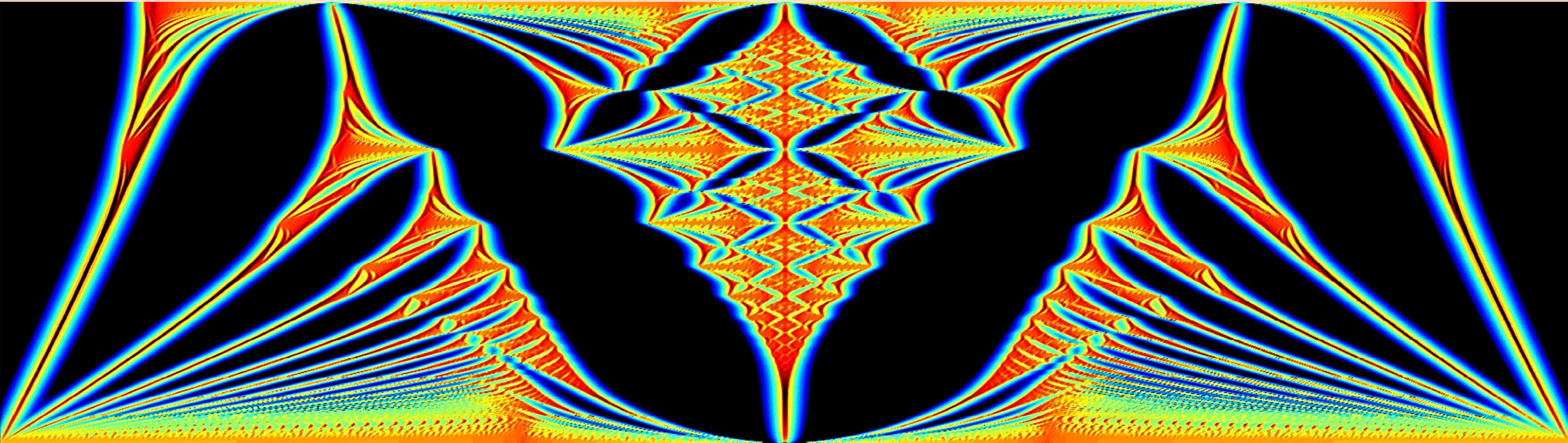
10th Popov Prize: The Solvability Complexity Index Hierarchy

Matthew J. Colbrook, University of Cambridge

FOCM Paris

20/06/2023

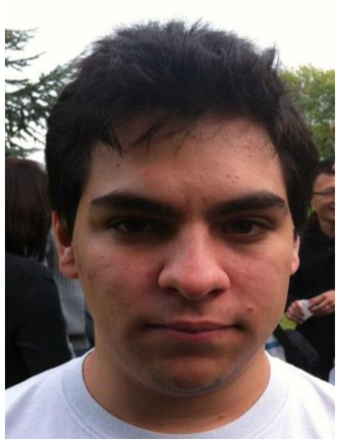


Special thanks to my collaborators!

Ben Adcock
Simon Fraser

Vegard Antun
Oslo

Lorna Ayton
Cambridge



Alex Bastounis
Leicester

Jonathan Ben-Artzi
Cardiff

Luca Gazdag
Oslo

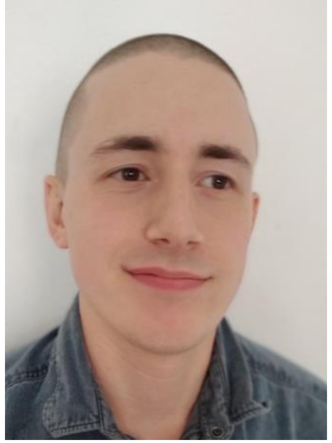
Anders Hansen
Cambridge

Andrew Horning
MIT
On job market soon!

Olavi Nevanlinna
Aalto

Max Neyra-Nesterenko
Simon Fraser

Markus Seidel
West Saxon



Alexei Stepanenko
Cambridge

Máté Szőke
Virginia Tech

Alex Townsend
Cornell

Outline

- Solvability Complexity Index Hierarchy and spectral problems.
- Example: Spectra with error control.
- Example: Smale’s 18th problem on limits of AI.
- Concluding remarks

Broad goal: classify difficulty of problems, prove optimality of algorithms, figure out what can and cannot be done computationally.

Classical infinite-dimensional spectral problem

$$A \text{"="} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad A\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk} x_k\right) e_j$$

Canonical basis vectors of $l^2(\mathbb{N})$

Also deal with PDEs, integral operators etc.

Finite-dimensional		Infinite-dimensional
Eigenvalues of $B \in \mathbb{C}^{n\times n}$	$\Longrightarrow$	Spectrum, $\mathrm{Sp}(A)$
$\{\lambda_j \in \mathbb{C}: \det(B - \lambda_j I) = 0\}$	$\Longrightarrow$	$\{\lambda \in \mathbb{C}: A - \lambda I \text{ is not invertible}\}$

Classical infinite-dimensional spectral problem

$$A \text{"="} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad A\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk} x_k\right) e_j$$

Canonical basis vectors of $l^2(\mathbb{N})$

Also deal with PDEs, integral operators etc.

Finite-dimensional	$\Longrightarrow$	**Infinite-dimensional**
Eigenvalues of $B \in \mathbb{C}^{n\times n}$	$\Longrightarrow$	Spectrum, $\mathrm{Sp}(A)$
$\{\lambda_j \in \mathbb{C}: \det(B - \lambda_j I) = 0\}$	$\Longrightarrow$	$\{\lambda \in \mathbb{C}: A - \lambda I \text{ is not invertible}\}$

*"Most operators that arise in practice are not presented in a representation in which they are diagonalized, and it is often very hard to locate even a single point in the spectrum. Thus, one often has to settle for numerical approximations. Unfortunately, there is a dearth of literature on this basic problem and, so far as we have been able to tell, **there are no proven [general] techniques**."* W. Arveson, Berkeley (1994)

What can go wrong?

Typical approach:

- **Matrix case ($l^2(\mathbb{N})$):** truncate to $\mathcal{P}_n A \mathcal{P}_n^* \in \mathbb{C}^{n\times n}$.
- **PDE on unbounded domain:** truncate domain then discretise.

two sources of error

Some key issues:

- Spectral pollution (evals accumulate at points not in $\mathrm{Sp}(A)$ as $n \to \infty$)
- Spectral invisibility.
- Dealing with essential spectra and continuous spectra.
- Stability, non-normality etc.
- Verification – can we compute spectral properties with error bounds?

Motivation

- **Applications:** Quantum mechanics, structural mechanics, optics, acoustics, statistical physics, number theory, matter physics, PDEs, data analysis, neural networks and AI, nuclear scattering, optics, computational chemistry, …
- **Specific open problems**, e.g., computational quantum mechanics (Schwinger 1960), (Digernes, Varadarajan, Varadhan, 1994):

> Given a self-adjoint Schrödinger operator $-\Delta + V$ on $\mathbb{R}$,
> can we approximate its spectrum from sampling V?

- **Verified computations:** Many **computer-assisted proofs** involve spectra. E.g., $E(Z) =$ ground state energy of $H = \sum_{k=1}^{N}\left(-\Delta_{x_k} - Z|x_k|^{-1}\right) + \sum_{j\leq k}\left|x_j - x_k\right|^{-1}$.

  Dirac-Schwinger conjecture: asymptotics of $E(Z)$ (Fefferman, Seco 1996)
- **Foundations:** What is computationally possible? Beyond spectra etc.

Not all spectral problems are equally hard …

Warm-up: bounded diagonal operators

$$A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & \ddots \end{pmatrix}$$

Assumption: Algorithm can query entries of A

Algorithm: $\Gamma_n(A) = \{a_1, a_2, \dots, a_n\} \to \mathrm{Sp}(A) = \overline{\{a_1, a_2, \dots\}}$ in Haus. Metric.

One-sided error control: $\Gamma_n(A) \subset \mathrm{Sp}(A)$

$$d_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\}$$

Optimal: Can't obtain $\hat{\Gamma}_n(A) \to \mathrm{Sp}(A)$ with $\mathrm{Sp}(A) \subset \hat{\Gamma}_n(A)$.

Warm-up: compact self-adjoint operators

classic method
"finite section"

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Algorithm: $\Gamma_n(A) = \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ converges to $\mathrm{Sp}(A)$ in Haus. Metric.

Question: Can we verify the output?

i.e., Does there exist some alg. $\hat{\Gamma}_n(A) \to \mathrm{Sp}(A)$ with $\hat{\Gamma}_n(A) \subset \mathrm{Sp}(A) + B_{2^{-n}}$?

Warm-up: compact self-adjoint operators

classic method
"finite section"

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Algorithm: $\Gamma_n(A) = \mathrm{Sp}(\mathcal{P}_n A \mathcal{P}_n^*)$ converges to $\mathrm{Sp}(A)$ in Haus. Metric.

Question: Can we verify the output?

i.e., Does there exist some alg. $\hat{\Gamma}_n(A) \to \mathrm{Sp}(A)$ with $\hat{\Gamma}_n(A) \subset \mathrm{Sp}(A) + B_{2^{-n}}$?

Answer: No algorithm can do this on whole class!

What about Jacobi operators?

$$A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & b_2 & a_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad b_k > 0, \qquad a_k \in \mathbb{R}$$

Non-trivial, e.g., spurious eigenvalues.

What about Jacobi operators?

$$A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & b_2 & a_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad b_k > 0, \qquad a_k \in \mathbb{R}$$

Non-trivial, e.g., spurious eigenvalues.

Sparse: finitely many non-zeros in each column

Enlarge class to **sparse normal operators** - surely now much harder?!

What about Jacobi operators?

$$A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & b_2 & a_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \qquad b_k > 0, \qquad a_k \in \mathbb{R}$$

Non-trivial, e.g., spurious eigenvalues.

Sparse: finitely many non-zeros in each column

Enlarge class to **sparse normal operators** - surely now much harder?!

Answer: $\exists \{\Gamma_n\}$ s.t. $\lim_{n\to\infty} \Gamma_n(A) = \mathrm{Sp}(A)$ and $\Gamma_n(A) \subset \mathrm{Sp}(A) + B_{2^{-n}}$,

for any sparse normal operator A

A curious case of limits

General bounded: $$A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Algorithm: $\exists\{\Gamma_{n_3,n_2,n_1}\}$ s.t. $\lim_{n_3\to\infty}\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_3,n_2,n_1}(A) = \mathrm{Sp}(A)$

Question: Can we do better?

- Hansen, *"On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators,"* **J. Amer. Math. Soc.**, 2011.

A curious case of limits

General bounded: $A=\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

Algorithm: $\exists\{\Gamma_{n_3,n_2,n_1}\}$ s.t. $\lim_{n_3\to\infty}\lim_{n_2\to\infty}\lim_{n_1\to\infty}\Gamma_{n_3,n_2,n_1}(A)=\mathrm{Sp}(A)$

Question: Can we do better?

Answer: No! Canonically embed problems such as:

Explains Arveson's lament!

Given $B\in\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, does B have a column with infinitely many 1's?

$\Longrightarrow$ lower bound on number of "successive limits" needed (indep. of comp. model).

- Hansen, *"On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators,"* **J. Amer. Math. Soc.**, 2011.
- C., *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* **Found. Comput. Math.**, 2022.

General algorithm: beyond recursion theory

Computational problem:

- Class of objects Ω (e.g., operators).
- Metric space $(\mathcal{M}, d)$ (e.g., Hausdorff metric).
- Thing we want to compute $\Xi: \Omega \to \mathcal{M}$.
- Info we can access, Λ a set of functions $\Omega \to \mathbb{C}$ (e.g., matrix entries).

General algorithm: map $\Gamma: \Omega \to \mathcal{M}$ such that for any $A \in \Omega$, $\exists$ a finite non-empty subset $\Lambda_\Gamma(A) \subseteq \Lambda$ such that

$$B \in \Omega, f(B) = f(A) \ \forall f \in \Lambda_\Gamma(A) \Rightarrow \Lambda_\Gamma(A) = \Lambda_\Gamma(B), \Gamma(A) = \Gamma(B)$$

A lower bound for general algorithms holds in **ALL** models of computation.

Solvability Complexity Index Hierarchy

- Δ_0: Solved in finite time (v. rare for cts problems).
- Δ_1: Solved in "one limit" with full error control:

$$d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}$$

- Δ_2: Solved in "one limit":

$$\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$$

- Δ_3: Solved in "two successive limits":

$$\lim_{n\to\infty} \lim_{m\to\infty} \Gamma_{n,m}(A) = \Xi(A)$$

⋮

Can work in *any* model. E.g., BSS machine, Turing machine, interval arithmetic, inexact input etc.

- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "*On the solvability complexity index hierarchy and towers of algorithms,*" preprint.
- Hansen, "*On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators,*" **J. Amer. Math. Soc.**, 2011.

