Solving Wiener–Hopf Type Problems Numerically
A Spectral Method Approach

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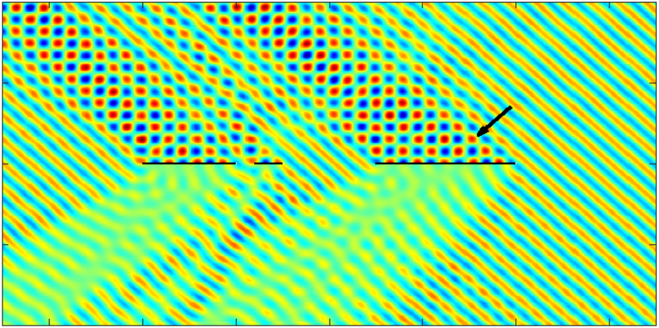

Scattering Problem

Acoustic 2D scattering governed by the Helmholtz equation

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + k_0^2 q = 0, \quad (x, y) \in \mathcal{D}.$$

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Typical boundary conditions on $\partial\mathcal{D}$:

- Zero normal velocity (Neumann: prescribed $\partial q/\partial n = q_n$)
- Continuity of pressure (Dirichlet: prescribed q)
- Impedance/porosity
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Sommerfeld radiation condition at infinity (radiates <u>to</u> infinity):

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial}{\partial r} - ik_0 \right) q(r, \theta) = 0$$

Crucial for well-posed problem (and important physically)!

Sketch of talk

- Building a method
- Rigid plate example
- Application I: elastic plates
- Application II: perforated screens and Robin BCs
- Open Problems

Take home message: method provides an efficient way of solving 2D scattering problems of interest in this community (and beyond)

Unified Transform

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$$\frac{\partial}{\partial x}\left(v\frac{\partial q}{\partial x} - q\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial q}{\partial y} - q\frac{\partial v}{\partial y}\right) = 0.$$

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Assuming everything converges, Green's theorem implies

$$\int_{\partial\mathcal{D}}\left[\left(v\frac{\partial q}{\partial x} - q\frac{\partial v}{\partial x}\right)dy - \left(v\frac{\partial q}{\partial y} - q\frac{\partial v}{\partial y}\right)dx\right] = 0.$$

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Choosing $v = \mathrm{e}^{-\mathrm{i}\beta(\lambda z + \frac{\bar{z}}{\lambda})}$ with $\beta = k_0/2$, $z = x + \mathrm{i}y$ gives

$$\int_{\partial\mathcal{D}} \mathrm{e}^{-\mathrm{i}\beta(\lambda z + \frac{\bar{z}}{\lambda})}\left[q_n + \beta\left(\lambda\frac{dz}{ds} - \frac{1}{\lambda}\frac{d\bar{z}}{ds}\right)q\right]ds = 0, \qquad \lambda \in \mathcal{C}(\mathcal{D}).$$

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View this as a (generalised) **Fourier transform** of the boundary integral equations.

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Using that solution anti-symmetric in y:

$$\frac{1}{4}[q](x,0) + \int_{\mathbb{R}} G(x - x')q_y(x',0)dx' = 0,$$

where G is the Green's function $G(x) = \mathrm{i}H_0^{(1)}(k_0\,|x|)/4$.

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Take FT with frequency $\omega = \beta\,(\lambda + 1/\lambda)$ to get

$$\int_{\mathbb{R}\setminus\gamma} \mathrm{e}^{-\mathrm{i}\beta x(\lambda + \frac{1}{\lambda})} q_y(x,0)dx$$
$$+ \int_{\gamma} \mathrm{e}^{-\mathrm{i}\beta x(\lambda + \frac{1}{\lambda})} \left[q_y(x,0) + \frac{\beta}{2}\left(\lambda - \frac{1}{\lambda}\right)[q](x,0)\right] dx = 0, \qquad \lambda \in \Lambda$$

$$\Lambda = (-1,0) \cup (1,\infty) \cup \{\mathrm{e}^{\mathrm{i}\theta} : \pi < \theta < 2\pi\}$$

Connections with Existing Techniques

To obtain WH equation set $\beta(\lambda + \frac{1}{\lambda}) = -\alpha$ and

$$K(\alpha) = \sqrt{\alpha^2 - k_0^2} = \beta\left(\lambda - \frac{1}{\lambda}\right).$$

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Why is this useful?

