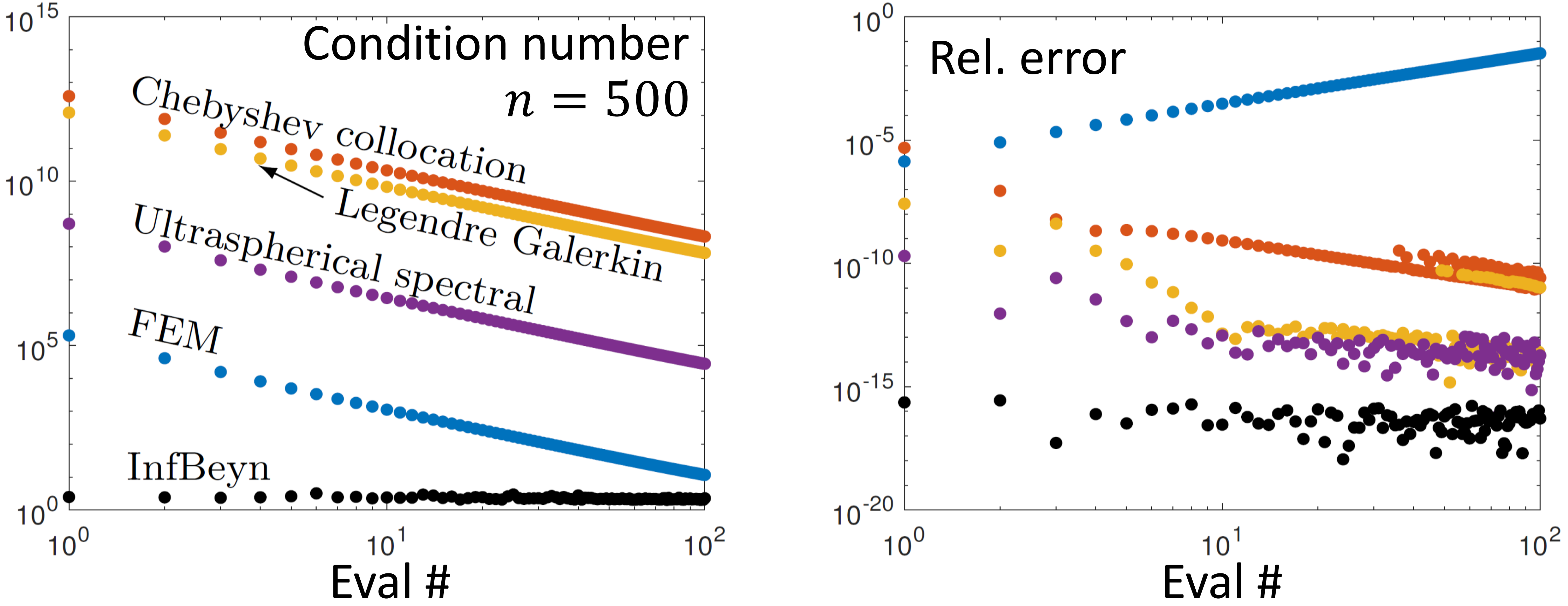
Example: E-functions resolved before e-values

e-vector subspace error ≈ 0.001, e-val error ≈ 40 (InfBeyn error < 10^{-12})



Example: Instability

$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \lambda u, \qquad u(0) = 0, \qquad u'(1) + \frac{\lambda}{\lambda - 1}u(1) = 0.$$



Example: Ghost essential spectra

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + k^2\left(\eta^2 - \mu(\lambda)\right)\phi = 0$$

$$\mu(\lambda) = \frac{\delta_+}{k^2} + \frac{\delta_-}{8k^2\lambda^2} + \frac{\lambda^2}{k^2}$$