Solvability Complexity Index Hierarchy

- Δ_0: Solved in finite time (v. rare for cts problems).
- Δ_1: Solved in "one limit" with full error control:

$$d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}$$

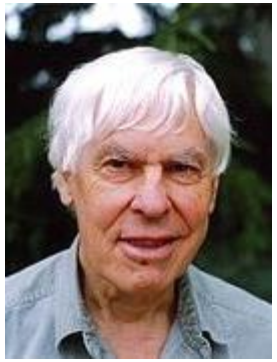
- Δ_2: Solved in "one limit":

$$\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$$

- Δ_3: Solved in "two successive limits":

$$\lim_{n\to\infty} \lim_{m\to\infty} \Gamma_{n,m}(A) = \Xi(A)$$

⋮



Steve Smale: *"Is there any purely [rational] iterative generally convergent algorithm for polynomial zero finding?"*

Curt McMullen: *"Yes, if the degree is three; no, if the degree is higher."*

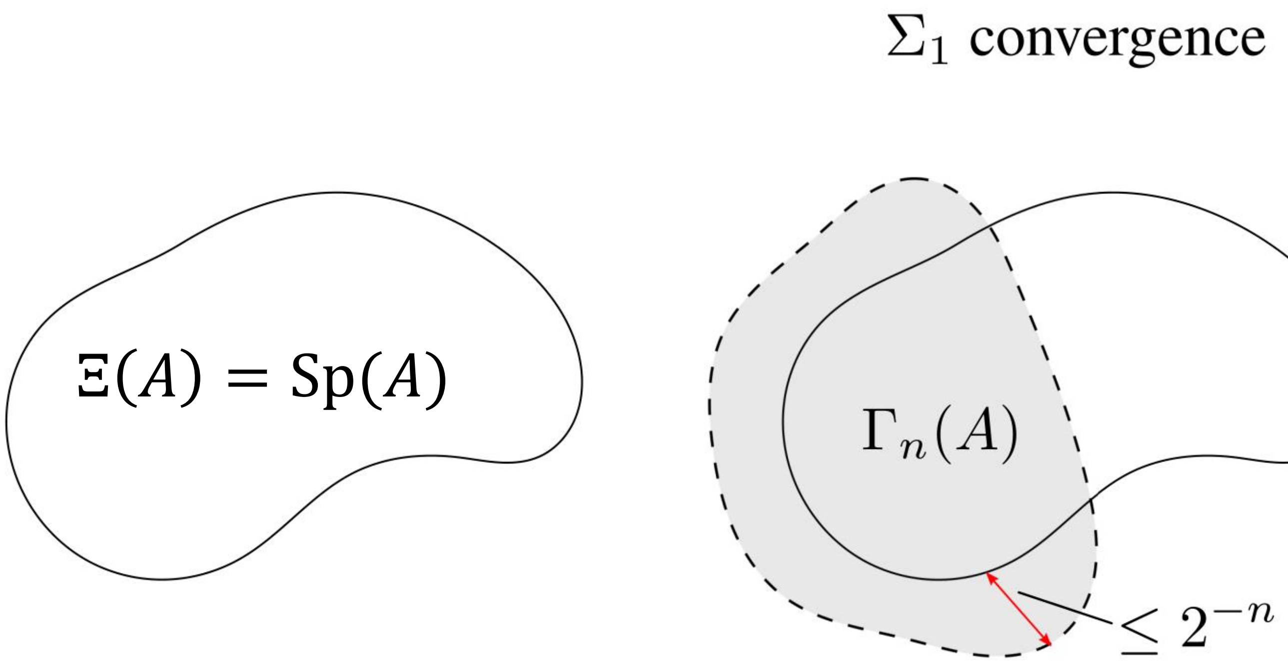
Peter Doyle & Curt McMullen: *"The problem can be solved using successive limits for the quartic and quintic, but not the sextic."*

Can work in *any* model. E.g., BSS machine, Turing machine, interval arithmetic, inexact input etc.

- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, *"On the solvability complexity index hierarchy and towers of algorithms,"* preprint.
- Hansen, *"On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators,"* **J. Amer. Math. Soc.**, 2011.
- McMullen, *"Families of rational maps and iterative root-finding algorithms,"* **Ann. of Math.**, 1987.
- Doyle, McMullen, *"Solving the quintic by iteration,"* **Acta Math.**, 1989.
- Smale, *"The fundamental theorem of algebra and complexity theory,"* **Bull. Amer. Math. Soc.**, 1981.

Error control for spectral problems

$$d_{\mathrm{H}}(X,Y)=\max\left\{\sup_{x\in X} d(x,Y),\sup_{y\in Y} d(y,X)\right\}$$

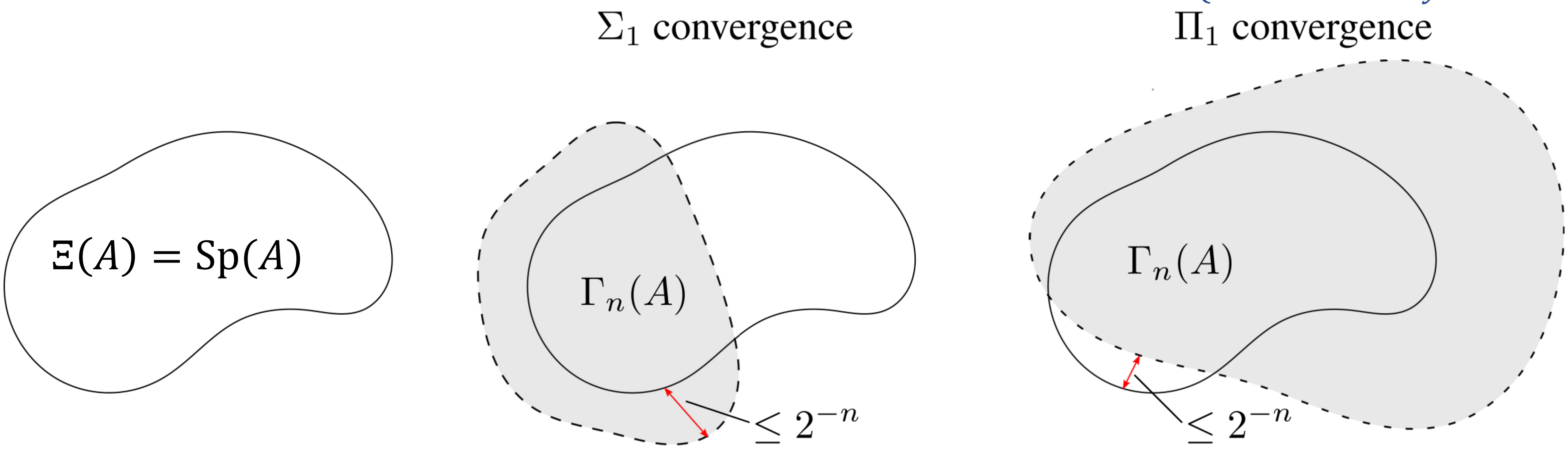


- Σ_1: $\exists$ alg. $\{\Gamma_n\}$ s.t. $\lim_{n\to\infty}\Gamma_n(A)=\Xi(A)$, $\max_{z\in\Gamma_n(A)}\mathrm{dist}\big(z,\Xi(A)\big)\leq 2^{-n}$

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss., University of Cambridge**, 2020.

Error control for spectral problems

$$d_{\mathrm{H}}(X,Y) = \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\}$$



- Σ_1: $\exists$ alg. $\{\Gamma_n\}$ s.t. $\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$, $\max_{z\in\Gamma_n(A)} \mathrm{dist}\big(z, \Xi(A)\big) \leq 2^{-n}$
- Π_1: $\exists$ alg. $\{\Gamma_n\}$ s.t. $\lim_{n\to\infty} \Gamma_n(A) = \Xi(A)$, $\max_{z\in\Xi(A)} \mathrm{dist}(z, \Gamma_n(A)) \leq 2^{-n}$

Such problems can be used in a proof!

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss., University of Cambridge**, 2020.

Sampler of results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

$$\Pi_0 = \Delta_0 = \Sigma_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$$\Delta_1 \subsetneq \Pi_1 \subsetneq \Delta_2, \quad \Delta_1 \subsetneq \Sigma_1 \subsetneq \Delta_2, \quad \Delta_2 \subsetneq \Pi_2 \subsetneq \Delta_3, \quad \Delta_2 \subsetneq \Sigma_2 \subsetneq \Delta_3, \quad \Delta_3 \subsetneq \Pi_3 \cdots, \quad \Delta_3 \subsetneq \Sigma_3 \cdots$$

Sampler of results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

Compact operators

$$\Pi_0 = \Delta_0 = \Sigma_0$$

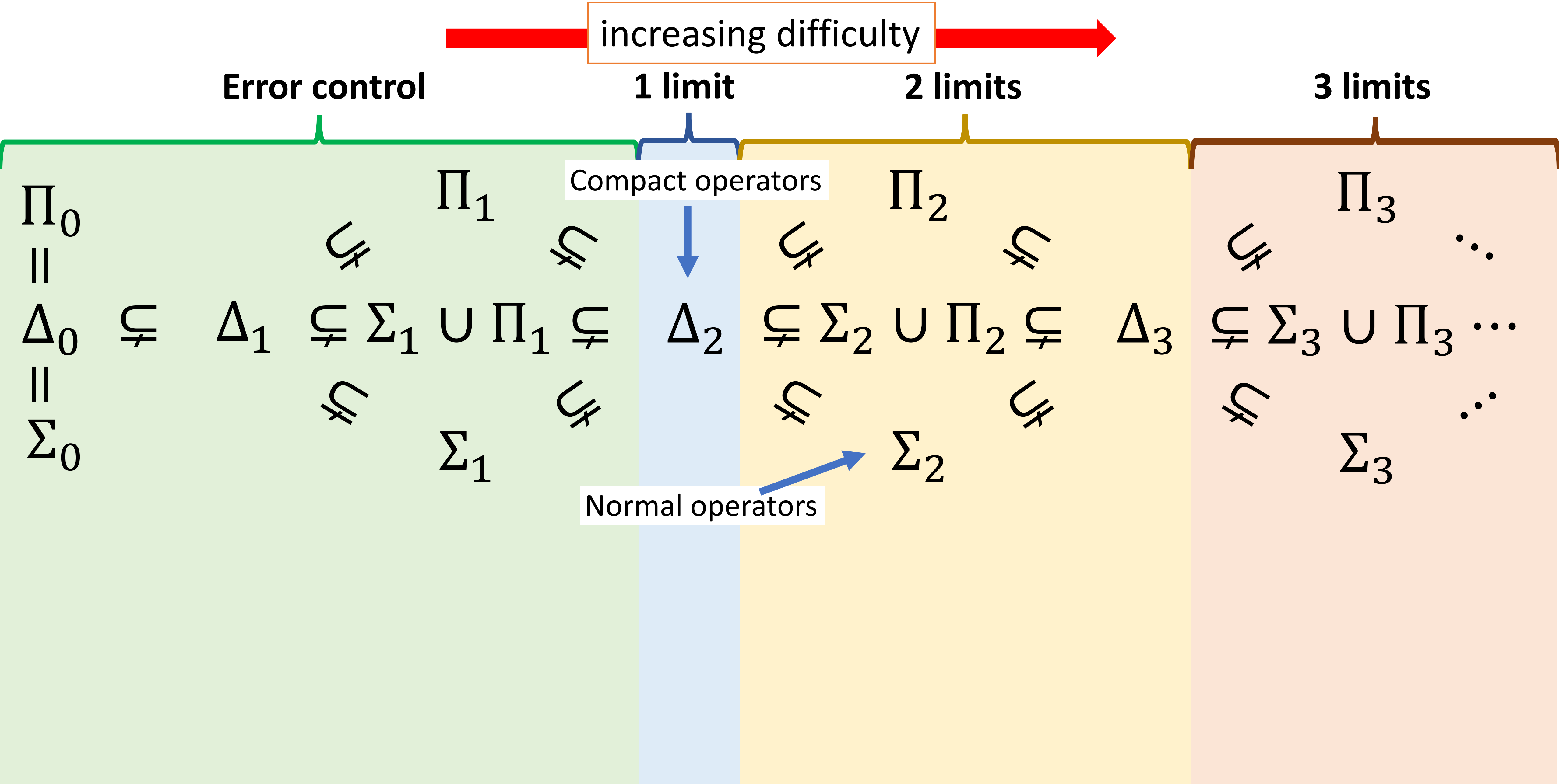
$$\Delta_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$$\Delta_1 \subsetneq \Pi_1 \subsetneq \Delta_2, \quad \Delta_1 \subsetneq \Sigma_1 \subsetneq \Delta_2$$

$$\Delta_2 \subsetneq \Pi_2 \subsetneq \Delta_3, \quad \Delta_2 \subsetneq \Sigma_2 \subsetneq \Delta_3$$

$$\Delta_3 \subsetneq \Pi_3 \cdots, \quad \Delta_3 \subsetneq \Sigma_3 \cdots$$

Sampler of results for bounded op. on $l^2(\mathbb{N})$



Sampler of results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

$\Pi_0 = \Delta_0 = \Sigma_0$

$$\Delta_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$\Delta_1 \subsetneq \Pi_1 \subsetneq \Delta_2$, $\Delta_1 \subsetneq \Sigma_1 \subsetneq \Delta_2$

$\Delta_2 \subsetneq \Pi_2 \subsetneq \Delta_3$, $\Delta_2 \subsetneq \Sigma_2 \subsetneq \Delta_3$

$\Delta_3 \subsetneq \Pi_3 \cdots$, $\Delta_3 \subsetneq \Sigma_3 \cdots$

Compact operators → Δ_2

"Sparse" operators → Π_2

Normal operators → Σ_2

Sampler of results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

$\Pi_0 = \Delta_0 = \Sigma_0$

$$\Delta_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

$\Delta_1 \subsetneq \Pi_1 \subsetneq \Delta_2$, $\Delta_1 \subsetneq \Sigma_1 \subsetneq \Delta_2$

$\Delta_2 \subsetneq \Pi_2 \subsetneq \Delta_3$, $\Delta_2 \subsetneq \Sigma_2 \subsetneq \Delta_3$

$\Delta_3 \subsetneq \Pi_3$, $\Delta_3 \subsetneq \Sigma_3$, $\cdots$

Compact operators → Δ_2

"Sparse" operators → Π_2

Normal operators → Σ_2

General operators → Π_3

Sampler of results for bounded op. on $l^2(\mathbb{N})$

increasing difficulty →

Error control | **1 limit** | **2 limits** | **3 limits**

Compact operators → Δ_2

"Sparse" operators → Π_2

Π_0, Π_1, Π_2, Π_3 …

$$\Pi_0 = \Delta_0 = \Sigma_0$$

$$\Delta_0 \subsetneq \Delta_1 \subsetneq \Sigma_1 \cup \Pi_1 \subsetneq \Delta_2 \subsetneq \Sigma_2 \cup \Pi_2 \subsetneq \Delta_3 \subsetneq \Sigma_3 \cup \Pi_3 \cdots$$

