

Residual Dynamic Mode Decomposition

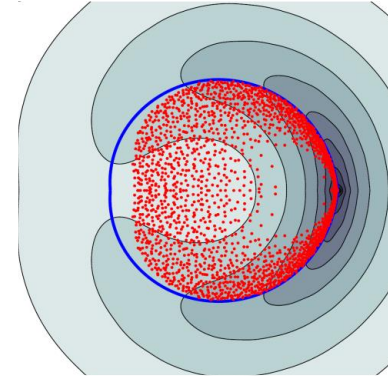
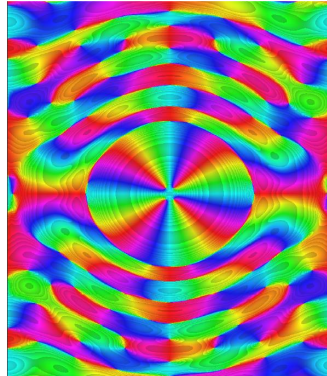
Robust and verified Koopmanism!

Matthew Colbrook (m.colbrook@damtp.cam.ac.uk)

University of Cambridge

Joint work with

Lorna Ayton (Cambridge), **Máté Szőke** (Virginia Tech), **Alex Townsend** (Cornell)



Maths



C., Townsend, “Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,” preprint.

C., Ayton, Szőke, “Residual Dynamic Mode Decomposition,” preprint.



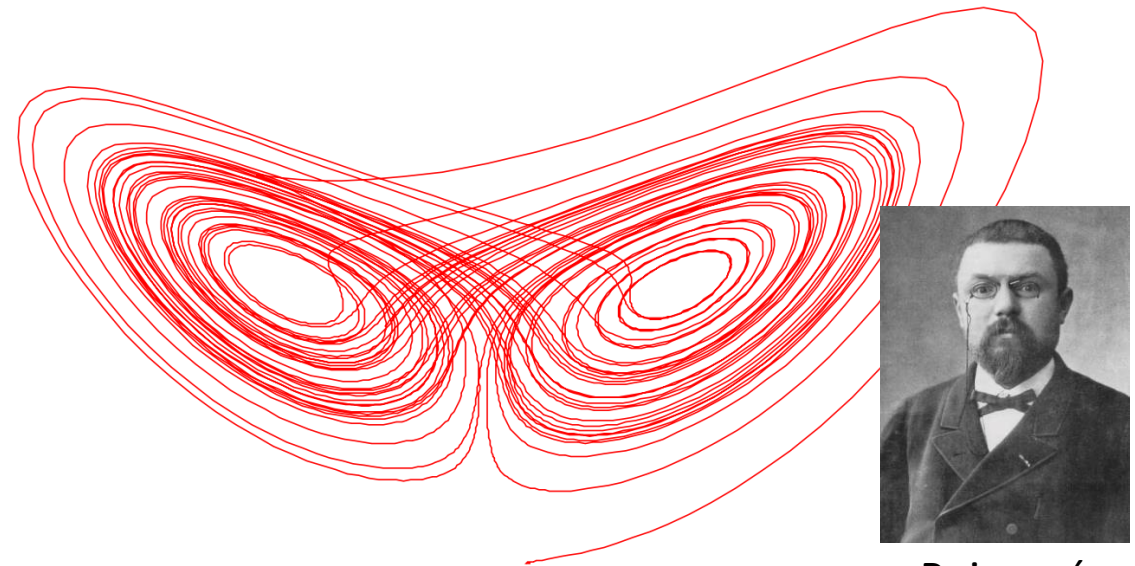
Applications

Data-driven dynamical systems

- State $x \in \Omega \subseteq \mathbb{R}^d$, **unknown** function $F: \Omega \rightarrow \Omega$ governs dynamics

$$x_{n+1} = F(x_n)$$

- **Goal:** Learn about system from data $\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^M$
 - **Data:** experimental measurements or numerical simulations
 - E.g., **used for** forecasting, control, design, understanding
- **Applications:** chemistry, climatology, electronics, epidemiology, finance, fluids, molecular dynamics, neuroscience, plasmas, robotics, video processing, etc.



Poincaré

Operator viewpoint

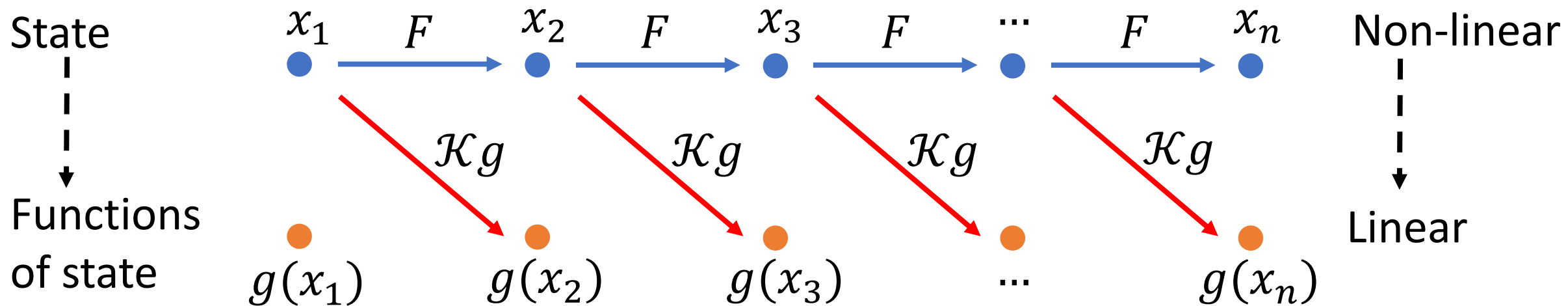
Koopman

von Neumann



- **Koopman operator** \mathcal{K} acts on functions $g: \Omega \rightarrow \mathbb{C}$

$$[\mathcal{K}g](x_n) = g(F(x_n)) = g(x_{n+1})$$
- \mathcal{K} is **linear** but acts on an **infinite-dimensional** space.



- Work in $L^2(\Omega, \omega)$ for positive measure ω , with inner product $\langle \cdot, \cdot \rangle$.

• Koopman, "Hamiltonian systems and transformation in Hilbert space," *Proc. Natl. Acad. Sci. USA*, 1931.

• Koopman, v. Neumann, "Dynamical systems of continuous spectra," *Proc. Natl. Acad. Sci. USA*, 1932.

Why is linear (much) easier?

$$x_{n+1} = F(x_n)$$

- Suppose $F(x) = Ax, A \in \mathbb{R}^{d \times d}, A = V\Lambda V^{-1}$.

- Set $\xi = V^{-1}x$,

$$\xi_n = V^{-1}x_n = V^{-1}A^n x_0 = \Lambda^n V^{-1}x_0 = \Lambda^n \xi_0$$

- Let $w^T A = \lambda w$, set $\varphi(x) = w^T x$,

$$[\mathcal{K}\varphi](x) = w^T Ax = \lambda \varphi(x)$$

Long-time dynamics
become trivial!



Eigenfunction

Much more general (**non-linear** and even **chaotic** F).

Koopman mode decomposition

generalized
eigenfunction of \mathcal{K}

eigenfunction of \mathcal{K}

$$g(x) = \sum_{\text{eigs } \lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) + \int_{[-\pi, \pi]_{\text{per}}} \phi_{\theta, g}(x) d\theta$$

$$g(x_n) = [\mathcal{K}^n g](x_0) = \sum_{\text{eigs } \lambda_j} c_{\lambda_j} \lambda_j^n \varphi_{\lambda_j}(x_0) + \int_{[-\pi, \pi]_{\text{per}}} e^{in\theta} \phi_{\theta, g}(x_0) d\theta$$

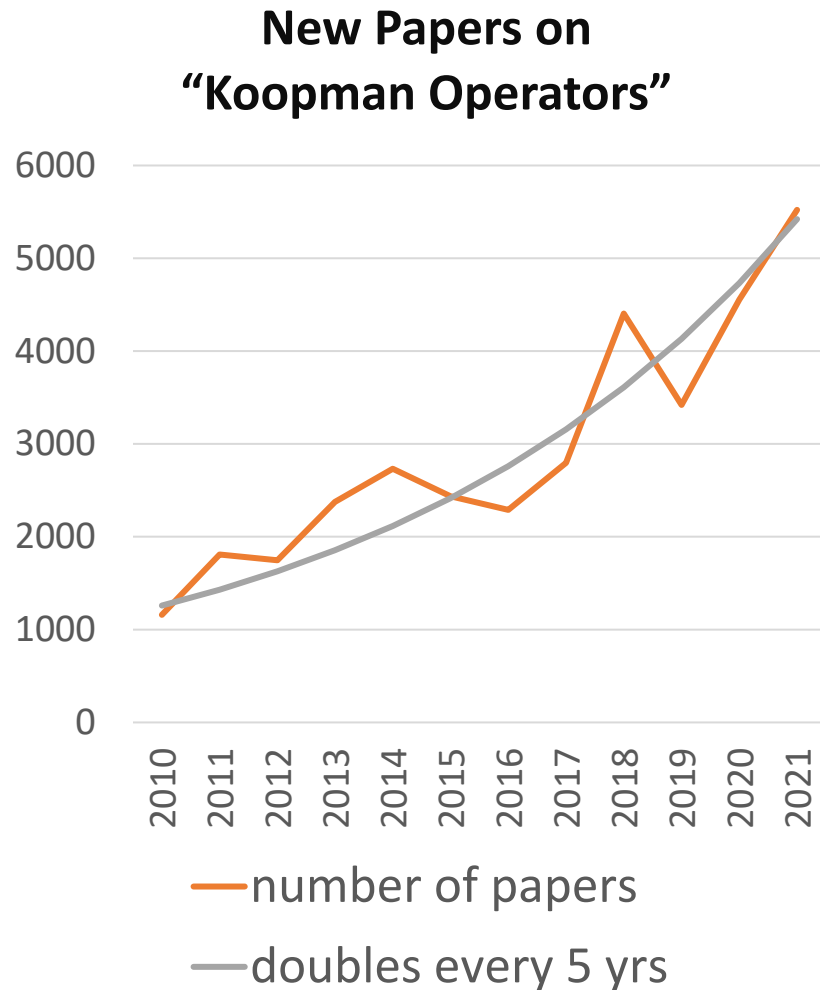
$$[\mathcal{K}g](x) = g(F(x))$$

Encodes: geometric features, invariant measures, transient behavior, long-time behavior, coherent structures, quasiperiodicity, etc.

GOAL: Data-driven approximation of \mathcal{K} and its spectral properties.

- Mezić, “Spectral properties of dynamical systems, model reduction and decompositions,” **Nonlinear Dynam.**, 2005.

Koopmania*: A revolution in the big data era?



≈35,000 papers over last decade!

BUT: Computing spectra in infinite dimensions is notoriously hard!

**Wikipedia: "its wild surge in popularity is sometimes jokingly called 'Koopmania'"*

Challenges of computing

$$\text{Spec}(\mathcal{K}) = \{\lambda \in \mathbb{C}: \mathcal{K} - \lambda I \text{ is not invertible}\}$$

Truncate: $\mathcal{K} \longrightarrow \mathbb{K} \in \mathbb{C}^{N_K \times N_K}$

- 1) **“Too much”:** Approximate spurious modes $\lambda \notin \text{Spec}(\mathcal{K})$
- 2) **“Too little”:** Miss parts of $\text{Spec}(\mathcal{K})$
- 3) **Continuous spectra.**

Verification: Is it right?

Build the matrix: Dynamic Mode Decomposition (DMD)

Given dictionary $\{\psi_1, \dots, \psi_{N_K}\}$ of functions $\psi_j: \Omega \rightarrow \mathbb{C}$,

$$\{x^{(m)}, y^{(m)} = F(x^{(m)})\}_{m=1}^M$$

$$\langle \psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_{N_K}(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_{N_K}(x^{(M)}) \end{pmatrix}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_{N_K}(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_{N_K}(x^{(M)}) \end{pmatrix}}_{\Psi_X} \right]_{jk}$$

$$\langle \mathcal{K}\psi_k, \psi_j \rangle \approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\begin{pmatrix} \psi_1(x^{(1)}) & \dots & \psi_{N_K}(x^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(x^{(M)}) & \dots & \psi_{N_K}(x^{(M)}) \end{pmatrix}}_{\Psi_X} \underbrace{\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_M \end{pmatrix}}_W \underbrace{\begin{pmatrix} \psi_1(y^{(1)}) & \dots & \psi_{N_K}(y^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_1(y^{(M)}) & \dots & \psi_{N_K}(y^{(M)}) \end{pmatrix}}_{\Psi_Y} \right]_{jk}$$

$$\mathcal{K} \longrightarrow \mathbb{K} = (\Psi_X^* W \Psi_X)^{-1} \Psi_X^* W \Psi_Y \in \mathbb{C}^{N_K \times N_K}$$

Recall open problems: too much, too little, continuous spectra, verification

- Schmid, "Dynamic mode decomposition of numerical and experimental data," **J. Fluid Mech.**, 2010.
- Rowley, Mezić, Bagheri, Schlatter, Henningson, "Spectral analysis of nonlinear flows," **J. Fluid Mech.**, 2009.
- Kutz, Brunton, Brunton, Proctor, "Dynamic mode decomposition: data-driven modeling of complex systems," **SIAM**, 2016.
- Williams, Kevrekidis, Rowley "A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition," **J. Nonlinear Sci.**, 2015.

Residual DMD (ResDMD): Approx. \mathcal{K} and $\mathcal{K}^*\mathcal{K}$

$$\begin{aligned}\langle \psi_k, \psi_j \rangle &\approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \psi_k(x^{(m)}) = \left[\underbrace{\Psi_X^* W \Psi_X}_G \right]_{jk} \\ \langle \mathcal{K}\psi_k, \psi_j \rangle &\approx \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})} = \left[\underbrace{\Psi_X^* W \Psi_Y}_{K_1} \right]_{jk} \\ \langle \mathcal{K}\psi_k, \mathcal{K}\psi_j \rangle &\approx \sum_{m=1}^M w_m \overline{\psi_j(y^{(m)})} \psi_k(y^{(m)}) = \left[\underbrace{\Psi_Y^* W \Psi_Y}_{K_2} \right]_{jk}\end{aligned}$$

Residuals: $g = \sum_{j=1}^{N_K} \mathbf{g}_j \psi_j, \quad \|\mathcal{K}g - \lambda g\|^2 \approx \mathbf{g}^* [K_2 - \lambda K_1^* - \bar{\lambda} K_1 + |\lambda|^2 G] \mathbf{g}$

-
- C., Townsend, “Rigorous data-driven computation of spectral properties of Koopman operators for dynamical systems,” preprint.
 - C., Ayton, Szóke, “Residual Dynamic Mode Decomposition,” **J. Fluid Mech.**, under minor rev.
 - Code: <https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition>

Quadrature with trajectory data

$$\text{E.g., } \langle \mathcal{K}\psi_k, \psi_j \rangle = \lim_{M \rightarrow \infty} \sum_{m=1}^M w_m \overline{\psi_j(x^{(m)})} \underbrace{\psi_k(y^{(m)})}_{[\mathcal{K}\psi_k](x^{(m)})}$$

Three examples:

- **High-order quadrature:** $\{x^{(m)}, w_m\}_{m=1}^M$ M -point quadrature rule.
 Rapid convergence. Requires free choice of $\{x^{(m)}\}_{m=1}^M$ and small d .
- **Random sampling:** $\{x^{(m)}\}_{m=1}^M$ selected at random.
 Large d . Slow Monte Carlo $O(M^{-1/2})$ rate of convergence. ← Most common
- **Ergodic sampling:** $x^{(m+1)} = F(x^{(m)})$.
 Single trajectory, large d . Requires ergodicity, convergence can be slow. ↘

ResDMD: avoiding “too much”

$$\text{res}(\lambda, \mathbf{g})^2 = \frac{\mathbf{g}^* [K_2 - \lambda K_1^* - \bar{\lambda} K_1 + |\lambda|^2 G] \mathbf{g}}{\mathbf{g}^* G \mathbf{g}}$$

eigenvectors

eigenvalues

Algorithm 1:

1. Compute $G, K_1, K_2 \in \mathbb{C}^{N_K \times N_K}$ and eigendecomposition $K_1 V = G V \Lambda$.
2. For each eigenpair (λ, \mathbf{v}) , compute $\text{res}(\lambda, \mathbf{v})$.
3. **Output:** subset of e-vectors $V_{(\varepsilon)}$ & e-vals $\Lambda_{(\varepsilon)}$ with $\text{res}(\lambda, \mathbf{v}) \leq \varepsilon$ ($\varepsilon = \text{input tol}$).

Theorem (no spectral pollution): Suppose quad. rule converges. Then

$$\limsup_{M \rightarrow \infty} \max_{\lambda \in \Lambda_{(\varepsilon)}} \|(\mathcal{K} - \lambda)^{-1}\|^{-1} \leq \varepsilon$$

ResDMD: avoiding “too much”

$$\text{res}(\lambda, \mathbf{g})^2 = \frac{\mathbf{g}^* [K_2 - \lambda K_1^* - \bar{\lambda} K_1 + |\lambda|^2 G] \mathbf{g}}{\mathbf{g}^* G \mathbf{g}}$$

eigenvectors

eigenvalues

Algorithm 1:

1. Compute $G, K_1, K_2 \in \mathbb{C}^{N_K \times N_K}$ and eigendecomposition $K_1 V = G V \Lambda$.
2. For each eigenpair (λ, \mathbf{v}) , compute $\text{res}(\lambda, \mathbf{v})$.
3. **Output:** subset of e-vectors $V_{(\varepsilon)}$ & e-vals $\Lambda_{(\varepsilon)}$ with $\text{res}(\lambda, \mathbf{v}) \leq \varepsilon$ ($\varepsilon = \text{input tol}$).

