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Operator Splitting for the Random Wave Moment Equations

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1. INTRODUCTION

Much work has been done in recent years on the scattering of waves in random media. Of particular importance are the moment equations, describing the statistical behavior of the wavefield, for most of which exact analytical solutions have not been found. Operator splitting has now been used for the fast and accurate numerical solution of the fourth moment [4] and may also be applied to the higher moments. The purpose of this note is to examine the accuracy of the method in terms of properties of the operators.

Operator splitting is particularly suitable in this case: the operators appearing in the equations are easily integrated separately, while their sum is not, especially for strongly scattering media. It turns out that for the most important class of moments, the symmetric ones, the commutator of the operators is small when acting on the solution space, and the method is thus highly accurate. The results here are not precisely quantitative but illustrate the broad behavior of the solution.

In Section 2 we give a brief description of the physical setting and moment equations, and the approximate solution used. The accuracy of this solution is examined in Section 3, and some motivation is given for the consideration of the commutativity of the operators.

2. MOMENT EQUATIONS AND APPROXIMATE SOLUTIONS

We describe briefly the mathematical and physical setting. Further details can be found in [4] and [7]. We will assume here that a monochromatic plane wave u, with wavenumber k, is incident on the half-space $(z > 0)$ of a two-dimensional medium (x, z). We suppose the medium has a refractive index $n(x,z) = \langle n \rangle + \mu n_1(x,z)$, where the angle-brackets denote an ensemble average, and n_1 is a Gaussian random variable with mean 0 and variance μ. (Time-dependence may be ignored since in most applications n changes very slowly relative to the wave-speed.) Denote by f the normalized autocorrelation function of n_1, so that $f(\xi) = \rho(\xi)/\rho(0)$, where

$$\rho(\xi) = \int_{-\infty}^{\infty} \langle n_1(x,z')n_1(x+\xi,z'') \rangle \ d(z'-z'').$$

Let L be the correlation length of the medium in the z-direction. We define the parameter $\Gamma = k^3\mu^2\rho(0)L^2$, which describes the strength of the scattering. The wavelength is assumed to be significantly shorter than L, and scattering therefore takes place mainly in a forward z-direction. (With this assumption one obtains a parabolic equation for u which involves a randomly variable multiplication operator. The numerical propagation of this wavefield has long been carried out by operator splitting methods; see [6,3,5].)

An important and much-studied class of problems which arises is that of the moments, which describe the averaged behavior of the wave and are governed by differential equations. In particular the fourth moment $m(z) = \langle u_1(z)u_2(z)u_3^*(z)u_4^*(z) \rangle$ (where $u_i(z)$ denotes

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$u(x_i, z)$) yields the scintillation index and spectrum of intensity fluctuations [7]. The fourth moment can be recast in terms of two transverse dimensions and obeys the equation:

$$\frac{\partial m}{\partial Z} = (A' + B')m \tag{1}$$

where

$$A' = -i\frac{\partial^2}{\partial\xi\partial\eta},\ B' = -2\Gamma\left[1 - g(\xi,\eta)\right],\ g(\xi,\eta) = f(\xi) + f(\eta) - \frac{1}{2}\{f(\xi+\eta) + f(\xi-\eta)\},$$

$\xi = (x_1 - x_2 - x_3 + x_4)/2L$, $\eta = (x_1 - x_2 + x_3 - x_4)/2L$, and $Z = z/kL^2$. This has the initial condition $m(\xi,\eta,0) \equiv 1$.

More generally, we consider the $2n$th *symmetric* moments m, given by $m(x_1,\ldots,x_n, y_1,\ldots,y_n) =< u_1\ldots u_n u^*_{1'}\ldots u^*_{n'} >$ where $u_i = u(x_i), u_{i'} = u(y_i)$, and x_i, y_i are co-ordinates on the x-axis. We can write the differential equation for such moments [1] as:

$$\frac{\partial m}{\partial z} = (A' + B').m, \tag{2}$$

$$\text{where } A' = \frac{1}{2k}\sum_{i=1}^{n}(\Delta_i - \Delta_i'),$$

$$\text{and } B' = B'(\{x_i\},\{y_i\}) = -\frac{k^2}{8}\sum_{i,j=1}^{n}\left[\rho(x_i - x_j) + \rho(y_i - y_j) - 2\rho(x_i - y_j)\right].$$

Here, $\Delta_i = \partial^2/\partial x_i^2$, and $\Delta_i' = \partial^2/\partial y_i^2$, and again the initial condition is $m \equiv 1$. In practice the dimension of this problem can always be reduced, by stationarity and other symmetries, as it has been for the fourth moment.