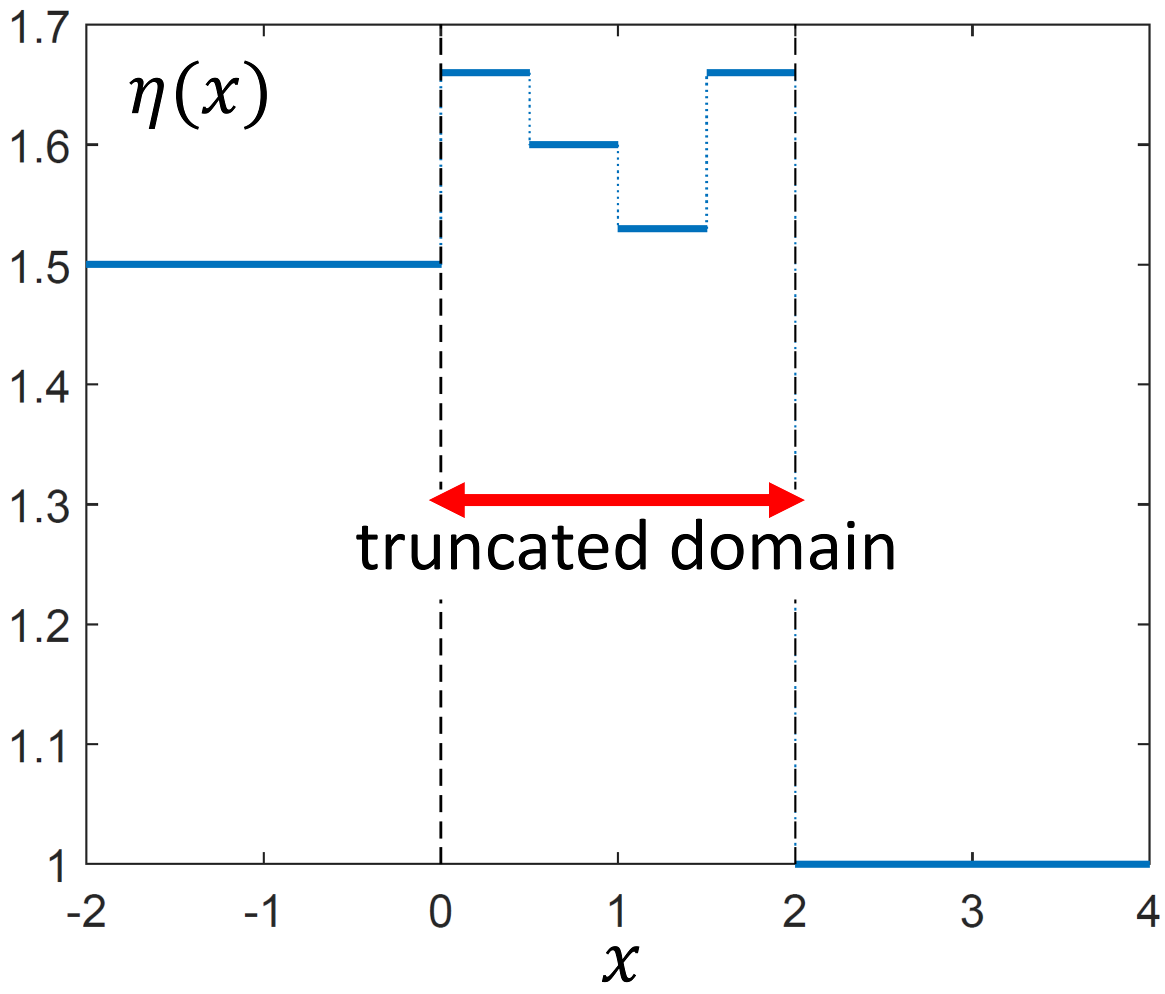
$$\frac{\mathrm{d}\phi}{\mathrm{d}x}(0) + \left(\frac{\delta_-}{2\lambda} - \lambda\right)\phi(0) = 0$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x}(2) + \left(\frac{\delta_-}{2\lambda} + \lambda\right)\phi(2) = 0$$

η corresponds to refractive index.

λ correspond to guided and leaky modes.

