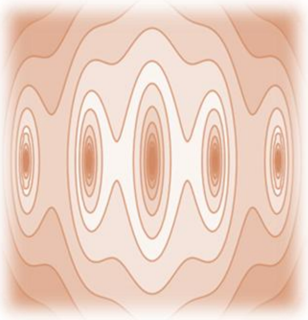

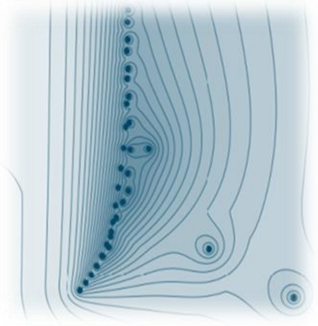

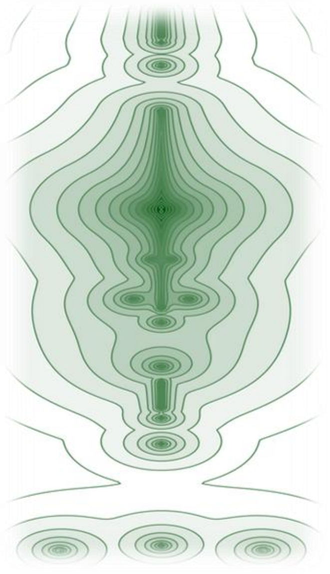

A Computational Framework for Infinite-Dimensional Nonlinear Spectral Problems

Matthew Colbrook

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21/05/2026

For papers and talk slides/videos, visit:
http://www.damtp.cam.ac.uk/user/mjc249/home.html

Motivating example throughout talk

- Klein–Gordon equation (relativistic spinless particle):

$$\left(-\nabla^2 + m^2 + \left[\frac{\partial}{\partial t} - ieq\right]^2\right)\psi = 0.$$

- Ansatz $\psi(x,t) = e^{izt} f(x)$:

$$(-\nabla^2 + m^2 - [eq - z]^2)f = 0.$$

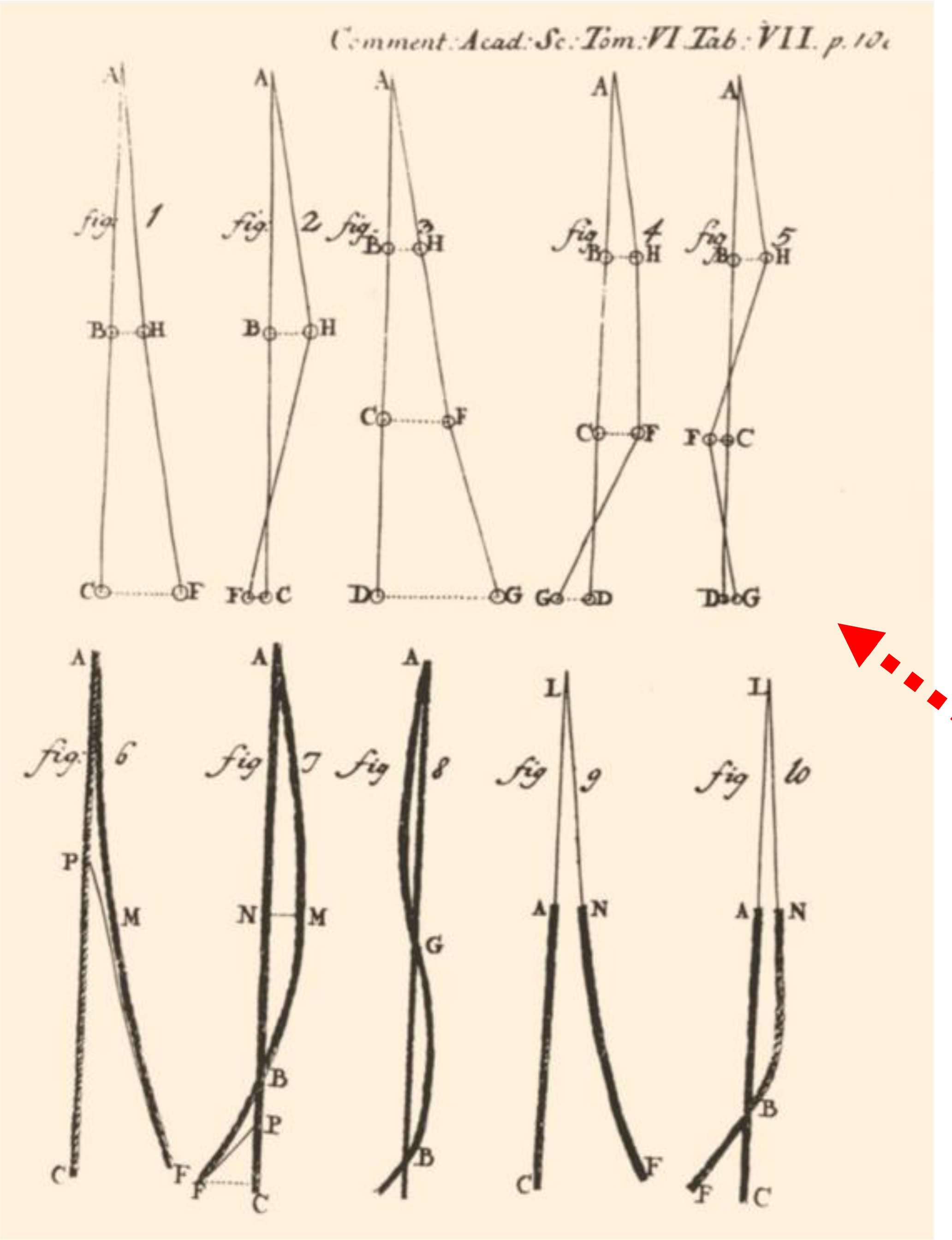
Nonlinear in spectral parameter z.

- Replace $-\nabla^2 + m^2$ by positive operator H_0 and eq by V:

$$T(z) = H_0 - (V - zI)^2.$$

- Lundberg, "Spectral and scattering theory for the Klein-Gordon equation," **Communications in Mathematical Physics**, 1973.
- Weder, "Scattering theory for the Klein-Gordon equation," **Journal of Functional Analysis**, 1978.

World’s 1st eigenvector was from a nonlinear spectral problem!



Modes of oscillations are roots of Bessel functions:

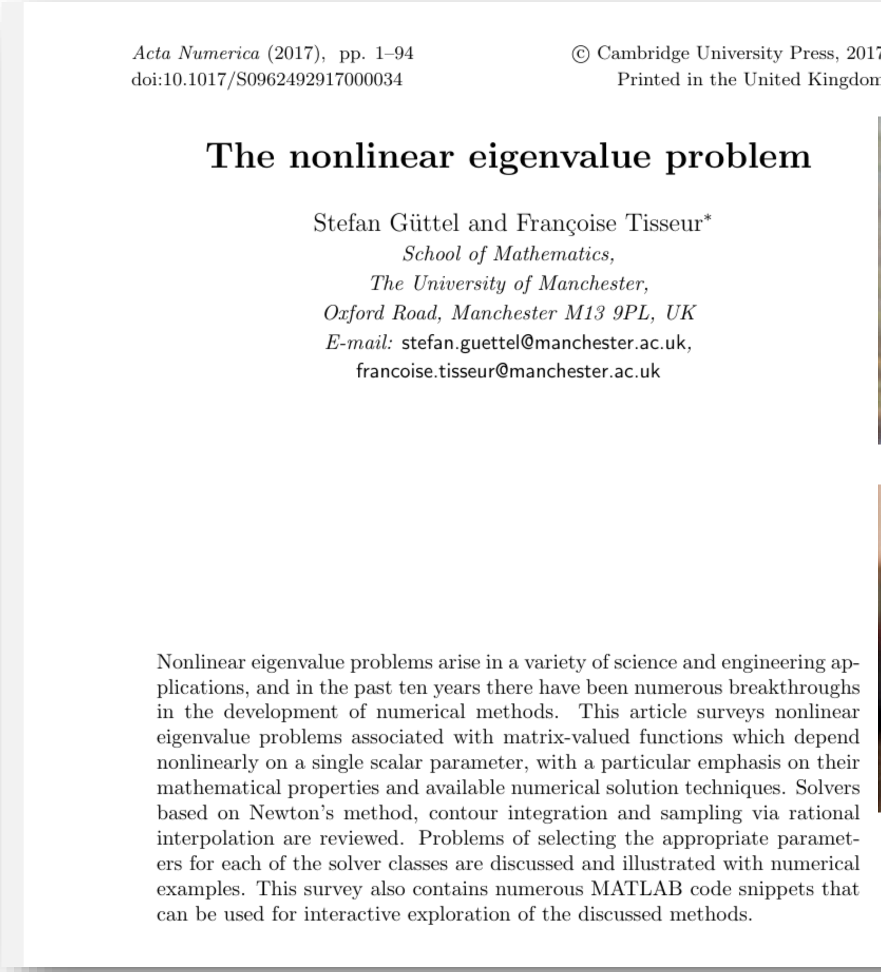
$$J_0(2/\sqrt{z}) = 0$$

From Daniel Bernoulli’s 1733 masterpiece *“Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae”*

Many applications of “nonlinear” spectra

- Mechanical vibrations
- Fluid stability
- Delay systems
- Photonics
- Resonances
- …

Well-studied in finite dimensions, but many applications arise from infinite dimensions.



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The nonlinear eigenvalue problem

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Nonlinear eigenvalue problems arise in a variety of science and engineering applications, and in the past ten years there have been numerous breakthroughs in the development of numerical methods. This article surveys nonlinear eigenvalue problems associated with matrix-valued functions which depend nonlinearly on a single scalar parameter, with a particular emphasis on their mathematical properties and available numerical solution techniques. Solvers based on Newton’s method, contour integration and sampling via rational interpolation are reviewed. Problems of selecting the appropriate parameters for each of the solver classes are discussed and illustrated with numerical examples. This survey also contains numerous MATLAB code snippets that can be used for interactive exploration of the discussed methods.

Example 1: Klein–Gordon equation

$$T(z) = H_0 - (V - zI)^2$$

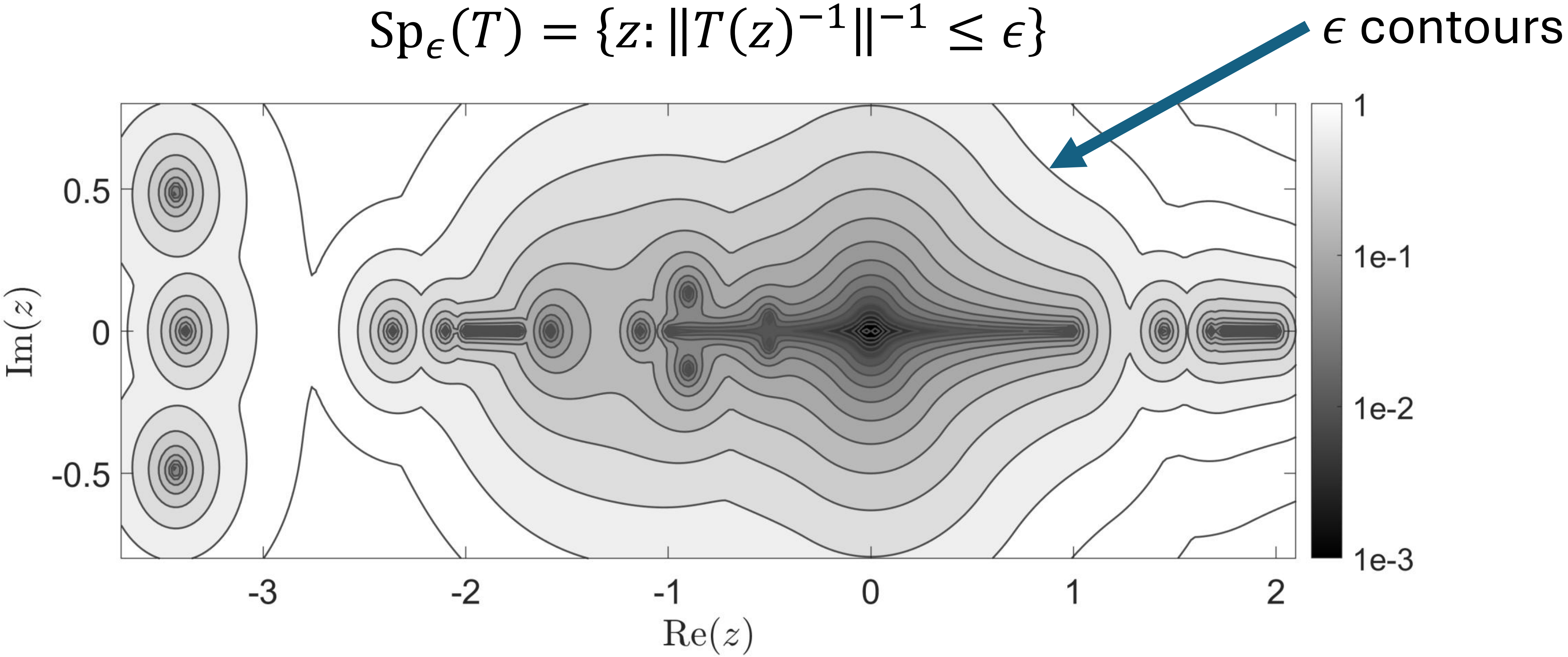
$$H_0 = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & \frac{3}{2} & 2 & \frac{1}{2} & & & \\ & & \frac{1}{2} & |2| & \frac{3}{2} & & \\ & & & \frac{3}{2} & 2 & \frac{1}{2} & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad [Vx]_n = -5\exp(-|n|)x_n, \quad x \in l^2(\mathbb{Z}), n \in \mathbb{Z}.$$

A nice bounded quadratic pencil.