Theorem (no spectral pollution): Suppose quad. rule converges. Then

$$\limsup_{M \rightarrow \infty} \max_{\lambda \in \Lambda_{(\varepsilon)}} \|(\mathcal{K} - \lambda)^{-1}\|^{-1} \leq \varepsilon$$

BUT: Typically, does not capture all of spectrum! (“too little”)

ResDMD: avoiding “too little”

$$\text{Spec}_\varepsilon(\mathcal{K}) = \bigcup_{\|\mathcal{B}\| \leq \varepsilon} \text{Spec}(\mathcal{K} + \mathcal{B}), \quad \lim_{\varepsilon \downarrow 0} \text{Spec}_\varepsilon(\mathcal{K}) = \text{Spec}(\mathcal{K})$$

Algorithm 2:

First convergent method for general \mathcal{K}

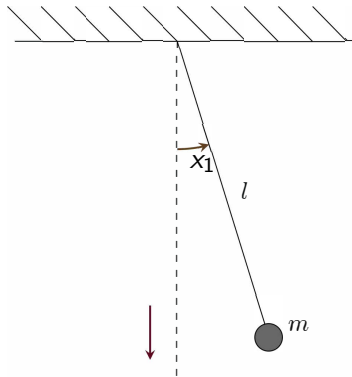
1. Compute $G, K_1, K_2 \in \mathbb{C}^{N_K \times N_K}$.
2. For z_k in comp. grid, compute $\tau_k = \min_{g = \sum_{j=1}^{N_K} \mathbf{g}_j \psi_j} \text{res}(z_k, g)$, corresponding g_k (gen. SVD).
3. **Output:** $\{z_k: \tau_k < \varepsilon\}$ (approx. of $\text{Spec}_\varepsilon(\mathcal{K})$), $\{g_k: \tau_k < \varepsilon\}$ (ε -pseudo-eigenfunctions).

Theorem (full convergence): Suppose the quadrature rule converges.

- **Error control:** $\{z_k: \tau_k < \varepsilon\} \subseteq \text{Spec}_\varepsilon(\mathcal{K})$ (as $M \rightarrow \infty$)
- **Convergence:** Converges locally uniformly to $\text{Spec}_\varepsilon(\mathcal{K})$ (as $N_K \rightarrow \infty$)

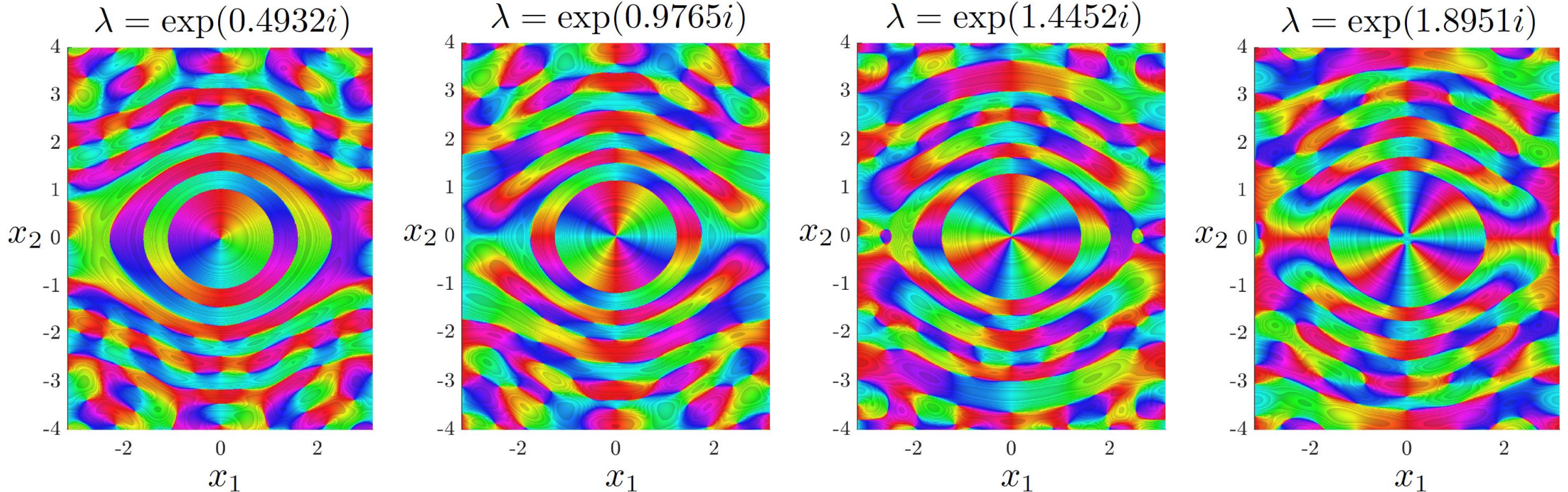
Example: non-linear pendulum

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin(x_1), \quad \Omega = [-\pi, \pi]_{\text{per}} \times \mathbb{R}$$



Computed pseudospectra ($\varepsilon = 0.25$). Eigenvalues of \mathbb{K} shown as dots (spectral pollution).

Approximate eigenfunctions



Colour represents complex argument, constant modulus shown as shadowed steps.
All residuals smaller than $\varepsilon = 0.05$ (made smaller by increasing N_K).

The Challenges

- ~~1) “Too much”: Approximate spurious modes $\lambda \notin \text{Spec}(\mathcal{K})$~~ ✓
- ~~2) “Too little”: Miss parts of $\text{Spec}(\mathcal{K})$~~ ✓
- 3) Continuous spectra.

Verification: Is it right?

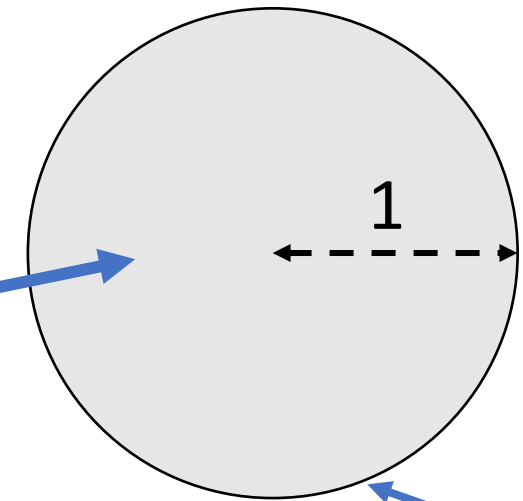
Setup for continuous spectra

No such assumption
was made in first part of talk!

Suppose system is measure-preserving (e.g., Hamiltonian, ergodic, post-transient etc.)

$$\Leftrightarrow \mathcal{K}^* \mathcal{K} = I \text{ (isometry)}$$

$$\Rightarrow \text{Spec}(\mathcal{K}) \subseteq \{z: |z| \leq 1\}$$



spectral
measure
supp. on
boundary

(NB: we consider unitary extensions via Wold decomposition.)

Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{Projector onto Span}(v_k)}}}{v_k v_k^*} \right) v, \quad Av = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{eigenvalues}}}{\lambda_k} v_k v_k^* \right) v, \quad v \in \mathbb{C}^n$$

Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{Projector onto Span}(v_k)}}{v_k v_k^*} \right) v, \quad Av = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{eigenvalues}}}{\lambda_k} v_k v_k^* \right) v, \quad v \in \mathbb{C}^n$$

Energy of “v” in each eigenvector: $\mu_v(\lambda_j) = \langle v_j v_j^* v, v \rangle = |v_j^* v|^2$

This is called the spectral measure with respect to a vector v .

Spectral decomposition of operators

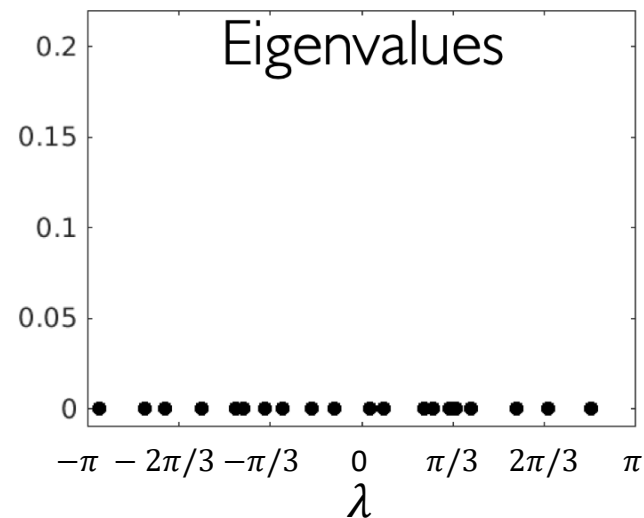
$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n v_k v_k^* \right) v, \quad Av = \left(\sum_{k=1}^n \lambda_k v_k v_k^* \right) v, \quad v \in \mathbb{C}^n$$

↑
Projector onto $\text{Span}(v_k)$
↑
eigenvalues

Energy of “v” in each eigenvector: $\mu_v(\lambda_j) = \langle v_j v_j^* v, v \rangle = |v_j^* v|^2$

This is called the spectral measure with respect to a vector v .



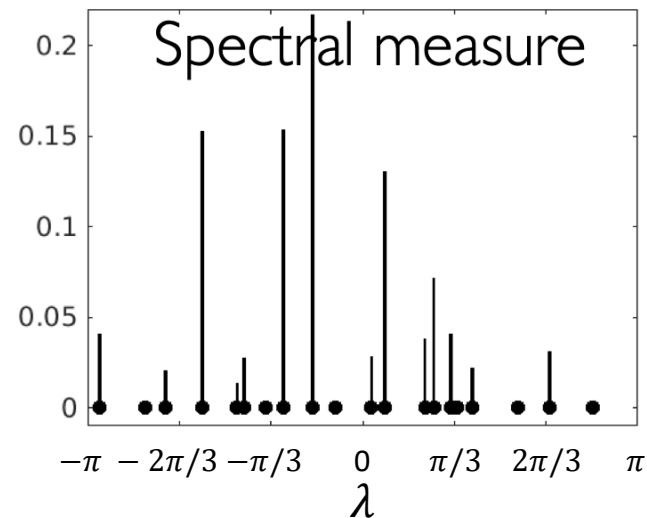
Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{Projector onto Span}(v_k)}}{v_k v_k^*} \right) v, \quad Av = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{eigenvalues}}}{\lambda_k v_k v_k^*} \right) v, \quad v \in \mathbb{C}^n$$

Energy of “v” in each eigenvector: $\mu_v(\lambda_j) = \langle v_j v_j^* v, v \rangle = |v_j^* v|^2$

This is called the spectral measure with respect to a vector v .



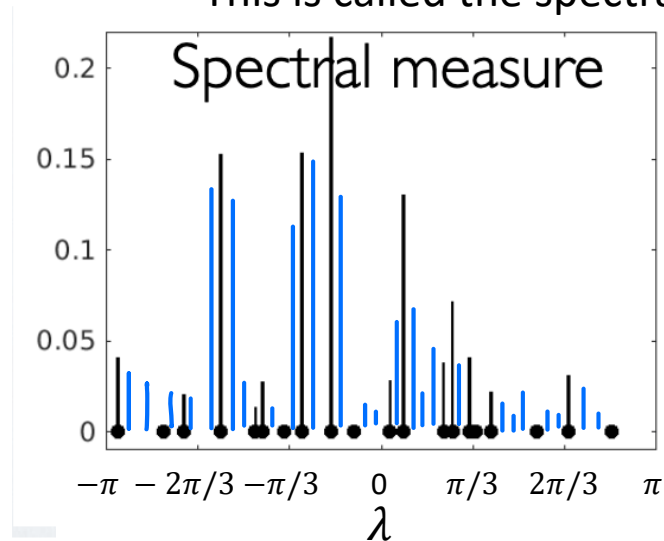
Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{Projector onto Span}(v_k)}}}{v_k v_k^*} \right) v, \quad Av = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{eigenvalues}}}{\lambda_k} v_k v_k^* \right) v, \quad v \in \mathbb{C}^n$$

Energy of “v” in each eigenvector: $\mu_v(\lambda_j) = \langle v_j v_j^* v, v \rangle = |v_j^* v|^2$

This is called the spectral measure with respect to a vector v .



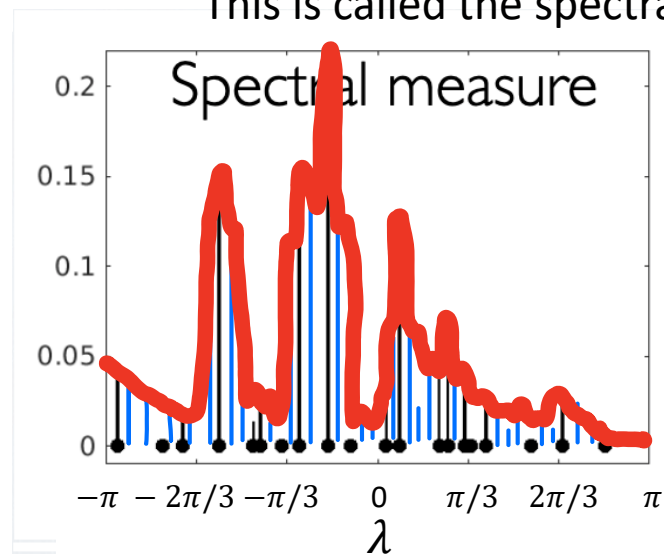
Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{Projector onto Span}(v_k)}}{v_k v_k^*} \right) v, \quad Av = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{eigenvalues}}}{\lambda_k v_k v_k^*} \right) v, \quad v \in \mathbb{C}^n$$

Energy of “v” in each eigenvector: $\mu_v(\lambda_j) = \langle v_j v_j^* v, v \rangle = |v_j^* v|^2$

This is called the spectral measure with respect to a vector v .



Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow O.N. basis of eigenvectors v_1, \dots, v_n :

$$v = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{Projector onto Span}(v_k)}}{v_k v_k^*} \right) v, \quad Av = \left(\sum_{k=1}^n \underset{\substack{\uparrow \\ \text{eigenvalues}}}{\lambda_k} v_k v_k^* \right) v, \quad v \in \mathbb{C}^n$$

Energy of “v” in each eigenvector: $\mu_v(\lambda_j) = \langle v_j v_j^* v, v \rangle = |v_j^* v|^2$

This is called the spectral measure with respect to a vector v .

\mathcal{K} is unitary \Rightarrow projection-valued measure ξ

$$g = \left(\int_{\mathbb{T}} d\xi(y) \right) g, \quad \mathcal{K}g = \left(\int_{\mathbb{T}} y d\xi(y) \right) g$$

Spectral measure $\nu_g(B) = \langle \xi(B)g, g \rangle$

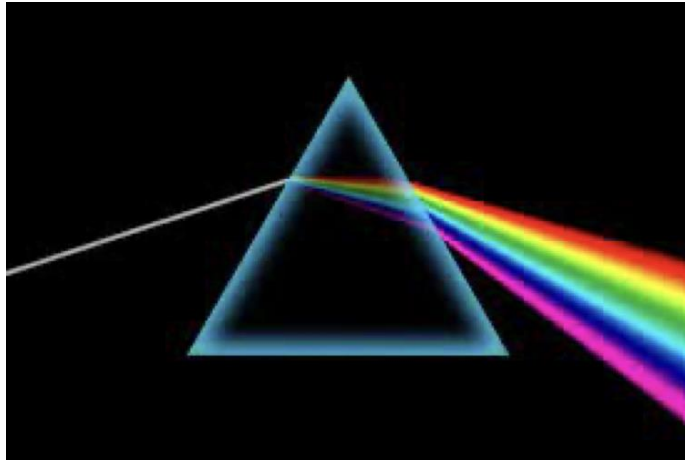
Spectral decomposition of operators

$A \in \mathbb{C}^{n \times n}$ normal \Rightarrow

\Rightarrow

O

White light contains a continuous spectrum



$\text{an}(v_k)$

eigenvector:

This is ca

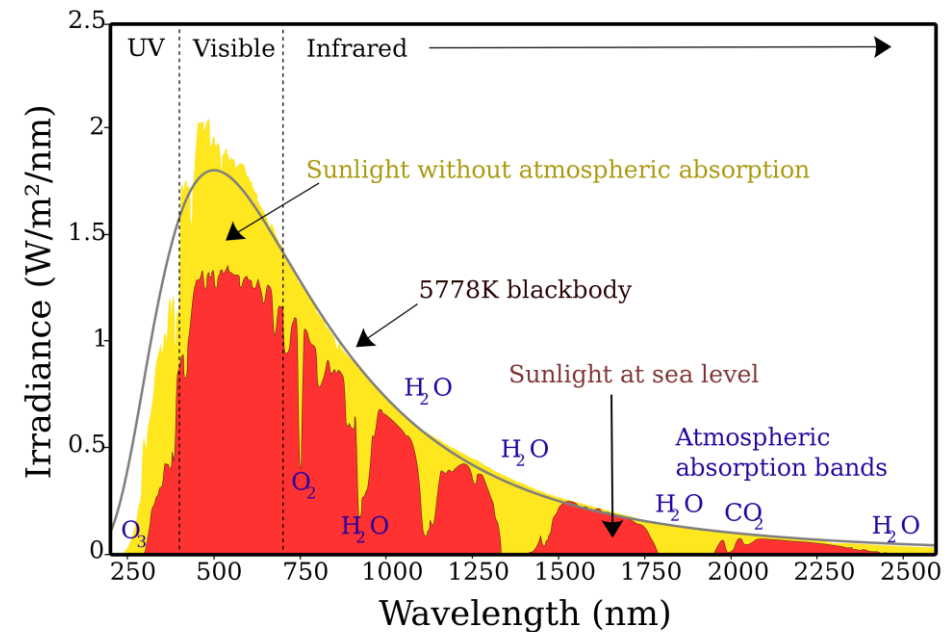
\Rightarrow

$$g = \left(\int_{\mathbb{T}} d\xi(y) \right) g,$$

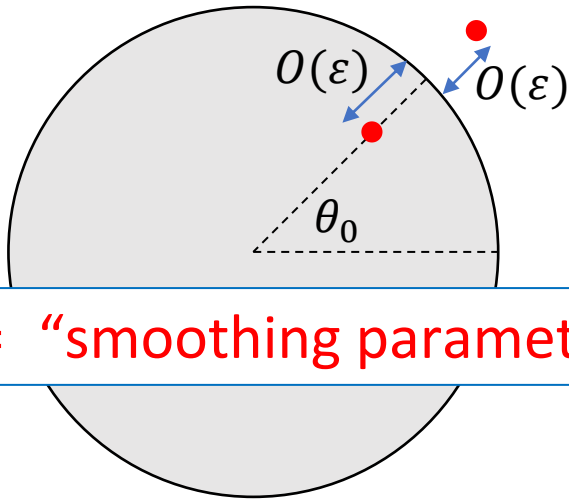
Spectral measure $\nu_g(B) = \langle \xi(B)g, g \rangle$

Often interesting to look at
the intensity of each wavelength

Spectrum of Solar Radiation (Earth)



Evaluating spectral measure



$\varepsilon =$ “smoothing parameter”

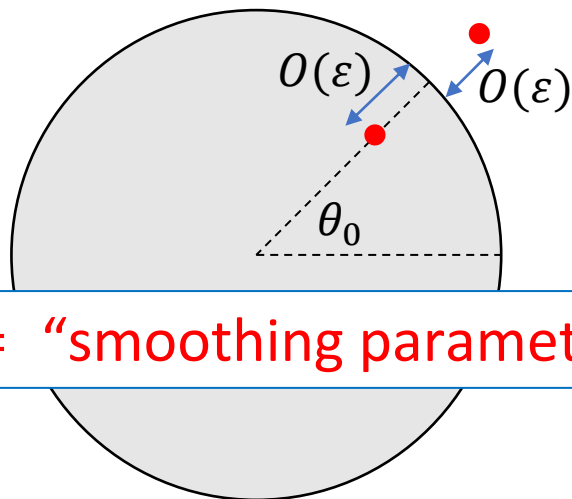
$$[P_\varepsilon * \nu_g](\theta_0) = \int_{[-\pi, \pi]_{\text{per}}} P_\varepsilon(\theta_0 - \theta) d\nu_g(\theta)$$

Smoothing convolution

Poisson kernel for
unit disk

$$P_\varepsilon(\theta_0) = \frac{1}{2\pi} \frac{(1 + \varepsilon)^2 - 1}{1 + (1 + \varepsilon)^2 - 2(1 + \varepsilon)\cos(\theta_0)}$$

Evaluating spectral measur



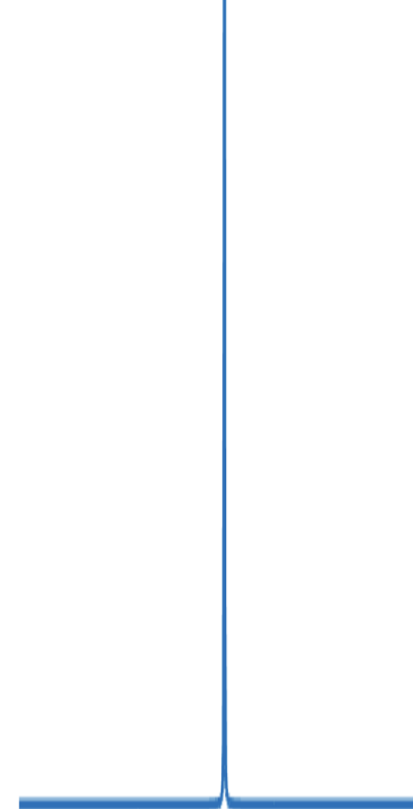
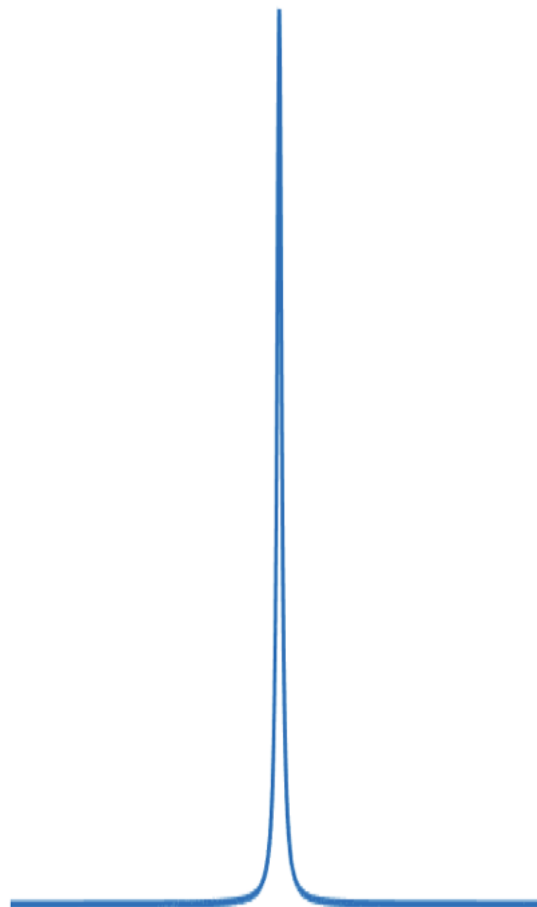
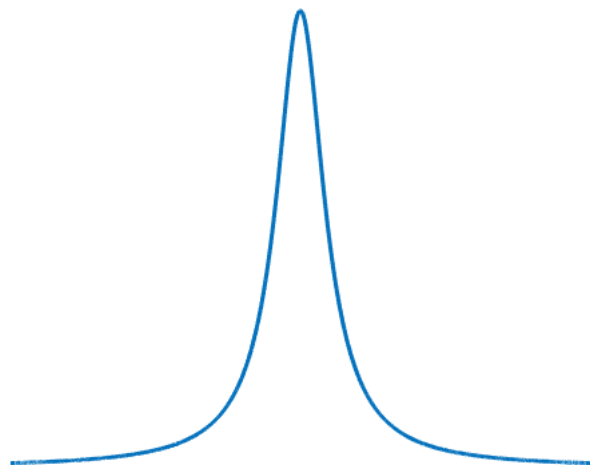
$$[P_\epsilon * \nu_g](\theta_0) = \int_{\mathbb{T}} P_\epsilon(\theta_0 - \theta) d\nu_g(\theta)$$

Smoothing cc

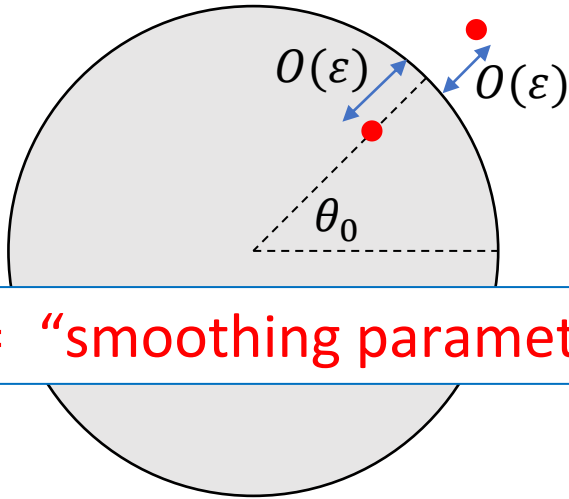
Poisso
unit di

$$\frac{1}{1 + \epsilon}$$

$\overline{0}$



Evaluating spectral measure



$\varepsilon =$ “smoothing parameter”

Smoothing convolution

$$[P_\varepsilon * \nu_g](\theta_0) = \int_{[-\pi, \pi]_{\text{per}}} P_\varepsilon(\theta_0 - \theta) d\nu_g(\theta)$$

Poisson kernel for
unit disk

$$P_\varepsilon(\theta_0) = \frac{1}{2\pi} \frac{(1 + \varepsilon)^2 - 1}{1 + (1 + \varepsilon)^2 - 2(1 + \varepsilon)\cos(\theta_0)}$$

$$[P_\varepsilon * \nu_g](\theta_0) = \mathcal{C}_g(e^{i\theta_0}(1 + \varepsilon)^{-1}) - \mathcal{C}_g(e^{i\theta_0}(1 + \varepsilon))$$

$$\mathcal{C}_g(z) = \int_{[-\pi, \pi]_{\text{per}}} \frac{e^{i\theta} d\nu_g(\theta)}{e^{i\theta} - z} = \begin{cases} \langle (\mathcal{K} - zI)^{-1} g, \mathcal{K}^* g \rangle, & \text{if } |z| > 1 \\ -z^{-1} \langle g, (\mathcal{K} - \bar{z}^{-1}I)^{-1} g \rangle, & \text{if } 0 < |z| < 1 \end{cases}$$

ResDMD computes
with error control

Example

$$\mathcal{K} = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_0\rho_1 & & & \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\alpha_0\rho_1 & & & \\ & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & \\ & \rho_2\rho_1 & -\alpha_1\rho_2 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & \ddots \\ & & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$\alpha_j = (-1)^j 0.95^{(j+1)/2}, \quad \rho_j = \sqrt{1 - |\alpha_j|^2}$$

Generalized shift, typical building block of many dynamical systems.

Fix N_K , vary ε : unstable!

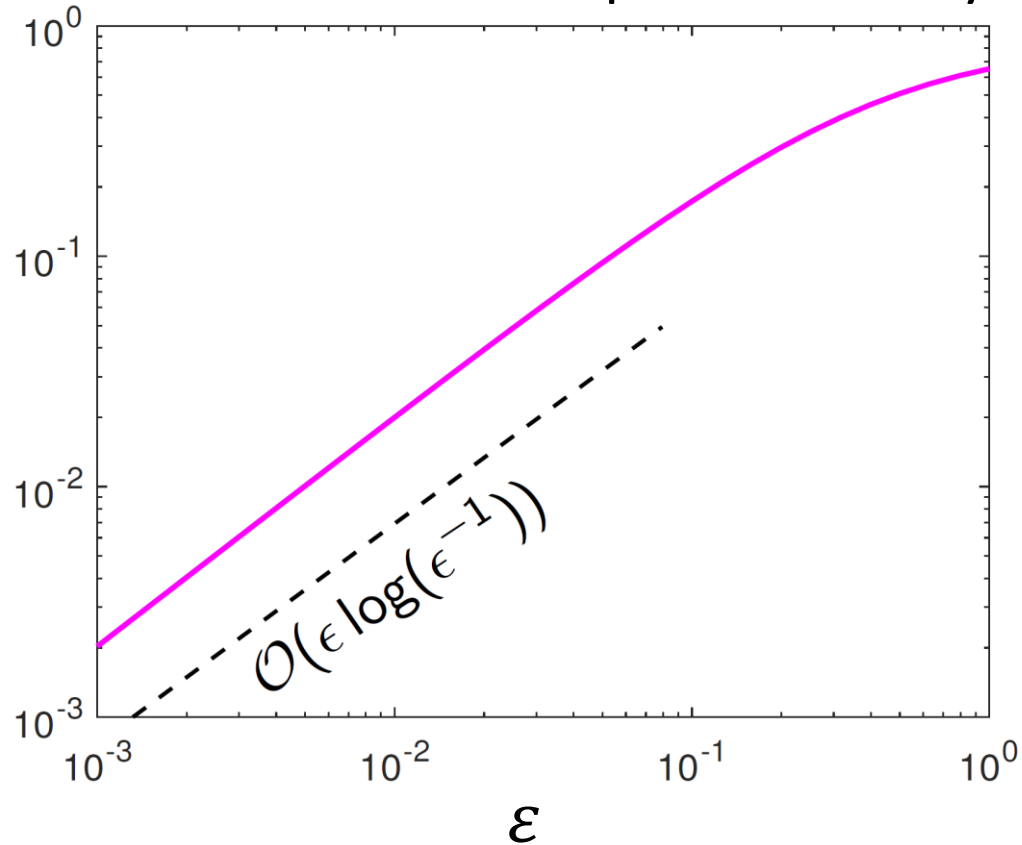
Fix ε , vary N_K : too smooth!

Adaptive: new matrix to compute residuals crucial

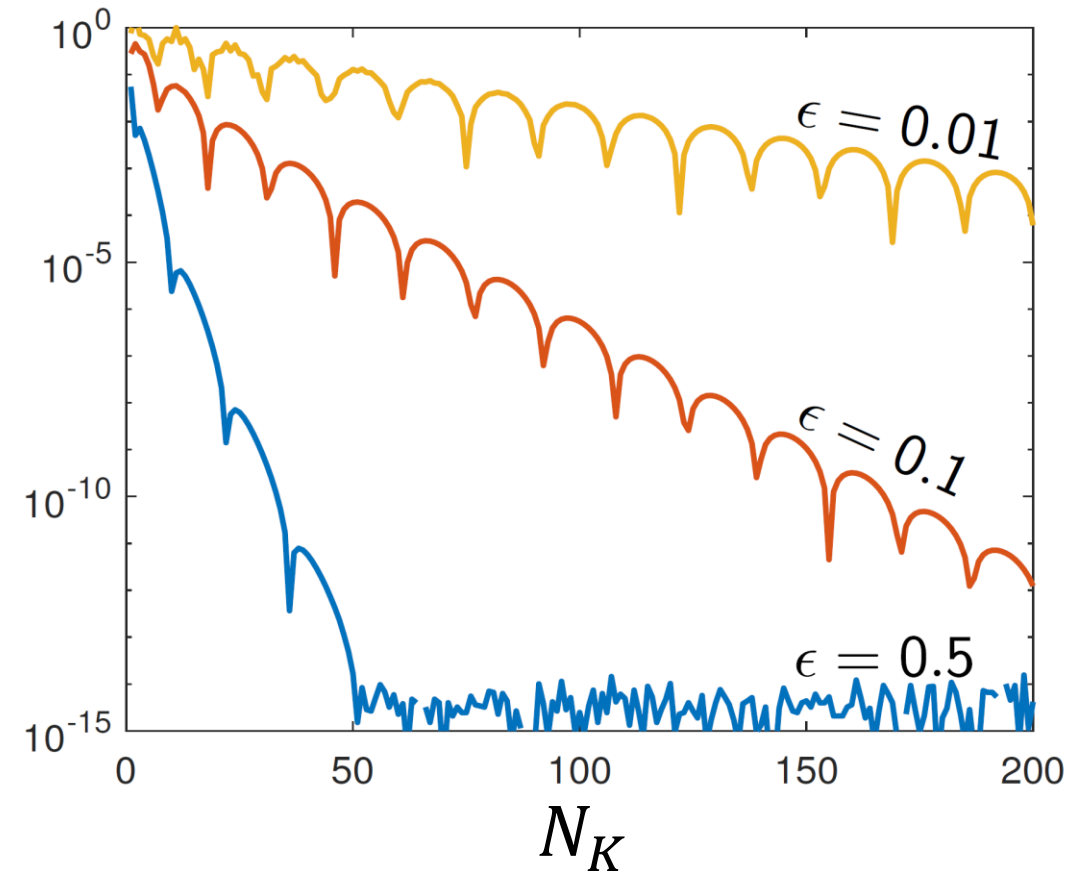
But ... slow convergence

Problem: As $\varepsilon \downarrow 0$, error is $O(\varepsilon \log(1/\varepsilon))$ and $N_K(\varepsilon) \rightarrow \infty$.

Pointwise error for spectral density



Error due to discretization

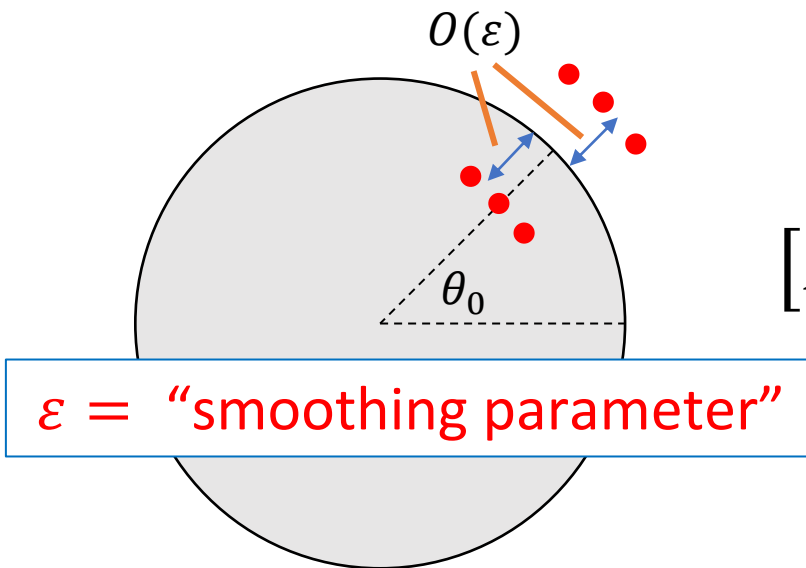


Small N_K critical in data-driven computations. Can we improve convergence rate?

High-order rational kernels

m th order rational kernels:

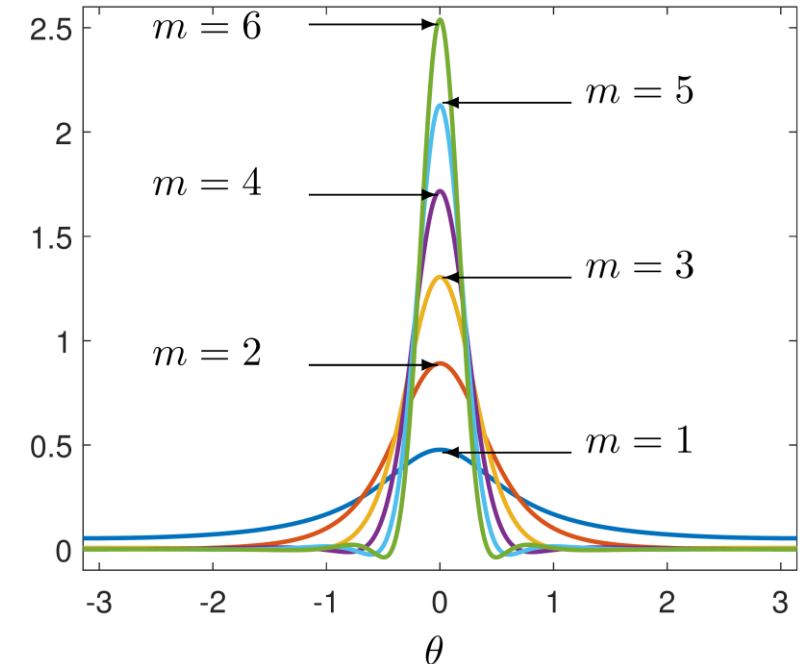
$$K_\varepsilon(\theta) = \frac{e^{-i\theta}}{2\pi} \sum_{j=1}^m \left[\frac{c_j}{e^{-i\theta} - (1 + \varepsilon \bar{z}_j)^{-1}} - \frac{d_j}{e^{-i\theta} - (1 + \varepsilon z_j)} \right]$$



ResDMD computes
with error control

$$[K_\varepsilon * v_g](\theta_0) = \sum_{j=1}^m \left[c_j \mathcal{C}_g(e^{i\theta_0}(1 + \varepsilon \bar{z}_j)^{-1}) - d_j \mathcal{C}_g(e^{i\theta_0}(1 + \varepsilon z_j)) \right]$$

Kernels



Smaller N_K (larger ε)

Convergence

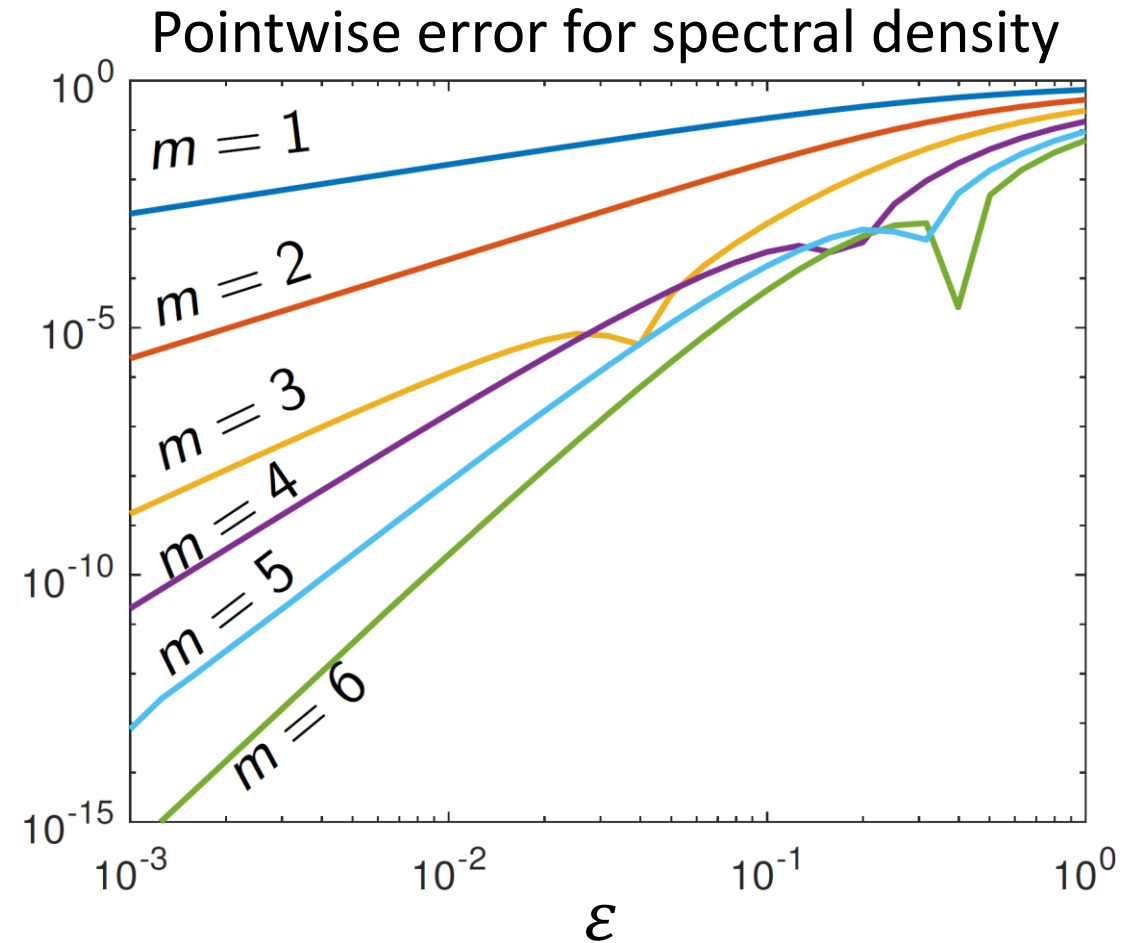
Theorem: Automatic selection of $N_K(\varepsilon)$ with $O(\varepsilon^m \log(1/\varepsilon))$ convergence:

- Density of continuous spectrum ρ_g .
(pointwise and L^p)
- Integration against test functions.
(weak convergence)

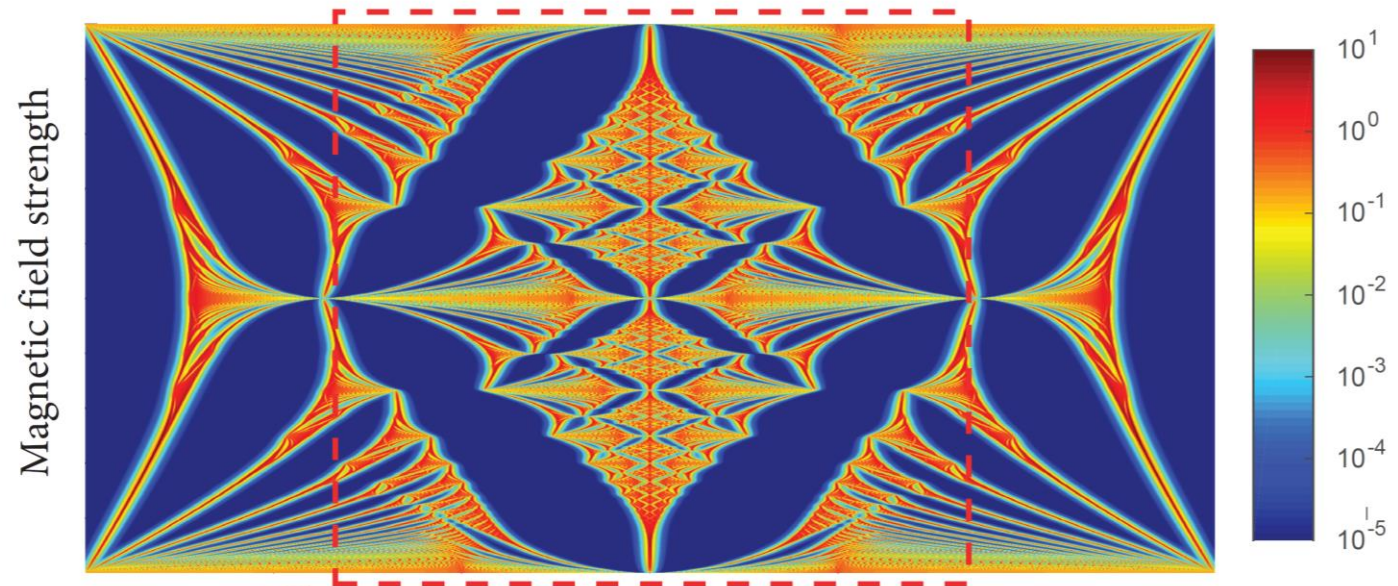
$$\int_{[-\pi, \pi]_{\text{per}}} h(\theta) [K_\varepsilon * \nu_g](\theta) \, d\theta$$

$$= \int_{[-\pi, \pi]_{\text{per}}} h(\theta) \, d\nu_g(\theta) + O(\varepsilon^m \log(1/\varepsilon))$$

Also recover discrete spectrum.



Spectral measures of self-adjoint operators



Horizontal slice = spectral measure at constant magnetic field strength.

Software package

SpecSolve available at <https://github.com/SpecSolve>
 Capabilities: ODEs, PDEs, integral operators, discrete operators.

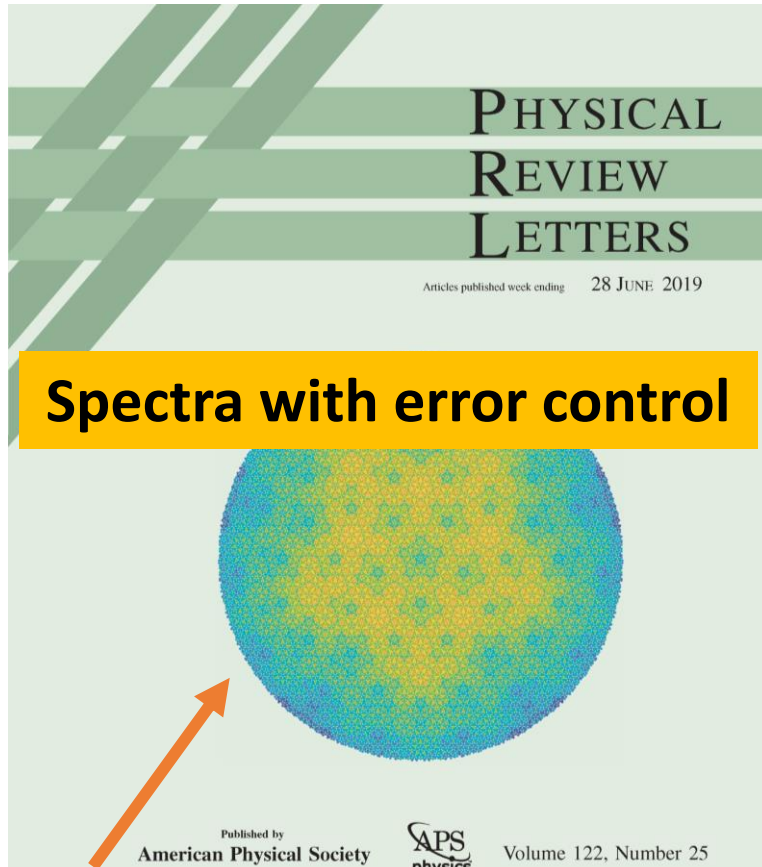
The Challenges

- ~~1) “Too much”: Approximate spurious modes $\lambda \notin \text{Spec}(\mathcal{K})$~~ ✓
- ~~2) “Too little”: Miss parts of $\text{Spec}(\mathcal{K})$~~ ✓
- ~~3) Continuous spectra.~~ ✓

Verification: Is it right?

Is it right?

The importance of verification

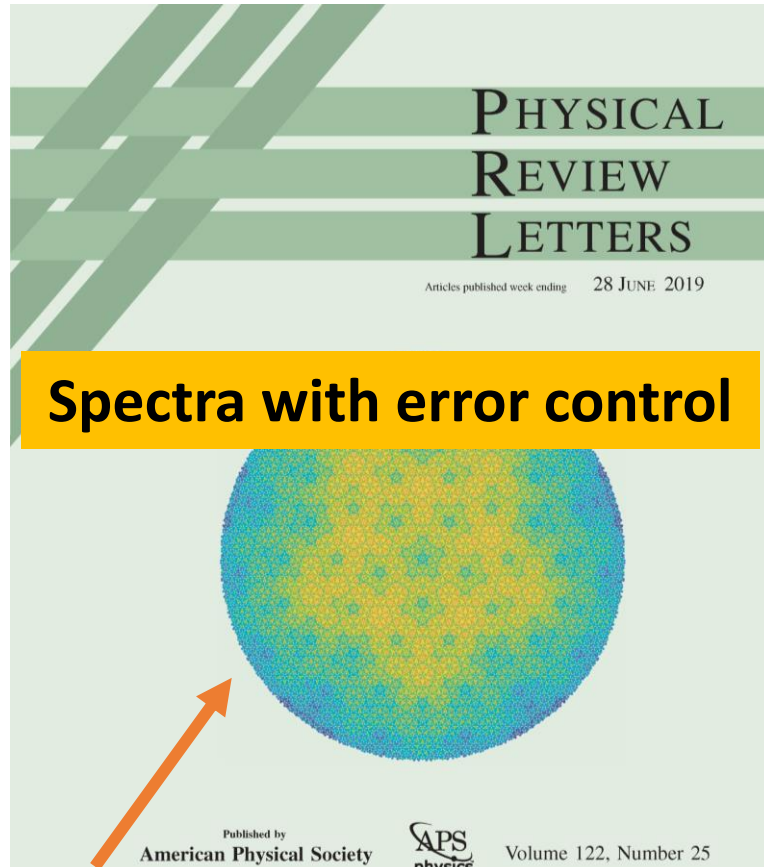


E.g., ground state of quasicrystal

- C., Roman, Hansen, “How to compute spectra with error control,” *Phys. Rev. Lett.*, 2019.

Is it right?

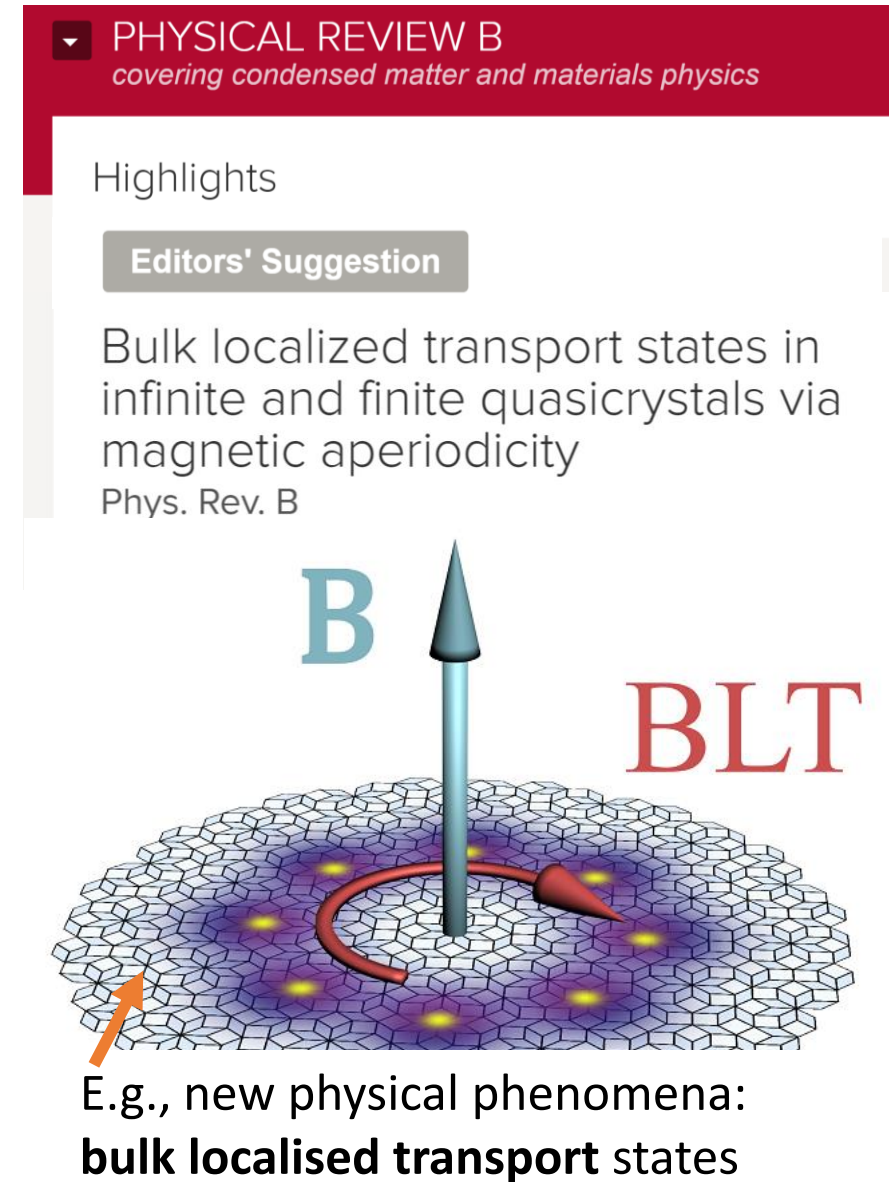
The importance of verification



E.g., ground state of quasicrystal

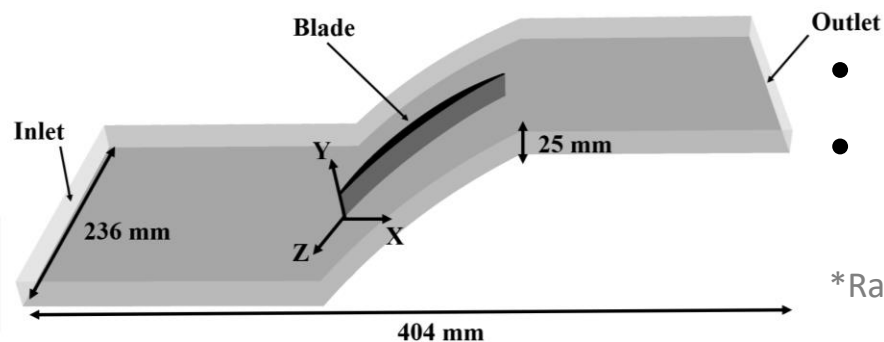
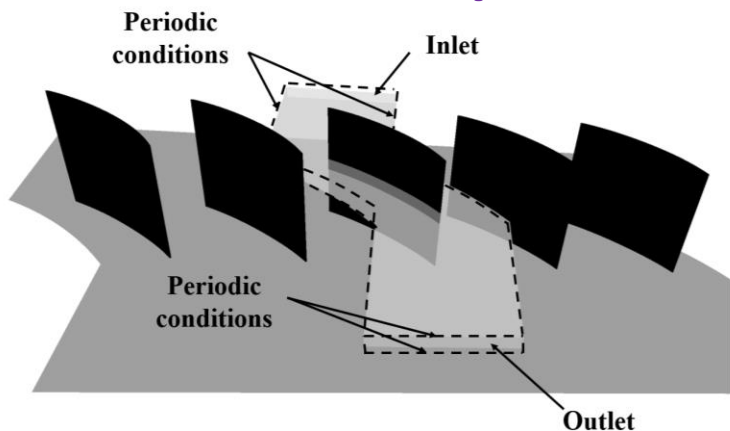


**Certainty in computed
spectral properties**



- C., Roman, Hansen, "How to compute spectra with error control," **Phys. Rev. Lett.**, 2019.
- Johnstone, C., Nielsen, Öhberg, Duncan, "Bulk Localised Transport States in Infinite and Finite Quasicrystals via Magnetic Aperiodicity," **Phys. Rev. B**, 2022.

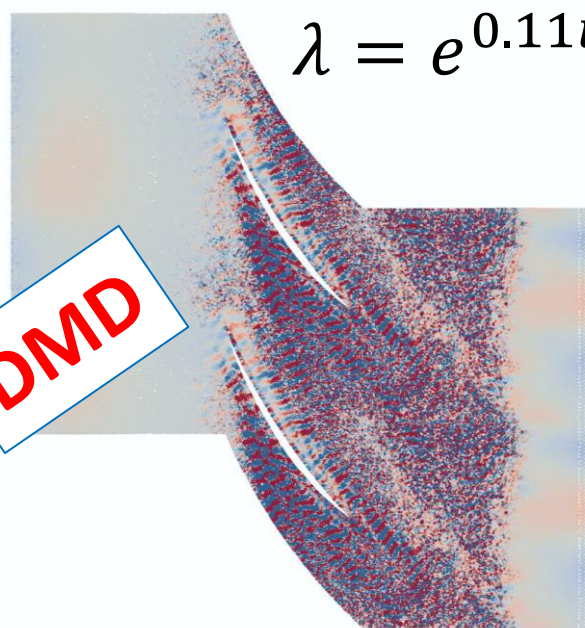
Example: Trustworthy computation for large d



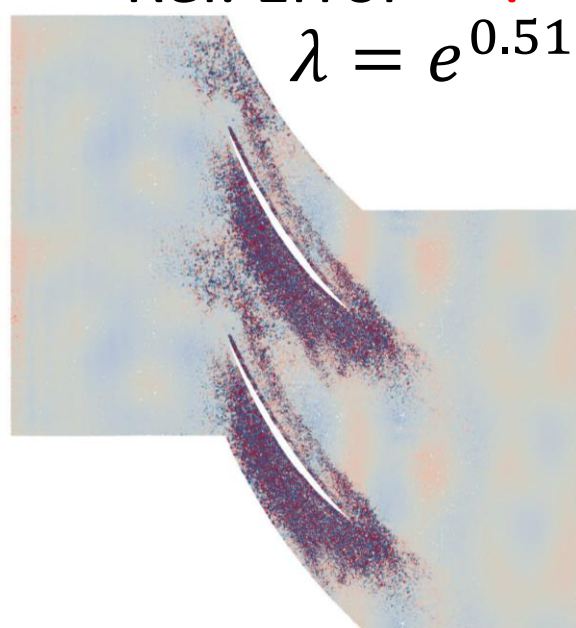
- Reynolds number $\approx 3.9 \times 10^5$
- Ambient dimension (d) $\approx 300,000$ (number of measurement points)

*Raw measurements provided by Stephane Moreau (Sherbrooke)

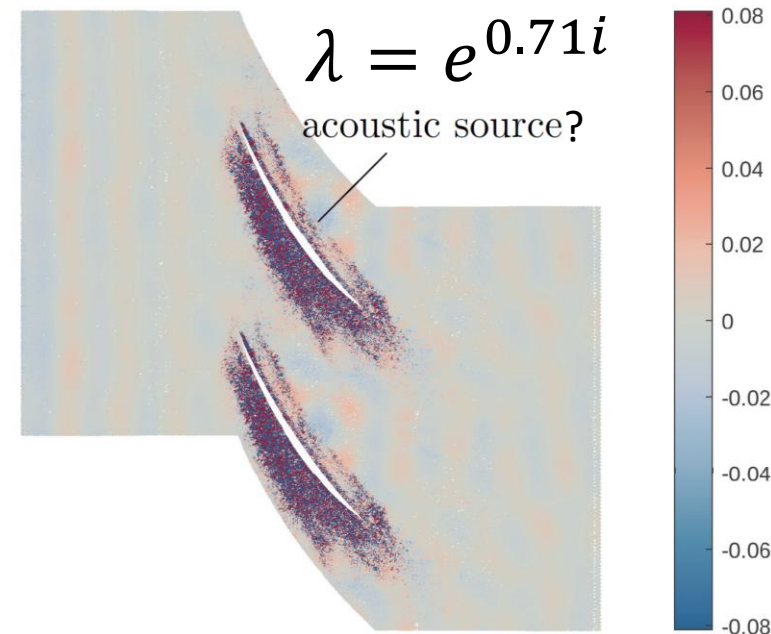
Rel. Error = ?
 $\lambda = e^{0.11i}$



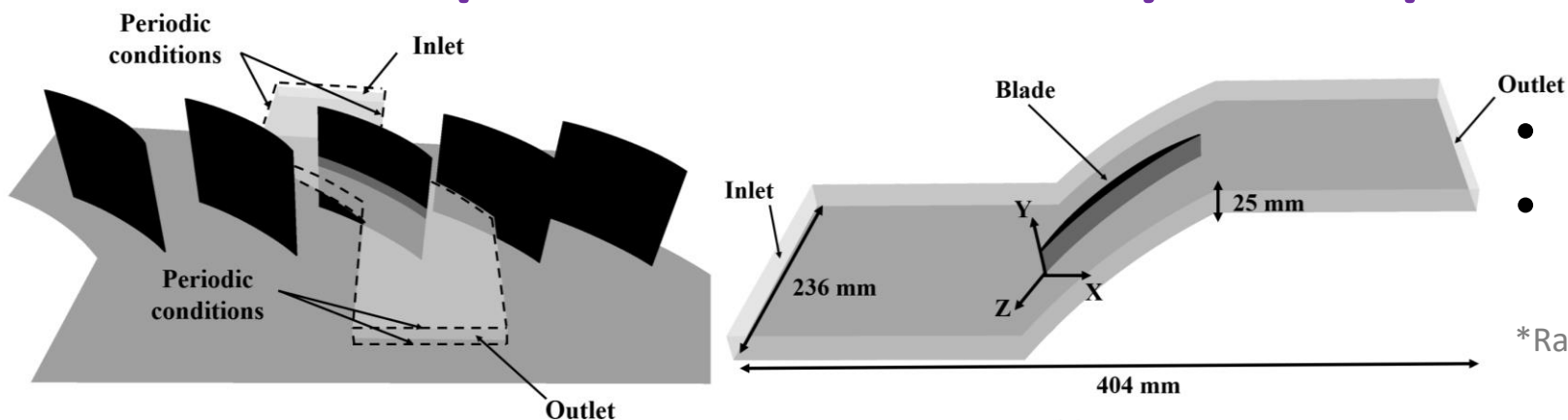
Rel. Error = ?
 $\lambda = e^{0.51i}$



Rel. Error = ?
 $\lambda = e^{0.71i}$
 acoustic source?



Example: Trustworthy computation for large d



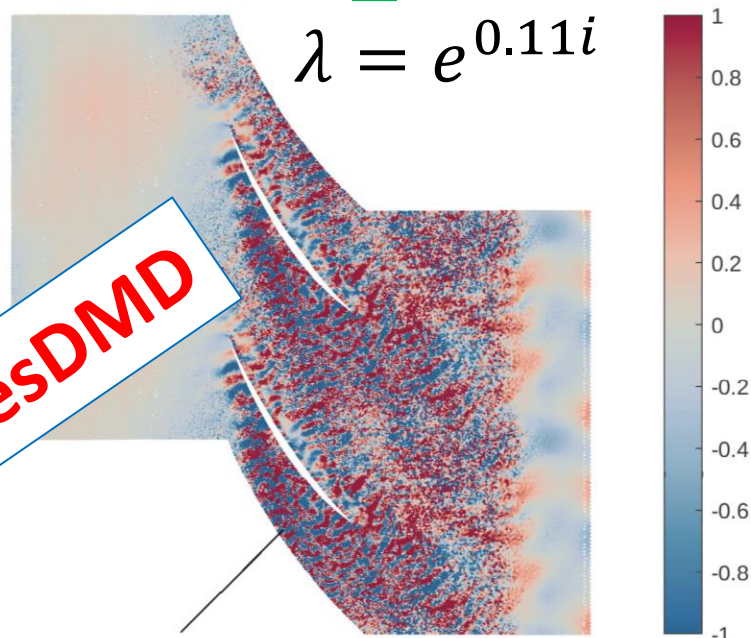
- Reynolds number $\approx 3.9 \times 10^5$
- Ambient dimension (d) $\approx 300,000$ (number of measurement points)

*Raw measurements provided by Stephane Moreau (Sherbrooke)

Rel. Error ≤ 0.0054

$$\lambda = e^{0.11i}$$

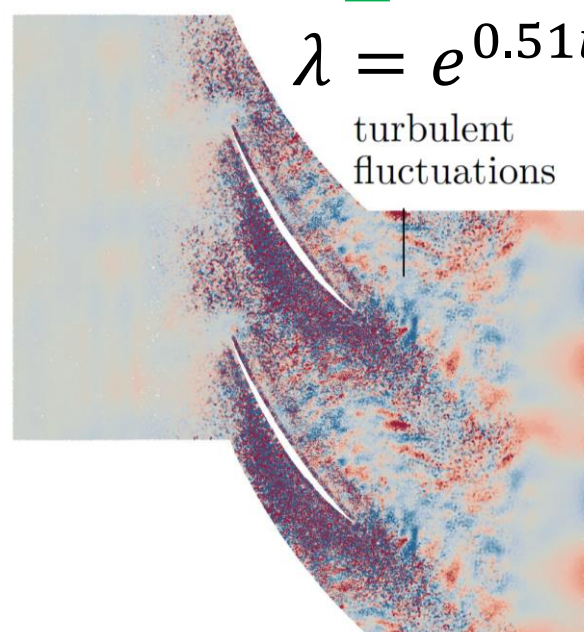
ResDMD



acoustic vibrations

Rel. Error ≤ 0.0128

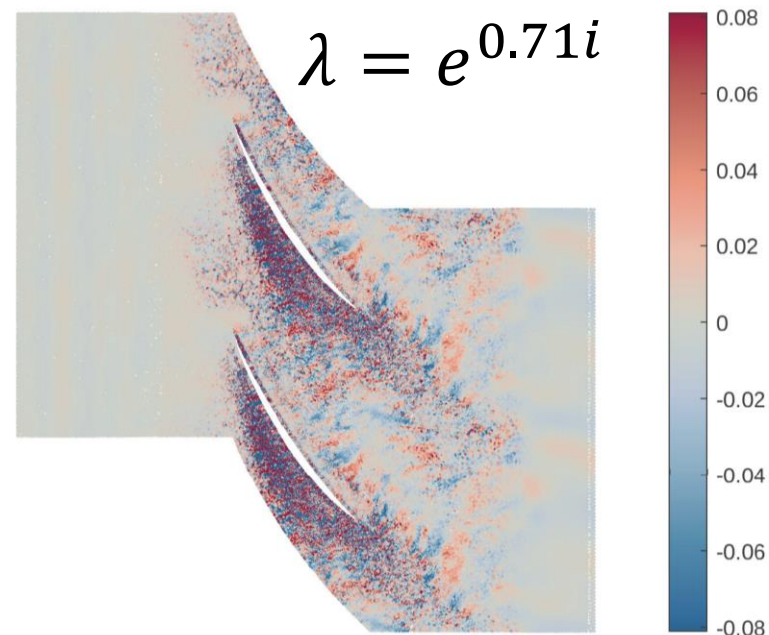
$$\lambda = e^{0.51i}$$



turbulent
fluctuations

Rel. Error ≤ 0.0196

$$\lambda = e^{0.71i}$$



Large d ($\Omega \subseteq \mathbb{R}^d$): robust and scalable

Popular to learn dictionary $\{\psi_1, \dots, \psi_{N_K}\}$

E.g., DMD with truncated SVD (linear dictionary, most popular), kernel methods (this talk), neural networks, etc.

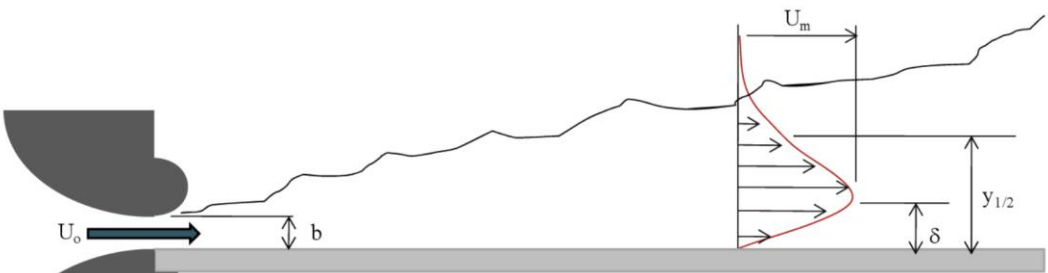
Q: Is discretisation $\text{span}\{\psi_1, \dots, \psi_{N_K}\}$ large/rich enough?

Above algorithms:

- Pseudospectra: $\{z_k : \tau_k < \varepsilon\} \subseteq \text{Spec}_\varepsilon(\mathcal{K})$ **error control**
- Spectral measures: $\mathcal{C}_g(z)$ and smoothed measures **adaptive check**

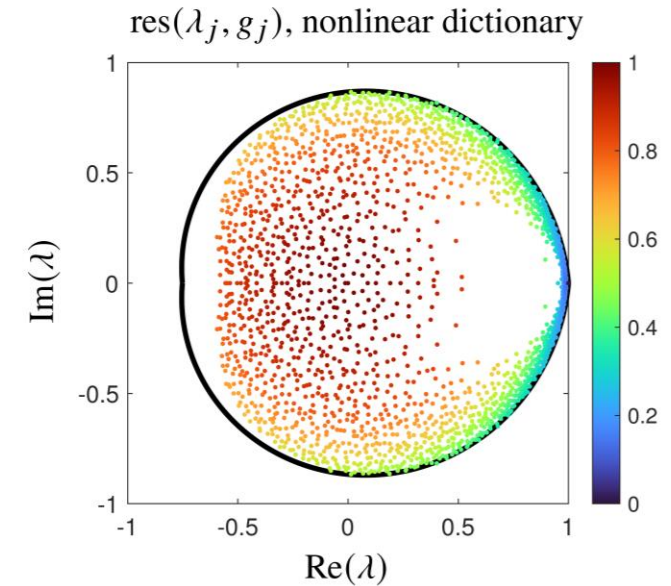
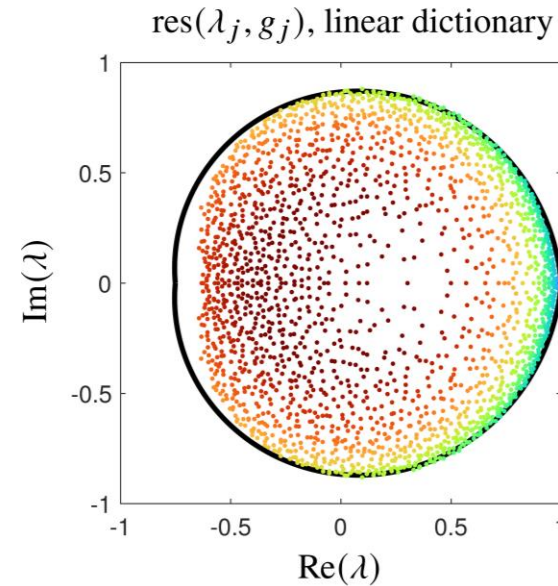
\Rightarrow Rigorously **verify** learnt dictionary $\{\psi_1, \dots, \psi_{N_K}\}$

Example: Verify the dictionary

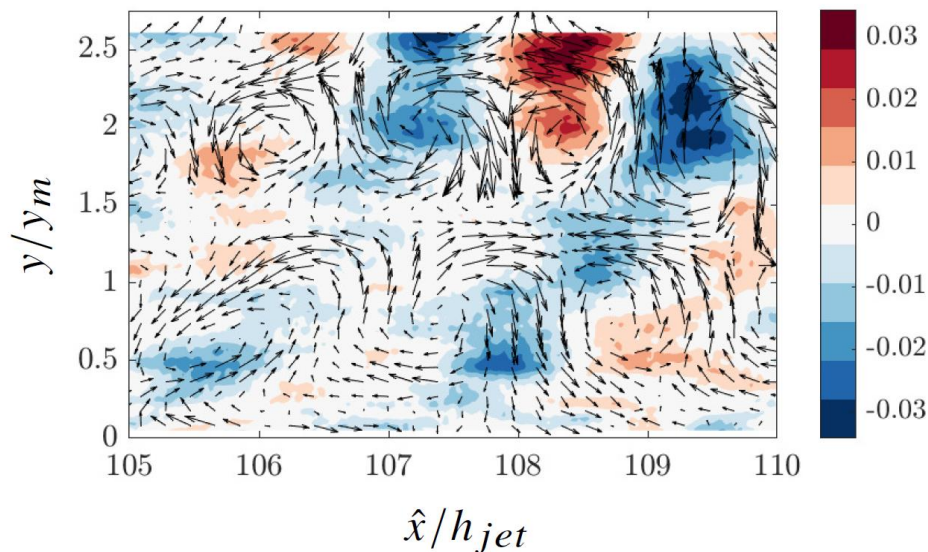


- Reynolds number $\approx 6.4 \times 10^4$
- Ambient dimension (d) $\approx 100,000$ (velocity at measurement points)