Discretized using FEM ($n = 129$, default)



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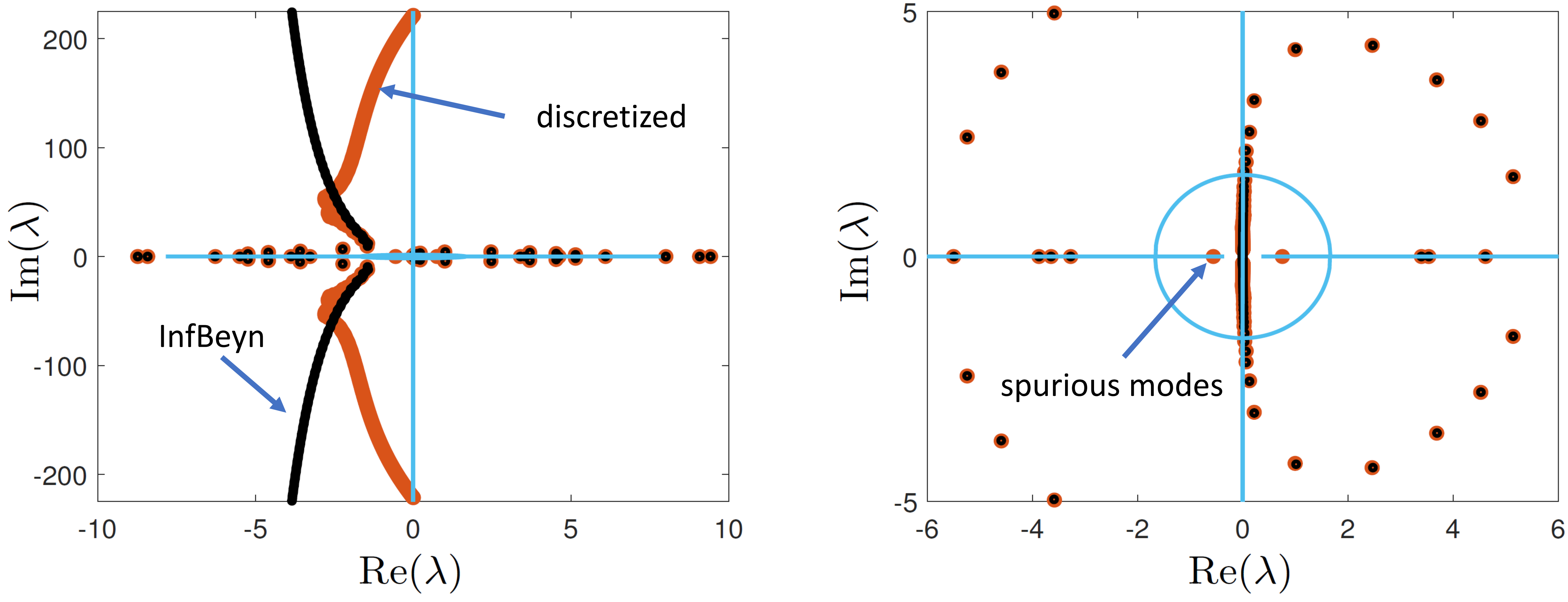