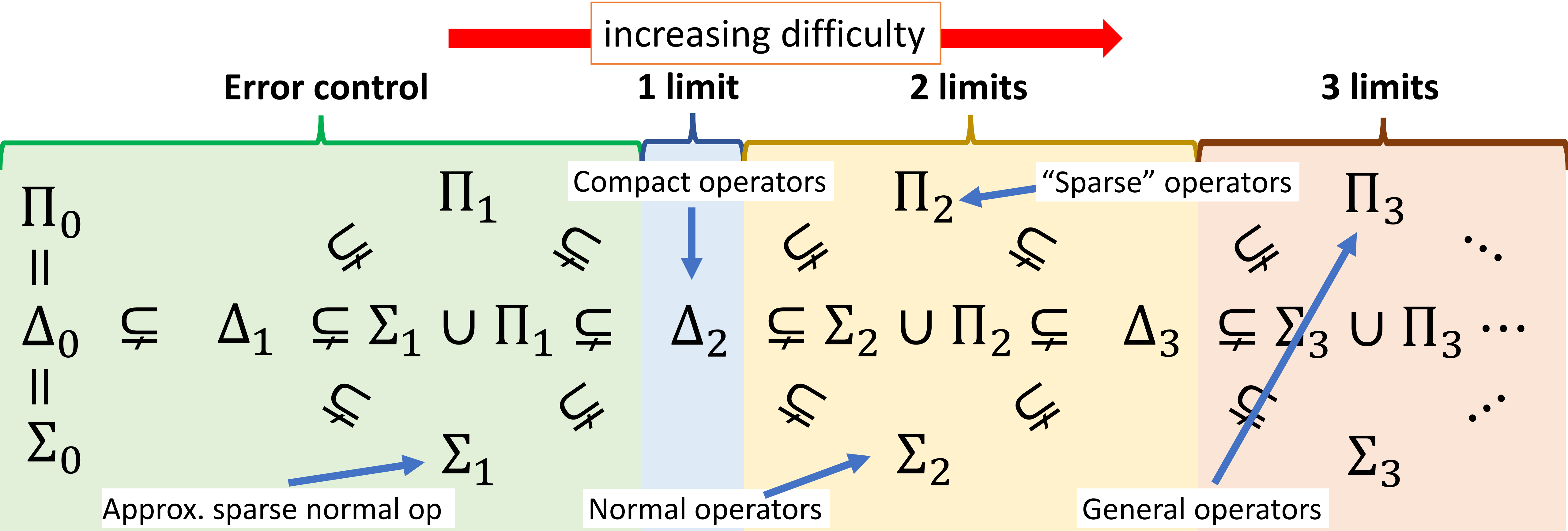
Σ_0, Σ_1, Σ_2, Σ_3 …

Approx. sparse normal op → Σ_1

Normal operators → Σ_2

General operators → Π_3

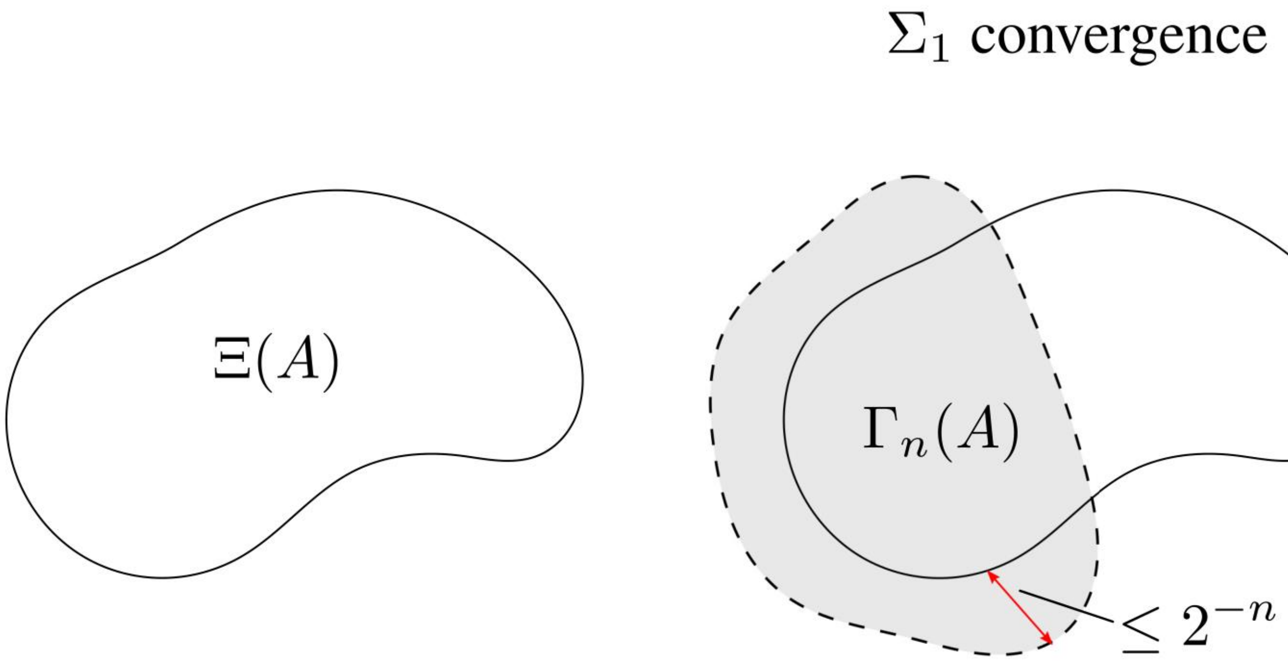
Sampler of results for bounded op. on $l^2(\mathbb{N})$



Zoo of problems: spectral type (pure point, absolutely continuous, singularly continuous), Lebesgue measure and fractal dimensions of spectra, discrete spectra, essential spectra, eigenspaces + multiplicity, spectral radii, essential numerical ranges, geometric features of spectrum (e.g., capacity), spectral gap problem, resonances ...

- C., *"The foundations of infinite-dimensional spectral computations,"* **PhD diss.**, University of Cambridge, 2020.
- C., *"On the computation of geometric features of spectra of linear operators on Hilbert spaces,"* **Found. Comput. Math.**, 2022.
- C., Hansen, *"The foundations of spectral computations via the solvability complexity index hierarchy,"* **J. Eur. Math. Soc.**, 2023.
- C., *"Computing spectral measures and spectral types,"* **Commun. Math. Phys.**, 2021.
- C., Horning, Townsend, *"Computing spectral measures of self-adjoint operators,"* **SIAM Rev.**, 2021.
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, *"On the solvability complexity index hierarchy and towers of algorithms,"* preprint.

Example 1: Σ_1 algorithm for spectra



Two reasons its hard!

$$A = \bigoplus_{r=1}^{\infty} J_{l_r}, \qquad J_{l_r} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Instability

$$\mathrm{Sp}(A) = \begin{cases} \{0\}, & \sup l_r < \infty \\ \{z : |z| \leq 1\}, & \text{otherwise} \end{cases}$$

No algorithm when given $\{l_r\}_{r=1}^{\infty}$ can determine if it is bounded.

$\Longrightarrow$ No algorithm computes spectra of gen. tridiagonal operators.

Two reasons its hard!

$$A = \oplus_{r=1}^{\infty} J_{l_r}, \qquad J_{l_r} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Instability

$$\mathrm{Sp}(A) = \begin{cases} \{0\}, & \sup l_r < \infty \\ \{z : |z| \leq 1\}, & \text{otherwise} \end{cases}$$

No algorithm when given $\{l_r\}_{r=1}^{\infty}$ can determine if it is bounded.

$\Longrightarrow$ No algorithm computes spectra of gen. tridiagonal operators.

Always have:

$$\|(A - z)^{-1}\|^{-1} \leq \mathrm{dist}(z, \mathrm{Sp}(A))$$

Two reasons its hard!

$$A = \bigoplus_{r=1}^{\infty} J_{l_r}, \qquad J_{l_r} = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Instability

$$\mathrm{Sp}(A) = \begin{cases} \{0\}, & \sup l_r < \infty \\ \{z : |z| \leq 1\}, & \text{otherwise} \end{cases}$$

No algorithm when given $\{l_r\}_{r=1}^{\infty}$ can determine if it is bounded.

$\Longrightarrow$ No algorithm computes spectra of gen. tridiagonal operators.

Always have:

$$\|(A-z)^{-1}\|^{-1} \leq \mathrm{dist}(z, \mathrm{Sp}(A))$$

Assume:

$$g(\mathrm{dist}(z, \mathrm{Sp}(A))) \leq \|(A-z)^{-1}\|^{-1}$$

known function

Two reasons its hard!

$$A = \bigoplus_{r=1}^{\infty} A_{l_r}, \qquad A_{l_r} = \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Info at ∞

$$\mathrm{Sp}(A) = \{0,2\}, \qquad \mathrm{Sp}\big(\mathrm{diag}(1,0,\ldots)\big) = \{0,1\}$$

More involved: choose $\{l_r\}_{r=1}^{\infty}$ to trick any supposed algorithm (try it!)

Two reasons its hard!

$$A = \bigoplus_{r=1}^{\infty} A_{l_r}, \qquad A_{l_r} = \begin{pmatrix} 1 & & & & 1 \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ 1 & & & & 1 \end{pmatrix} \in \mathbb{C}^{l_r \times l_r}$$

Info at ∞

$$\mathrm{Sp}(A) = \{0,2\}, \qquad \mathrm{Sp}\big(\mathrm{diag}(1,0,\dots)\big) = \{0,1\}$$

More involved: choose $\{l_r\}_{r=1}^{\infty}$ to trick any supposed algorithm (try it!)

Assume:

We have access (Λ) to inner products

$\langle Ae_j, e_i \rangle, \qquad \langle Ae_j, Ae_i \rangle, \qquad \langle A^*e_j, A^*e_i \rangle$

Sketch of method

$\mathcal{P}_n$ = orthog-projection onto span$\{e_1, \dots, e_n\}$
$\mathcal{P}_n: l^2(\mathbb{N}) \longrightarrow \mathbb{C}^n$, $\mathcal{P}_n^*: \mathbb{C}^n \longrightarrow l^2(\mathbb{N})$

Spectra through injection moduli (smallest singular value)

$$\sigma_{\inf}(T) = \inf\{\|Tv\| : v \in \mathfrak{D}(T), \|v\| = 1\}$$

$$\|(A-z)^{-1}\|^{-1} = \min\{\sigma_{\inf}(A-z), \sigma_{\inf}(A^* - \bar{z})\}$$

$$\sqrt{\sigma_{\inf}(\mathcal{P}_n(A-z)^*(A-z)\mathcal{P}_n^*)} = \sigma_{\inf}([A-z]\mathcal{P}_n^*) \downarrow \sigma_{\inf}(A-z)$$

$$g^{-1}\left(\sqrt{\sigma_{\inf}(\mathcal{P}_n[A-z]^*[A-z]\mathcal{P}_n^*)}\right) \downarrow g^{-1}(\|(A-z)^{-1}\|^{-1}) \geq \mathrm{dist}\big(z, \mathrm{Sp}(A)\big)$$

$$\|(A-z)^{-1}\|^{-1} \geq g(\mathrm{dist}(z, \mathrm{Sp}(A)))$$

Error control!

Final ingredient: adaptive search for local minimisers.

What did we do?

See conditions to make possible!

Often, infinite-dimensional solve-then-discretise needed

- **Lower bound:** embed a problem of known difficulty.

  Now have canonical ways to do this.

  Holds regardless of computational model.

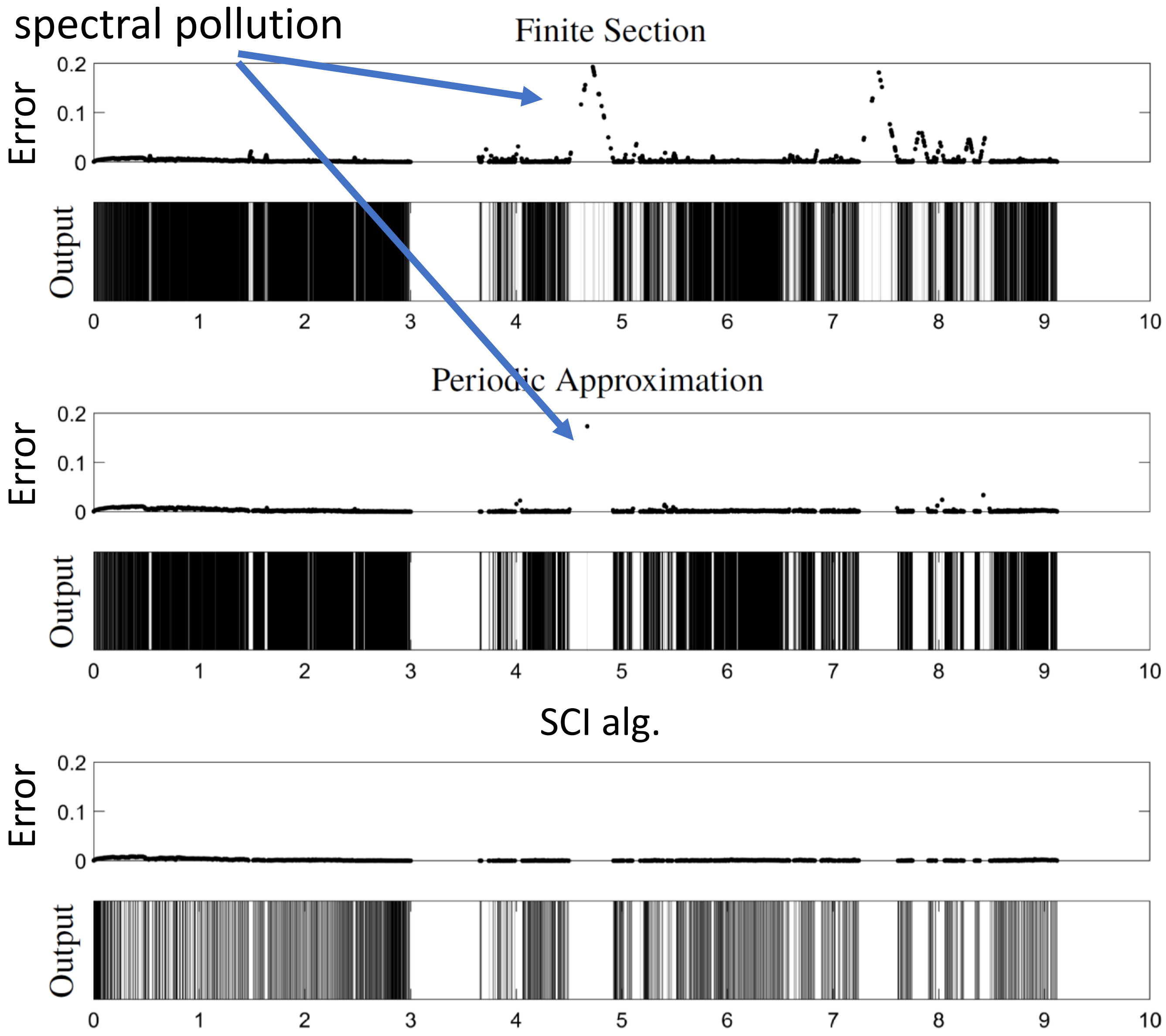
- **Upper bound:** build an algorithm.