What could possibly go wrong?

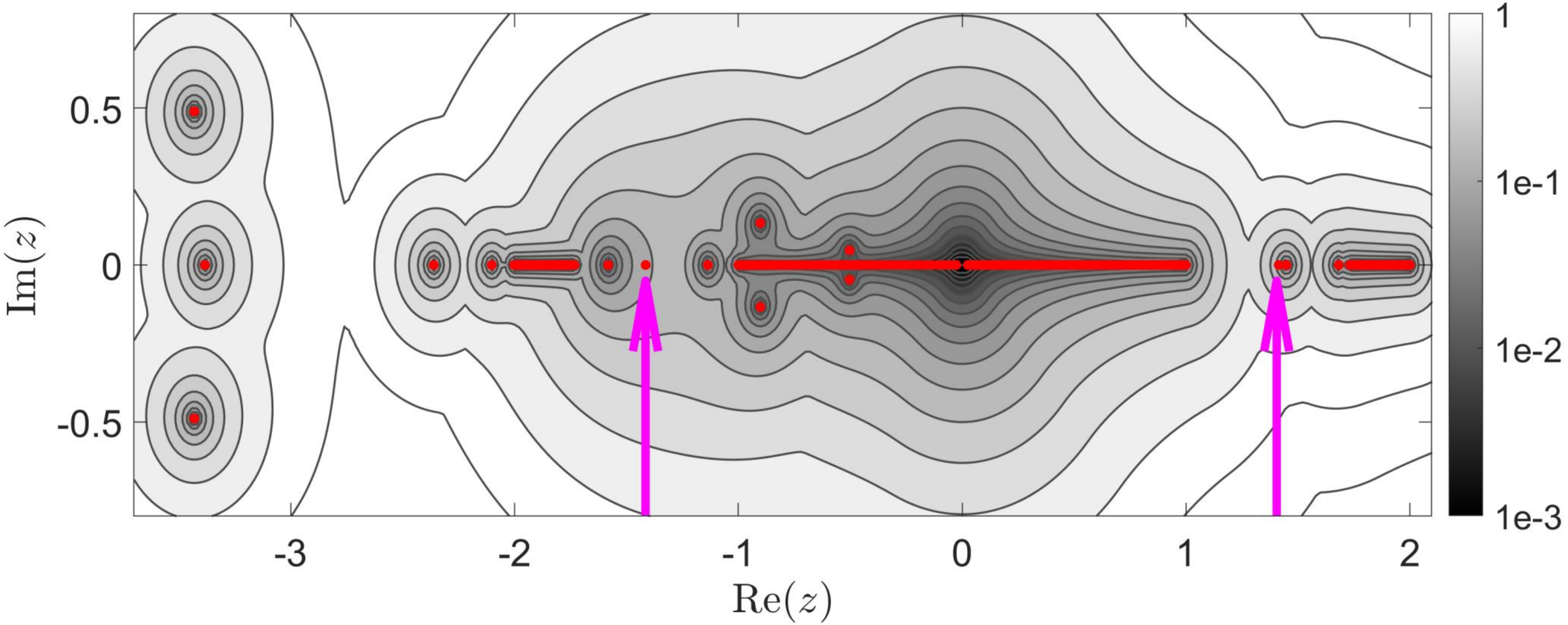
Example 1: Klein–Gordon equation

$$\mathrm{Sp}_\epsilon(T) = \{z : \|T(z)^{-1}\|^{-1} \leq \epsilon\}$$



Example 1: Klein–Gordon equation

Truncate to $\text{span}\{e_{-100}, \ldots, e_{100}\}$ – spurious eigenvalues!



Can we develop a systematic computational theory in infinite dimensions?

Want:

- Convergence to underlying applicational problem.
  i.e., no truncation or discretization artifacts
- Practical.

Part 1: Spectra and pseudospectra when $z \mapsto T(z)$ is continuous.

Part 2: Discrete spectrum when $z \mapsto T(z)$ is holomorphic.

An operator is a subspace

- **Fact of life:** Most applications involve differential operators.

An operator is a subspace

- **Fact of life:** Differential operators are unbounded.

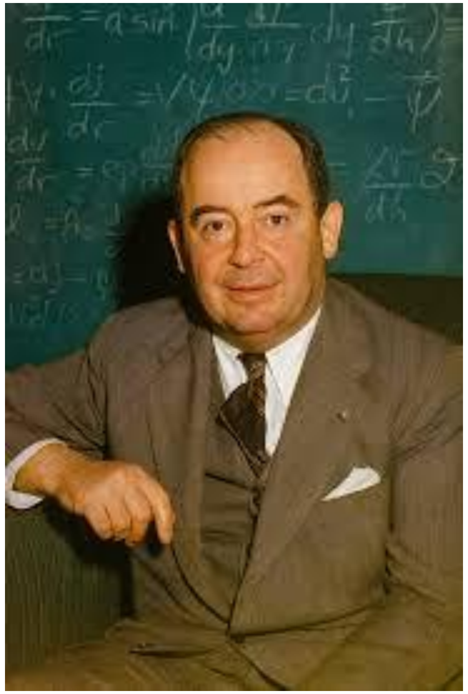
An operator is a subspace

- **Fact of life:** Differential operators are unbounded.
- $\mathcal{H}_1, \mathcal{H}_2$ separable Hilbert spaces (countable o.n. basis)
- Operator $A: \mathcal{D}(A) \to \mathcal{H}_2$ with domain $\mathcal{D}(A) \subset \mathcal{H}_1$

$$\text{graph}(A) = \{(x, Ax): x \in \mathcal{D}(A)\} \subset \mathcal{H}_1 \times \mathcal{H}_2$$

A is *closed* when $\text{graph}(A)$ is a closed subspace.

John von Neumann



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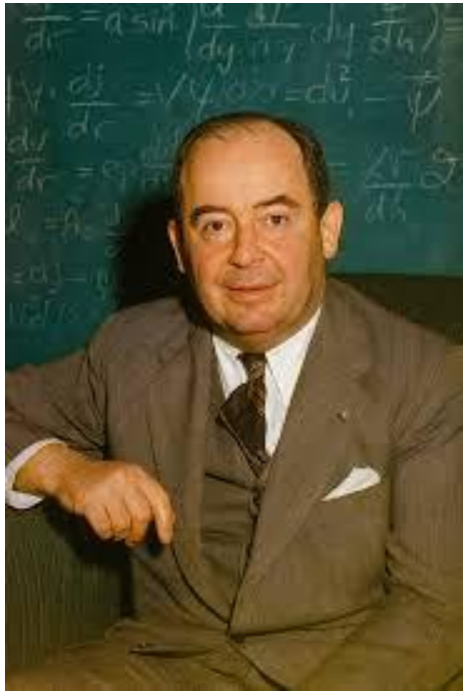

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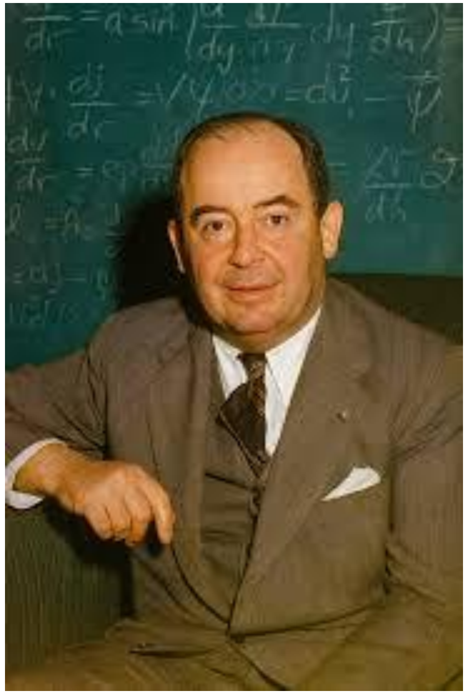
Not closed: $A = \frac{d^2}{dx^2}$ on $[-1,1]$ with no BCs

Closed: $A = \frac{d^2}{dx^2}$ on $[-1,1]$, $\mathcal{D}(A) = \{u \in H^2(-1,1), u(-1) = u(1) = 0\}$

An operator is a subspace

John von Neumann



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Why am I telling you this?

1. Correct setting (spectra don't make sense otherwise).
2. Compare operators with different $\mathcal{D}(A)$ (e.g, z-dependent BCs).
3. We can use subspace theory to compute pseudospectra...

The setting

- $U \subset \mathbb{C}$ a bounded domain (non-empty, connected open set)
- $T: U \to \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, i.e., each $T(z)$ closed linear operator.

Spectrum: $\mathrm{Sp}(T) = \{z \in U: T(z)^{-1} \text{ doesn't exist}\}$

Pseudospectrum: $\mathrm{Sp}_\epsilon(T) = \{z \in U: \|T(z)^{-1}\|^{-1} \leq \epsilon\}$

Assume: $\mathrm{graph}(T(z))$ varies continuously in z.

Implies: $z \mapsto \|T(z)^{-1}\|^{-1}$ continuous on U.

Every problem you meet in the “wild” satisfies this!

What type of convergence do we want?

For $\Gamma_n(T) \subset \mathbb{C}$ to approximate $\mathrm{Sp}(T) \subset \mathbb{C}$ as $n \to \infty$:

- **No spectral pollution:** Every $z \in U$ with $\liminf_{n\to\infty} \mathrm{dist}\big(z, \Gamma_n(T)\big) = 0$ must have $z \in \mathrm{Sp}(T)$. **NO BAD GUYS**
- **No spectral invisibility:** Every $z \in \mathrm{Sp}(T)$ must have $\limsup_{n\to\infty} \mathrm{dist}\big(z, \Gamma_n(T)\big) = 0$. **NO GOOD GUYS ARE MISSED**

Local uniform convergence on U.

Bullet proof algorithm: $\mathrm{Sp}_\epsilon(T)$ to $\mathrm{Sp}(T)$

$\sigma_{\inf}(A) = \inf\{\|Ax\| : x \in \mathcal{D}(A), \|x\| = 1\}$

Fact: $\|T(z)^{-1}\|^{-1} = \min\{\sigma_{\inf}(T(z)), \sigma_{\inf}(T(z)^*)\}$

- ***Idea:*** Approximate $\sigma_{\inf}(T(z))$.
  Let $\mathcal{P}_N$ (possibly z-dependent) be orthogonal projection onto subspace V_N. Natural density and domain assumptions on V_N.