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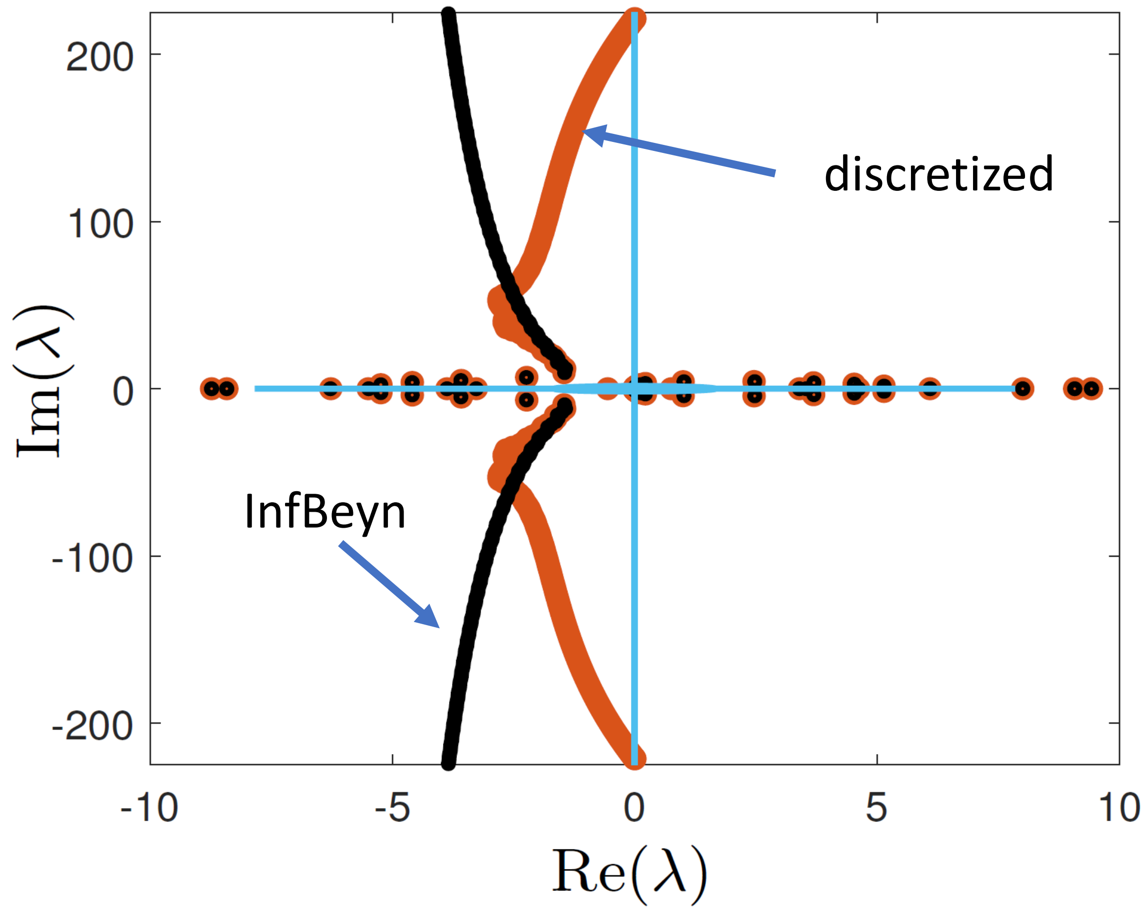