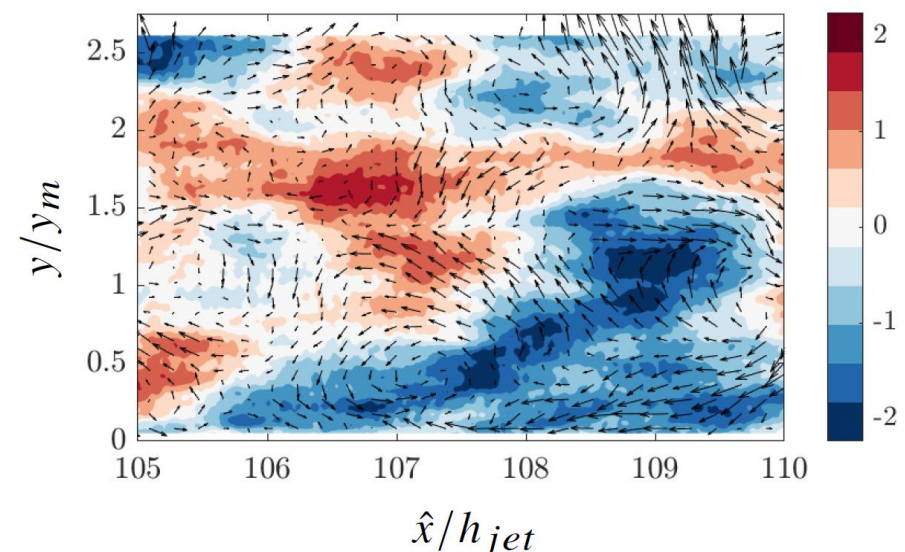
*Raw measurements provided by Máté Szőke (Virginia Tech)



$$\lambda = 0.9439 + 0.2458i, \text{ error} \leq 0.0765$$

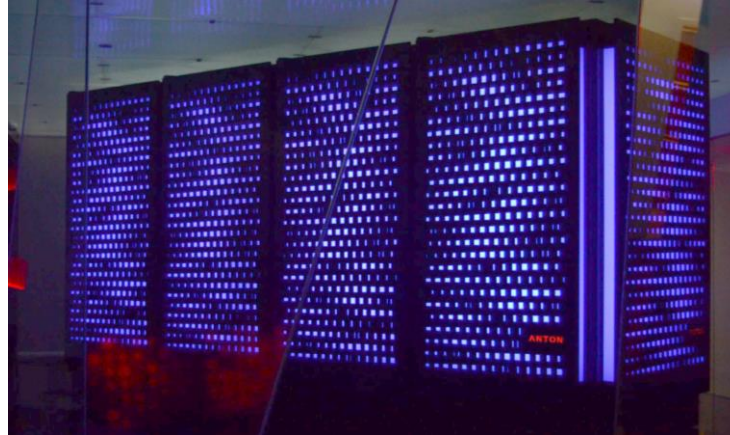
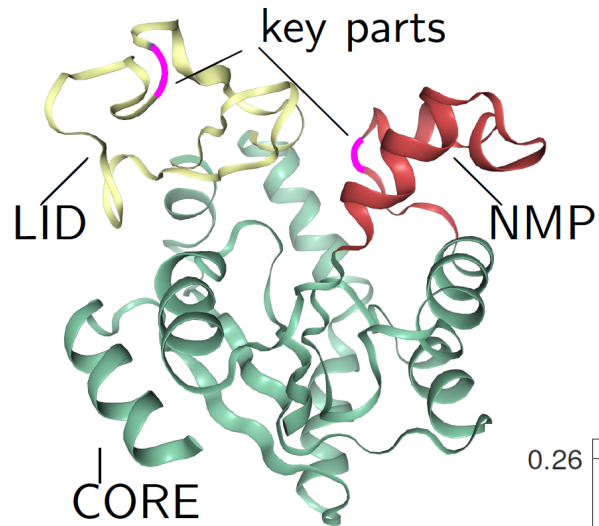


$$\lambda = 0.8948 + 0.1065i, \text{ error} \leq 0.1105$$



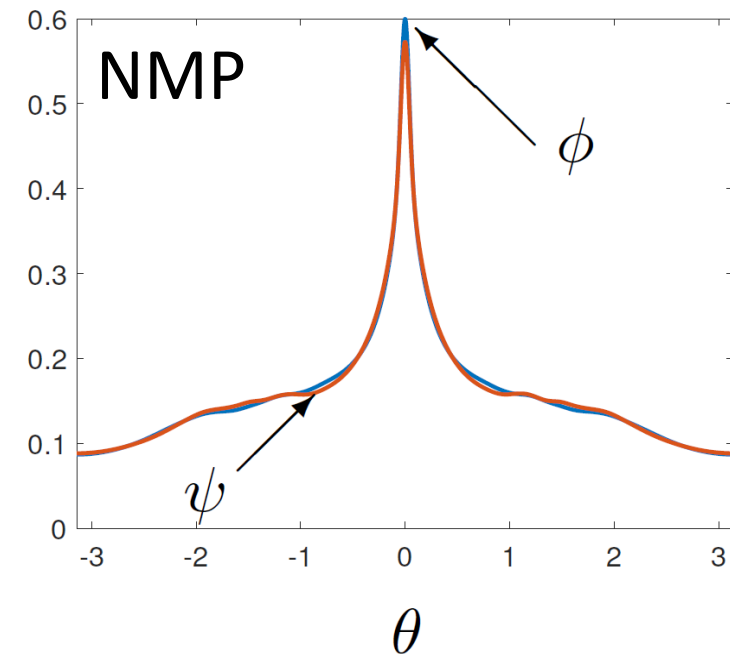
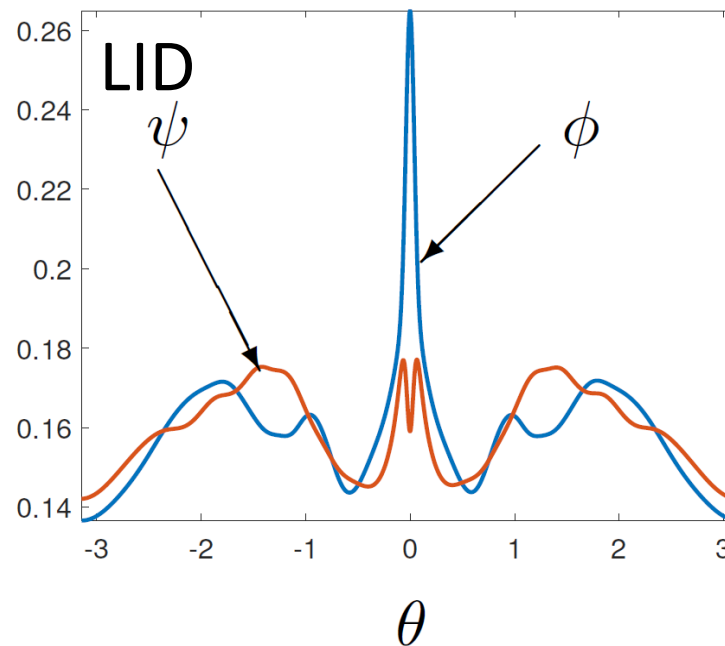
Example: Spectral measures in large d

Adenylate Kinase

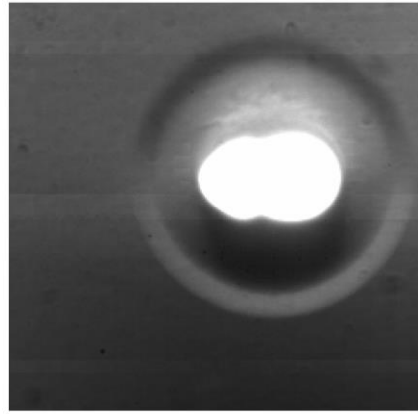
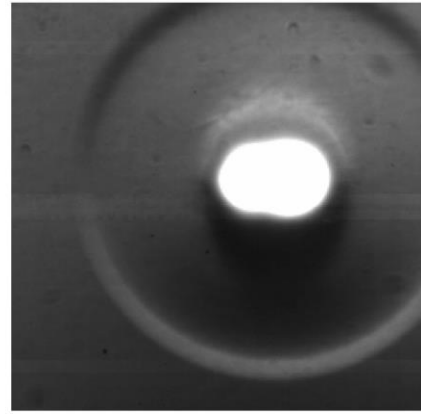
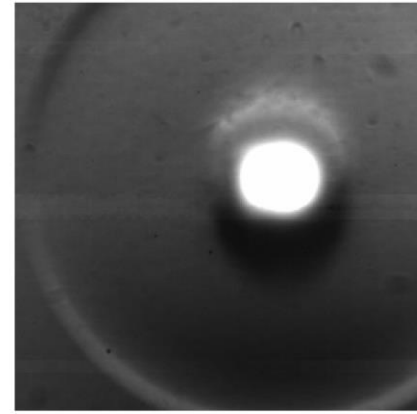
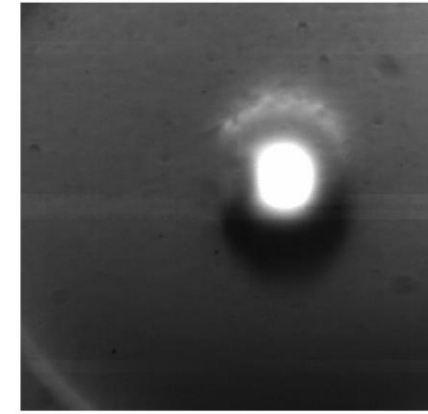
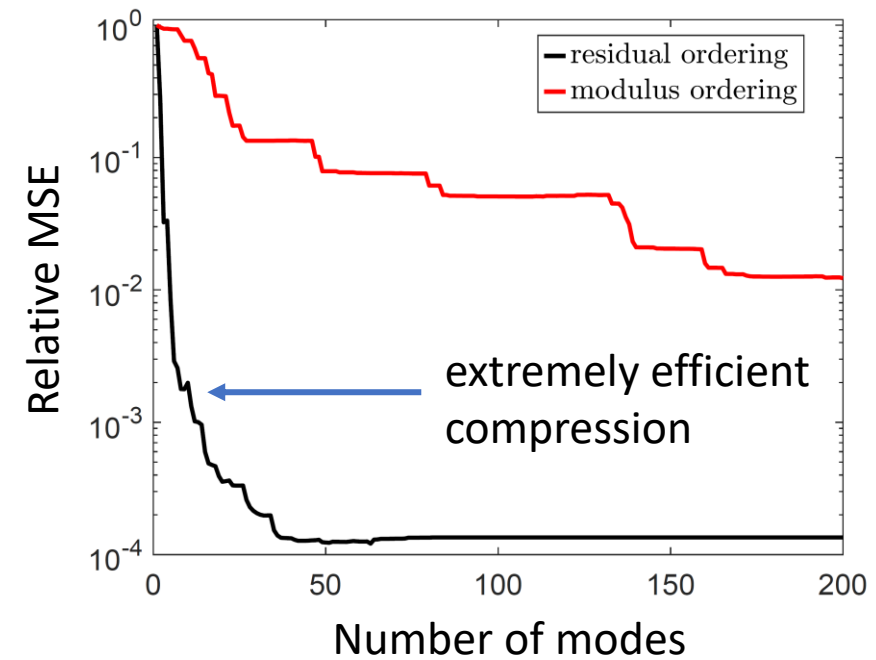
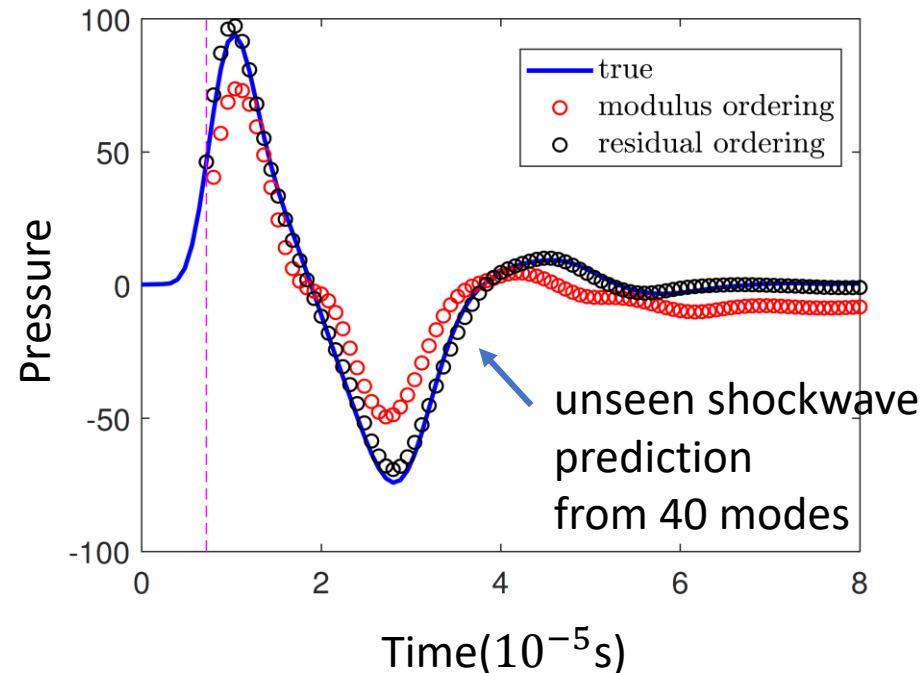


- Ambient dimension (d) $\approx 20,000$ (positions and momenta of atoms)
- 6th order kernel (spec res 10^{-6})

*Dataset: www.mdanalysis.org/MDAnalysisData/adk_equilibrium.html



Example: Trustworthy Koopman mode decomposition

a) $t = 5 \mu s$ b) $t = 10 \mu s$ c) $t = 15 \mu s$ d) $t = 20 \mu s$ 

- C., Ayton, Szőke, “Residual Dynamic Mode Decomposition,” **J. Fluid Mech.**, under minor rev.

Wider programme

SCI provides needed assumptions



- Infinite-dimensional computational analysis \Rightarrow **Practical and rigorous algorithms.**
- Solvability Complexity Index \Rightarrow **Classify difficulty of problems, prove algorithms are optimal.**
- **Extends to:** Foundations of AI, optimization, computer-assisted proofs, and PDEs etc.

DATA SCIENCE + NUMERICAL ANALYSIS

-
- C., "On the computation of geometric features of spectra of linear operators on Hilbert spaces," **Found. Comput. Math.**, to appear.
 - C., "Computing spectral measures and spectral types," **Comm. Math. Phys.**, 2021.
 - C., Horning, Townsend "Computing spectral measures of self-adjoint operators," **SIAM Rev.**, 2021.
 - C., Roman, Hansen, "How to compute spectra with error control," **Phys. Rev. Lett.**, 2019.
 - C., Hansen, "The foundations of spectral computations via the solvability complexity index hierarchy," **J. Eur. Math. Soc.**, 2022.
 - C., Antun, Hansen, "The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem," **Proc. Natl. Acad. Sci. USA**, 2022.
 - C., "Computing semigroups with error control," **SIAM J. Numer. Anal.**, 2022.
 - C., "The mpEDMD Algorithm for Data-Driven Computations of Measure-Preserving Dynamical Systems," preprint.
 - Ben-Artzi, C., Hansen, Nevanlinna, Seidel, "On the solvability complexity index hierarchy and towers of algorithms," arXiv, 2020.
 - Smale, "The fundamental theorem of algebra and complexity theory," **Bull. Amer. Math. Soc.**, 1981, 36 pp.
 - McMullen, "Families of rational maps and iterative root-finding algorithms," **Ann. of Math.**, 1987, 27 pp.

Example: Barriers of deep learning

PNAS

RESEARCH ARTICLE | APPLIED MATHEMATICS | FULL ACCESS

The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale's 18th problem

Matthew J. Colbrook, Vegard Antun, and Anders C. Hansen

March 16, 2022 | 119(12) e2107151110

Significance

Instability is the Achilles' heel of training algorithms finding good solutions. This foundational issue has been a century on the limits of AI. We demonstrate limitations of deep neural networks. Despite numerous successes, only in specific cases do they provide a classification theory on which suitable conditions—rare and a number of hidden layers.

IEEE Spectrum FOR THE TECHNOLOGY INSIDER

NEWS ARTIFICIAL INTELLIGENCE

Some AI Systems May Be Impossible to Compute

New research suggests there are limitations to what deep neural networks can do

BY CHARLES Q. CHOI | 30 MAR 2022 | 4 MIN READ



they get things wrong, but in a new study, AI generally suffers from a mathematical paradox.

“There are fundamental limits inherent in mathematics and, similarly, AI algorithms can't exist for certain problems”

—Matthew Colbrook

UNIVERSITY OF CAMBRIDGE

Study at Cambridge About the University Research at Cambridge

Research

Research home News Our people Spotlights About research Business and enterprise

Mathematical paradox demonstrates the limits of AI

Newsjournal of the Society for Industrial and Applied Mathematics

Volume 55/ Issue 4 May 2022

Protecting Privacy with Synthetic Data

By Matthew R. Francis

Researchers across every scientific discipline need complete and reliable data sets to draw trustworthy conclusions. However, publishing all data from a given study can be undesirable. For example, medical data in particular include personal information that—if published in full—would violate patients' privacy and potentially expose them to harm. Similarly, many

studies in the social sciences include demographic or geographical data that could easily be exploited by unscrupulous parties. In short, researchers must strike a delicate balance between publishing enough data to verify their conclusions and protecting the privacy of the people involved. Unfortunately, multiple studies have shown that simply anonymizing the data—by removing individuals' names before publication, for instance—is insufficient, as outsiders can use context clues to reconstruct missing information and expose research subjects. “We want to generate synthetic data for public release to replace the original data set,” Bei Jiang of the University of Alberta said. “When we design our framework, we have this main goal in mind: we want to produce the same inference results as in the original data set.”

In contrast with falsified data, which is one of the deadliest scientific sins, researchers can generate synthetic data directly from original data sets. If the construction process is done properly, other scientists can then analyze this synthetic data and trust that their conclusions are no different from what they would have obtained with full access to the original raw data—ideally, at least. “When you [create] synthetic data, what does it mean to be private yet realistic?” Sebastian Gamba

of the University of Quebec in Montreal asked. “It's still an open research question.” During the 2022 American Association for the Advancement of Science Annual Meeting, which took place virtually in February, Jiang and Gamba each presented formal methods for the generation of synthetic data that ensure privacy. Their models draw from multiple fields to address challenges in the era of big data, where the stakes are higher than ever. “There is always a trade-off between utility and risk,” Jiang said. “If you want to protect people [who] are at a higher risk, then you perturb their data. But the utility will be lowered the more you perturb. A better approach is to account for their risks to begin with.”