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- **Ingredient:** Search for z s.t. $\min\{\sigma_{\inf}(T(z)\mathcal{P}_N), \sigma_{\inf}(T(z)^*\mathcal{P}_N)\} < \epsilon$.
- **Output:** $\Gamma_N^\epsilon(T)$ with $\Gamma_N^\epsilon(T) \subset \mathrm{Sp}_\epsilon(T)$ and $\lim_{N\to\infty} \Gamma_N^\epsilon(T) = \mathrm{Sp}_\epsilon(T)$.

$$\lim_{\epsilon \downarrow 0} \lim_{N\to\infty} \Gamma_N^\epsilon(T) = \mathrm{Sp}(T)$$

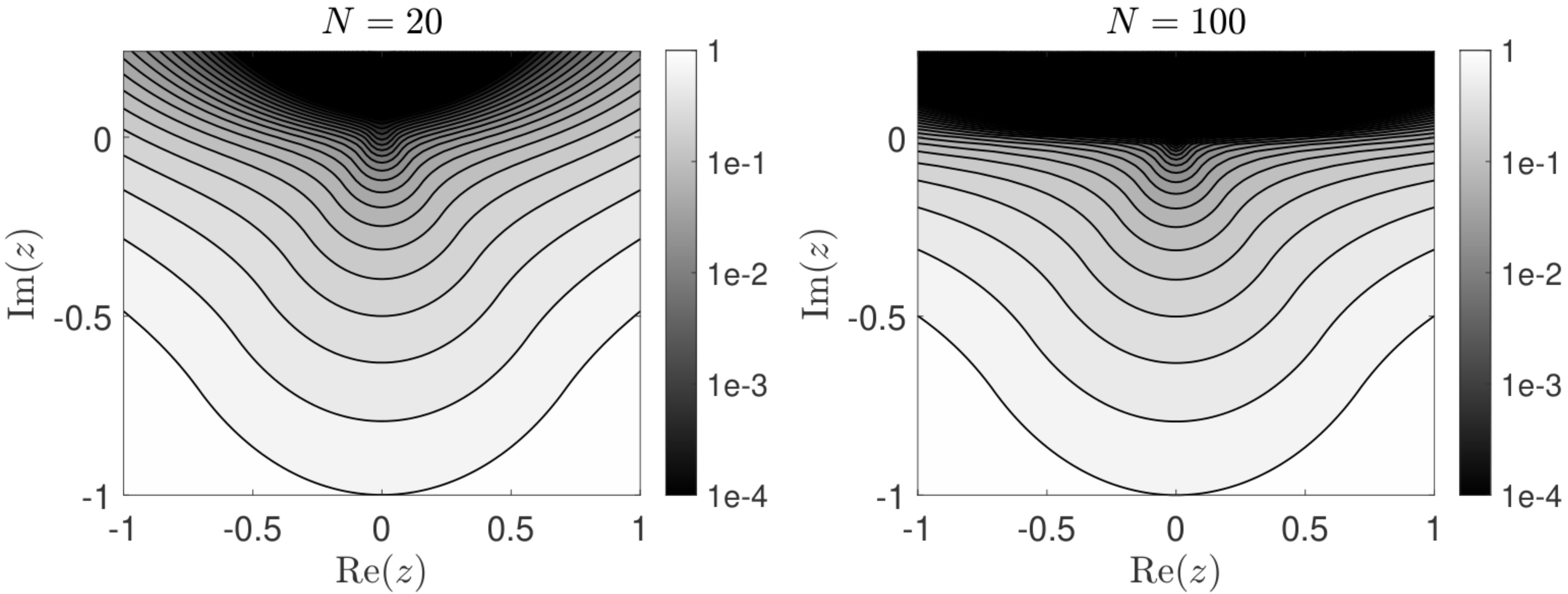
Example 2: Wave equation with acoustic BCs

$$T(z)u = u'' + z^2 u, \quad \mathcal{D}(T(z)) = \{u \in \mathcal{W}^{2,2}(\mathbb{R}_{>0}) : -u'(0) + \mathrm{i}zu(0) = 0\}$$

$$\phi_n(x) = L_n(x)e^{-x/2}, \quad L_n(x) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)^n x^n, \quad n = 0, 1, 2, \ldots$$

$$\hat{\phi}_n(x) = \phi_n(x) + \alpha_n \phi_0(x), \quad \alpha_n = -(2\mathrm{i}z + 2n + 1)/(2\mathrm{i}z + 1), \quad n \in \mathbb{N}.$$

+ QR to compute $\sigma_{\inf}(T(z)\mathcal{P}_N)$



Weird things happen with truncation

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Truncate to $[0, L]$ and rescale to $[0,1]$:

$$S(\lambda)u = u'' + (2\pi\lambda)^2 u, \quad \mathcal{D}(S(\lambda)) = \{u \in \mathcal{W}^{2,2}((0,1)) : -u'(0) + \mathrm{i}2\pi\lambda u(0) = u(1) = 0\}$$

$\mathrm{Sp}(S) = \emptyset$  

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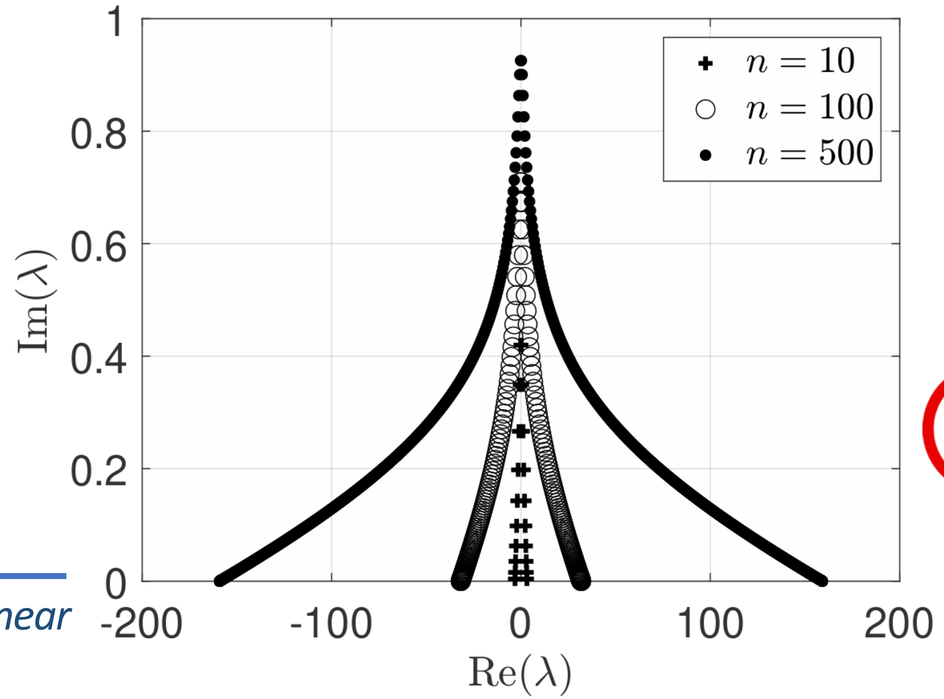
FEM-discretised version

in NLEVP collection



?

Betcke, Higham, Mehrmann, Schröder, Tisseur, *"NLEVP: A collection of nonlinear eigenvalue problems,"* **ACM Transactions on Mathematical Software**, 2013.

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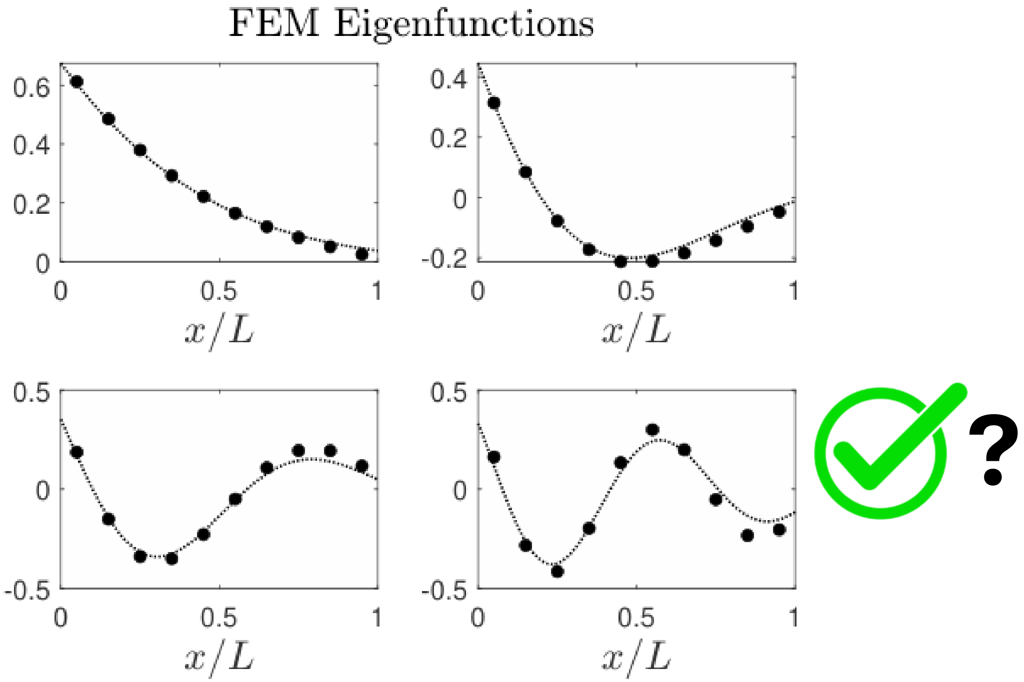

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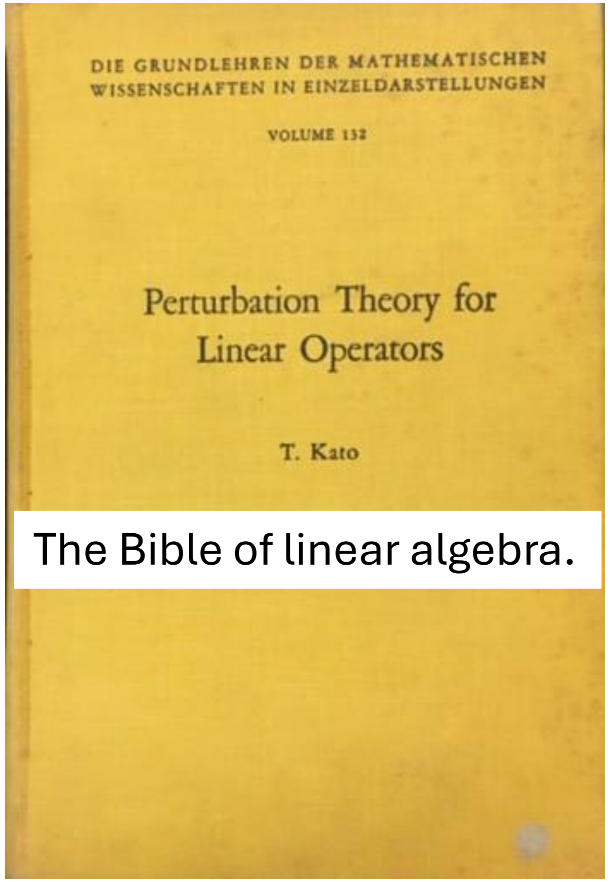
Holomorphic families

- If $T(z)$ bounded, means local power series:

$$T(z) = \sum_{k=0}^{\infty} (z - z_0)^k T_k\,.$$



The Bible of linear algebra.

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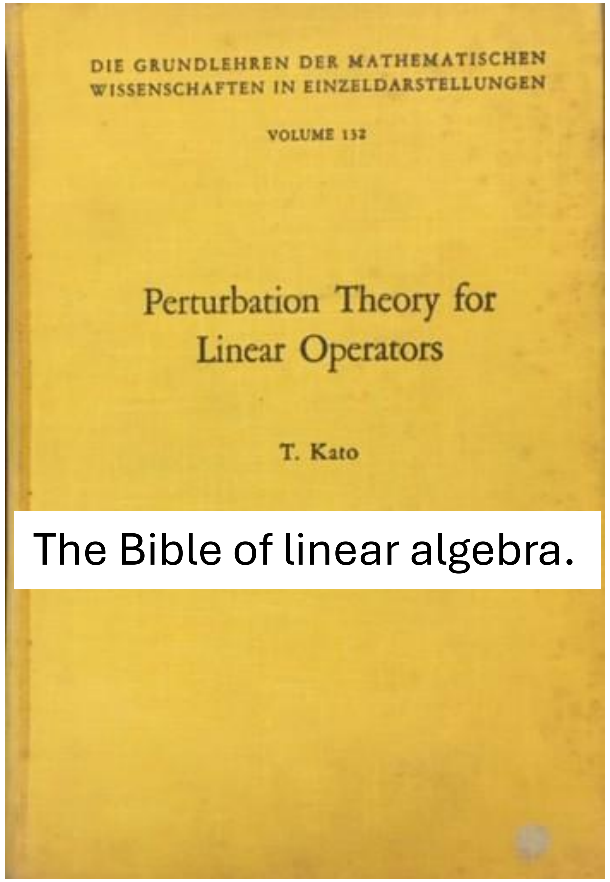

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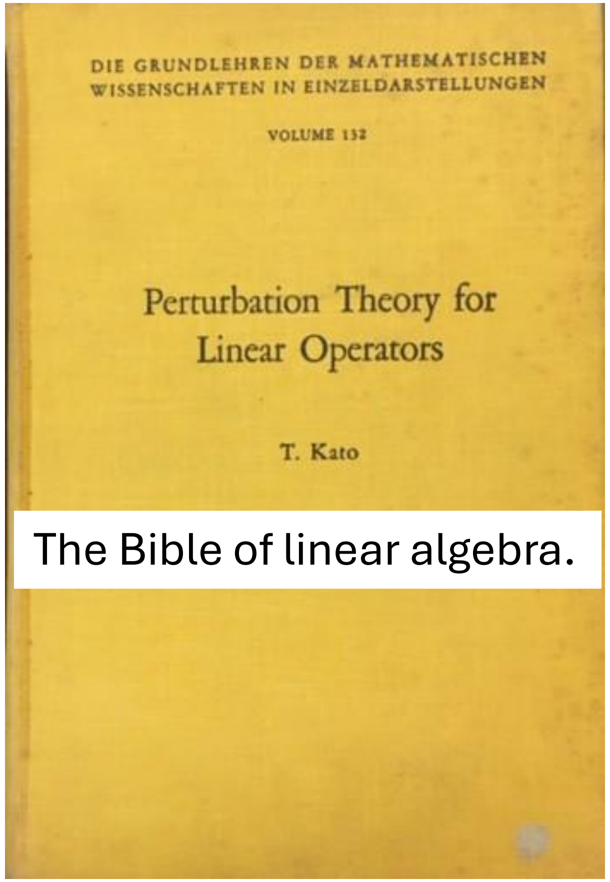
- For unbounded T, use a trick of Rellich:

$$A: U \to \mathcal{B}(\mathcal{H}_3, \mathcal{H}_1), \quad B: U \to \mathcal{B}(\mathcal{H}_3, \mathcal{H}_2)$$

bounded holomorphic with $T(z)A(z) = B(z)$.