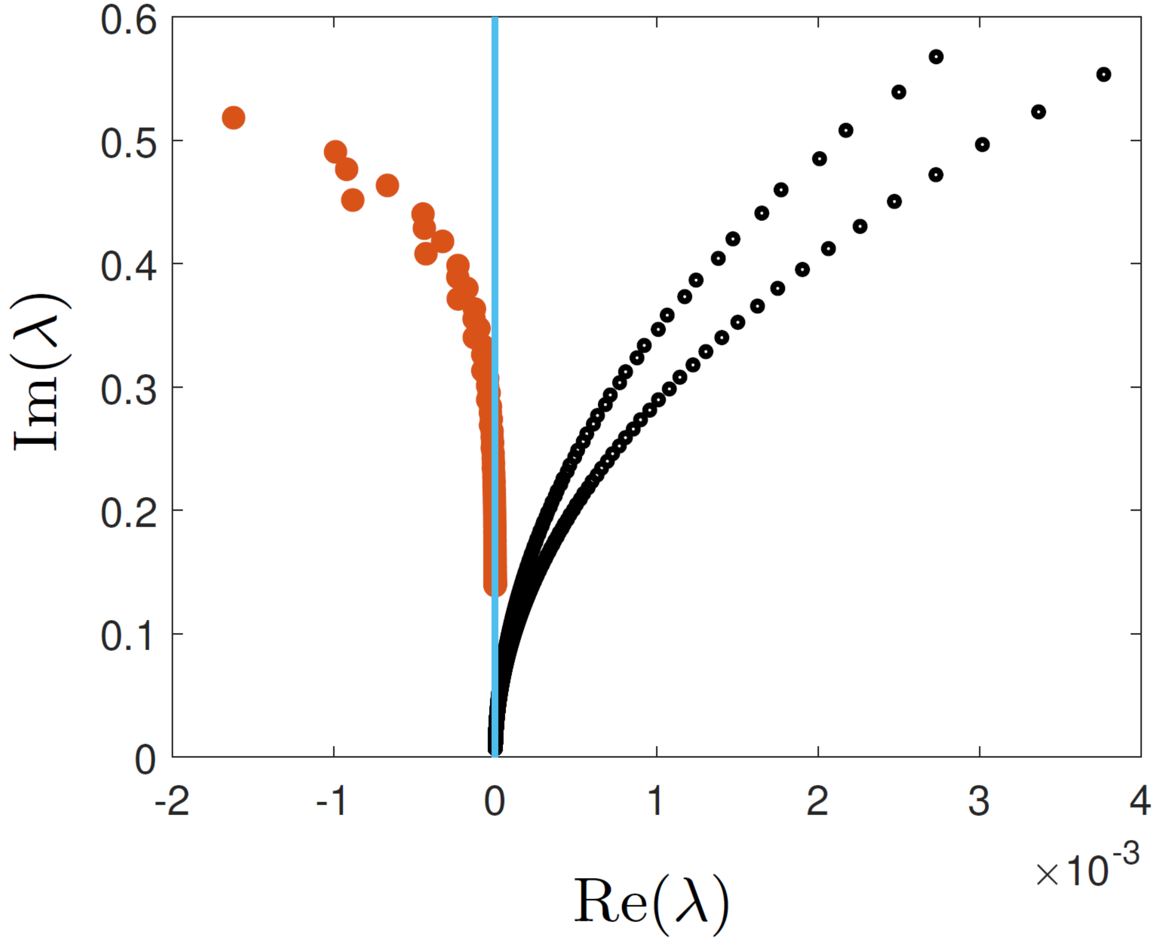

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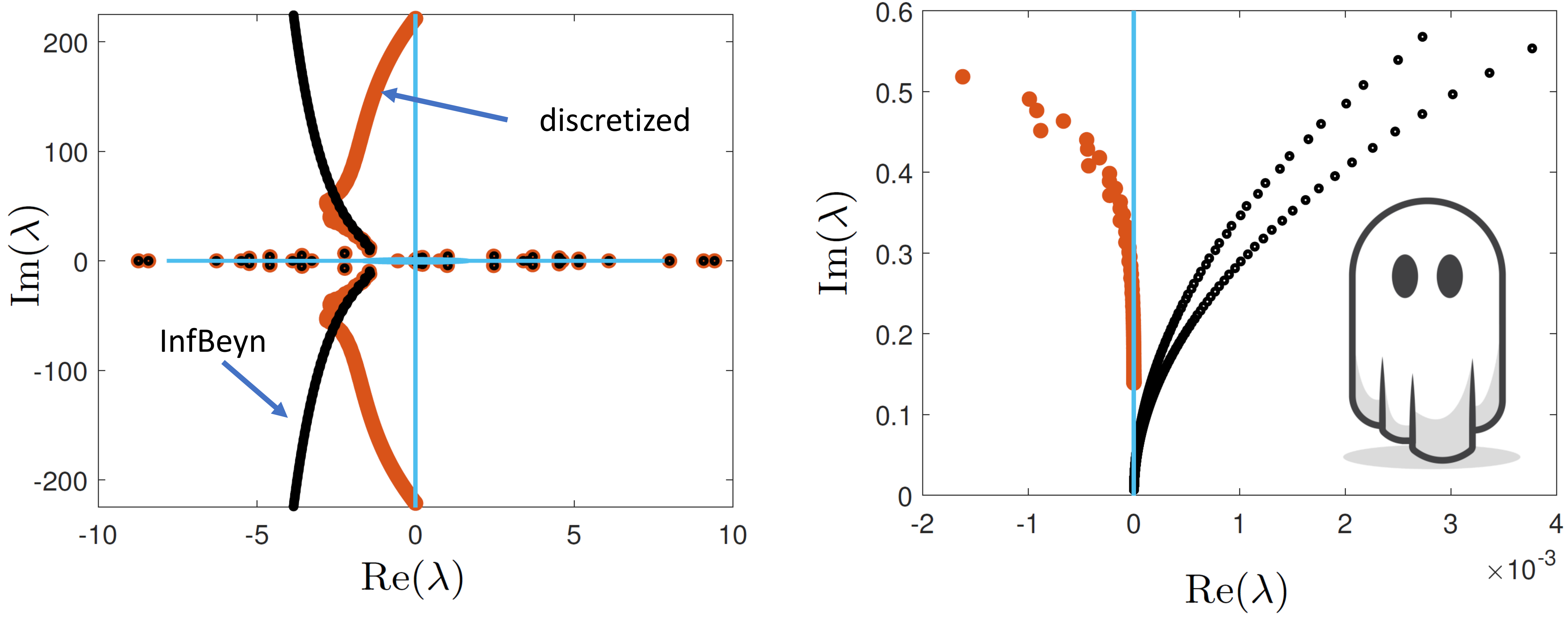