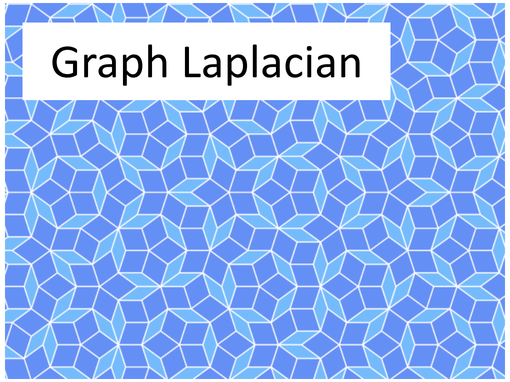
  Problem dependent.

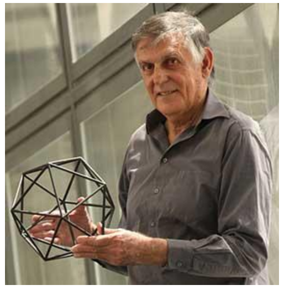
  Typically involves resolvent $(A - z)^{-1}$ for spectral problems.

NB: One can show without g or $\langle Ae_j, Ae_i\rangle$, $\langle A^*e_j, A^*e_i\rangle$, $\mathrm{SCI} \geq 2$.

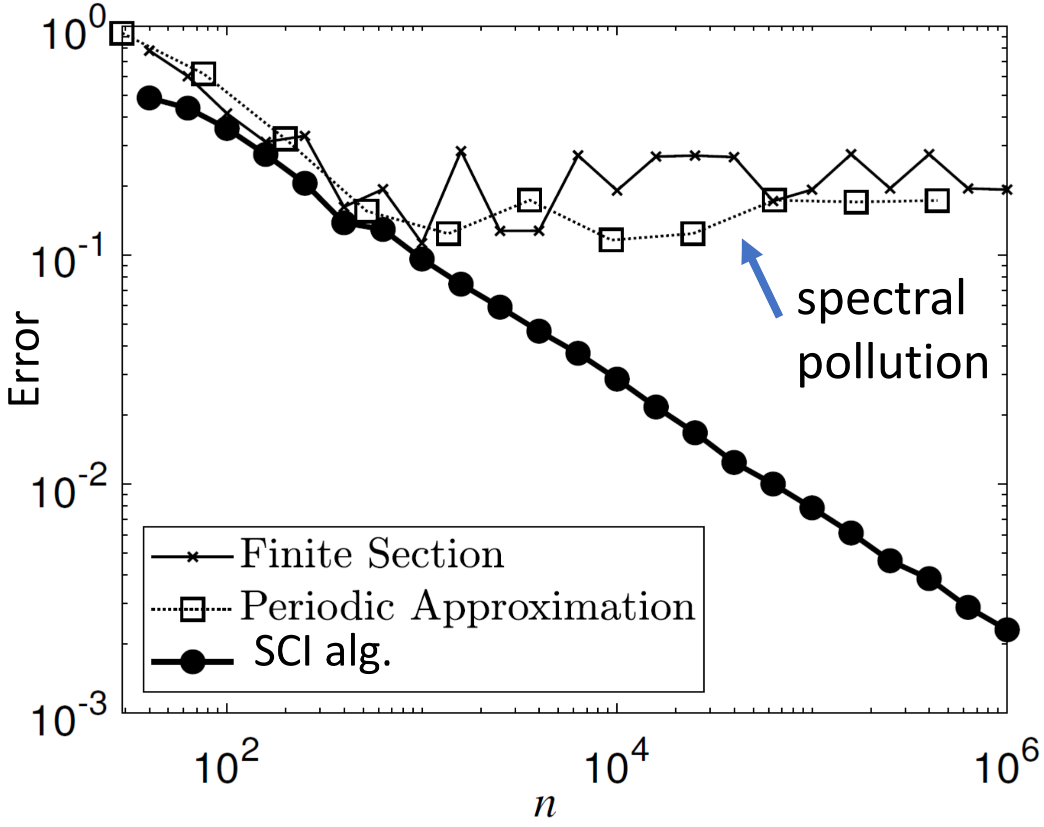
Example: Quasicrystal



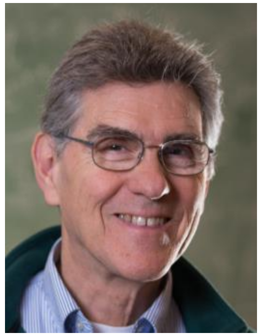




Dan Shechtman (Nobel Prize in Chemistry 2011.)



Example with non-trivial g

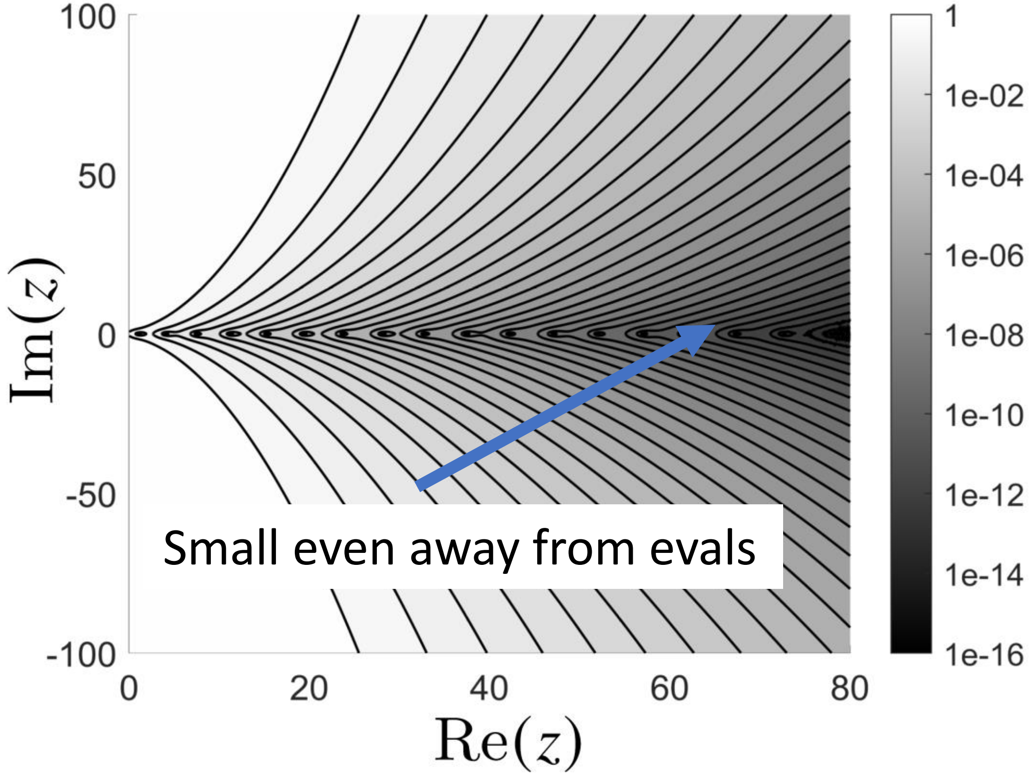


Carl Bender

Michael Berry

$$T = -\frac{d^2}{dx^2} + ix^3 \text{ on } \mathbb{R}$$

$\|(T-z)^{-1}\|^{-1}$



j	E_j to 30 digits with int. arith.
1	1.156 267 071 988 113 293 799 219 177 999 9
2	4.109 228 752 809 651 535 843 668 478 561 3
3	7.562 273 854 978 828 041 351 809 110 631 4
4	11.314 421 820 195 804 402 233 783 948 426 9
5	15.291 553 750 392 532 388 181 630 791 751 9
6	19.451 529 130 691 728 314 686 111 714 104 4
7	23.766 740 435 485 819 131 558 025 968 789 9
8	28.217 524 972 981 193 297 595 053 878 268 9
9	32.789 082 781 862 957 492 447 371 485 046 3
10	37.469 825 360 516 046 866 428 873 594 530 5
100	627.694 712 248 436 511 352 673 702 901 153 6

- C., *"INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS, Foundations, Algorithms, and Modern Applications.,"* **CUP**, to appear.

Differential operators on $L^2(\mathbb{R}^d)$

Theorem: Ω: class of self-adjoint diff. operators on $L^2(\mathbb{R}^d)$

$$T = \sum_{k\in\mathbb{Z}_{\geq 0}^d, |k|\leq N} c_k(x)\, \partial^k$$

- $C_0^\infty(\mathbb{R}^d)$ a core of T.
- $\{c_k\}$ poly bounded, locally bounded total variation.

Can access (to arbitrary precision):

- $\{c_k(q)\}$ for $q \in \mathbb{Q}^d$.
- Polynomial that bounds $\{c_k\}$ on $\mathbb{R}^d$.

(a) Know $\|c_k\|_{\mathrm{TV}([-n,n]^d)} \leq b_n \Longrightarrow \{\mathrm{Sp}, \Omega\} \in \Sigma_1$.

(b) Know $\|c_k\|_{\mathrm{TV}([-n,n]^d)} = O(b_n) \Longrightarrow \{\mathrm{Sp}, \Omega\} \in \Delta_2 \backslash (\Sigma_1 \cup \Pi_1)$.

- C., Hansen, *"The foundations of spectral computations via the solvability complexity index hierarchy,"* **J. Eur. Math. Soc.**, 2022

Differential operators on $L^2(\mathbb{R}^d)$

Theorem: Ω: class of self-adjoint diff. operators on $L^2(\mathbb{R}^d)$

$$T = \sum_{k\in\mathbb{Z}_{\geq 0}^d, |k|\leq N} c_k(x)\, \partial^k$$

- $C_0^\infty(\mathbb{R}^d)$ a core of T.
- $\{c_k\}$ poly bounded, locally bounded total variation.

Can access (to arbitrary precision):

- $\{c_k(q)\}$ for $q \in \mathbb{Q}^d$.
- Polynomial that bounds $\{c_k\}$ on $\mathbb{R}^d$.

(a) Know $\|c_k\|_{\mathrm{TV}([-n,n]^d)} \leq b_n \Longrightarrow \{\mathrm{Sp}, \Omega\} \in \Sigma_1$.

(b) Know $\|c_k\|_{\mathrm{TV}([-n,n]^d)} = O(b_n) \Longrightarrow \{\mathrm{Sp}, \Omega\} \in \Delta_2 \backslash (\Sigma_1 \cup \Pi_1)$.

Sampling schemes to construct matrix.

Extends to other domains, singular coefficients etc.

Verifiable (→ (a))

Not verifiable (→ (b))

• C., Hansen, "*The foundations of spectral computations via the solvability complexity index hierarchy,*" **J. Eur. Math. Soc.**, 2022

Example 2: Smale's 18th problem "What are the limits of AI?"