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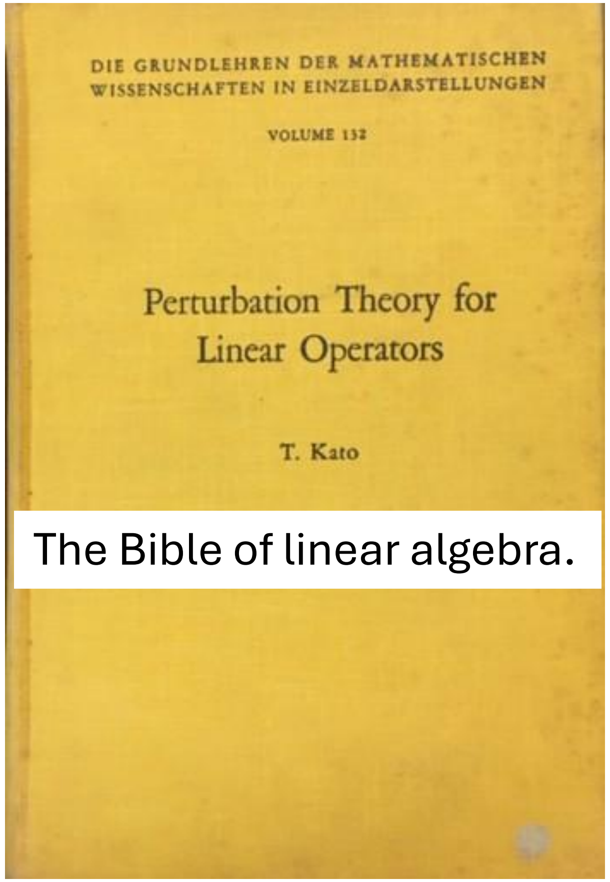
bounded holomorphic with $T(z)A(z) = B(z)$.
- Lift results from the bounded case...



The Bible of linear algebra.

Keldysh's theorem (Laurent expansion of $T(z)^{-1}$)

Assume: T holomorphic, $\mathrm{Sp}(T) \neq U$, $\mathcal{D}(T(z))$ dense + $T(z)$ Fredholm $\forall z \in U$.

Consequence: Let $\lambda \in \mathrm{Sp}(T)$. Then λ is an isolated eigenvalue + there exists

- V_λ: quasimatrix of m_λ generalised eigenvectors in $\mathcal{H}_1$;
- W_λ: quasimatrix of m_λ (left) generalised eigenvectors in $\mathcal{H}_2$;
- J_λ: $m_\lambda \times m_\lambda$ Jordan matrix;
- Bounded holomorphic family $R_\lambda(z)$ s.t. near λ;

$$T(z)^{-1} = R_\lambda(z) + V_\lambda(zI - J_\lambda)^{-1} W_\lambda^*.$$

Combine if finitely many evals in U: Stack the matrices:

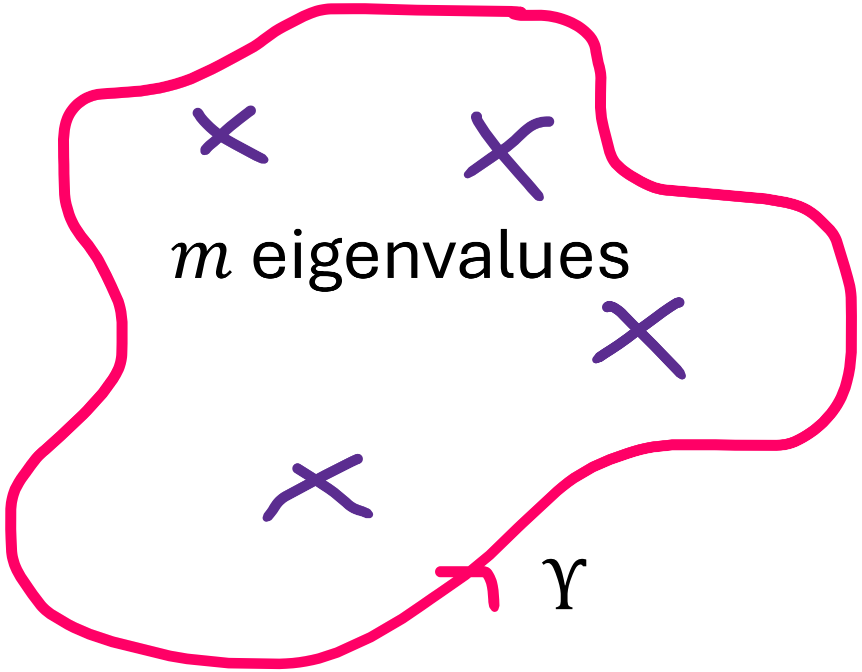
$$T(z)^{-1} = R(z) + V(zI - J)^{-1} W^*.$$

Mstislav Keldysh

Contour methods: → finite linear problem

$T(z)^{-1} = R(z) + V(zI - J)^{-1}W^*$

Random quasimatrices: $F = (f_1 \cdots f_l)$, $G = (g_1 \cdots g_r)$ (see later!)



$h(z)$ holomorphic

$$M_h = \frac{1}{2\pi i}\int_\Upsilon h(z)\, F^*T(z)^{-1}G\, \mathrm{d}z$$
$$= F^*Vh(J)W^*G \in \mathbb{C}^{l\times r}$$

Contour methods: → finite linear problem

$$h_{i,j}(z) = h_{\text{left},i}(z) h_{\text{right},j}(z), \quad i = 1, \dots, p, \quad j = 1, \dots, q$$

$$B_0 = \begin{pmatrix} M_{h_{1,1}} & \cdots & M_{h_{1,q}} \\ \vdots & & \vdots \\ M_{h_{p,1}} & \cdots & M_{h_{p,q}} \end{pmatrix} \in \mathbb{C}^{pl \times qr}, \quad B_1 = \begin{pmatrix} M_{h_{1,1}\cdot(z-z_0)} & \cdots & M_{h_{1,q}\cdot(z-z_0)} \\ \vdots & & \vdots \\ M_{h_{p,1}\cdot(z-z_0)} & \cdots & M_{h_{p,q}\cdot(z-z_0)} \end{pmatrix} \in \mathbb{C}^{pl \times qr}$$

$$V_p = \begin{pmatrix} F_{\text{left}}^* V h_{\text{left},1}(J) \\ \vdots \\ F_{\text{left}}^* V h_{\text{left},p}(J) \end{pmatrix} \in \mathbb{C}^{pl \times m}, \quad W_q^* = \begin{pmatrix} h_{\text{right},1}(J) W^* G_{\text{right}} & \cdots & h_{\text{right},q}(J) W^* G_{\text{right}} \end{pmatrix} \in \mathbb{C}^{m \times qr}$$

Factorised form: $B_0 = V_p W_q^*, \qquad B_1 = V_p (J - z_0 I) W_q^*$

Contour methods: → finite linear problem

Lemma: *Suppose* V_p *and* W_q *have rank* m. *Let* $B_0 = U_0 \Sigma_0 V_0^*$ *be a compact SVD. Then* $X = U_0^* V_p$ *and* $Y = V_0^* W_p$ *are invertible and*

$$S(z) = U_0^*[B_1 - (z - z_0)B_0]V_0 = X(J - zI)^{-1}X^{-1}\Sigma_0, (Y^* = X^{-1}\Sigma_0)$$

- Reduces to finite-dimensional linear problem S!
- Similar result recovers eigenvectors V and W.
- **In practice:** M_h approximated using quadrature, system solves $F^*T(z)^{-1}G$ (embarrassingly parallelisable).

$$M_h = \frac{1}{2\pi i}\int_\Upsilon h(z)\, F^*T(z)^{-1}G \,\mathrm{d}z$$

Unifies methods in the literature

Paper	Corresponding parameters
Sakurai and Sugiura 2003	Polynomial h, $p = q, l = r = 1$
Asakura, Sakurai, Tadano, Ikegami and Kimura 2009	Polynomial h, $p = q, l = r$
Beyn 2012	Polynomial h, $p = q$, no sketching on left
Yokota and Sakurai 2013	Polynomial h, $p = q, l = r$
Brennan, Embree and Gugercin 2023	Rational h (connections with system realisation)

- Rank condition generically holds for sufficiently large p, q.
- Other methods use iteration (not in this talk).

Key Questions:

- How does error in approximating M_h affect results?
- How does use of random vectors affect results?

Perturbation analysis

$M_h = \frac{1}{2\pi i}\int_\Upsilon \; h(z)\, F^* T(z)^{-1} G \,\mathrm{dz}$ approximated by $\widetilde{M_h}$

$\Rightarrow \widetilde{B_0} \approx \widetilde{U_0}\widetilde{\Sigma_0}\widetilde{V_0^*}$ (m-truncated SVD), $\widetilde{B_1}$, $\tilde{S}(z) = \widetilde{U_0^*}\left[\widetilde{B_1} - (z - z_0)\widetilde{B_0}\right]\widetilde{V_0}$.

Assume: $\left\|B_0 - \widetilde{B_0}\right\| \leq \epsilon_B \|B_0\|$, $\left\|B_1 - \widetilde{B_1}\right\| \leq \epsilon_B \|B_1\|$.