Conclusion

- Discretization can cause serious issues.
- **InfBeyn** overcomes these in regions of discrete spectra: **convergent, stable, efficient**.
- Compute pseudospectra with explicit **error control** (generic pencils, even with essential spectra!)

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Code: **https://github.com/MColbrook/infNEP**

- C., Townsend, *"Avoiding discretization issues for nonlinear eigenvalue problems"*, preprint.

A final shameless plug!

New methods for spectra on surfaces!

Gustav Conradie

Dan Fortunato

A final shameless plug!

Everybody else's fav dinosaur

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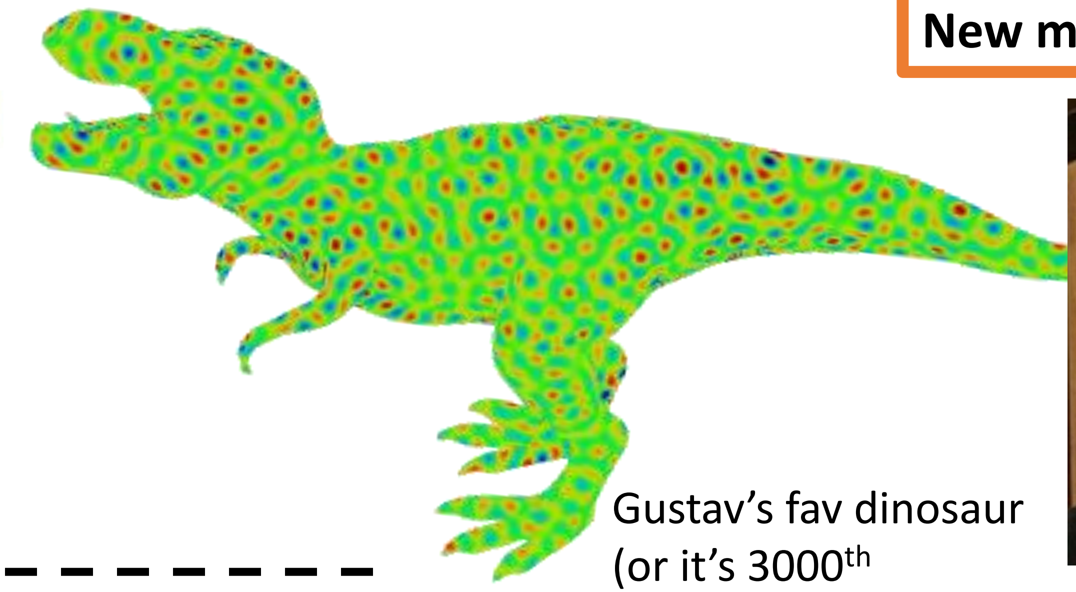

Gustav's fav dinosaur (or it's 3000th Laplace–Beltrami efun)