In applications (see [4]) the medium may typically be described, for example, by correlation functions of the form $\rho(\xi) = e^{-\xi^2}$ (Gaussian medium) or $\rho(\xi) = (1 + |\xi|)e^{-|\xi|}$ (fourth order power law medium). In particular ρ is always even in ξ and monotonic in $|\xi|$. We will also assume that ρ has continuous first and second derivatives. This excludes the class of 'fractal' media, which any discretization inevitably fails to describe properly. From these assumptions and the symmetries in equation (2) follow certain elementary identities: in particular $m(\{x_i\},\{y_i\}) = m(P\{x_i\}, Q\{y_i\}) = m(-\{x_i\},-\{y_i\}) = m^*(\{y_i\},\{x_i\})$ for any permutations P and Q. Furthermore, m is bounded and continuous. Where we need to think in terms of specific spaces we may restrict the moments to some compact set, with the square-integral norm.

2.1. Formal solution and approximation. Denote by $[X, Y]$ the commutator $XY - YX$ of operators X and Y. The solution of (2) can be written formally as

$$m(z + \Delta z) = \exp\left[\int_z^{z+\Delta z}(A' + B')dz\right].m(z) \tag{3}$$

since $\left[A_z' + B_z', \int_{z_1}^{z_2}(A' + B')dz\right] = 0$ for all z, z_1, z_2. (To see why this condition is sufficient, write the exponential as the series $\sum_{n=0}^{\infty}\frac{C^n}{n!}$, where $C = \int(A' + B')$, take derivatives on both sides of the equation, and compare with Eq. 2). We can write equation 3 as

$$m(z + \Delta z) = \exp(A + B).m(z) \tag{4}$$

where, in general, A and B depend on Δz, and here, in particular, $A = \Delta z.A'$, $B = \Delta z.B'$. We write the following approximation for (4):

$$m(z + \Delta z) \approx \exp(A).\exp(B).m(z) \tag{5}$$

It is easy to show that this has error of order $(\Delta z)^2$. However, expanding and comparing terms we see that the error is a function of the commutator (see e.g. [2]) and is in fact, to first order, $[A,B]/2$. In general the accuracy of operator-splitting depends far more crucially on $[A,B]$ than on the simple quantity Δz. The main object of this note is to emphasize this point and to show that in our case the behavior of $[A,B]$ leads to high accuracy.

Unless $[A,B]=0$, in which case (5) is exact, some subtlety is needed in characterizing $[A,B]$ as "small." The natural measure for bounded operators, $\|[A,B]\|/(2\|A\|\|B\|)$, does not apply here, since we have an (unbounded) differential operator. On the other hand the requirement that $\|[A,B]\|/(2\|ABm\|)$ be small is too strong, since $[A,B]$ may be small when acting on the solution space even if $\|[A,B]\|$ is large; this is the situation in our case. The quantity we will have in mind is $\|[A,B]m\|/(2\|ABm\|)$. With the norm this is a global quantity, which reflects overall accuracy. There is another aspect to the accuracy: we are interested only in a particular small region of the solution, and we can examine the point-wise error there.

The implementation of the numerical solution for the fourth moment using equation 5 is described in [4]. To summarize briefly: a finite region of the (ξ,η)−plane is first discretized. $\exp(A)$ has a simple explicit solution using the Fourier transform, which is approximated computationally with the Fast Fourier Transform, and $\exp(B)$ is already in explicit form. The boundary conditions, which depend on lower-order moments, are handled analytically. This computational scheme is fast and accurate up to very large scattering strength Γ. On the other hand the discretized operators A and B have, respectively, purely imaginary and purely real eigenvalues; accurate, stable solutions for $A+B$ become increasingly unattainable as Γ becomes large.

3. Results

As indicated, when the operators A and B nearly commute, the approximation (5) is accurate. In general $[A,B]$ itself will not be small, but in our case it can be shown that $[A,B]m$ is small for all solutions m. Roughly speaking the values of m are clustered around a certain subset of coordinates; $[A,B]m$ is zero on that set, and $\|[A,B]m\|$ is therefore small.

It may be illustrative to consider an extreme example, in which $[A,B]P=0$ for a non-trivial projection P : Denote by C_P the operator PCP for any operator C, and by Q the complement $1-P$ of P. Let A and B be operators such that $A=A_Q+P$, $B=B_Q+P$, where $[A_Q,B_Q]$ is large. Note that $\exp(zP)=Q+\exp(z)P$. Now let f_0 be an initial condition. Then, if $f_0=Pf_0$, we have that $[A,B]f_0=0$. Also, $f(z)=\exp(\int_0^z(A+B))=\exp(z(A_Q+B_Q))\exp(2zP)f_0=\exp(2z)f_0$, and therefore $f(z)=Pf(z)$ for all z. Thus for an initial condition in subspace $range(P)$ the solution stays there and the splitting of A and B is exact; but if any component of the initial condition is in $null(P)$ splitting may be very inaccurate.