Consequence: If λ an eigenvalue of $\tilde{S}$ then

$$\|(J - \lambda I)^{-1}\|^{-1} \leq \left\|V_p^\dagger\right\| \left\|W_q^\dagger\right\| \left(4\|B_1\| \frac{\sigma_1(B_0)}{\sigma_m(B_0)} + 2|\lambda - z_0| \|B_0\| + \|B_1\|\right) \epsilon_B.$$

Choosing random F and G

- Positive, self-adjoint, trace-class operator C on $\mathcal{H}$.
- Orthonormal basis of e-vectors ψ_β, e-values $c_\beta^2 \geq 0, \sum_\beta c_\beta^2 < \infty$.
- **Karhunen–Loève** expansion:

$g = \sum_\beta c_\beta \zeta_\beta \psi_\beta$, $\zeta_\beta = \frac{1}{\sqrt{2}}(a_\beta + ib_\beta) \sim$ i.i.d. standard complex Gaussian

Write $g \sim \mathrm{GP}(0, C)$, C the *covariance operator*.

$$F_{\text{left}} = (f_1 \cdots f_l), f_j \sim \mathrm{GP}(0, C_{\text{left}}), \quad G_{\text{right}} = (g_1 \cdots g_r), g_j \sim \mathrm{GP}(0, C_{\text{right}})$$

Sketching doesn't degrade things

$$F_{\text{left}} = (f_1 \cdots f_l)$$
$$f_j \sim \text{GP}(0, C_{\text{left}})$$

Lemma: *Let $p \in \mathbb{N}$ and, for each $j = 1, \ldots, p$, let $A_j \in \mathbb{C}^{m \times m}$. Let L be a quasimatrix with $m \geq \text{rank}(V) \geq 2$ columns in a separable (possibly finite-dimensional) Hilbert space $\mathcal{H}$. Define the sketched and unsketched matrices*

$$A_{\text{sketch}} = \begin{pmatrix} F_{\text{left}}^* V A_1 \\ \vdots \\ F_{\text{left}}^* V A_p \end{pmatrix} L^*, \qquad A = \begin{pmatrix} V A_1 \\ \vdots \\ V A_p \end{pmatrix} L^*.$$

Let $\mathcal{P}$ denote the orthogonal projection onto the column space of V, and assume that

$$\text{rank}(\sqrt{C_{\text{left}}}\mathcal{P}^*) = \text{rank}(V) \leq l, \qquad \text{rank}(A) = m.$$

Then, for every $\epsilon > 0$ satisfying

$$\epsilon < \frac{1}{2\pi}\left(\frac{6.298}{l - \text{rank}(V) + 1}\right)^{2(l - \text{rank}(V)+1)},$$

we have

$$\mathbb{P}\left(\frac{\sigma_1(A_{\text{sketch}})}{\sigma_m(A_{\text{sketch}})} \geq \frac{\sigma_1(A)}{\sigma_m(A)} \frac{\sigma_1(\sqrt{C_{\text{left}}}\mathcal{P}^*)}{\sigma_{\text{rank}(V)}(\sqrt{C_{\text{left}}}\mathcal{P}^*)} \frac{6.298}{(2\pi\epsilon)^{\frac{1}{2(l-\text{rank}(V)+1)}}} \frac{l}{l - \text{rank}(V) + 1}\right) \leq \epsilon.$$

Set $l = \text{rank}(V) + 5$,

$$\frac{\sigma_1(A_{\text{sketch}})}{\sigma_m(A_{\text{sketch}})} \leq 1.7l \cdot \frac{\sigma_1(A)}{\sigma_m(A)} \cdot \frac{\sigma_1(\sqrt{C_{\text{left}}}\mathcal{P}^*)}{\sigma_{\text{rank}(V)}(\sqrt{C_{\text{left}}}\mathcal{P}^*)}$$

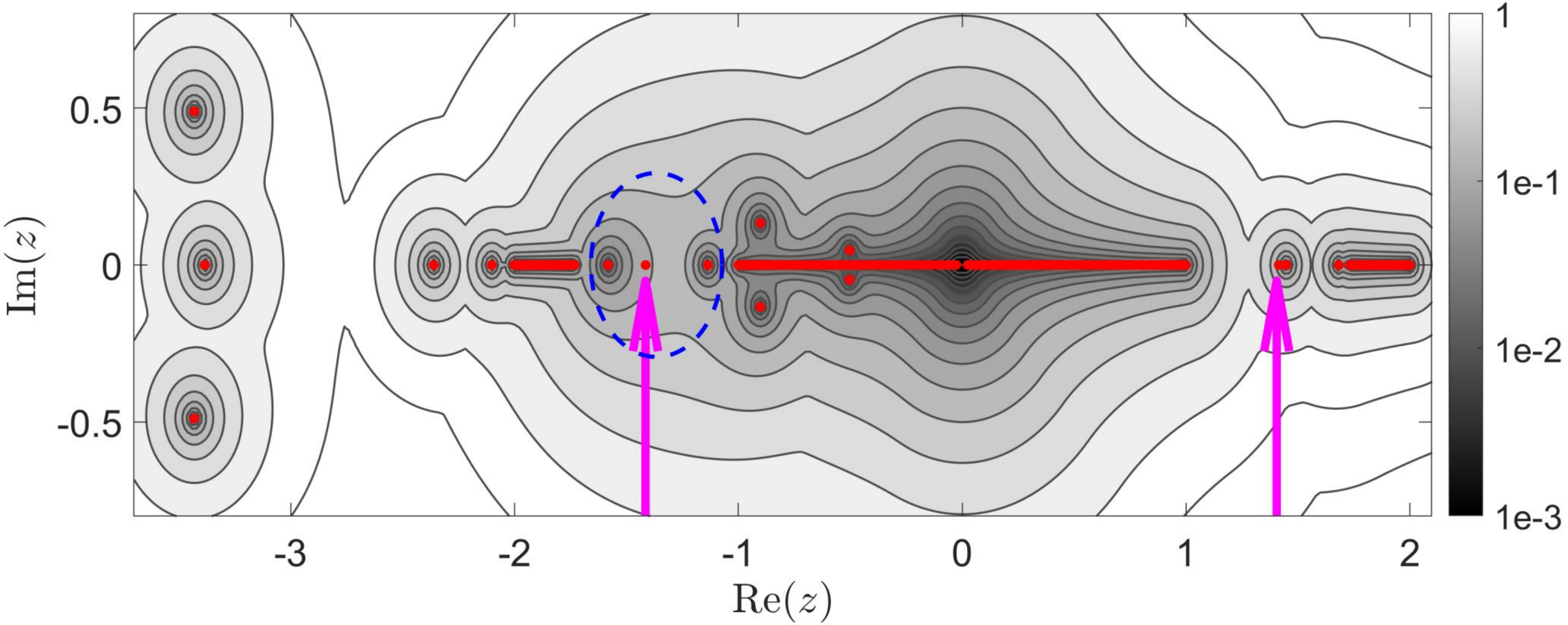
with probability > 0.999.

Apply twice to control $\|V_p^\dagger\|\|W_q^\dagger\|$ and $\frac{\sigma_1(B_0)}{\sigma_m(B_0)}$.

Stability no worse than unsketched matrix (depends on T & chosen h)

Example 3: Above matters in practice

Blue contour that contains 2 true eigenvalues...



Example 3: Above matters in practice

KL expansion:

$$\sum_{\beta} c_\beta \zeta_\beta \psi_\beta$$

- Choose C with $\psi_\beta = e_\beta$ (canonical basis of $\ell^2(\mathbb{Z})$).
- $N = 300$ (601 basis functions), trapezoidal rule.

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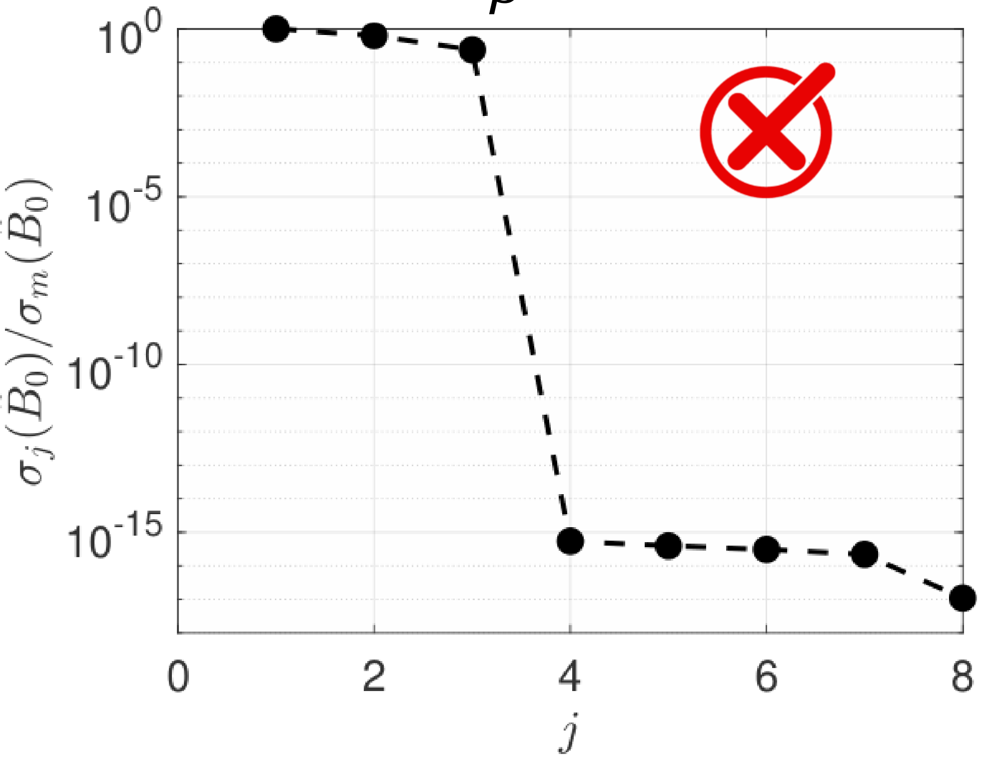
$c_\beta = 1$



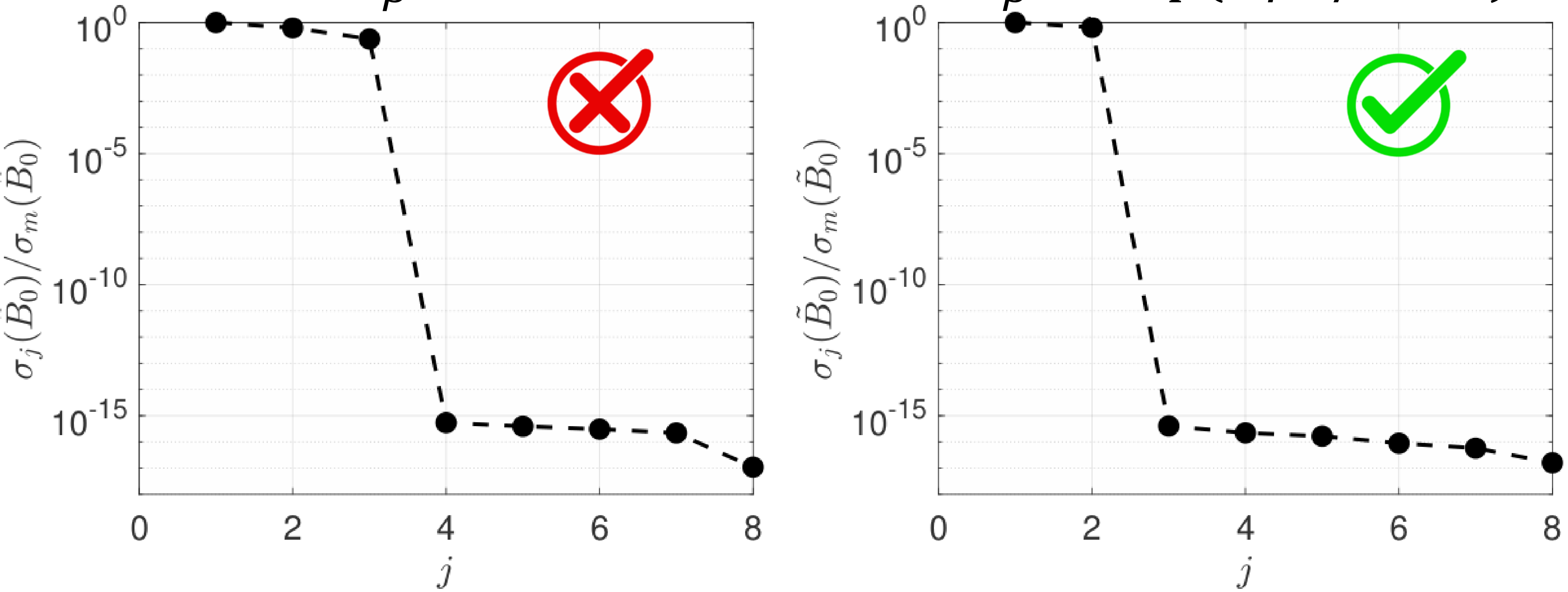
Example 3: Above matters in practice

KL expansion:

$$\sum_{\beta} c_\beta \zeta_\beta \psi_\beta$$

- Choose C with $\psi_\beta = e_\beta$ (canonical basis of $\ell^2(\mathbb{Z})$).
- $N = 300$ (601 basis functions), trapezoidal rule.

$c_\beta = 1$ $\qquad$ $c_\beta = \exp(-\beta^2/1000)$



But I thought we had convergence?!?!

But I thought we had convergence?!?!

- $c_\beta = 1$ does not lead to a convergent series $\sum_\beta c_\beta \zeta_\beta \psi_\beta$

  Analysis only holds if F_{left} and G_{right} in the correct Hilbert space.

- Eigenvectors v_N corresponding to pollution converge weakly to 0:

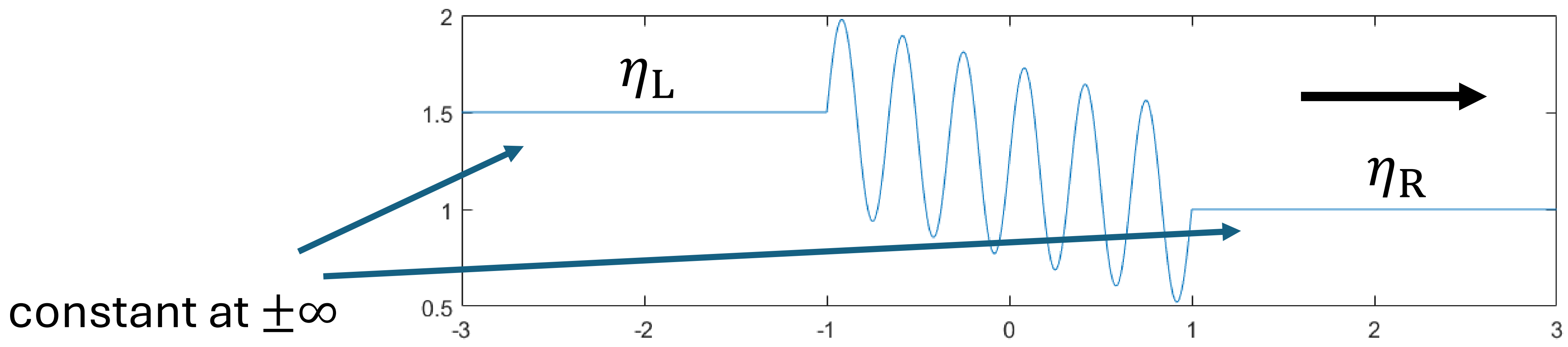
$$\lim_{N\to\infty} \langle v_N, u \rangle = 0 \quad \forall u \in \ell^2(\mathbb{Z}).$$

When we sketch and truncate the SVD of $\widetilde{B_0}$ we suppress them!

$\Longrightarrow$ **With care, contour methods defend against pollution!**

Example 4: Resonances

Planar waveguide with refractive index η:



Modes $\sim \phi(x)\exp(ik(nx - t))$ described by

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + k^2[\eta(x)^2 - z^2]\phi(x) = 0, \qquad x \in \mathbb{R}.$$

Take $k = 11$, consider ***resonances*** – physical solutions $\notin L^2(\mathbb{R})$.
(very useful, very hard to compute!)

Example 4: Resonances

Truncate to $[-1,1]$, use Dirichlet to Neumann map:

$$\phi(x) = \begin{cases} \exp(\mathrm{i}\alpha_{\mathrm{L}}(z)(x+1))\phi(-1), & x < -1 \\ \exp(-\mathrm{i}\alpha_{\mathrm{R}}(z)(x-1))\phi(1), & x > 1 \end{cases}$$

$$\alpha_{\mathrm{L}}(z) = k\sqrt{\eta_{\mathrm{L}}^2 - z^2},$$

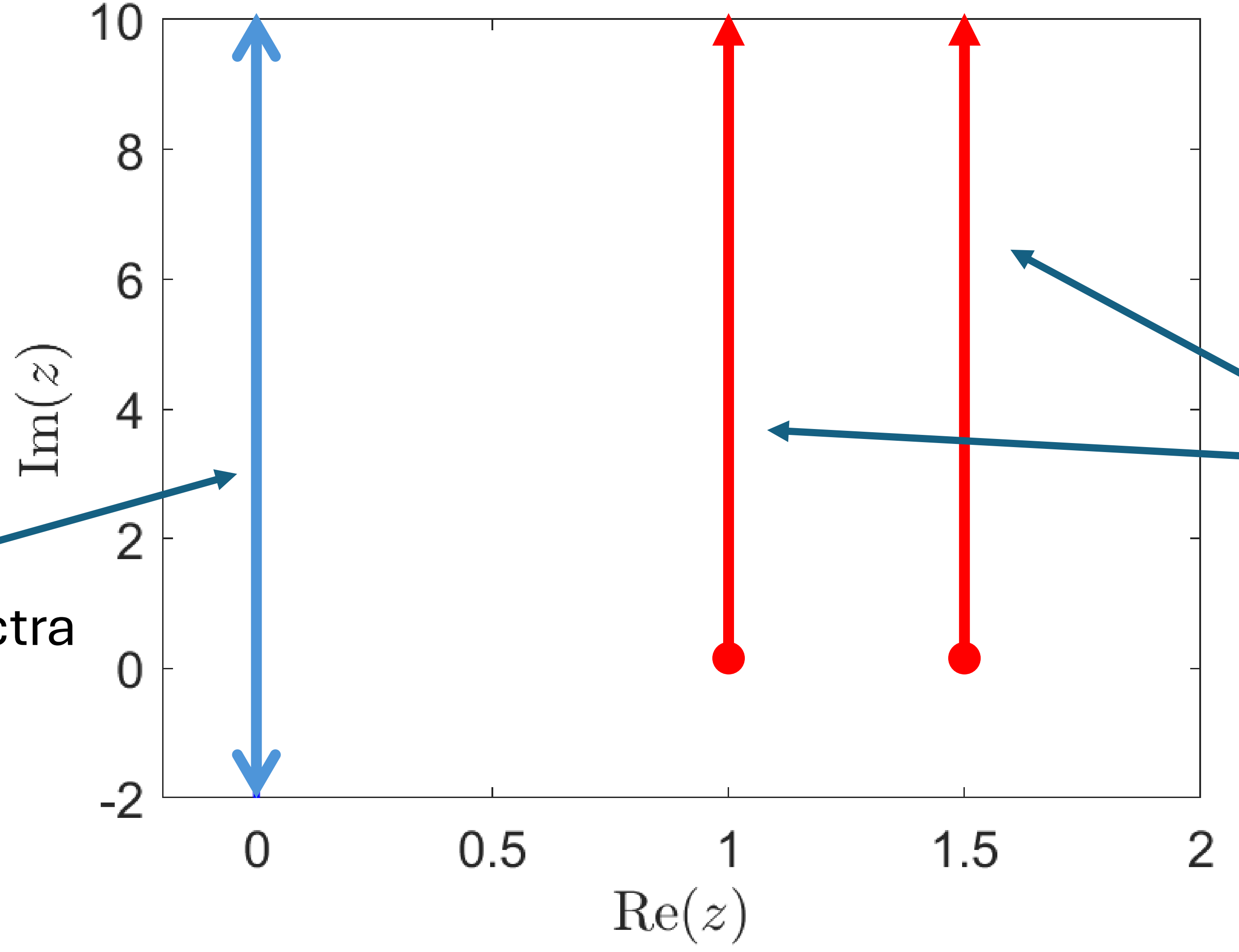
$$\alpha_{\mathrm{R}}(z) = k\sqrt{\eta_{\mathrm{R}}^2 - z^2}.$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x) + k^2([\eta(x)]^2 - z^2)\phi(x) = 0, \quad x \in (-1, 1),$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}x}(-1) - \mathrm{i}\alpha_{\mathrm{L}}(z)\phi(-1) = 0, \quad \frac{\mathrm{d}\phi}{\mathrm{d}x}(1) + \mathrm{i}\alpha_{\mathrm{R}}(z)\phi(1) = 0.$$

Nonlinearity in boundary conditions!