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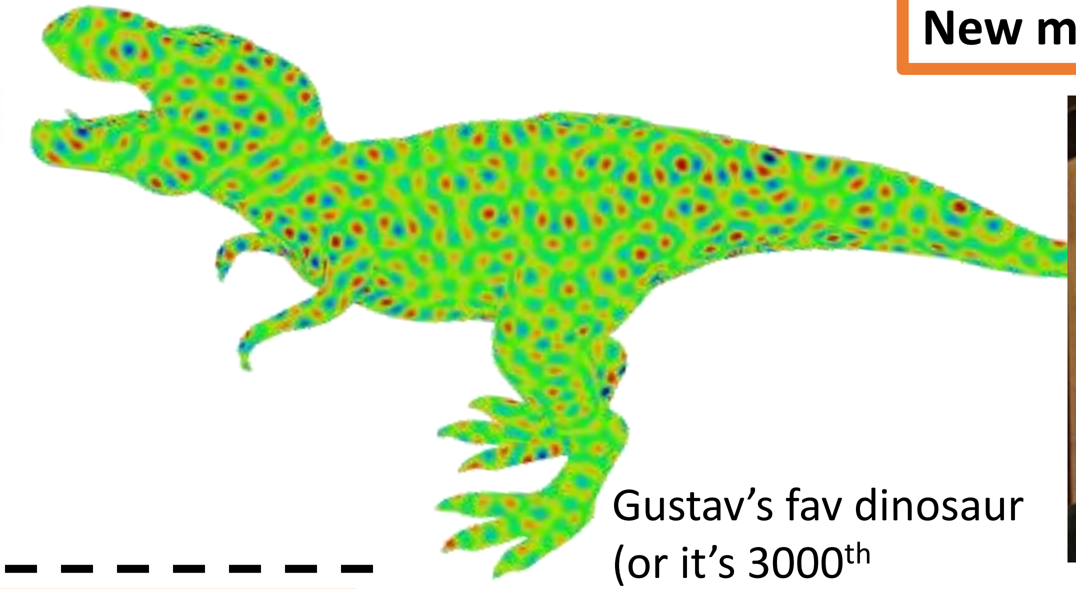
A final shameless plug!

Everybody else's fav dinosaur



Gustav's fav dinosaur (or it's 3000th Laplace–Beltrami efun)

New methods for spectra on surfaces!

Gustav Conradie

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New methods for generalized eigenfunctions!

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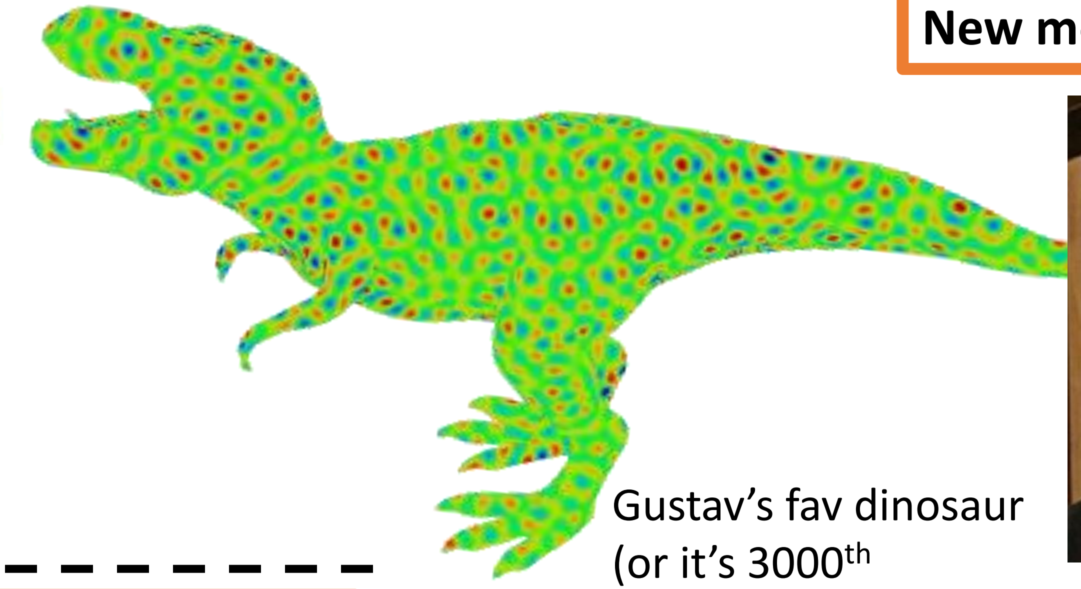