Unfortunately, malicious actors have access to the same algorithmic tools as researchers. Therefore, protection of confidentiality also involves testing synthetic data against the types of attacks that such players might utilize. “In practice, this helps one really understand the translation between an abstract privacy parameter and a practical guarantee,” Gamba said. In other words, the robustness of a formal mathematical model is irrelevant if the model is not well implemented.

Differential Privacy Made Simple

Gamba and his collaborators turned to differential privacy: a powerful mathematical formulation that in principle is the best available technique for securing confidentiality. However, the approach is also complex and difficult to implement without a high degree of statistical knowledge. To smooth the

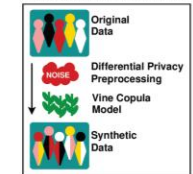


Figure 1. In addition to adding noise to the data set, Sebastian Gamba's differential privacy-based method involves using an information theory algorithm to obtain synthetic data that—in principle—shields the privacy of the people involved. Figure courtesy of the author.

Figure 2. Researchers can protect privacy by performing a full statistical analysis on the original data set, then using a message-passing algorithm called multiple imputation to construct a synthetic data set that has exactly the same statistical characteristics. Figure courtesy of the author.

Proving Existence Is Not Enough: Mathematical Paradoxes Unravel the Limits of Neural Networks in Artificial Intelligence

By Vegard Antun, Matthew J. Colbrook, and Anders C. Hansen

The impact of deep learning (DL), neural networks (NNs), and artificial intelligence (AI) over the last decade has been profound. Advances in computer vision and natural language processing have yielded smart speakers in our homes, driving assistance in our cars, and automated diagnoses in medicine. AI has also rapidly entered scientific computing. However, overwhelming amounts of empirical evidence [3, 8] suggest that modern AI is often non-robust (unstable), may generate hallucinations, and can produce nonsensical output with high levels of prediction confidence (see Figure 1). These issues present a serious concern for AI use within legal frameworks. As stated by the European Commission's Joint Research Centre, “In the light of the recent advances in AI, the serious negative consequences of its use for EU citizens and organizations have led to multiple initiatives [...] Among the identified requirements, the concepts of robustness and explainability of AI systems have emerged as key elements for a future regulation.”

Robustness and trust of algorithms lie at the heart of numerical analysis [9]. The lack of robustness and trust in AI is hence the Achilles' heel of DL and has become a serious political issue. Classical approximation theorems show that a continuous function can be approximated arbitrarily well by a NN [5]. Therefore, stable problems that are described by stable functions can be solved stably with a NN. These results inspire the following fundamental question: Why does DL lead to unstable methods and AI-generated hallucinations, even in scenarios where we can prove that stable and accurate NNs exist?

Our main result reveals a serious issue for certain problems: while stable and accurate NNs may provably exist, no training algorithm can obtain them (see Figure 2). A similar program on the boundaries of AI is necessary. Stephen Smale already suggested such a program in the 18th problem on his list of mathematical problems for the 21st century. What are the limits of AI? [11].

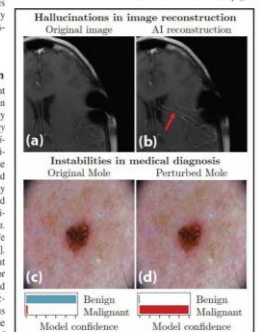


Figure 1. Hallucinations in image reconstruction and instabilities in medical diagnosis. (a) The correct, original image from the 2020 Ischaemic Challenge. (b) Reconstruction by an artificial intelligence (AI) method that produces an incorrect detail (AI-generated hallucination). (c) Dermaloscopic image of a benign melanocytic nevus, along with the diagnostic probability computed by a deep neural network (NN). (d) Combined image of the nevus with a slight perturbation and the diagnostic probability from the same deep NN. One diagnosis is clearly incorrect, but can an algorithm determine which one? Figures 1a and 1b are courtesy of the 2020 Ischaemic Challenge [10], and 1c and 1d are courtesy of [8].

- C., Antun, Hansen, “The difficulty of computing stable and accurate neural networks: On the barriers of deep learning and Smale’s 18th problem,” Proc. Natl. Acad. Sci. USA.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS
1000 PENNSYLVANIA AVE, SUITE 900
PHILADELPHIA, PA 19104-2888 USA

Interested? Get in touch (e.g., I'll be around this afternoon)!

“One of great joys of doing mathematics is working with inspiring and brilliant people!”

- Arie Iserles

Some future directions:

- ResDMD + control \Rightarrow error control?
- Embed & learn symmetries (e.g., check out the algorithm mpEDMD).
- Forecasting with error bounds.
- Koopmanism meets neural nets (and vice versa).
- Foundations results for dynamical systems (i.e., impossibility results)?
- Further barriers in deep learning.
- Functional analysis meets data science meets numerical analysis!

Opportunities to collaborate, visit Cambridge, grad students & beyond!

Summary: rigorous data-driven Koopmanism!

- “Too much” or “Too little”

Idea: New matrix for residual \Rightarrow **ResDMD** for computing spectra.

- Continuous spectra and spectral measures:

Idea: Convolution with rational kernels via resolvent and **ResDMD**.

- Is it right?

Idea: Use **ResDMD** to verify computations. E.g., learned dictionaries.

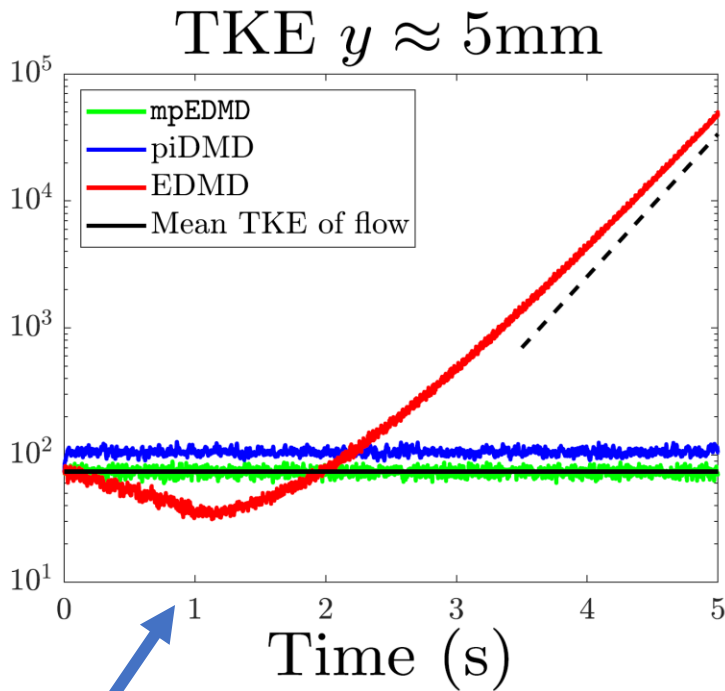
Code:

<https://github.com/MColbrook/Residual-Dynamic-Mode-Decomposition>

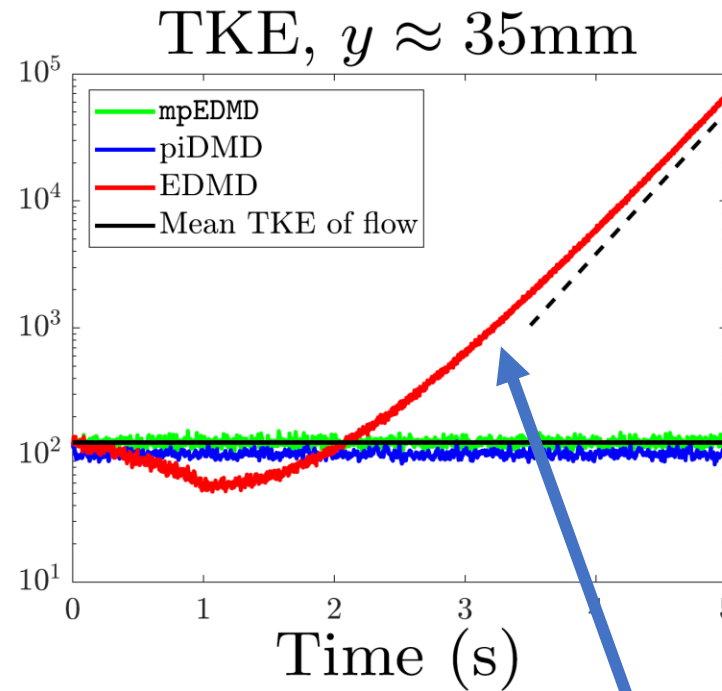
Additional slides...

measure-preserving EDMD...

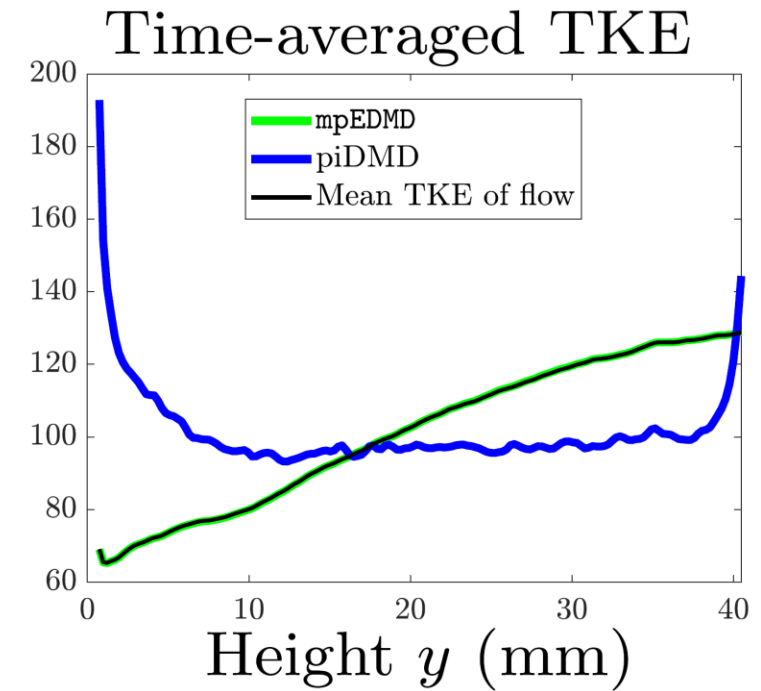
- Polar decomposition of \mathcal{K} . Easy to combine with any DMD-type method!
- Converges for spectral measures, spectra, Koopman mode decomposition.
- Measure-preserving discretization for arbitrary measure-preserving systems.




Snapshots collected over 1s



EDMD unstable!



Solvability Complexity Index Hierarchy

Class $\Omega \ni A$, want to compute $\Xi: \Omega \rightarrow (\mathcal{M}, d)$  metric space

- Δ_0 : Problems solved in finite time (v. rare for cts problems).

- Δ_1 : Problems solved in “one limit” with full error control:

$$d(\Gamma_n(A), \Xi(A)) \leq 2^{-n}$$

- Δ_2 : Problems solved in “one limit”:

$$\lim_{n \rightarrow \infty} \Gamma_n(A) = \Xi(A)$$

- Δ_3 : Problems solved in “two successive limits”:

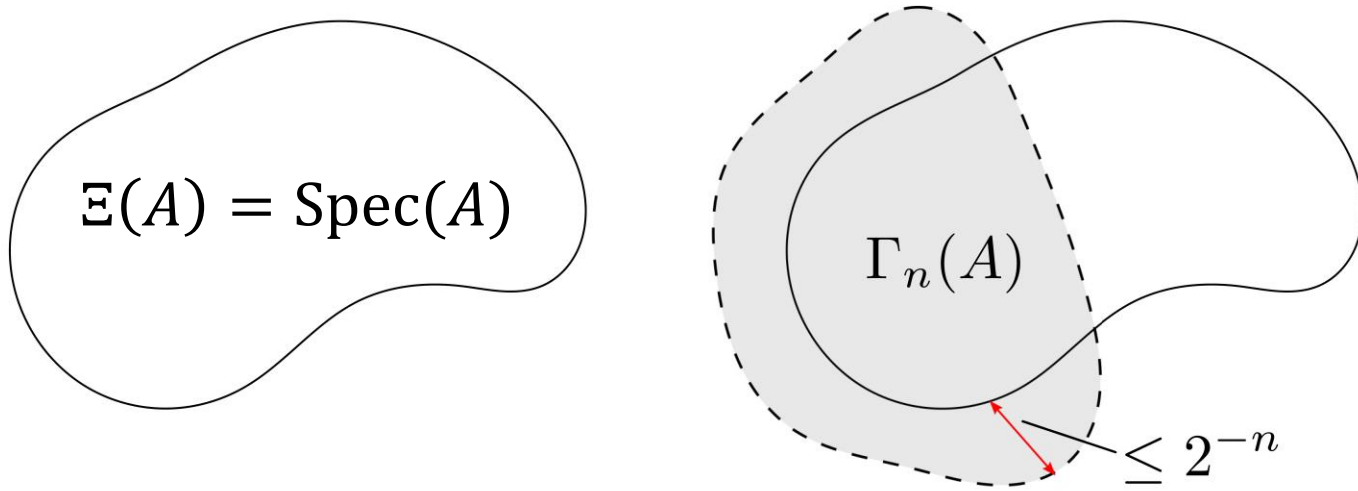
$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \Gamma_{n,m}(A) = \Xi(A)$$

⋮

-
- Ben-Artzi, C., Hansen, Nevanlinna, Seidel, “*On the solvability complexity index hierarchy and towers of algorithms*,” preprint.
 - Hansen, “*On the solvability complexity index, the n -pseudospectrum and approximations of spectra of operators*,” **J. Amer. Math. Soc.**, 2011.
 - McMullen, “*Families of rational maps and iterative root-finding algorithms*,” **Ann. of Math.**, 1987.
 - Doyle, McMullen, “*Solving the quintic by iteration*,” **Acta Math.**, 1989.
 - Smale, “*The fundamental theorem of algebra and complexity theory*,” **Bull. Amer. Math. Soc.**, 1981.

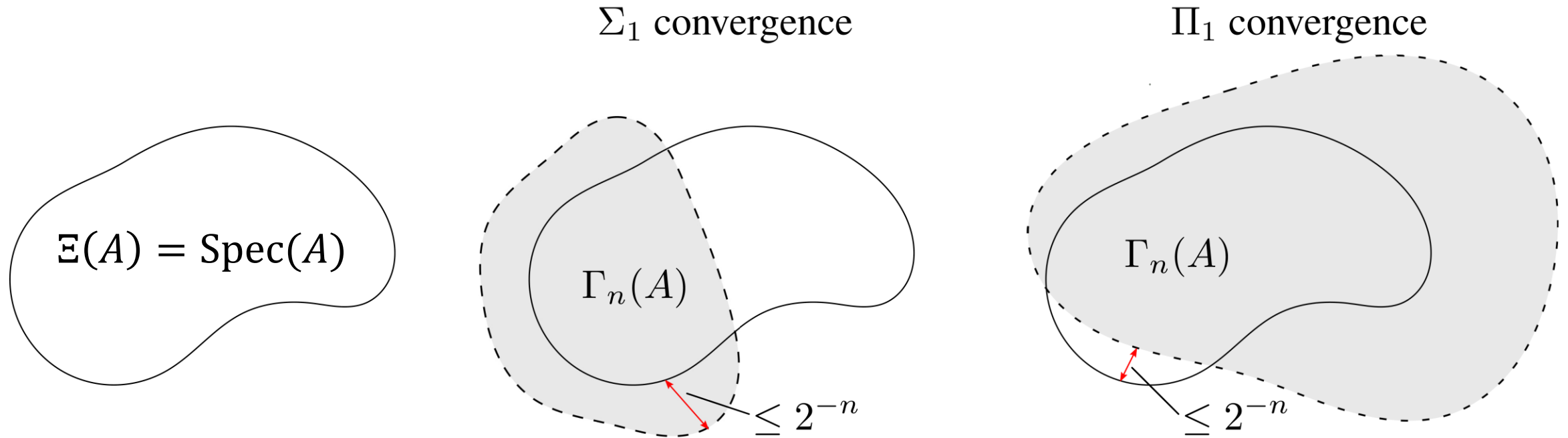
Error control for spectral problems

Σ_1 convergence



- $\Sigma_1: \exists \text{ alg. } \{\Gamma_n\} \text{ s.t. } \lim_{n \rightarrow \infty} \Gamma_n(A) = \Xi(A), \max_{z \in \Gamma_n(A)} \text{dist}(z, \Xi(A)) \leq 2^{-n}$

Error control for spectral problems



- $\Sigma_1: \exists \text{ alg. } \{\Gamma_n\} \text{ s.t. } \lim_{n \rightarrow \infty} \Gamma_n(A) = \Xi(A), \max_{z \in \Gamma_n(A)} \text{dist}(z, \Xi(A)) \leq 2^{-n}$
- $\Pi_1: \exists \text{ alg. } \{\Gamma_n\} \text{ s.t. } \lim_{n \rightarrow \infty} \Gamma_n(A) = \Xi(A), \max_{z \in \Xi(A)} \text{dist}(z, \Gamma_n(A)) \leq 2^{-n}$

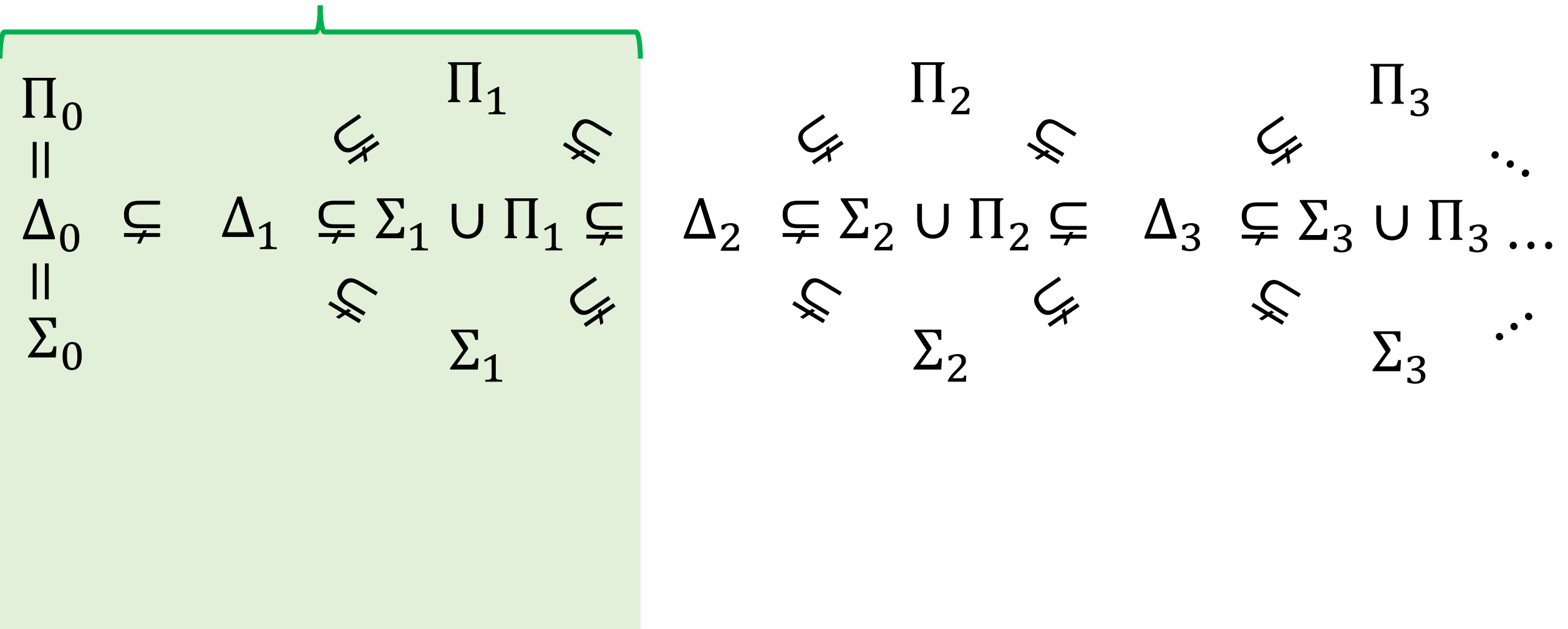
Such problems can be used in a proof!