"Very often, the creation of a technological artifact precedes the science that goes with it. The steam engine was invented before thermodynamics. Thermodynamics was invented to explain the steam engine, essentially the ***limitations*** *of it.* ***What we are after is the equivalent of thermodynamics for intelligence.****"* Yann LeCun

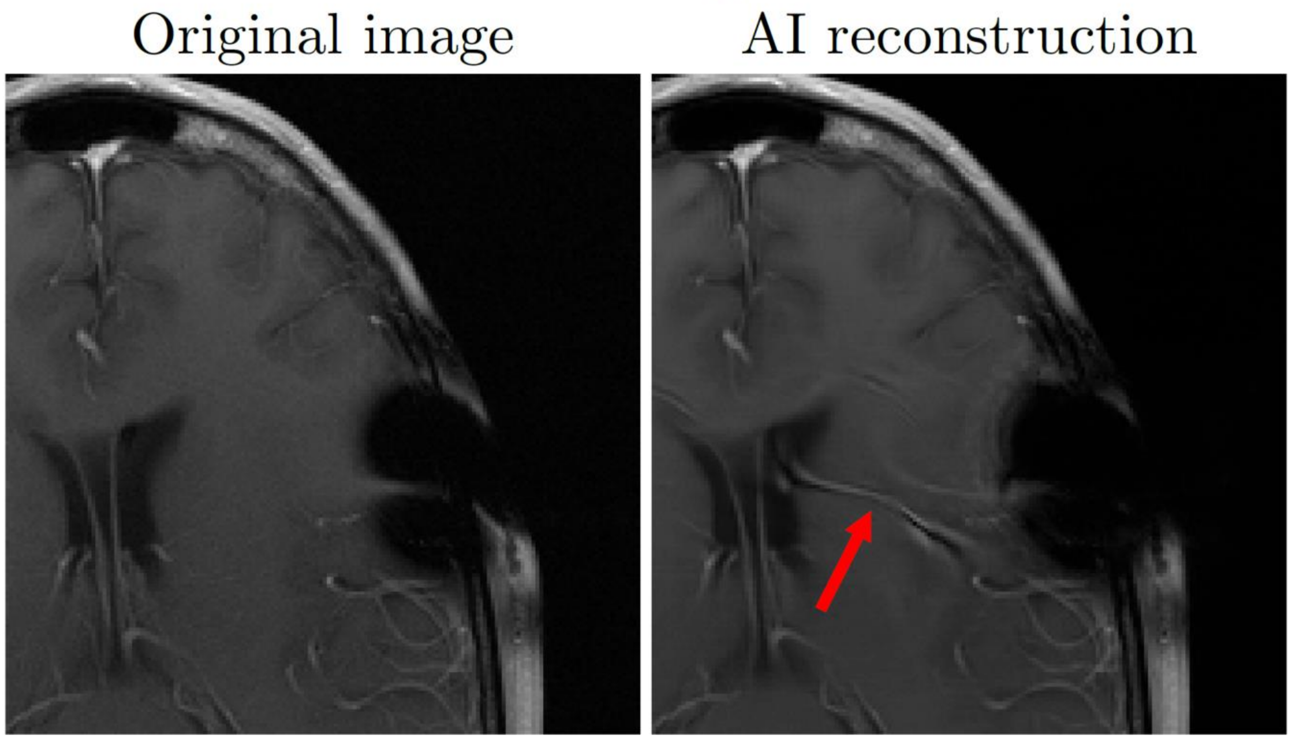
Lower bounds: Use SCI embedding techniques

randomized sequential general algorithms ⇢ **capture adaptive and probabilistic choice of training data**

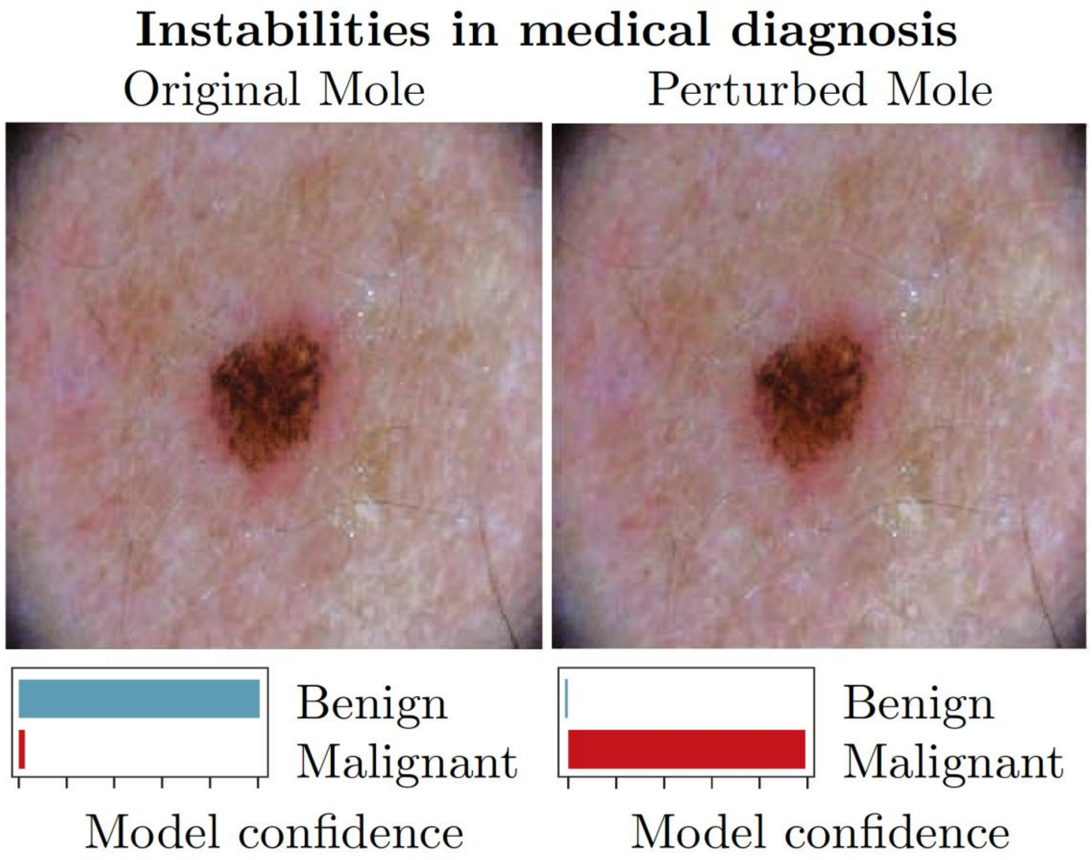
**S. Smale's list of problems for the 21st century (requested by V. Arnold), inspired by Hilbert's list*

Problem: hallucinations and instability

"**AI hallucination**", from Facebook and NYU's ***FastMRI challenge*** 2020.



"Such hallucinatory features are not acceptable and especially problematic if they mimic normal structures that are either not present or actually abnormal."



Finlayson et al., *"Adversarial attacks on medical machine learning,"* **Science**, 2019.

When can we make AI robust and trustworthy?

Example of the limits of deep learning

Paradox: "Nice" linear inverse problems where a *stable* and *accurate* neural network for image reconstruction exists, but it can never be trained!

E.g., suppose we want to solve (holds for much more general problems)

$$\min_{x \in \mathbb{C}^N} \|x\|_{l^1} + \lambda \|Ax - y\|_{l_2}^2$$

$A \in \mathbb{C}^{m \times N}$ (modality, $m < N$), $S = \{y_K\}_{K=1}^{R}$ (samples)

Arises when given $y \approx Ax + e$.

Enforce condition numbers bounded by 1.

Input data Λ

$$A \in \mathbb{C}^{m \times N} \text{ (modality, } m < N\text{)}, \qquad S = \{y_k\}_{k=1}^{R} \text{(samples)}$$

In practice, A not known exactly or stored to finite precision.

Assume access to $\{y_{n,k}\}_{k=1}^{R}$ and A_n (rational approx, e.g., floats) such that

$$\|y_{n,k} - y_k\| \leq 2^{-n}, \qquad \|A_n - A\| \leq 2^{-n}, \qquad n \in \mathbb{N}.$$

Training set for $(A, S) \in \Omega$:

$$\iota_{A,S} = \{(y_{n,k}, A_n): k = 1, \ldots, R \text{ and } n \in \mathbb{N}\}.$$

In a nutshell: allow access to arbitrary precision training data.

Question: Given a collection Ω of (A, S), does there exist a neural network approximating the solution map, and can it be trained by an algorithm?

What could go wrong?

$$\min_{x\in\mathbb{C}^N} \|x\|_{l^1} + \lambda\|Ax - y\|_{l_2}^2$$

Question: Given a collection Ω of (A, S), does there exist a neural network approximating the solution map, and can it be trained by an algorithm?

What could go wrong?

1. **Non-existence:** No neural network approximates solution map Ξ.

What could go wrong?

$$\min_{x\in\mathbb{C}^N} \|x\|_{l^1} + \lambda\|Ax - y\|_{l_2}^2$$

Question: Given a collection Ω of (A, S), does there exist a neural network approximating the solution map, and can it be trained by an algorithm?

What could go wrong?

1. ~~**Non-existence:** No neural network approximates solution map Ξ.~~

What could go wrong?

$$\min_{x \in \mathbb{C}^N} \|x\|_{l^1} + \lambda \|Ax - y\|_{l_2}^2$$

Question: Given a collection Ω of (A, S), does there exist a neural network approximating the solution map, and can it be trained by an algorithm?

What could go wrong?

1. ~~**Non-existence:** No neural network approximates solution map Ξ.~~
2. **Non-trainable:** $\exists$ neural network that approximates Ξ, but it can't be trained.

What could go wrong?

$$\min_{x \in \mathbb{C}^N} \|x\|_{l^1} + \lambda \|Ax - y\|_{l_2}^2$$

Question: Given a collection Ω of (A, S), does there exist a neural network approximating the solution map, and can it be trained by an algorithm?

What could go wrong?

1. ~~**Non-existence:** No neural network approximates solution map Ξ.~~
2. **Non-trainable:** $\exists$ neural network that approximates Ξ, but it can't be trained.
3. **Not practical:** $\exists$ neural network that approximates Ξ, and training algorithm. However, any training algorithm needs prohibitively many samples.

Fundamental barriers

Paradox: "Nice" linear inverse problems where a *stable* and *accurate* neural network for image reconstruction exists, but it can never be trained!

Theorem: Pick positive integers $n \geq 3$ and M. Class of problems such that:

- (**Not trainable**) No algorithm (even random) can train a neural network with $\boldsymbol{n}$ **digits** of accuracy over dataset with prob. $> 1/2$.
- (**Not practical**) $\boldsymbol{n-1}$ **digits** of accuracy possible over dataset, but any training algorithm requires **arbitrarily large training data**.
- (**Trainable and practical**) $\boldsymbol{n-2}$ digits of accuracy possible over dataset via training algorithm using $\boldsymbol{M}$ **training data**.

Any computational model

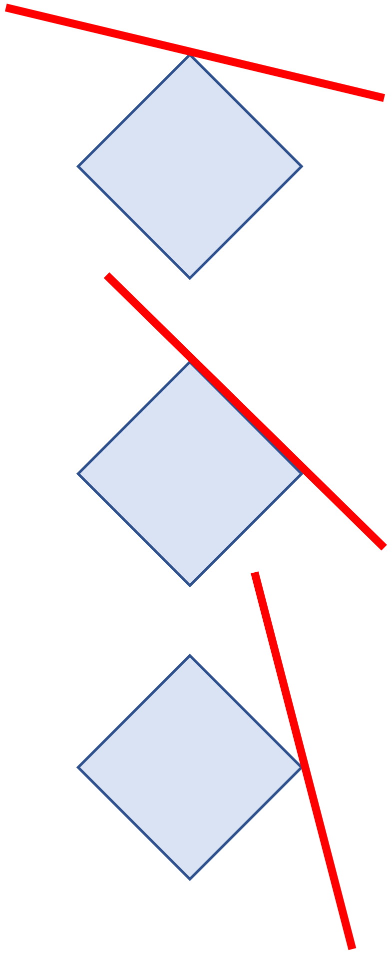
Holds for any architecture, any precision of training data.

$\Longrightarrow$ **Classification theory telling us what can and cannot be done**

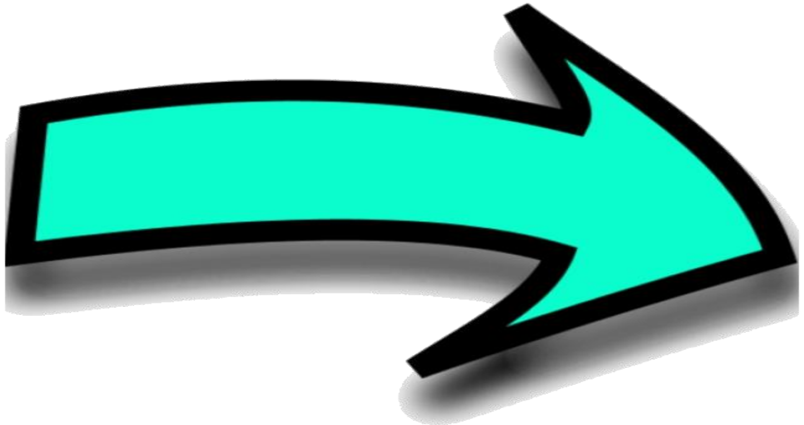
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.
- Antun, C., Hansen, *"Proving Existence Is Not Enough: : Mathematical Paradoxes Unravel the Limits of Neural Networks in Artificial Intelligence,"* **SIAM News**, May 2022.
- Choi, *"Some AI Systems May Be Impossible to Compute,"* **IEEE Spectrum**, March 2022.

Idea of mechanism

Low-dimensional
phase transitions



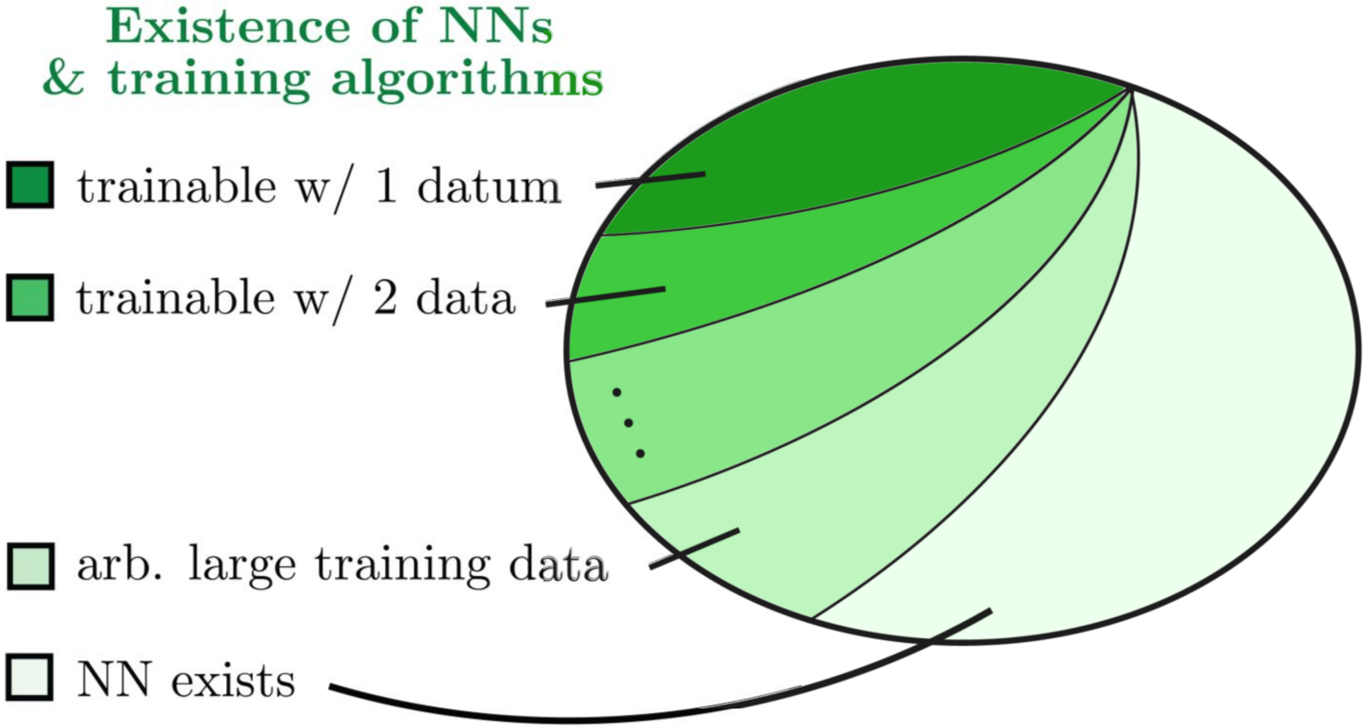
SCI embedding into
well-conditioned problems.



Randomised general algorithm
to capture training.

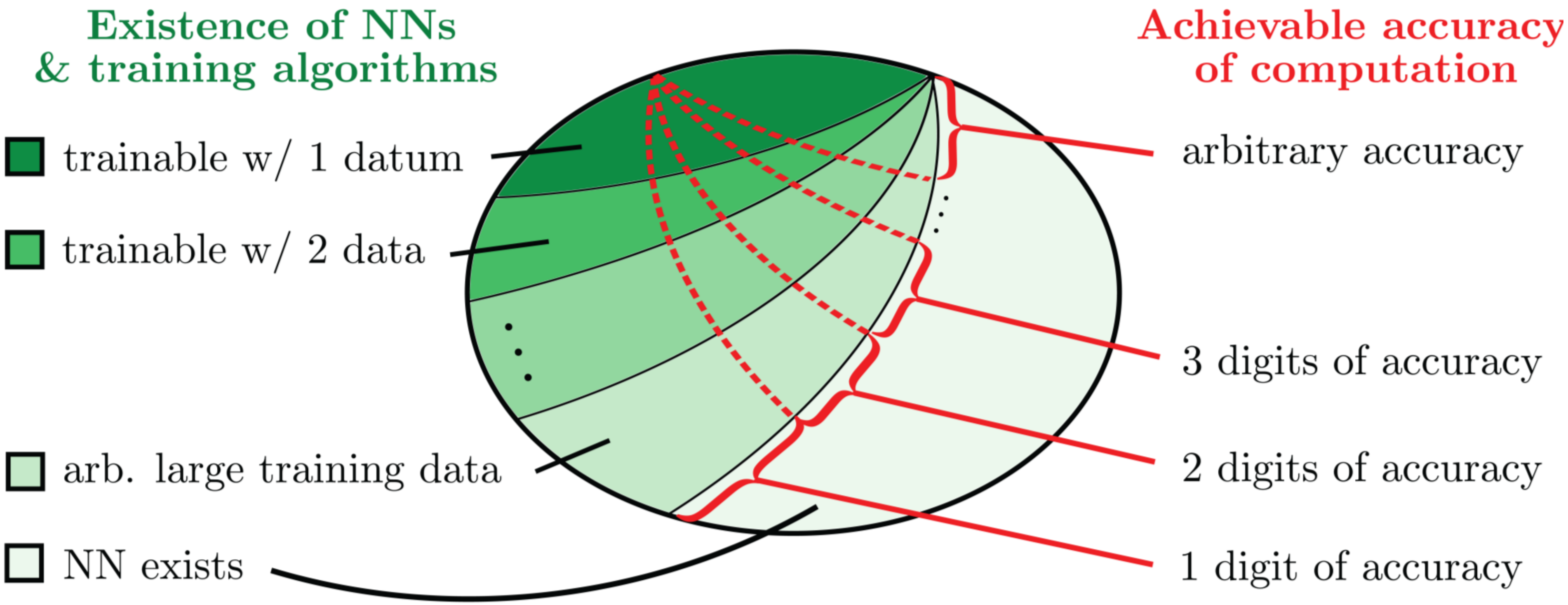
Workhorse lemma
(applies to other problems).

The world of neural networks



Given a problem and conditions, where does it sit in this diagram?

The world of neural networks



Given a problem and conditions, where does it sit in this diagram?

Example counterpart theorem

Certain conditions: stable neural networks trained with exponential accuracy.
E.g., *approximate sharpness inequality*:

$$\min_{x \in \mathbb{C}^N} f(x) \quad \text{s.t.} \quad \|Ax - y\| \leq \varepsilon$$

$$\text{dist}(x, \text{solution set}) \leq \left(\frac{[f(x) - f^*] + [\|Ax - y\| - \varepsilon] + \delta}{\alpha}\right)^{1/\beta}$$

(α, β) unknown

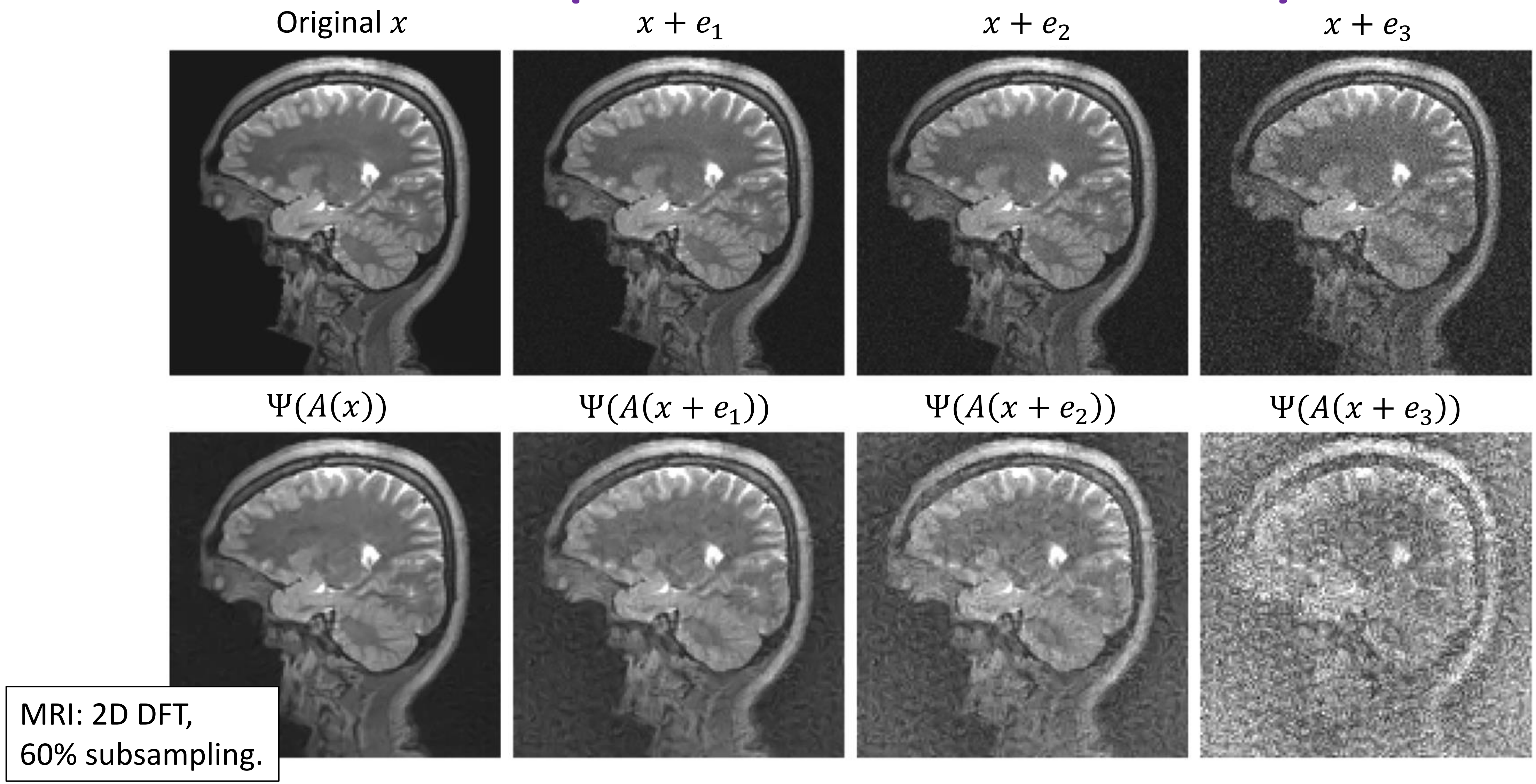
Fast **I**terative **RE**started **NET**works (FIRENETs)
(unrolled primal-dual with novel restart scheme)

Theorem: Training algorithm under above assumption produces *stable* neural networks φ_n of width $O(N)$, depth $O(n)$, guaranteed worst bound optimal in (α, β). E.g., $\beta = 1$,

$$\text{dist}(\varphi_n(y), \text{solution set}) \lesssim e^{-n} + \delta$$

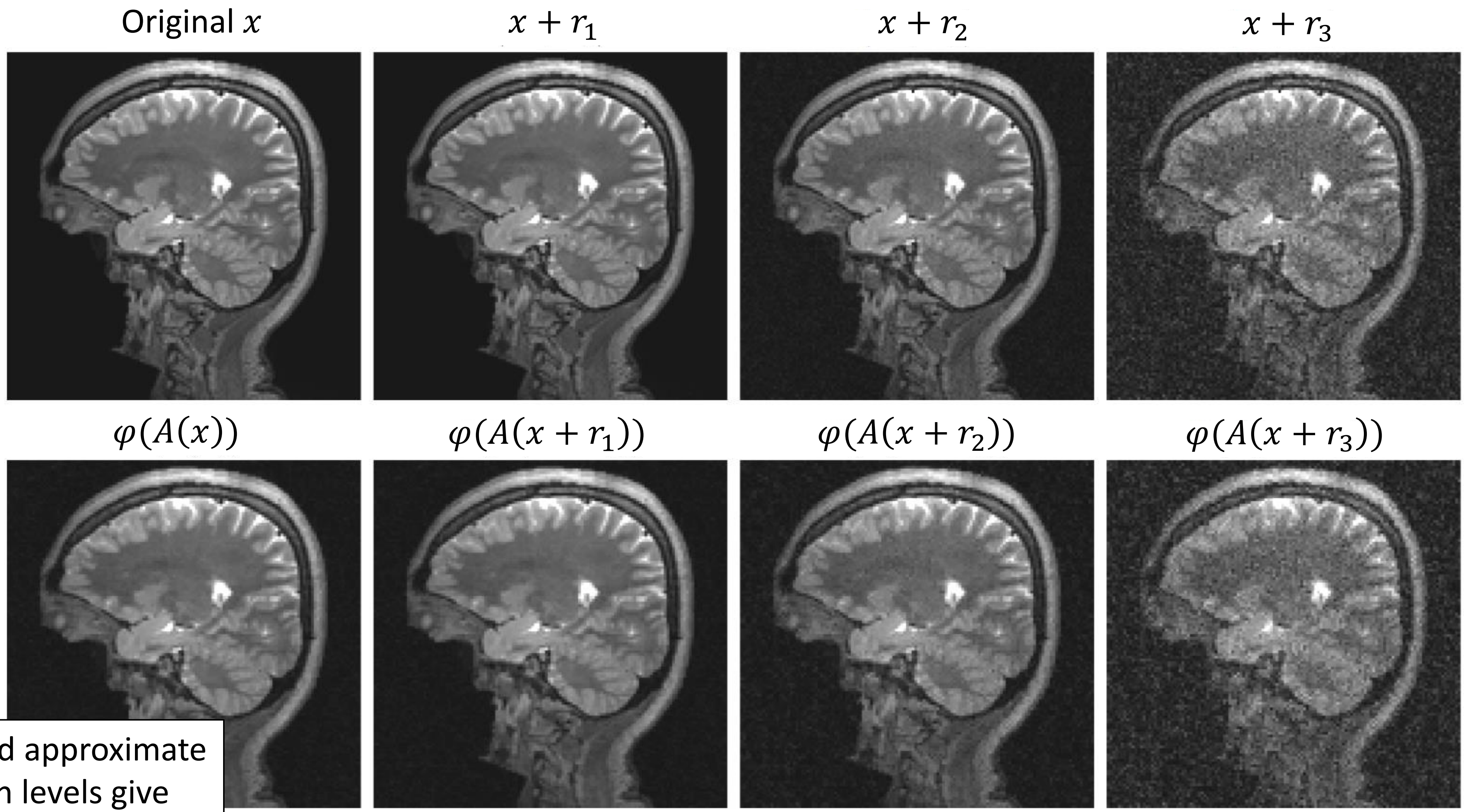
- C., Antun, Hansen, *"The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,"* **PNAS**, 2022.
- Adcock, C., Neyra-Nesterenko, *"Restarts subject to approximate sharpness: A parameter-free and optimal scheme for first-order methods"*, preprint.

Example of severe instability



- Zhu et al., "*Image reconstruction by domain-transform manifold learning,*" **Nature,** 2018.
- Antun et al., "*On instabilities of deep learning in image reconstruction and the potential costs of AI,*" **PNAS**, 2020.

FIRENET: provably stable (even to adversarial examples) and accurate



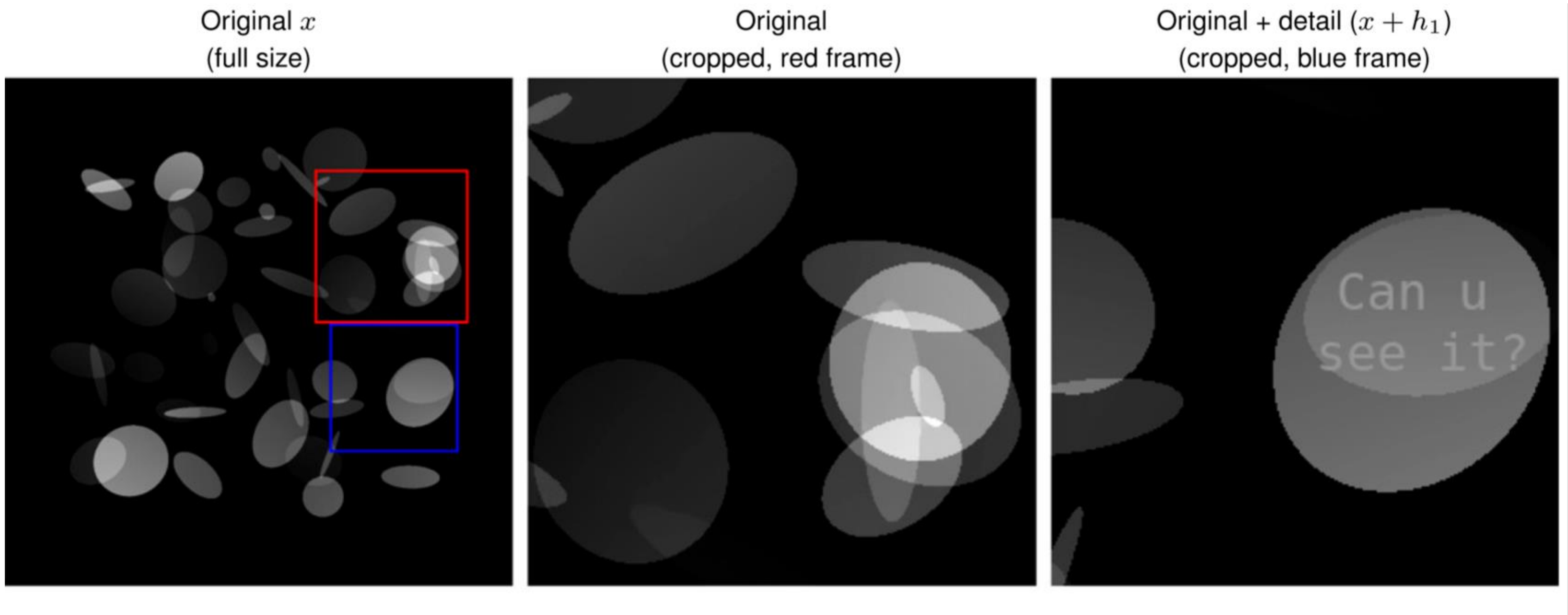
Sampling and approximate sparseness in levels give approximate sharpness

- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.

Stability vs accuracy

2D DFT,
15% subsampling.

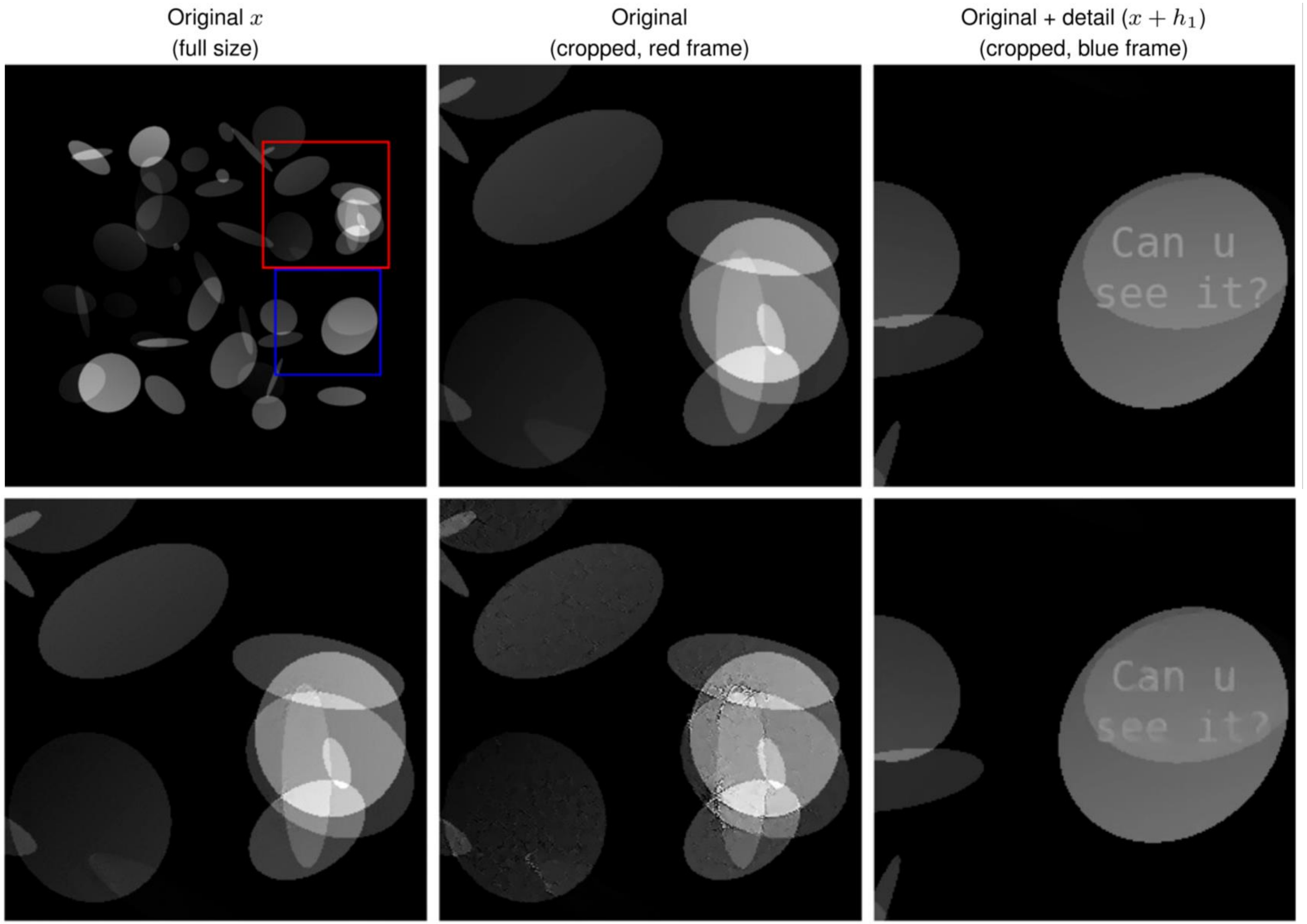
All networks trained on 5000 images of ellipses



- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.

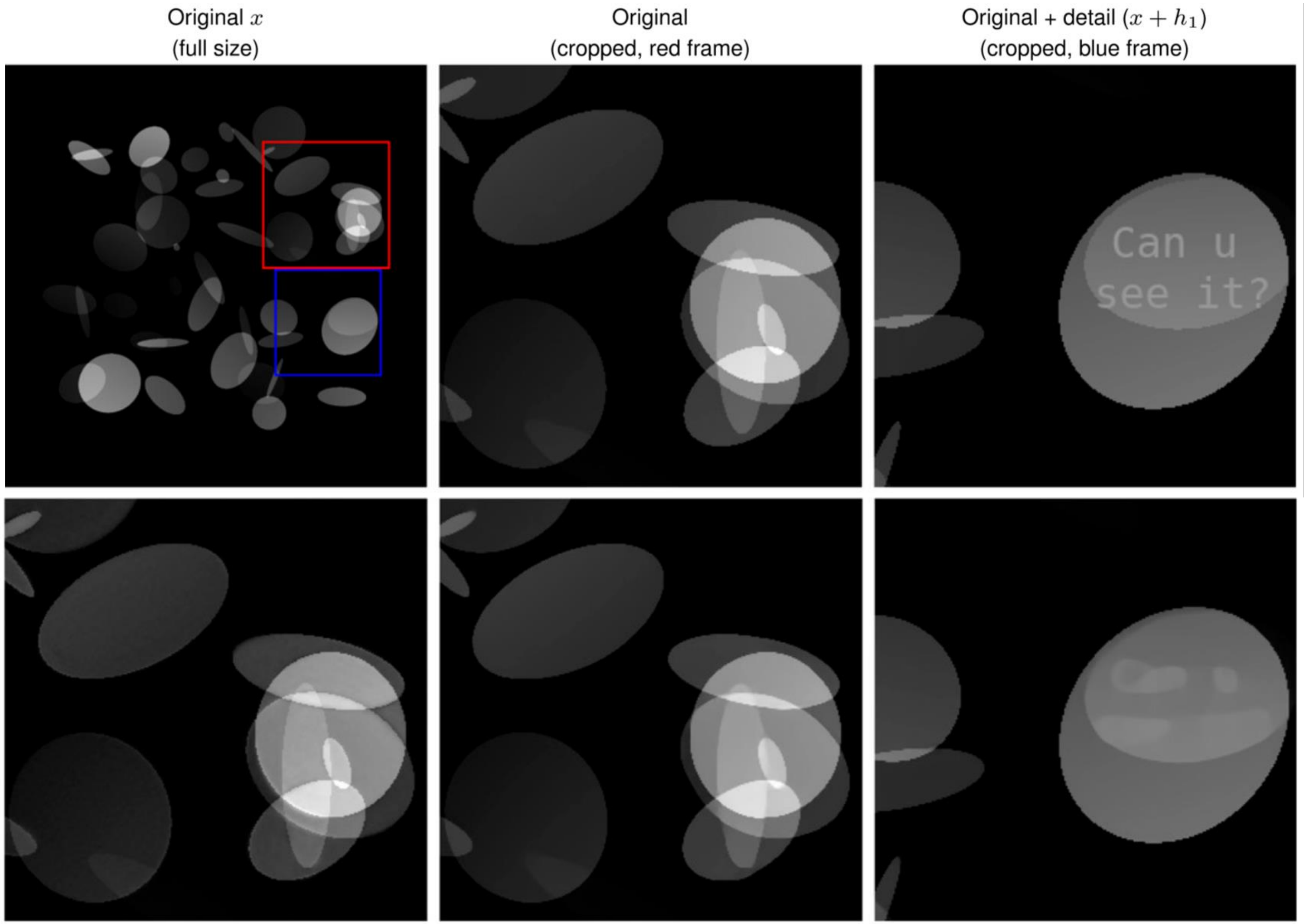
U-Net with no noise: accurate but unstable

U-Net: standard neural network architecture for imaging. Approx 4 million parameters.



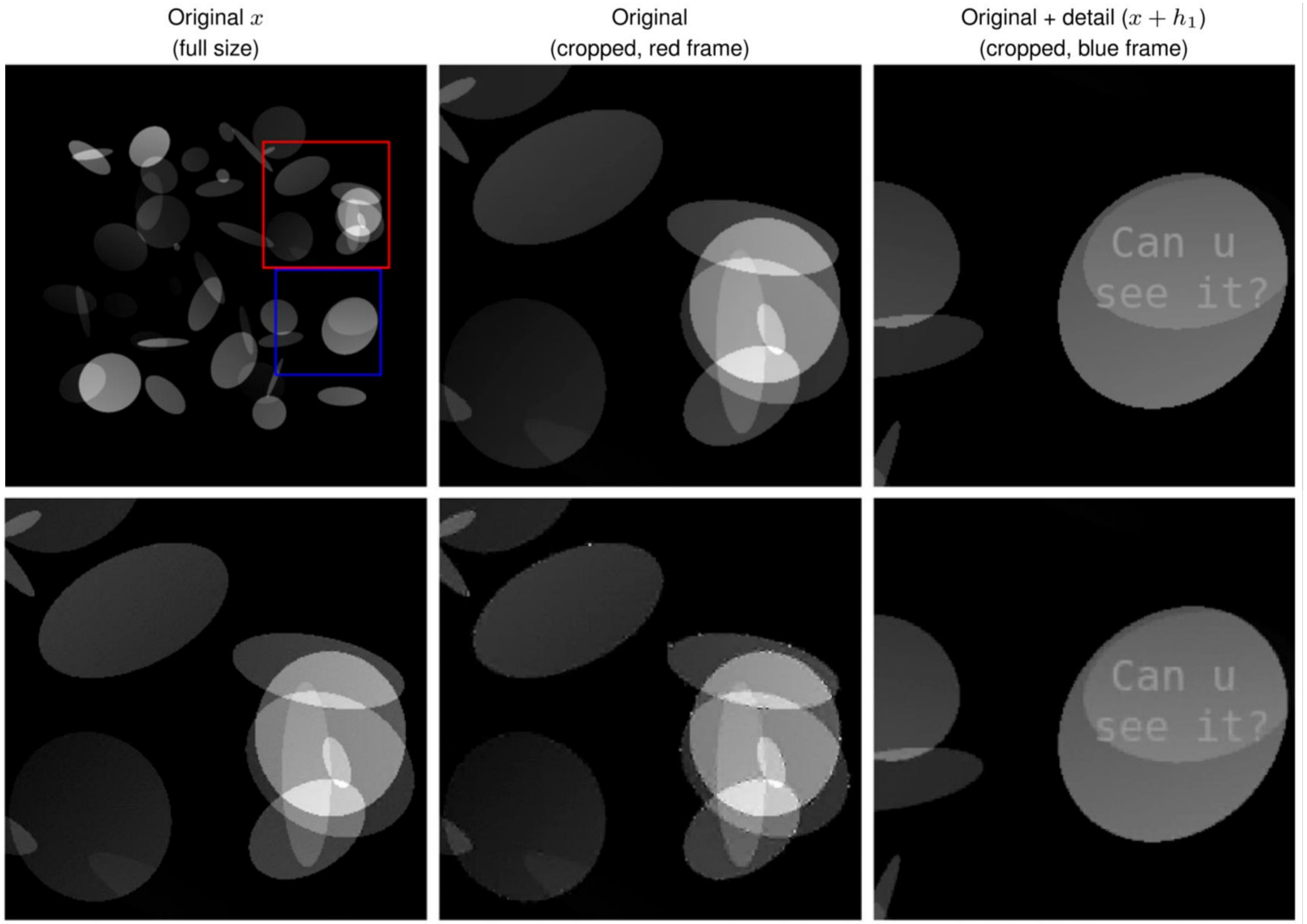
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.