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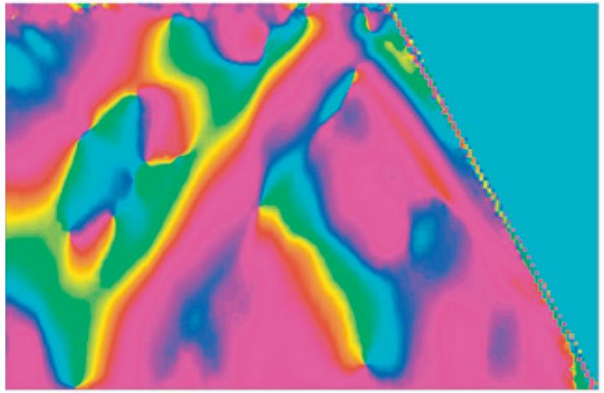
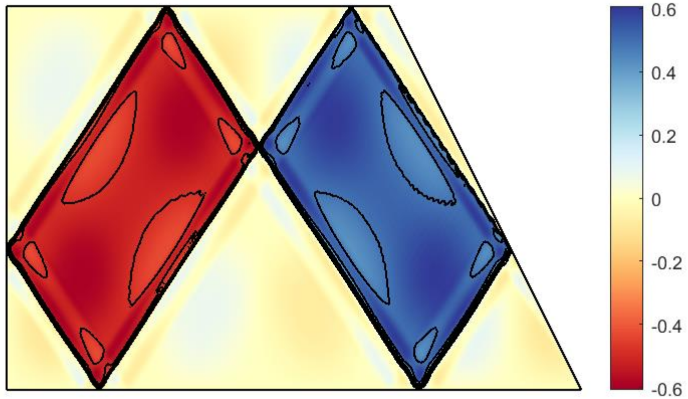


Internal wave experiment from Hazewinkel et al., 2010



Computed generalized eigenfunction (note the sharp discontinuity)

References

[1] Colbrook, Matthew J., and Alex Townsend. "Avoiding discretization issues for nonlinear eigenvalue problems." arXiv preprint arXiv:2305.01691 (2023).

[2] Colbrook, Matthew J. "Computing semigroups with error control." SIAM Journal on Numerical Analysis 60.1 (2022): 396-422.

[3] Colbrook, Matthew J., and Lorna J. Ayton. "A contour method for time-fractional PDEs and an application to fractional viscoelastic beam equations." Journal of Computational Physics 454 (2022): 110995.

[4] Colbrook, Matthew. The foundations of infinite-dimensional spectral computations. Diss. University of Cambridge, 2020.

[5] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). Computing Spectra--On the Solvability Complexity Index Hierarchy and Towers of Algorithms. arXiv preprint arXiv:1508.03280.

[6] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem." Proceedings of the National Academy of Sciences 119.12 (2022): e2107151119.

[7] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." SIAM review 63.3 (2021): 489-524.

[8] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.

[9] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society (2022).

[10] Colbrook, Matthew J. "Computing spectral measures and spectral types." Communications in Mathematical Physics 384 (2021): 433-501.

[11] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." Numerische Mathematik 143 (2019): 17-83.

[12] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." Foundations of Computational Mathematics (2022): 1-82.

[13] Colbrook, Matthew. "Pseudoergodic operators and periodic boundary conditions." Mathematics of Computation 89.322 (2020): 737-766.

[14] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." arXiv preprint arXiv:2111.14889 (2021).

[15] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[16] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." SIAM Journal on Numerical Analysis 61.3 (2023): 1585-1608.

[17] Johnstone, Dean, et al. "Bulk localized transport states in infinite and finite quasicrystals via magnetic aperiodicity." Physical Review B 106.4 (2022): 045149.

[18] Colbrook, Matthew J., et al. "Computing spectral properties of topological insulators without artificial truncation or supercell approximation." IMA Journal of Applied Mathematics 88.1 (2023): 1-42.

[19] Colbrook, Matthew J., and Andrew Horning. "Specsolve: spectral methods for spectral measures." arXiv preprint arXiv:2201.01314 (2022).

[20] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Resolvent-based techniques for computing the discrete and continuous spectrum of differential operators." XXI Householder Symposium on Numerical Linear Algebra. 2020.

[21] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective."