The Unified Transform

A New Tool for Scattering Problems

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Sketch of talk

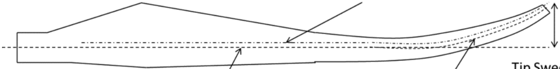
- Motivation
- Methodology I: scattering problem and unified transform
- Rigid plate example
- Methodology II: elastic plates
- Application: elastic plate extensions

Motivation

- **Application:** A big problem in aero-acoustics is noise reduction [1, 2, 3].
- **Current challenge:** developing **fast** and **accurate** numerical tools for scattering problems.
  $\rightarrow$ predict effect of physical parameters and external forces.
- Can we model complicated boundary conditions such as **elasticity**? (this is <u>difficult</u> via traditional methods)

Elastic $\rightarrow$ absorbs energy $\rightarrow$ reduced noise

Wind Turbines



Airport Noise

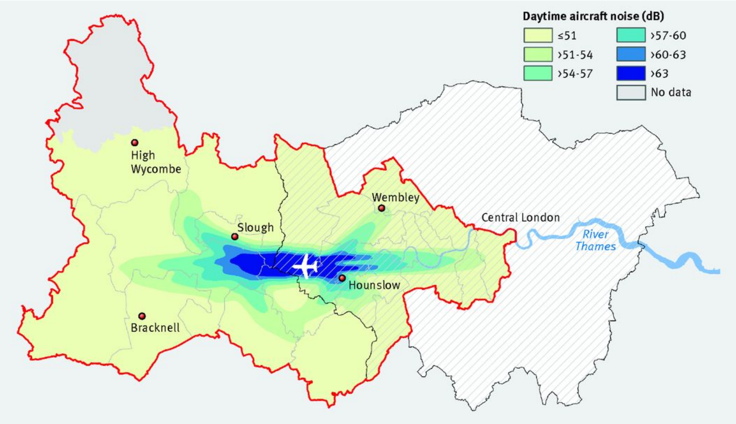


Figure: Noise levels (annual average) near Heathrow - a major health concern (source: The BMJ 2013;347:f5432).

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- Elastic plate deformation (more on this later)

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Sommerfeld radiation condition at infinity (radiates <u>to</u> infinity):

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial}{\partial r} - ik_0 \right) q(r,\theta) = 0$$

Unified Transform

Let q, v solve the Helmholtz equation in domain $\mathcal{D}$, then

$$\frac{\partial}{\partial x}\left(v\frac{\partial q}{\partial x} - q\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial q}{\partial y} - q\frac{\partial v}{\partial y}\right) = 0.$$

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Assuming everything converges, Green's theorem implies

$$\int_{\partial\mathcal{D}}\left[\left(v\frac{\partial q}{\partial x} - q\frac{\partial v}{\partial x}\right)dy - \left(v\frac{\partial q}{\partial y} - q\frac{\partial v}{\partial y}\right)dx\right] = 0.$$

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Choosing $v = \mathrm{e}^{-\mathrm{i}\beta(\lambda z + \frac{\bar{z}}{\lambda})}$ with $\beta = k_0/2$, $z = x + \mathrm{i}y$ gives

$$\int_{\partial\mathcal{D}} \mathrm{e}^{-\mathrm{i}\beta(\lambda z + \frac{\bar{z}}{\lambda})} \left[q_n + \beta\left(\lambda\frac{dz}{ds} - \frac{1}{\lambda}\frac{d\bar{z}}{ds}\right) q\right] ds = 0, \qquad \lambda \in \mathcal{C}(\mathcal{D}).$$

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View this as a **Fourier transform** of the boundary integral equations.