3.1. Operator-splitting for the symmetric moments.

We consider again the moments $m=<u_1\dots u_n u_1^*,\dots u_n^*>$ whose equations are given by (2). Let X be the vector space $R^n\times R^n$, and let Ω be the submanifold of elements of the form $(\{x_i\},P\{x_i\})$ for any permutation P, where $i=1,\dots n$. Write $dist(\underline{x},\Omega)=\inf_{\underline{y}\in\Omega}\|\underline{x}-\underline{y}\|$ for $\underline{x}\in X$, where the norm is the usual L^2-norm on R^{2n}. In practice Ω is the only region of the solution which is of interest. In the fourth moment, for example, $m(\Omega)$ gives the scintillation index and spectrum of intensity fluctuations. We now have:

Lemma 1. *m has effective support on Ω in the following sense:*

(1) *If $\underline{x}=(x_1,\dots,x_n,y_1,\dots,y_n)$ then $|m(\underline{x})|$ falls to zero as $dist(\underline{x},\Omega)$ increases.*
(2) *$|m|$ reaches a maximum at every point on Ω with respect to some transverse direction.*

Proof:

(1) It is sufficient to consider $\underline{x}=(x',x_2,\dots,x_n,x_1,\dots,x_n)$ and to assume that $dist(\underline{x},\Omega)=|x'-x_1|=\alpha$, say. Then for large α, $u(x')$ is statistically independent of all $u(x_i)$'s, and $m\approx<u(x')><u_1|u_2|^2\dots|u_n|^2>=0$ since $<u>=0$ everywhere.

(2) Again, it suffices to consider the neighborhood of a point $\underline{x} = (x_1, x_2, \ldots, x_n, x_1, \ldots, x_n)$ where x_1 is far from $x_2, \ldots, x_n$. Now m is real on Ω. Let m_1, m_2 be the real and imaginary parts of m respectively, so that $|m| = \sqrt{m_1^2 + m_2^2}$. If $\underline{x}' = (x_1+\xi, x_2, \ldots, x_n, x_1, \ldots, x_n)$ and $\underline{x}'' = (x_1, \ldots, x_n, x_1+\xi, x_2, \ldots, x_n)$ then $m(\underline{x}') \approx \langle u(x_1)u^*(x_1+\xi)\rangle \langle |u_2|^2 \ldots |u_n|^2\rangle = m^*(\underline{x}'')$. So m_1 is even in ξ about $\underline{x}$ and $\partial m_1/\partial\xi = 0$. Thus at $\underline{x}$, $\partial|m|/\partial\xi = \frac{1}{|m|}\sqrt{m_1\frac{\partial m_1}{\partial\xi} + m_2\frac{\partial m_2}{\partial\xi}} = 0$.

We now have the following:

LEMMA 2. *$[A, B] = 0$ on Ω, i.e. $([A, B]f)(\underline{x}) = 0$ for all $\underline{x}$ in Ω.*

PROOF: First, by the symmetry $B(\{x_i\}, \{y_i\}) = B(\{y_i\}, \{x_i\})$, $\frac{\partial B}{\partial x_i} = \frac{\partial B}{\partial y_i} = 0$ on Ω and $\frac{\partial^2 B}{\partial x_i^2} = \frac{\partial^2 B}{\partial y_i^2}$ on Ω for all i. Now,

$$[A, B]f = \frac{(\Delta z)^2}{2k}\Big(\sum_i [\Delta_i, B]f - \sum_i [\Delta_i', B]f\Big)$$

and

$$[\Delta_i, B]f = \frac{\partial^2 B}{\partial x_i^2}f \; + \; 2\frac{\partial B}{\partial x_i}\frac{\partial f}{\partial x_i} \; = \; \frac{\partial^2 B}{\partial x_i^2}f$$

on Ω, and similarly for $[\Delta_i', B]f$. So $[A, B]f(\Omega) = 0$ for all f.

An analogous calculation must be made for the fourth moment whose simplified equation, given by equation (1), is the form used in [4] and elsewhere. (Here Ω corresponds exactly to the axes $\xi = 0, \eta = 0$):

LEMMA 3. *$[A, B]m = 0$ along the ξ, η axes.*

PROOF: With A, B as given, $[A, B]m = B_{\xi\eta}m + B_\xi m_\eta + B_\eta m_\xi$ (where the subscripts denote differentiation). We will consider the axis $\eta = 0$:

B is constant along this axis, so $B_\eta = 0$ there.
B is also symmetrical about the axis, i.e.,
B is even in ξ, so that $B_\xi = 0$.

Similarly, since ρ is an even function, its derivative is odd, so B_η is even in ξ, and thus $B_{\xi\eta} = 0$.

Since $[A, B]$ acts locally, we also have $[A, B]m \approx 0$ for regions bounded away from Ω. It follows that $[A, B]$ is small on the solution space, as required.

This result can be replaced by a weaker, but more precise statement: the local, step-wise truncation error on Ω is zero to second order in Δz. In addition many other terms appearing in the expansion of the error disappear on Ω. For example the functions $B[A, B]m$, AB^2m, and all terms left-multiplied by $[A, B]$ vanish there. Consequently the nonzero factors of third- and higher-order terms in Δz become relatively small, a fact borne out by numerical convergence tests.

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