Example 4: Resonances



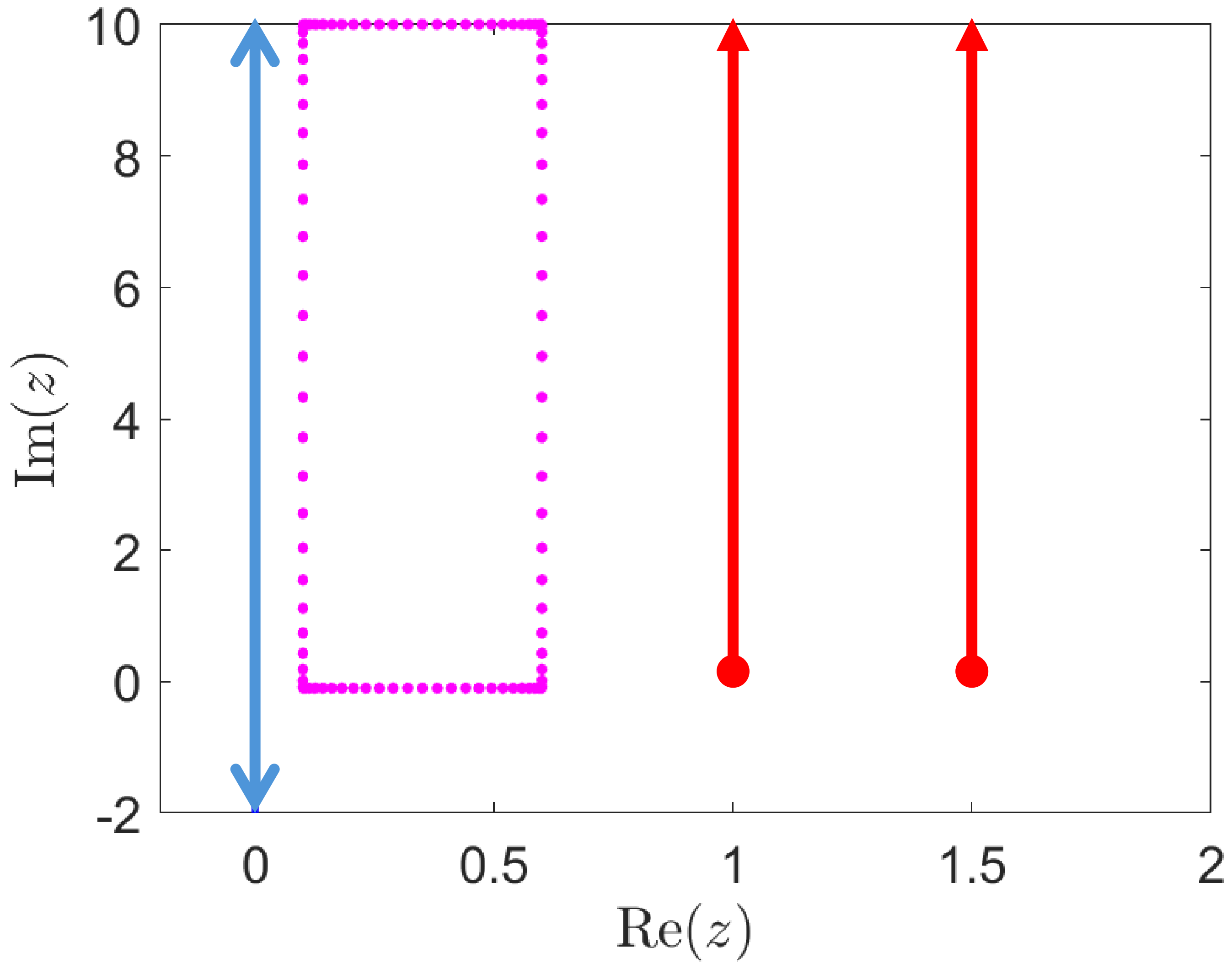
Wall of cts spectra

Branch cuts for $\alpha_{\mathrm{L}}, \alpha_{\mathrm{R}}$

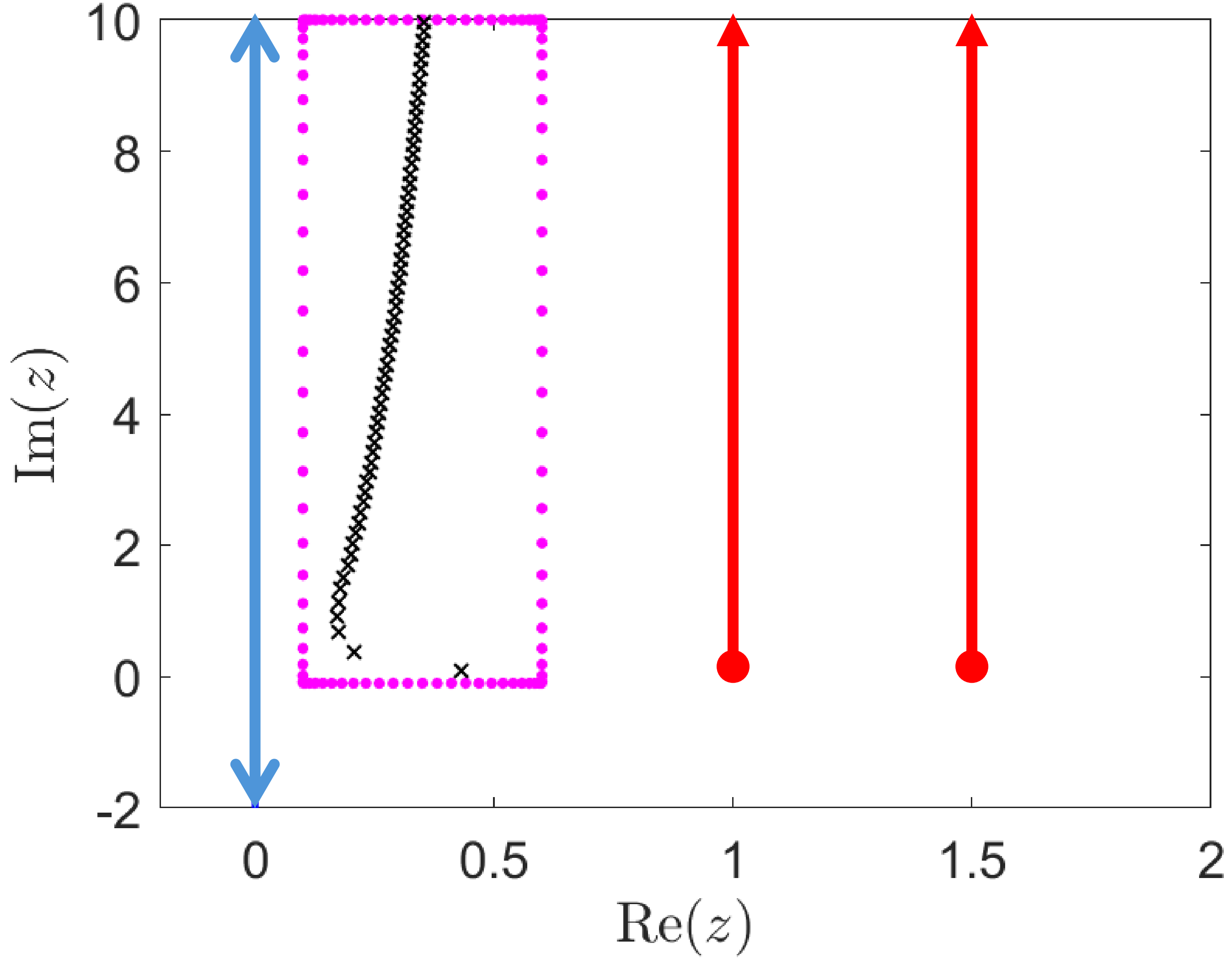
$$\alpha_{\mathrm{L}}(z) = k\sqrt{\eta_{\mathrm{L}}^2 - z^2},$$

$$\alpha_{\mathrm{R}}(z) = k\sqrt{\eta_{\mathrm{R}}^2 - z^2}.$$

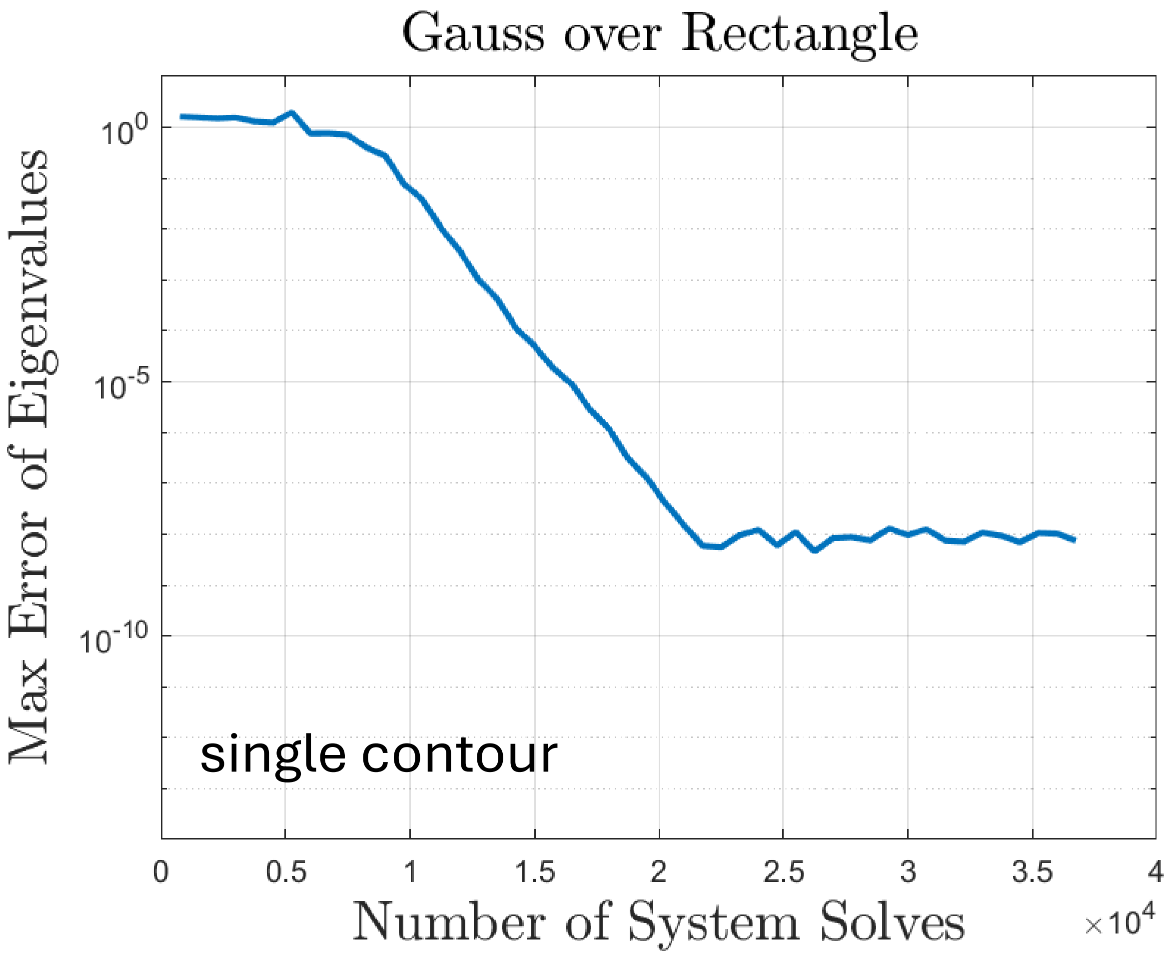
Example 4: Resonances



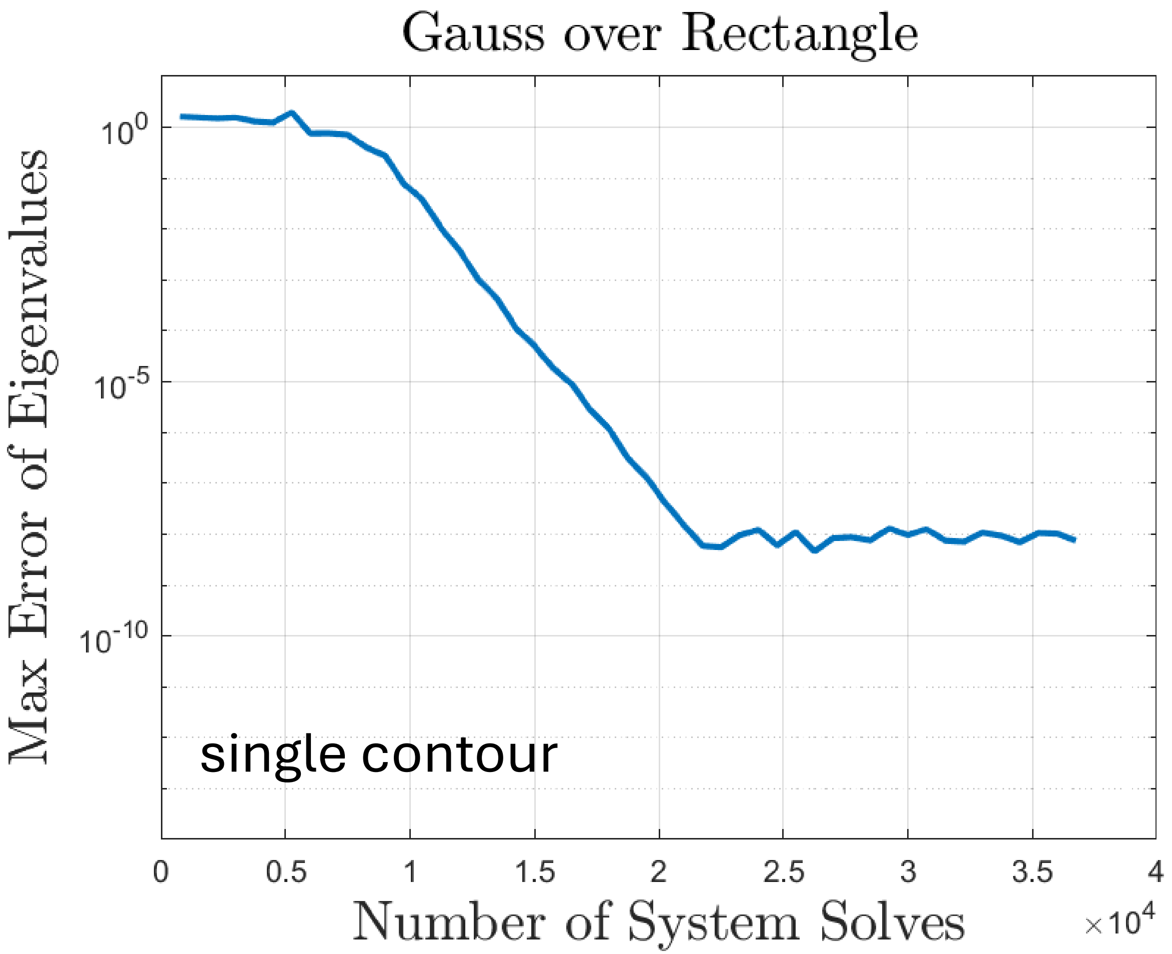
Example 4: Resonances

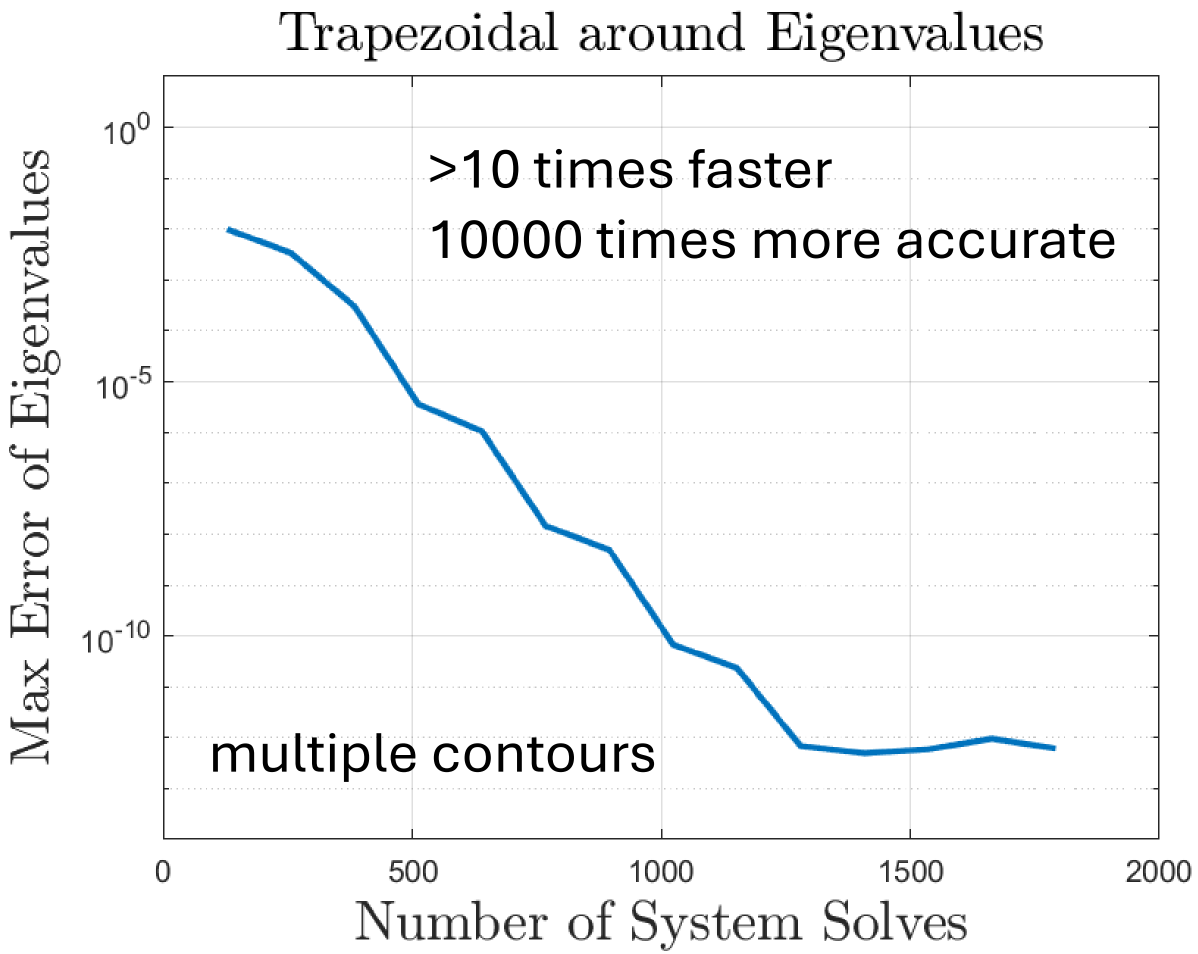


Example 4: Resonances

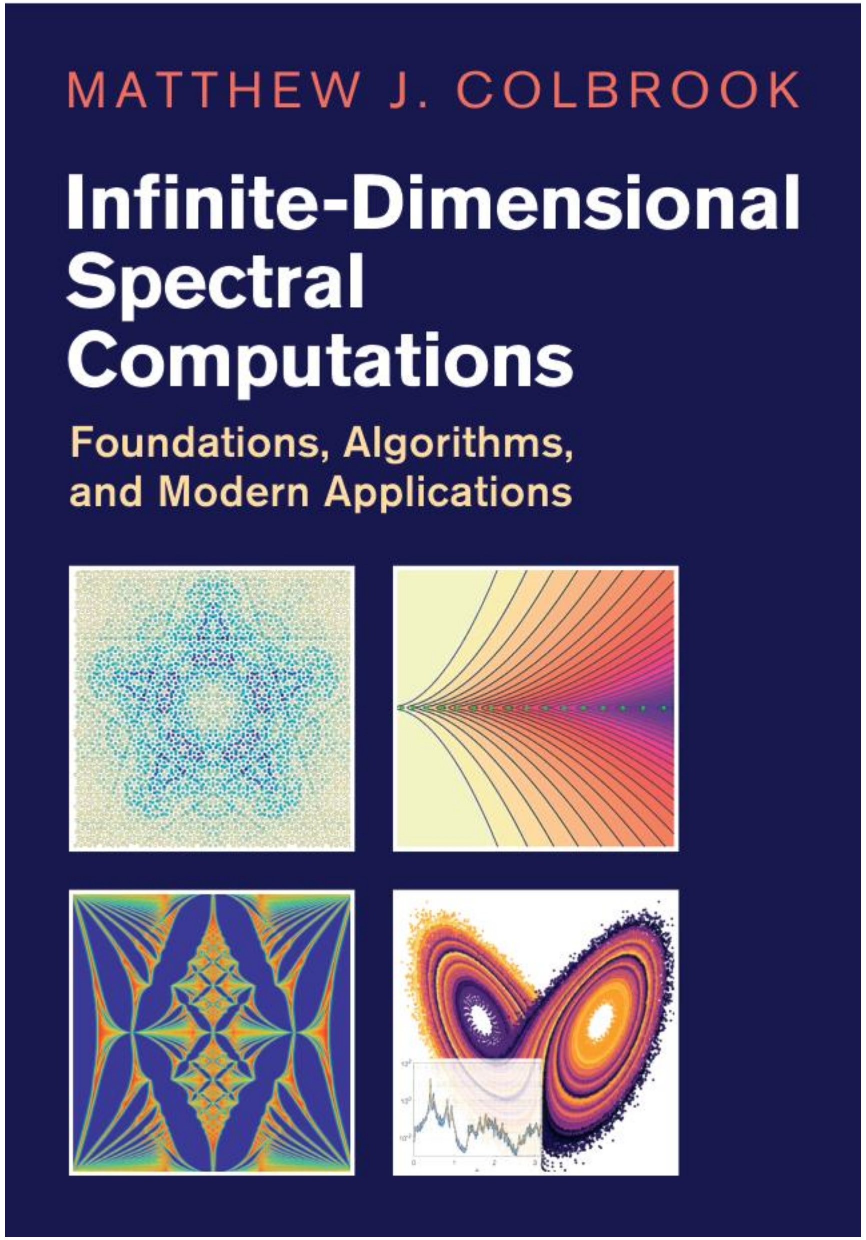


Example 4: Resonances





Shameless plug: CUP book out August 2026...



Contents

100s of:

- Classifications
- Algorithms
- Examples (full code)
- Exercises (full solutions)

If something of interest – speak to me!

This talk

Conclusion: Systematic treatment of nonlinear spectral problems in infinite dimensions

Discretization often causes issues. Bullet proof + practical algorithms exist.

- Singular values → $\mathrm{Sp}(T)$ (verified $\mathrm{Sp}_\epsilon(T)$) computed when $\mathrm{graph}(T(z))$ cts in z
- Keldysh's theorem: Laurent expansion when $z \mapsto T(z)$ holomorphic
- Contour methods use Keldysh + Cauchy to compute e-values and e-vectors:
  1) Stable up to intrinsic properties of T and scalar functions h.
  2) Sketching does not degrade stability (not surprising).
  3) Must be done with trace class covariance (surprising).
  4) Often better to use multiple smaller contours to post-process.

Good choice largely open problem!

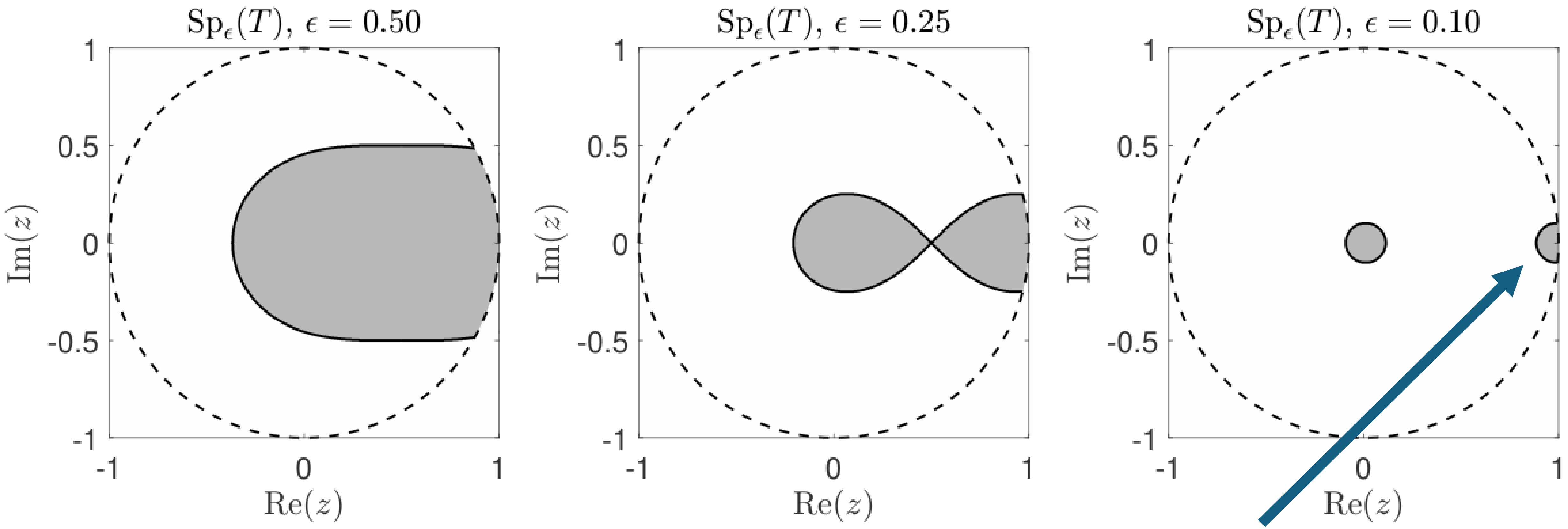
1,2,4 apply to finite dimensions too!

Also in this programme: *Classify problem difficulty + prove algorithms are optimal.*
Holomorphic operators of type A allow Newton's method.

- C., Drysdale, "Universal Methods for Nonlinear Spectral Problems," **Journal of Spectral Theory**, 2026.
- C, "INFINITE-DIMENSIONAL SPECTRAL COMPUTATIONS: Foundations, Algorithms, and Modern Applications," **CUP**, to appear.

Bonus example

$$U = \{z\colon |z| < 1\}, \quad T(z) = z(1 - z), \quad \mathrm{Sp}(T) = \{0\}.$$



Convergence isn't globally uniform

Bonus example: Nonlinear shift on $\ell^2(\mathbb{Z})$

$$T(z) = S - f(z)S^* = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & -f(z) & 0 & 1 & \\ & & -f(z) & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Exercise (for Marcus Webb):

a) $\mathrm{Sp}(T) = \{z: |f(z)| = 1\}$.

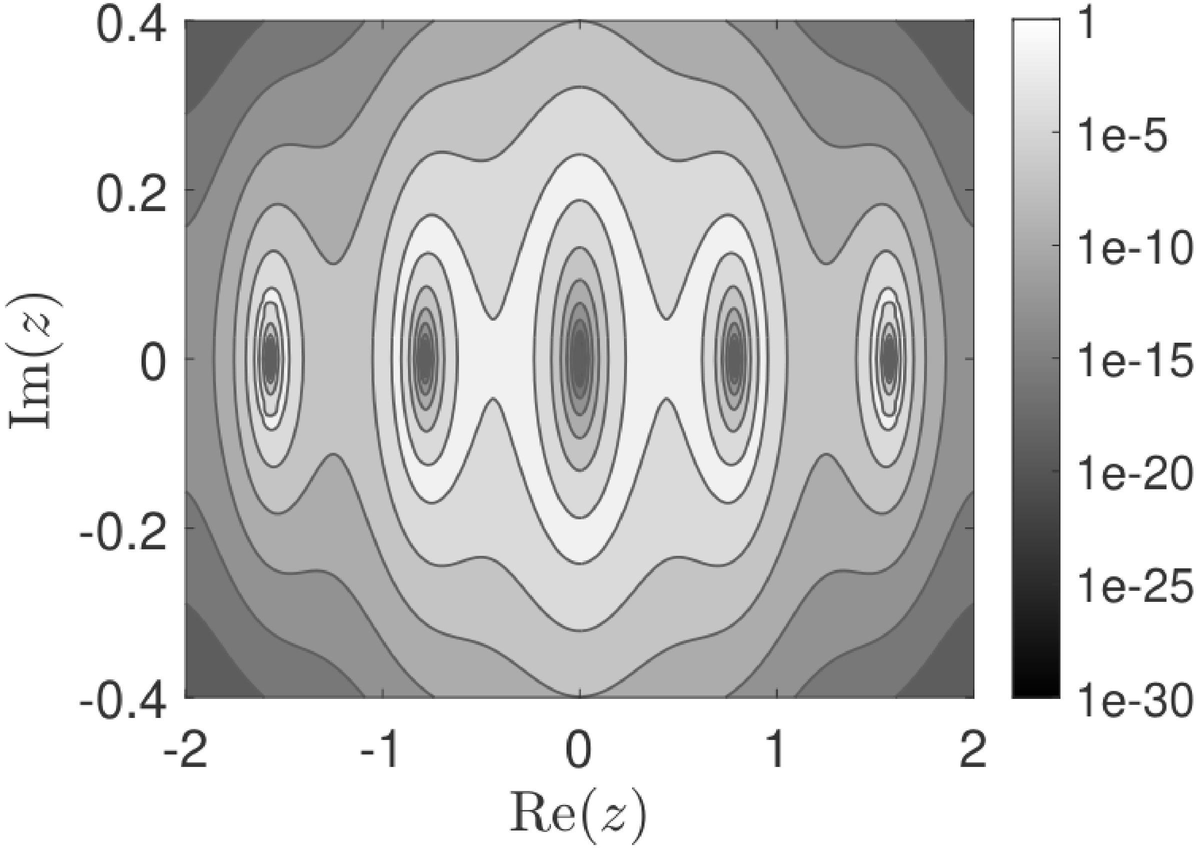
b) Truncate to $k \times k$ Toeplitz operator T_k:

$$\mathrm{Sp}(T_k) = \begin{cases} \mathbb{C}, & \text{odd } k \\ \{z: f(z) = 0\}, & \text{even } k \end{cases}$$

Bonus example: Nonlinear shift on $\ell^2(\mathbb{Z})$

$f(z) = \sin(4z)(|z|^2 + 1)$
$V_N = \mathrm{span}\{e_{-N}, \dots, e_N\}$

Pseudospectra of T_{40}



Bullet proof algorithm, using
$\sigma_{\inf}(T(z)\mathcal{P}_N) = \sigma_{\inf}(\mathcal{P}_{N+1}T(z)\mathcal{P}_N)$