Building a Numerical Method

Idea: Expand boundary values in a suitable basis:

$$q(s) = \sum_{j=1}^{N} a_j S_j(s), \quad q_n(s) = \sum_{j=1}^{N} b_j T_j(s)$$

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Let $\hat{f}(\lambda) = \int_{\partial\mathcal{D}} \mathrm{e}^{-\mathrm{i}\beta(\lambda z(s) + \frac{\bar{z}(s)}{\lambda})} f(s) ds$ and evaluate at λ_i:

$$\sum_j a_j \beta \left(\lambda \frac{dz}{ds} - \frac{1}{\lambda} \frac{d\bar{z}}{ds} \right) \hat{S}_j(\lambda_i) + b_j \hat{T}_j(\lambda_i) = 0.$$

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Linear system, row i evaluation at λ_i (Fourier collocation):

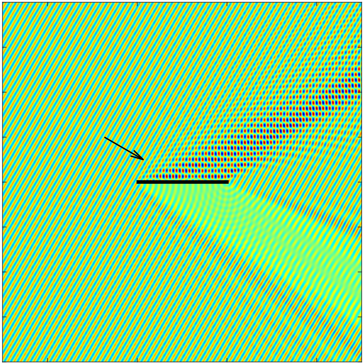
$$\begin{pmatrix} \text{Matrix formed} \\ \text{from combinations} \\ \text{of} \\ \hat{S}_j(\lambda_i) \text{ and } \hat{T}_j(\lambda_i) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \\ b_1 \\ \vdots \\ b_N \end{pmatrix} = 0$$

Advantages/extensions

Can generalise to separable PDEs and curved boundaries [4].
Some advantages of the method:

- Fast (couple of seconds for hundreds of basis functions).
- Easy to use and code (can be automated [5, 6]).
- Boundary based (dimensional reduction).
- Avoid evaluations of singular integrals (that arise in other methods such as BEM).
- Flexible choice of bases...

Single Rigid Plate (Analytic Solution Known)



$\mathcal{D}_1 : y > 0$

$\mathcal{D}_2 : y < 0$

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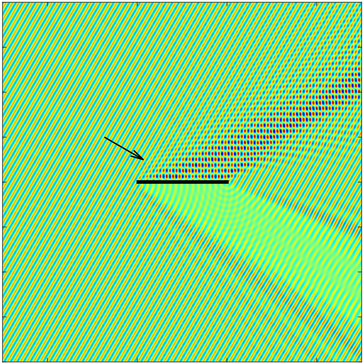

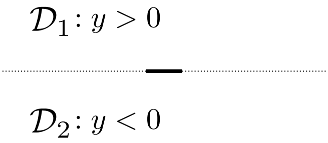

Use the symmetry...

For $\lambda \in (-1,0) \cup (1,\infty) \cup \{e^{i\theta} : \pi < \theta < 2\pi\}$:

$$\int_{-\infty}^{0} e^{-i\beta x(\lambda+\frac{1}{\lambda})} q_y(x,0)dx + \int_{1}^{\infty} e^{-i\beta x(\lambda+\frac{1}{\lambda})} q_y(x,0)dx$$

$$+ \int_{0}^{1} e^{-i\beta x(\lambda+\frac{1}{\lambda})} \frac{\beta}{2}\left(\lambda - \frac{1}{\lambda}\right)[q](x,0)dx = \int_{0}^{1} e^{-i\beta x(\lambda+\frac{1}{\lambda})} \frac{\partial q_I}{\partial y}(x,0)dx.$$

Technical Details

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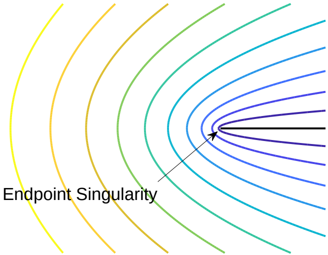

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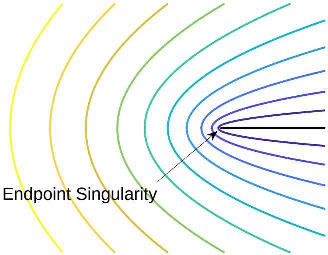

To capture endpoint singularities, expand $[q]$ in terms of weighted Chebyshev polynomials:

$$\sqrt{1-(2x-1)^2} \cdot U_n(2x-1).$$

Rapid Convergence!

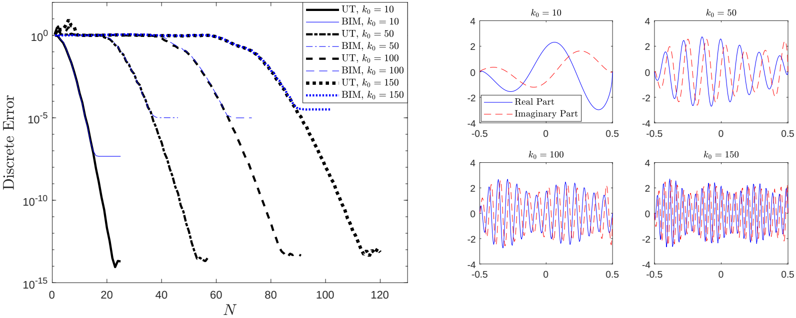

Figure: Left: Maximum relative error. UT denotes unified transform, BIM denotes boundary integral method of [7]. Right: Analytic solutions $[q](x,0)$ for different k_0.

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$$\left(\frac{\partial^4}{\partial x^4} - \frac{k_0^4}{\Omega_i^4}\right)\eta_i = -\frac{\epsilon_i}{\Omega_i^6}k_0^3[q] \quad \text{on} \quad \gamma_i.$$

ϵ_i =fluid loading (0.0021 for aluminium in air),
Ω_i =ratio of the bending wavenumber and the acoustic wavenumber (wobbliness),
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- Kinematic condition ($\eta_i = 0$ if γ_i rigid):

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- At endpoint $x = x_0$ of plate, either $\eta(x_0) = \eta'(x_0) = 0$ (clamped) or $\eta''(x_0) = \eta'''(x_0) = 0$ (free).

How to Cope? Vibrational Modes!

Idea: Expand η_i in eigenfunctions of ∇^4 subject to correct BCs:

$$\nabla^4 f_j = d_j^4 f_j, \quad \text{clamped/free at endpoints.}$$

$$\text{Expand: } \eta_i(x) \approx \sum_{j=1}^{N} a_{i,j} f_j(x),$$

$$\Rightarrow \frac{\partial q}{\partial y}(x) \approx -\frac{\partial q_I}{\partial y}(x) + \sum_{j=1}^{N} k_0^2 a_{i,j} f_j(x)$$

$$[q](x) = -\frac{\Omega_i^6}{k_0^3 \epsilon_i} \left(\frac{\partial^4}{\partial x^4} - \frac{k_0^4}{\Omega_i^4} \right) \eta_i(x)$$

$$\approx -\frac{\Omega_i^6}{k_0^3 \epsilon_i} \sum_{j=1}^{N} a_{i,j} \left(d_j^4 - \frac{k_0^4}{\Omega_i^4} \right) f_j(x).$$

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Compute f_j, d_j using standard spectral methods (very easy).

Easy to compute Fourier transforms:

$$(\lambda^4 - d_j^4) \int_a^b \mathrm{e}^{\mathrm{i}\lambda x} f_j(x) dx = (\mathrm{i}\lambda)^3 [\mathrm{e}^{\mathrm{i}\lambda x} f(x)]_{x=a}^b - (\mathrm{i}\lambda)^2 [\mathrm{e}^{\mathrm{i}\lambda x} f'(x)]_{x=a}^b$$
$$+ \mathrm{i}\lambda [\mathrm{e}^{\mathrm{i}\lambda x} f''(x)]_{x=a}^b - [\mathrm{e}^{\mathrm{i}\lambda x} f'''(x)]_{x=a}^b.$$

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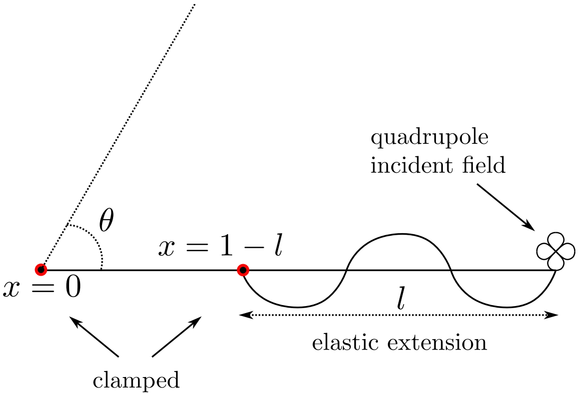
Upshot: Fast and accurate method able to cope with multiple plates with different physical parameters and geometric configurations. Mixture of elastic rigid plates etc. Can even cope with porous elastic plates.

Elastic Plate Extensions



Far-field Noise

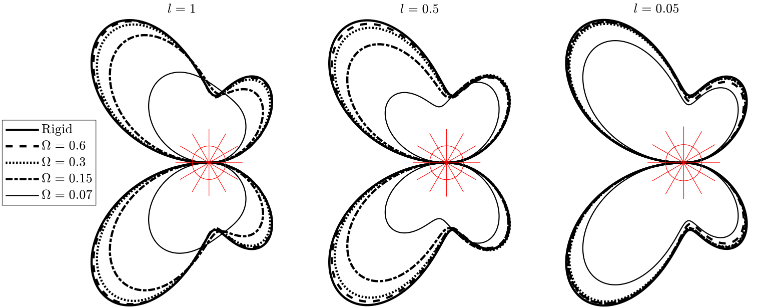


Figure: Far-field directivity for $k_0 = 5$, $\epsilon = 0.0021$ and different l.

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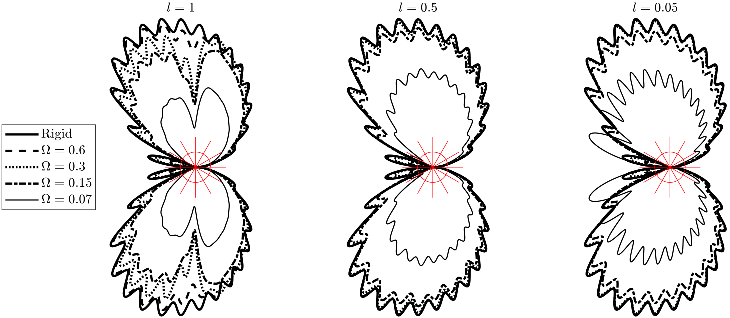

Figure: Far-field directivity for $k_0 = 50$, $\epsilon = 0.0021$ and different l.

Radiated Power

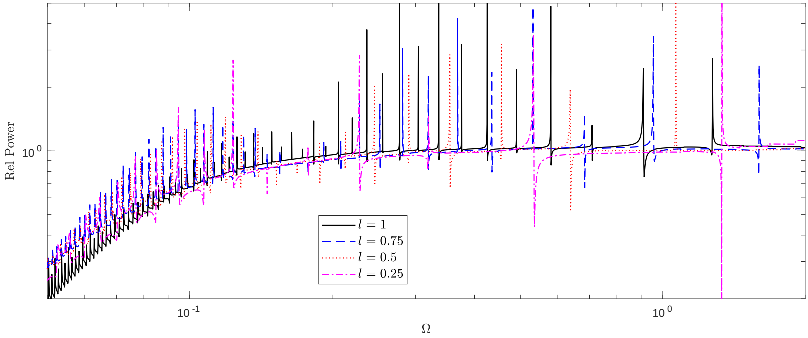

Figure: Relative power level as a function of Ω for $k_0 = 10$, $\epsilon = 0.0021$.

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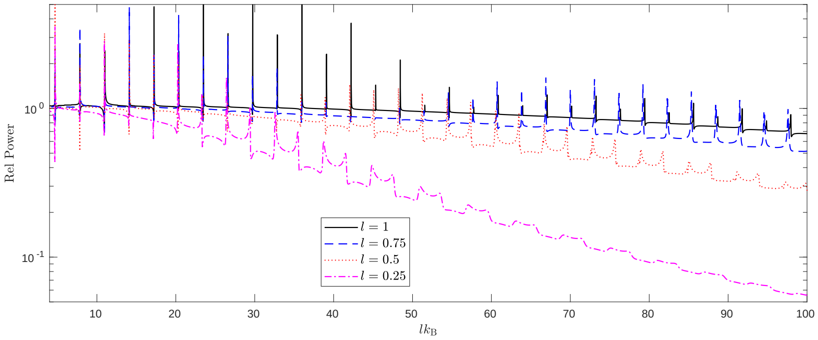

Figure: Relative power level as a function of lk_B ($k_B = k_0/\Omega$).

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- Different length extensions should be used depending on the frequencies to be reduced.
- Future work: consider aerodynamic impact of elastic extensions to balance acoustic and aerodynamic considerations.

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All of this can be extended to more complicated geometries [5].

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