Small sample of classification theorems

Increasing difficulty



Error control

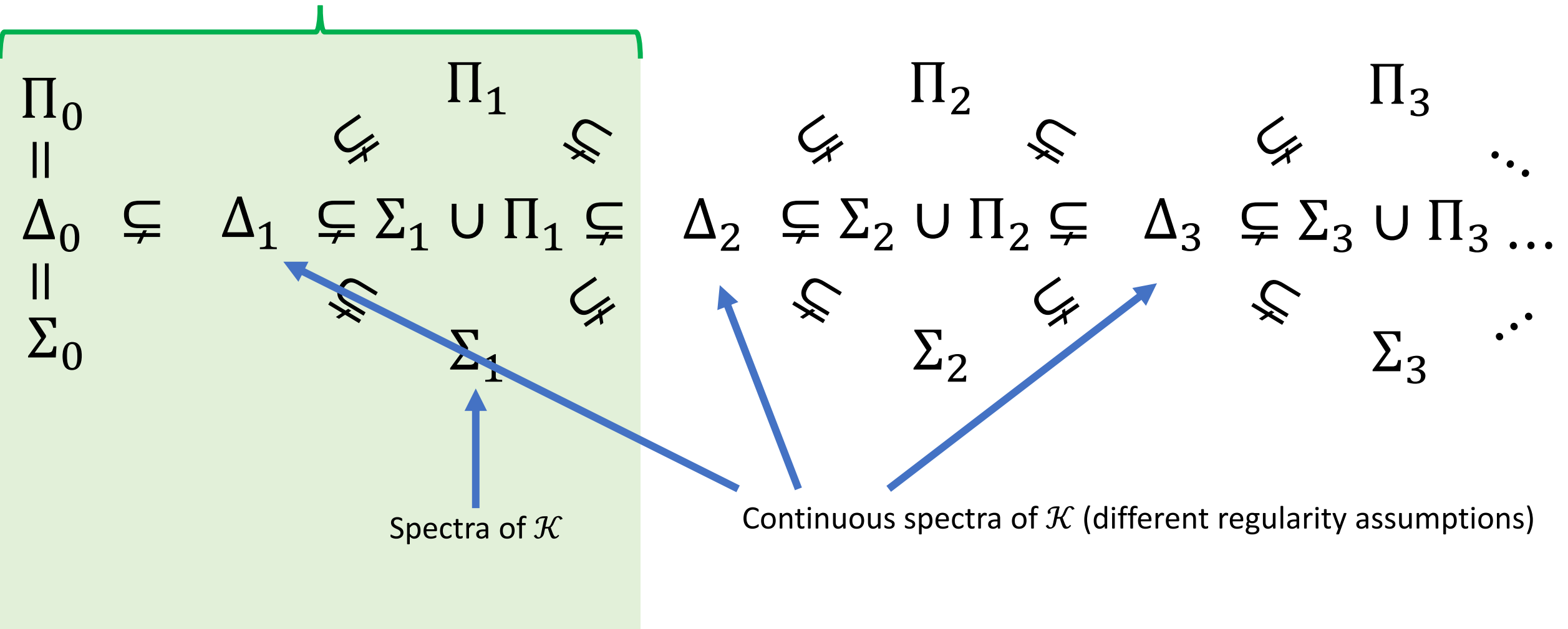


Small sample of classification theorems

Increasing difficulty



Error control



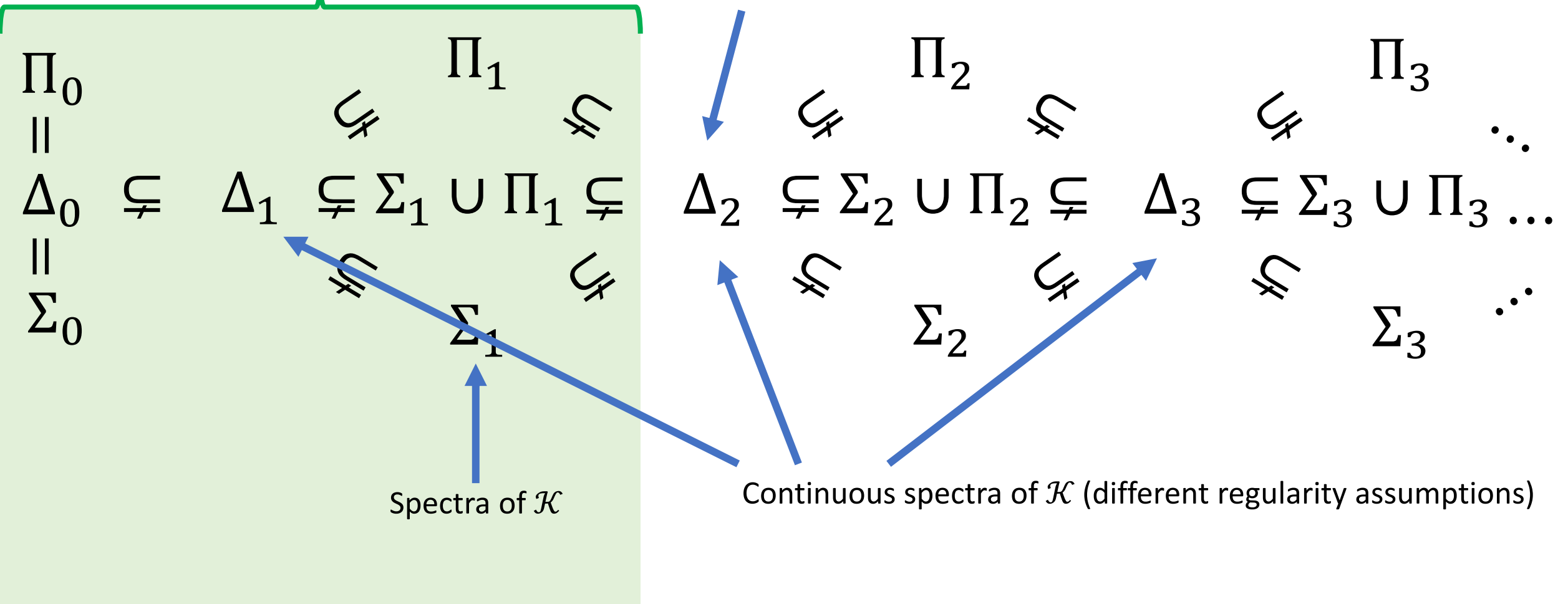
Small sample of classification theorems

Increasing difficulty



Error control

Spectra of compact operators

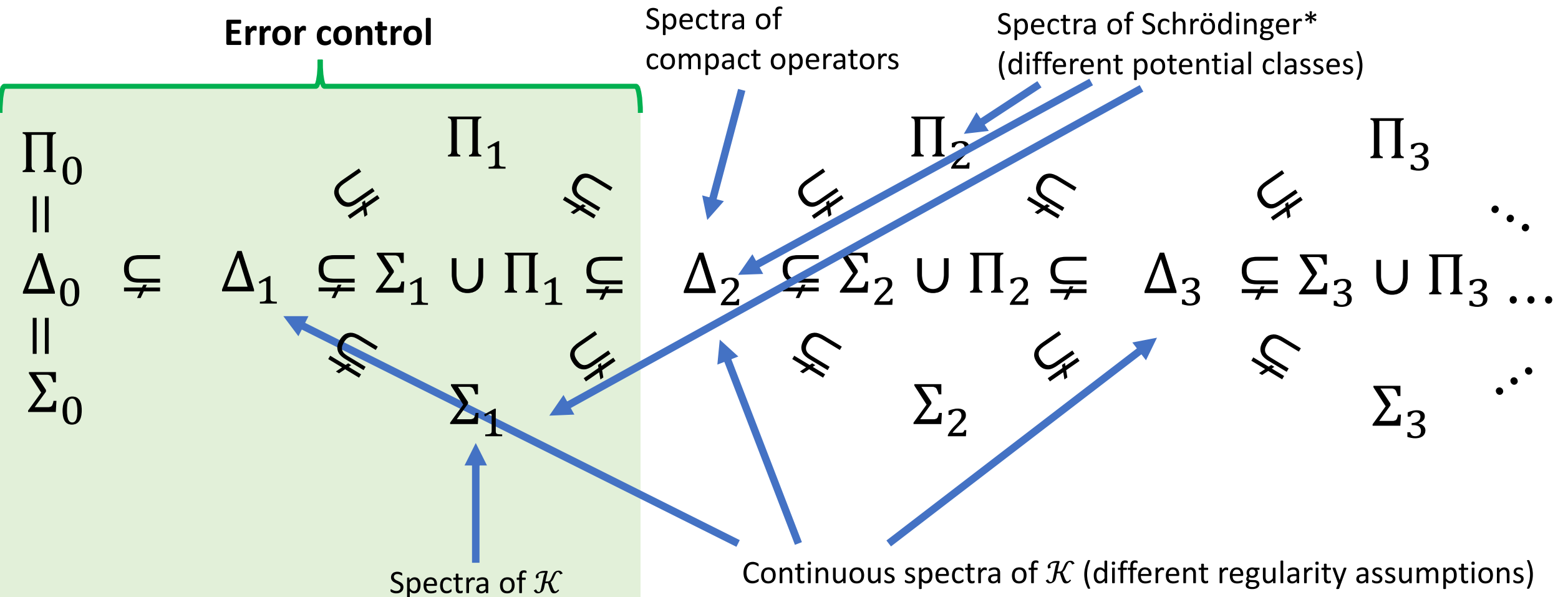


Small sample of classification theorems

Increasing difficulty



Error control



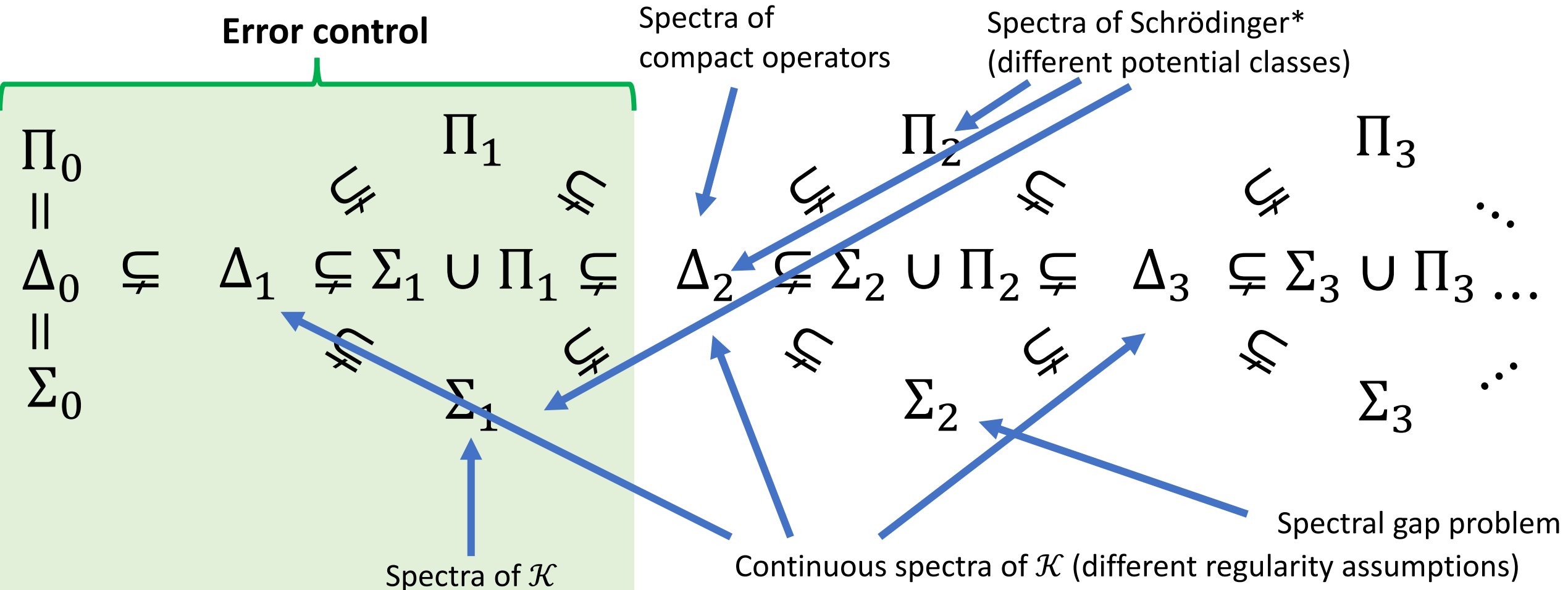
*Open problem of Schwinger: "The special canonical group," "Unitary operator bases," PNAS, 1960.

Small sample of classification theorems

Increasing difficulty



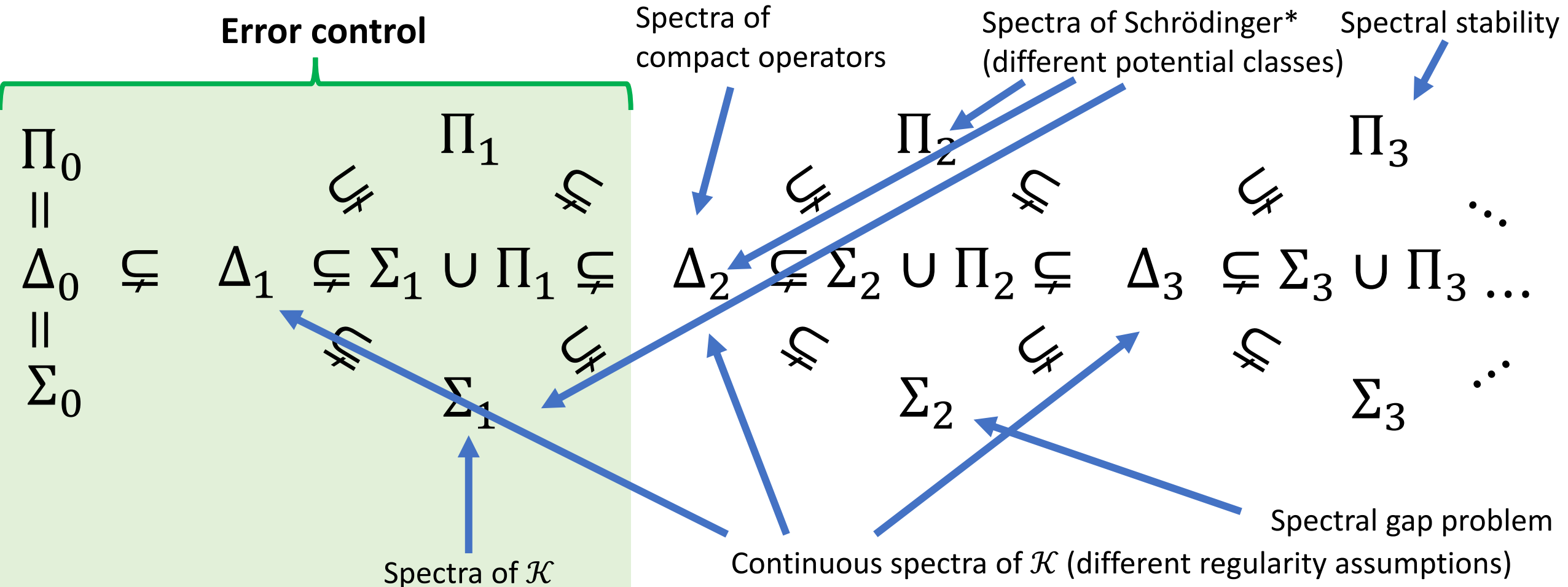
Error control



*Open problem of Schwinger: "The special canonical group," "Unitary operator bases," PNAS, 1960.

Small sample of classification theorems

Increasing difficulty



*Open problem of Schwinger: "The special canonical group," "Unitary operator bases," PNAS, 1960.