- Avoids singular integrals (convolution $\rightarrow$ multiplication).
- Generalise WH equation to arbitrary domains (and in fact separable PDEs).
- For some problems, allows analytic study for first time (where it is difficult to see how to use WH).
- WH for 3D problems? (More on this later.)

Building a Numerical Method

Main idea: Expand boundary values in a suitable basis:

$$q(s) = \sum_{j=1}^{N} a_j S_j(s), \quad q_n(s) = \sum_{j=1}^{N} b_j T_j(s)$$

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$$\sum_j a_j \beta \left(\lambda \frac{dz}{ds} - \frac{1}{\lambda} \frac{d\bar{z}}{ds} \right) \hat{S}_j(\lambda_i) + b_j \hat{T}_j(\lambda_i) = 0.$$

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Linear system, row i evaluation at λ_i (Fourier collocation):

$$\begin{pmatrix} \text{Matrix formed} \\ \text{from combinations} \\ \text{of} \\ \hat{S}_j(\lambda_i) \text{ and } \hat{T}_j(\lambda_i) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \\ b_1 \\ \vdots \\ b_N \end{pmatrix} = 0$$

Extensions/Advantages

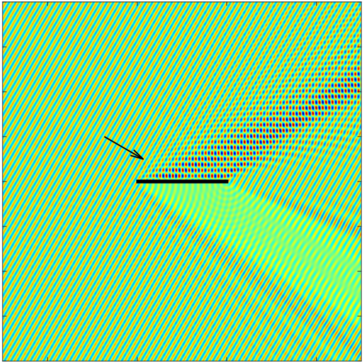
Can generalise to:

- Separable PDEs and curved boundaries [C. 2018].
- Arbitrary non-convex domains [C., Flyer & Fornberg 2018].
- Unbounded domains [C., Ayton & Fokas 2019].
- Fast evaluation in interior [C., Fokas & Hashemzadeh 2019].

Some advantages of the method:

- Fast (order of a second for hundreds of basis functions).
- Easy to use and code, can be automated.
- Boundary based (dimensional reduction).
- Avoid evaluations of singular integrals (that arise in other methods such as BEM).
- Flexible choice of bases...

Easy Example: Single Rigid Plate



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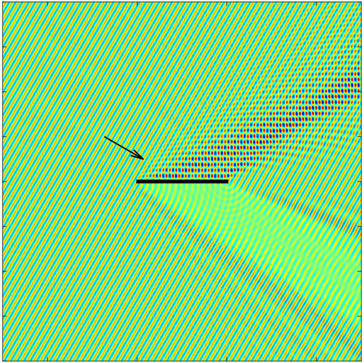

For $\lambda \in (-1,0) \cup (1,\infty) \cup \{e^{i\theta} : \pi < \theta < 2\pi\}$:

$$\int_{-\infty}^{-1} e^{-i\beta x(\lambda+\frac{1}{\lambda})} q_y(x,0)dx + \int_{1}^{\infty} e^{-i\beta x(\lambda+\frac{1}{\lambda})} q_y(x,0)dx$$

$$+ \int_{-1}^{1} e^{-i\beta x(\lambda+\frac{1}{\lambda})} \frac{\beta}{2}\left(\lambda - \frac{1}{\lambda}\right)[q](x,0)dx = \int_{-1}^{1} e^{-i\beta x(\lambda+\frac{1}{\lambda})} \frac{\partial q_I}{\partial y}(x,0)dx.$$

Technical Details

Suitable basis can be predicted from geometry/boundary conditions of the problem (interesting physics).

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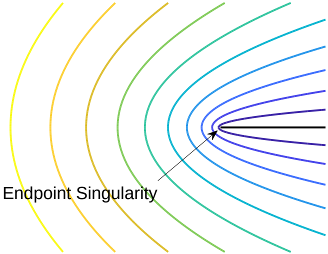

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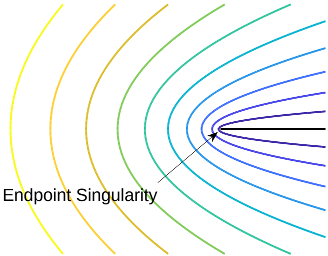
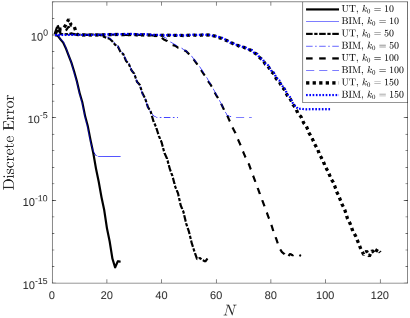


To capture endpoint singularities, expand $[q]$ in terms of weighted Chebyshev polynomials:

$$\sqrt{1-x^2} \cdot U_n(x).$$

Rapid Convergence!



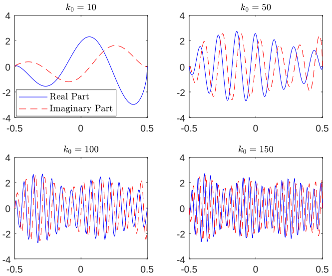


Figure: Left: Maximum relative error. UT denotes unified transform, BIM denotes boundary integral method of [Nigro 2017]. Right: Analytic solutions $[q](x, 0)$ for different k_0.

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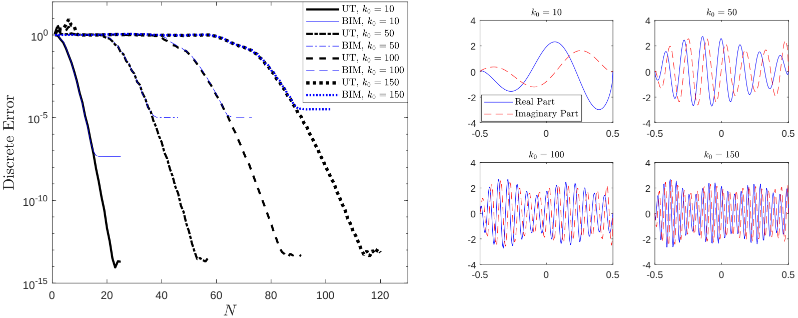

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Much more complicated geometries possible
(e.g. arrays of wedges, polygons, curved $\partial\mathcal{D}$ etc.)

Application I: elastic plates [C. & Ayton 2019]

- **Application:** A big problem in aero-acoustics is noise reduction (e.g. yesterday's talk on owls).
- **Current challenge:** developing **fast** and **accurate** numerical tools for scattering problems.
(saw some approaches yesterday)
$\to$ predict effect of physical parameters and external forces.
- Can we model complicated boundary conditions such as **elasticity**? (this is <u>difficult</u> via WH)

Elastic $\to$ absorbs energy $\to$ reduced noise

Elastic Problem

- $q_I \rightsquigarrow K$ collinear plates $\gamma_1, \gamma_2, ..., \gamma_K \rightsquigarrow q$.

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- If plate γ_i elastic, denote plate deformation by η_i then

$$\left(\frac{\partial^4}{\partial x^4} - \frac{k_0^4}{\Omega_i^4}\right)\eta_i = -\frac{\epsilon_i}{\Omega_i^6}k_0^3[q] \quad \text{on} \quad \gamma_i.$$

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- Kinematic condition ($\eta_i = 0$ if γ_i rigid):

$$k_0^2\eta_i = \frac{\partial q_I}{\partial y} + \frac{\partial q}{\partial y} \quad \text{on} \quad \gamma_i.$$

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- At endpoint $x = x_0$ of plate, either $\eta(x_0) = \eta'(x_0) = 0$ (clamped) or $\eta''(x_0) = \eta'''(x_0) = 0$ (free).

How to Cope? Vibrational Modes!

Main idea: Expand η_i in eigenfunctions of ∇^4 subject to correct BCs:

$$\nabla^4 f_j = d_j^4 f_j, \quad \text{clamped/free at endpoints.}$$

$$\text{Expand: } \eta_i(x) \approx \sum_{j=1}^{N} a_{i,j} f_j(x),$$

$$\Rightarrow \frac{\partial q}{\partial y}(x) \approx -\frac{\partial q_I}{\partial y}(x) + \sum_{j=1}^{N} k_0^2 a_{i,j} f_j(x)$$

$$[q](x) = -\frac{\Omega_i^6}{k_0^3 \epsilon_i} \left(\frac{\partial^4}{\partial x^4} - \frac{k_0^4}{\Omega_i^4} \right) \eta_i(x)$$

$$\approx -\frac{\Omega_i^6}{k_0^3 \epsilon_i} \sum_{j=1}^{N} a_{i,j} \left(d_j^4 - \frac{k_0^4}{\Omega_i^4} \right) f_j(x).$$

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Compute f_j, d_j using standard spectral methods (very easy).

Easy to compute Fourier transforms:

$$\begin{aligned}(\lambda^4 - d_j^4)\int_a^b \mathrm{e}^{\mathrm{i}\lambda x} f_j(x)dx &= (\mathrm{i}\lambda)^3[\mathrm{e}^{\mathrm{i}\lambda x} f(x)]_{x=a}^b - (\mathrm{i}\lambda)^2[\mathrm{e}^{\mathrm{i}\lambda x} f'(x)]_{x=a}^b \\ &+ \mathrm{i}\lambda[\mathrm{e}^{\mathrm{i}\lambda x} f''(x)]_{x=a}^b - [\mathrm{e}^{\mathrm{i}\lambda x} f'''(x)]_{x=a}^b.\end{aligned}$$

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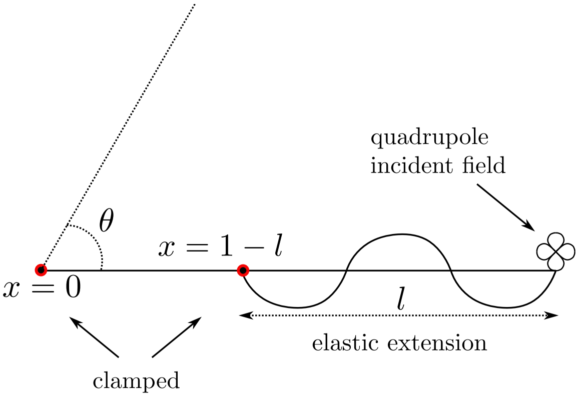
Upshot:

- Fast and accurate.
- Cope with multiple bodies with different physical parameters and geometric configurations.
- Can add porosity.

Elastic Plate Extensions



Far-field Noise

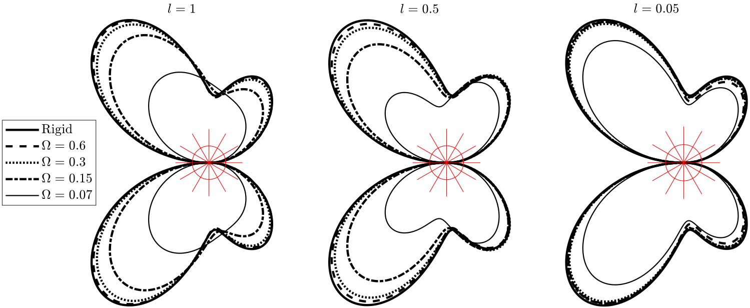


Figure: Far-field directivity for $k_0 = 5$, $\epsilon = 0.0021$ and different l.

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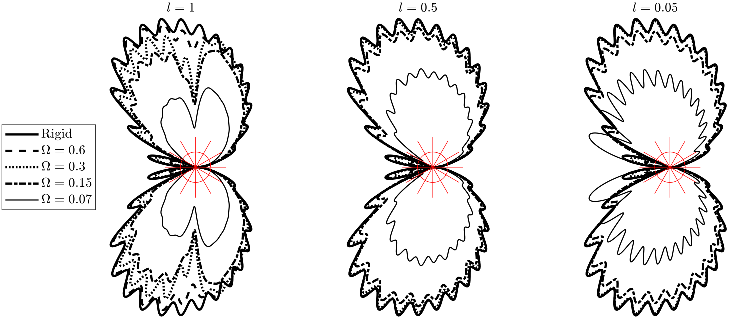

Figure: Far-field directivity for $k_0 = 50$, $\epsilon = 0.0021$ and different l.

Radiated Power

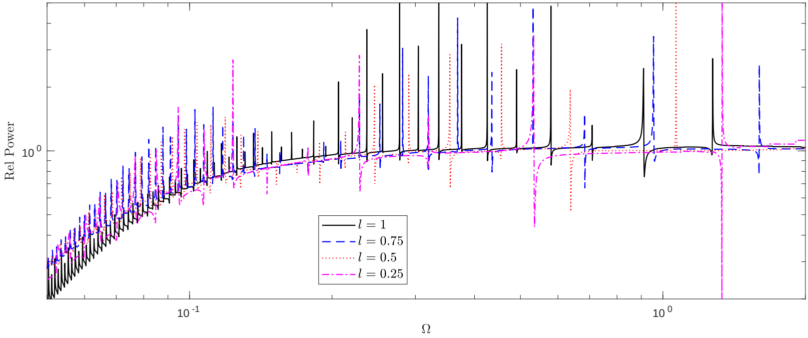

Figure: Relative power level as a function of Ω for $k_0 = 10$, $\epsilon = 0.0021$.

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- Different length extensions should be used depending on the frequencies to be reduced.
- Future work: consider aerodynamic impact of elastic extensions to balance acoustic and aerodynamic considerations. Extensions to 3D and elastic spheres.

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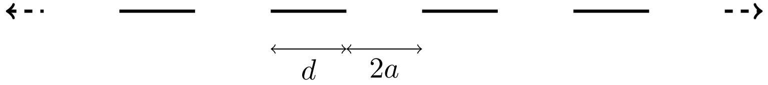

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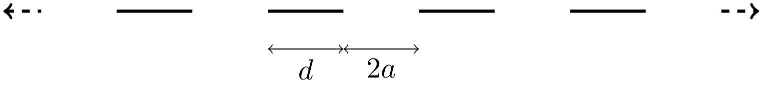

If $a \ll d \ll k_0^{-1}$ then expect homogenised BC [Lamb 1895, Leppington 1977, Howe 1998] with

$$\frac{\partial q}{\partial y}(x,0) + \frac{\partial q_I}{\partial y}(x,0) = \mu(x)[q](x,0)$$

and

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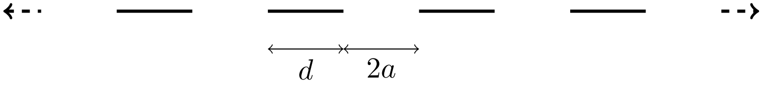

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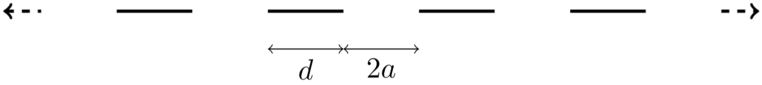

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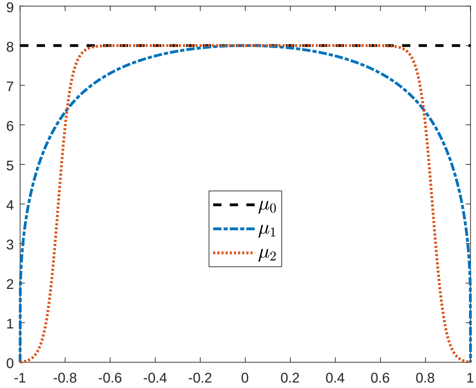
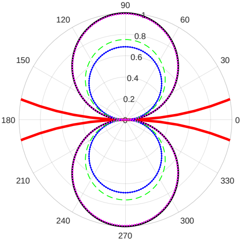
- How to take μ to zero at endpoints and does this matter?



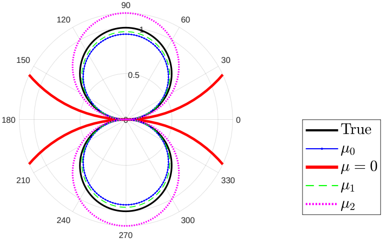
9
8
7
6
5
4
3
2
1
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
μ_0
μ_1
μ_2

Preliminary Results

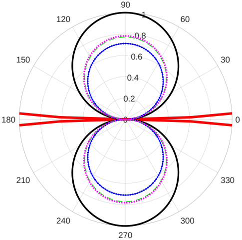
(a) $d/a = 10, (k_0 d)^{-1} = 14.5$



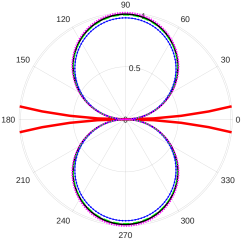
(b) $d/a = 100, (k_0 d)^{-1} = 12.7$



(c) $d/a = 10, (k_0 d)^{-1} = 59.5$



(d) $d/a = 100, (k_0 d)^{-1} = 50.95$



Conclusions

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- Can be viewed as a Fourier transform version of boundary integral methods (collocation in Fourier space).
- Easier to use and more accurate than boundary integral methods (e.g. no singular integrals).
- Suitable basis can capture difficult boundary conditions such as coupling to plate deformation.

Open Problems

- Can this be efficiently implemented in 3D, leading to multi-dimensional quasi-WH method? Analysis of singularities of solution would be a challenge!

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- Non-linear boundary conditions (e.g. via Newton iteration).
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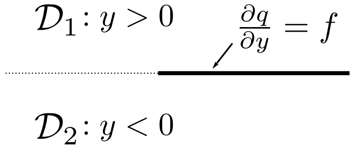
- Is there a unified (pun intended) way of viewing everything?

Infinite Plate Example

e.g. Modified Helmholtz $k_0 \to \mathrm{i}k_0$ (decay at infinity)



$$\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\beta x(\lambda - \frac{1}{\lambda})} \frac{\partial q}{\partial y}(x,0)dx - \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\beta x(\lambda - \frac{1}{\lambda})} \frac{\beta}{2}\left(\lambda + \frac{1}{\lambda}\right)[q](x,0)dx$$
$$= -\int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\beta x(\lambda - \frac{1}{\lambda})} f(x)dx, \qquad \lambda \in \mathbb{R}_-.$$

Basis functions: modified Laguerre (with exponentially decaying weight) to capture singular behaviour.

Infinite Plate Example

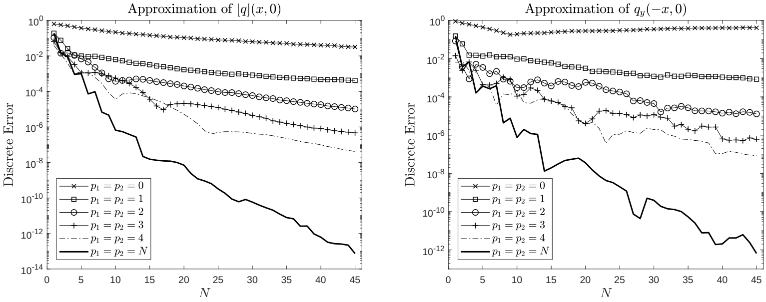
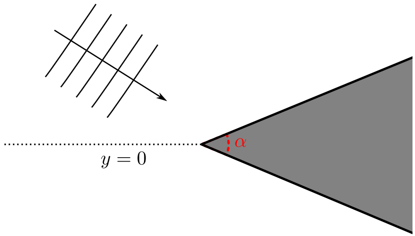
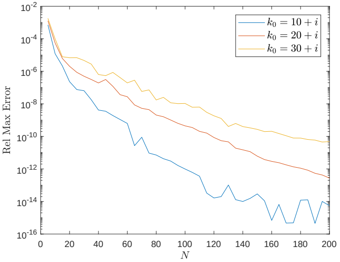


Figure: $p =$'Number of singular functions in basis'. Left: Convergence of computed Dirichlet values. Right: Convergence of computed Neumann values.

Wedge Example

e.g. Helmholtz (add some decay at infinity)





3D

Suppose

$$\nabla^2 q = 0 \qquad \forall (x,y,z) \in \mathcal{D}$$

then Green's theorem gives

$$\int_{\partial\mathcal{D}} \left(v\frac{\partial q}{\partial n} - q\frac{\partial v}{\partial n} \right) ds = 0$$

for v some other solution to Laplace.
Pick v as

$$v(x,y,z) = e^{-i(\lambda x+\mu y)+\sqrt{\lambda^2+\mu^2}z}$$

for $\lambda, \mu \in \mathbb{C}$.
Then

$$\int_{\partial\mathcal{D}} e^{-i(\lambda x+\mu y)+\sqrt{\lambda^2+\mu^2}z} \left(\frac{\partial q}{\partial n} - q\left[\sqrt{\lambda^2+\mu^2}\frac{\partial z}{\partial n} - i\lambda\frac{\partial x}{\partial n} - i\mu\frac{\partial y}{\partial n} \right] \right) ds$$
$$= 0.$$

3D

Initial results on separable domains look similar (rapid convergence with suitable basis functions etc.).

Challenges: Integrations over 2D surfaces can be tricky for $|\lambda|, |\mu|$ large (ideally want analytic form for given basis functions), study of singularities in 3D harder,...

Current Ideas (more welcome!):

- Domain decomposition and iterative solvers - becomes more like BEM in Fourier space.
- Couple with domain transform methods.

Curved Boundary and Separable PDE

PDE in divergence form:

$$\nabla \cdot (\alpha \nabla u) + \nabla \cdot (\beta u) + \gamma u = 0.$$

Domain $\mathcal{D}$ a curvilinear polygon, corners $\{z_j\}_1^n$ with the side Γ_j, joining z_j to z_{j+1} parametrised by

$$[-1, 1] \ni t \to (x_j(t), y_j(t)) \in \mathbb{R}^2.$$

Let v be a solution of adjoint, n outward normal,

$$\int_{\partial \mathcal{D}} u\big[(n \cdot \beta)v - n \cdot (\alpha^T \nabla v)\big] + v\big[n \cdot (\alpha \nabla u)\big] ds = 0.$$

One parameter family of solutions $v \to$ Global Relation.