U-Net with noise: stable but inaccurate



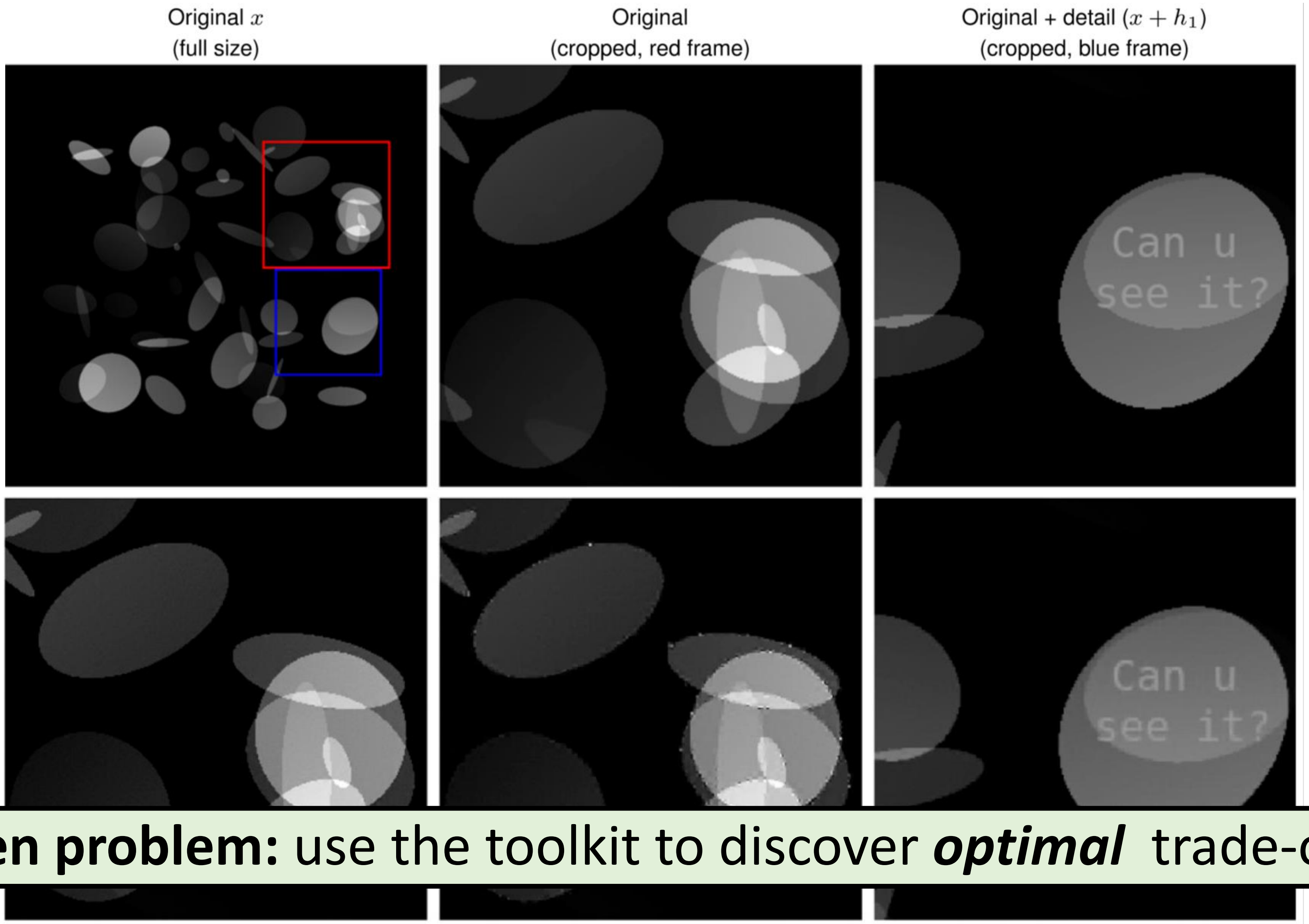
- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.

FIRENET: balances stability and accuracy?



- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.

FIRENET: balances stability and accuracy?



Open problem: use the toolkit to discover ***optimal*** trade-offs.

- C., Antun, Hansen, "*The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem,*" **PNAS**, 2022.

Why study this hierarchy?

- Optimality: understand boundaries of what’s possible.
- Lower bounds $\Longrightarrow$ spot assumptions needed to lower SCI.
- Upper bounds $\Longrightarrow$ new algorithms and methods.

FOUNDATIONS $\longleftrightarrow$ METHODS

- $\Sigma_1 \cup \Pi_1 \Longrightarrow$ computer-assisted proofs.
- Much of computational literature not sharp!

Remarks:

- Can use any model of computation.
- Existing hierarchies (e.g., arithmetic, Baire etc.) included as particular cases.

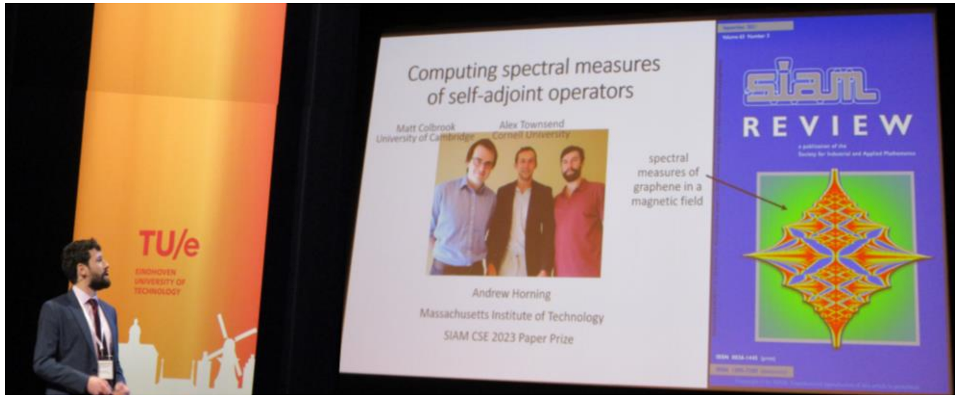
SCI throughout computational mathematics

- Resonances ($\Delta_2 \backslash \Sigma_1 \cup \Pi_1$)
  - Ben-Artzi, Marletta, Rösler, *"Computing scattering resonances,"* **J. Eur. Math. Soc.**, 2022.
  - Ben-Artzi, Marletta, Rösler, *"Computing the sound of the sea in a seashell,"* **Found. Comput. Math.**, 2022.
- Optimisation and regularisation
  - Bastounis, Hansen, Vlačić, *"The extended Smale's 9th problem,"* preprint.
  - C., Gazdag, Bastounis, Hansen, *"On phase transitions and approximation thresholds in semidefinite programs,"* preprint.
- Data-driven Koopmanism for dynamical systems (40,000 papers 2013-2023)
  - C., Townsend, *"Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,"* **CPAM**, to appear.
  - C., Ayton, Szőke, *"Residual Dynamic Mode Decomposition,"* **J. Fluid Mech.**, 2023.
- Polynomial root finding ($\Delta_2, \Delta_3, \Delta_4$)
  - Smale, *"The fundamental theorem of algebra and complexity theory,"* **Bull. Amer. Math. Soc.**, 1981.
  - McMullen, *"Families of rational maps and iterative root-finding algorithms,"* **Ann. of Math.**, 1987.
  - Doyle, McMullen, *"Solving the quintic by iteration,"* **Acta Math.**, 1989.

SCI throughout computational mathematics

- Computer-assisted proofs ($\Sigma_1 \cup \Pi_1 \backslash \Delta_1$)
  - Fefferman, Seco, *"Aperiodicity of the Hamiltonian flow in the Thomas-Fermi potential,"* **Rev. Mat. Iberoamericana**, 1993.
  - Fefferman, Seco, *"Interval arithmetic in quantum mechanics,"* **Applications of interval computations**, 1996.
  - Hales, *"A proof of the Kepler conjecture,"* **Ann. of Math.**, 2005.
  - Hales et al., *"A formal proof of the Kepler conjecture,"* **Forum Math. Pi**, 2017.
- Spectral measures and continuous spectra (Δ_1 or Δ_2 depending on $(\mathcal{M}, d)$)
  - Webb, Olver, *"Spectra of Jacobi operators via connection coefficient matrices,"* **CIMP**, 2021.
  - C., Horning, Townsend, *"Computing spectral measures of self-adjoint operators,"* **SIAM Rev.**, 2021.

Shameless plug: tune in tomorrow at 14:30 Room 105 for NONLINEAR SPECTRAL PROBLEMS!



Andrew Horning delivering plenary on computing spectral measures at SIAMCSE23

Summary

SCI hierarchy is a tool for discovering the foundations of computation.

Example 1: The zoo of spectral problems.

- Many spectral problems in infinite dimensions are impossible.
  Some are more impossible than others!
- New suite of "infinite-dimensional" algorithms for spectral problems.
  Rigorous, optimal, practical.

Example 2: Need for foundations in AI/deep learning (Smale's 18th prob).

- **Paradox:** "Nice" problems where stable, accurate NNs exist but can't be trained!
  Trainability depends on desired accuracy, training data.
- Conditions ⇒ FIRENETs exp. convergence + stable (even w.r.t. adversarial attacks).

Could this framework be useful in your area?

References

[1] Colbrook, Matthew J., Vegard Antun, and Anders C. Hansen. "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem." Proceedings of the National Academy of Sciences 119.12 (2022): e2107151119.

[2] Antun, V., M. J. Colbrook, and A. C. Hansen. "Proving existence is not enough: Mathematical paradoxes unravel the limits of neural networks in artificial intelligence." SIAM News 55.4 (2022): 1-4.

[3] Colbrook, Matthew, Vegard Antun, and Anders Hansen. "Mathematical paradoxes unearth the boundaries of AI." TheScienceBreaker 8.3 (2022).

[4] Adcock, Ben, Matthew J. Colbrook, and Maksym Neyra-Nesterenko. "Restarts subject to approximate sharpness: A parameter-free and optimal scheme for first-order methods." arXiv preprint arXiv:2301.02268 (2023).

[5] Colbrook, Matthew J. "WARPd: A linearly convergent first-order primal-dual algorithm for inverse problems with approximate sharpness conditions." SIAM Journal on Imaging Sciences 15.3 (2022): 1539-1575.

[6] Colbrook, Matthew. The foundations of infinite-dimensional spectral computations. Diss. University of Cambridge, 2020.

[7] Ben-Artzi, J., Colbrook, M. J., Hansen, A. C., Nevanlinna, O., & Seidel, M. (2020). Computing Spectra--On the Solvability Complexity Index Hierarchy and Towers of Algorithms. arXiv preprint arXiv:1508.03280.

[8] Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society (2022).

[9] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Computing spectral measures of self-adjoint operators." SIAM review 63.3 (2021): 489-524.

[10] Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.

[11] Colbrook, Matthew J. "On the computation of geometric features of spectra of linear operators on Hilbert spaces." Foundations of Computational Mathematics (2022): 1-82.

[12] Colbrook, Matthew J. "Computing spectral measures and spectral types." Communications in Mathematical Physics 384 (2021): 433-501.

[13] Colbrook, Matthew J., and Anders C. Hansen. "On the infinite-dimensional QR algorithm." Numerische Mathematik 143 (2019): 17-83.

[14] Colbrook, Matthew. "Pseudoergodic operators and periodic boundary conditions." Mathematics of Computation 89.322 (2020): 737-766.

[15] Colbrook, Matthew J., and Alex Townsend. "Avoiding discretization issues for nonlinear eigenvalue problems." arXiv preprint arXiv:2305.01691 (2023).

[16] Colbrook, Matthew, Andrew Horning, and Alex Townsend. "Resolvent-based techniques for computing the discrete and continuous spectrum of differential operators." XXI Householder Symposium on Numerical Linear Algebra. 2020.

[17] Johnstone, Dean, et al. "Bulk localized transport states in infinite and finite quasicrystals via magnetic aperiodicity." Physical Review B 106.4 (2022): 045149.

[18] Colbrook, Matthew J., et al. "Computing spectral properties of topological insulators without artificial truncation or supercell approximation." IMA Journal of Applied Mathematics 88.1 (2023): 1-42.

[19] Colbrook, Matthew J., and Andrew Horning. "Specsolve: spectral methods for spectral measures." arXiv preprint arXiv:2201.01314 (2022).

[20] Colbrook, Matthew J., and Alex Townsend. "Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems." arXiv preprint arXiv:2111.14889 (2021).

[21] Colbrook, Matthew J., Lorna J. Ayton, and Máté Szőke. "Residual dynamic mode decomposition: robust and verified Koopmanism." Journal of Fluid Mechanics 955 (2023): A21.

[22] Colbrook, Matthew J. "The mpEDMD algorithm for data-driven computations of measure-preserving dynamical systems." SIAM Journal on Numerical Analysis 61.3 (2023): 1585-1608.

[23] Brunton, Steven L., and Matthew J. Colbrook. "Resilient Data-driven Dynamical Systems with Koopman: An Infinite-dimensional Numerical Analysis Perspective."

[24] Colbrook, Matthew J. "Computing semigroups with error control." SIAM Journal on Numerical Analysis 60.1 (2022): 396-422.

[25] Colbrook, Matthew J., and Lorna J. Ayton. "A contour method for time-fractional PDEs and an application to fractional viscoelastic beam equations." Journal of Computational Physics 454 (2022): 110995.