PREDICTION OF LONG-TERM COASTAL EVOLUTION USING MOMENT EQUATIONS

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Abstract: Morphodynamic models can play a key role in assessing the likely impact of coastal developments. A major goal is the prediction of predicting coastal evolution over a regional scale. The deterministic approach to morphological prediction was pioneered by Pelnard-Considere (1956) who presented a one-line equation to forecast changes in coastline position. Subsequent models have extended this to multiple depth contours. Such models, however, do not forecast accurately the variability likely to be experienced in real situations.

One alternative to the deterministic modelling approach is to employ stochastic forecasting methods, such as Monte Carlo methods. A key input to this type of model is the statistical description of the wave climate or other driving forces. However, to obtain physically meaningful results very many realisation are needed.

In this paper we formulate and solve moment equations for the shoreline position. These equations describe the averaged or long-term solution, and its dependence on wave-climate; and eliminates the need for computationally intensive Monte-Carlo simulations.

INTRODUCTION

The prediction of the long-term movement of the shoreline is now an important issue for coastal engineers, However, the availability of observational data and our understanding of long-term morphological evolution limit the ability to make reliable predictions. This is compounded by the large computational effort required to predict changes in coastal morphology using the deterministic dynamical equations for fluid flow and sediment transport over even relatively short periods. Much of the research on coastal erosion and flooding has consequently been oriented towards coastal protection and harbour structures.

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Progress has been achieved with simplified models which predict longshore or cross-shore transport (e.g. Pelnard-Considere 1956, Bakker 1969, Perlin & Dean 1979, Nairn & Southgate 1993, Larson et al. 1997, Hanson et al 1998). The long-shore transport equation derived by Pelnard-Considere (1956) was, for example, subsequently extended to account for variations in wave direction and sediment transport along the shore (Larson et al. 1997). Over the past decade such models have become widely used in prediction of shoreline changes over periods of months or years. Bakker (1969) and Perlin & Dean (1979) developed models that included additional contours while parameterising the cross-shore transport, but these have not found their way into general practice. In practice, designers rely upon in situ measurements, or transform a deepwater wave climate to the shore, to derive an inshore wave climate representative of the site. Solutions to the one-line model, using time-averaged boundary conditions, are then taken to provide a first-order forecast of the time-averaged beach position. Wave conditions required to drive a one-line model are often available only as summary statistics. LeMehaute & Soldate (1979) addressed the problem of using wave statistics to drive the one-line model and proposed a procedure for constructing representative wave conditions. More recently, Perlin & Kit (1999) used a similar method to investigate the sensitivity of coastal response to local wave direction variation. Synthetic wave conditions were used by Vrijling & Meyer (1992) to perform Monte Carlo simulations with a one-line model to estimate the variability of shoreline position near a port. Reeve & Fleming (1997) used a one-line model and historical shoreline positions to infer the distribution of time-averaged sediment sources and thence to estimate likely future shoreline movement.

While providing an indication of the typical position of a beach and the sensitivity of this to variations in the boundary conditions, none of these approaches provides a direct method for determining the mean and variance of the shoreline in the presence of variable wave conditions. However, by taking averages of the underlying equations, it is possible to derive moment equations, which govern the evolution of key quantities such as the mean beach position and its long-term variance, which provides a measure of the statistical reliability of the solution. Such equations have long been used in wave propagation in random media, but have not previously been developed for this problem. In this paper, we summarise the derivation of equations governing the evolution of the first and second moments (mean and autocovariance) of shoreline position, present general solutions for these equations and provide solutions for specific example cases.

MOMENT EQUATIONS

First moment equation

We will summarise here the derivation of the first moment equation, which describes the mean shoreline, and indicate how the variablity of this can be investigated via the second moment equation. The starting point is the governing equation for the evolution of the beach profile in Larson, eq. (7):

$$y_t = \varepsilon y_{xx} \tag{1}$$

where $\varepsilon(t)$ varies randomly as a function of time, and subscripts denote differentiation. It is convenient to write this as

$$y_t = Ay. \tag{2}$$

Using the equation above we can formulate evolution equations, for the averaged coastal profile and its variance. This will allow the evaluation of the effect of features such as breakwaters and the study of spatial variability. Such moment equations have long been used in seismology, ocean acoustics and radio astronomy, in a different form, but have not previously been applied to this problem.

The first moment gives the mean value of y as it evolves with time. In order to form the evolution equation we must specify the statistics of ε and we then require an expression for the evolution of y itself. We will assume that ε has stationary statistics and can therefore be written

$$\varepsilon =< \varepsilon > +\delta(t) \tag{3}$$

where the mean $< \varepsilon >$ is constant and the perturbation δ has mean zero and stationary statistics, which are known. For convenience will will also assume that δ is Gaussian distributed. (Here and below the angled brackets denote an ensemble average, taken over all possible realisations of the random function ε.)

Taking Fourier transforms of each side of eq. (1) we get

$$(\hat{y})_t \ = \ -\nu^2 \widehat{(\varepsilon y)} \ \equiv \ -\nu^2 \varepsilon \ \hat{y}. \tag{4}$$

where $\hat{y}(\nu, t)$ is the Fourier transform of y with respect to x. This has the following solution over any time step $[t, t+\tau]$:

$$\hat{y}(\nu, t+\tau) = \exp\left[-\nu^2 \int_t^{t+\tau} \varepsilon(t')dt'\right] \ \hat{y}(\nu, t). \tag{5}$$

Taking the average of this equation and substituting (3) into the result we get

$$< \hat{y}(\nu, t+\tau) >= e^{-\nu^2 \int\limits_t^{t+\tau} <\varepsilon> dt'} < e^{-\nu^2 \int\limits_t^{t+\tau} \delta(t')dt'} > \quad \hat{y}(\nu, t). \qquad (6)$$

Since δ is Gaussian distributed and has stationary statistics in time, and $< \varepsilon >$ is constant, after some manipulation (see Papoulis 1987) this can be written

$$< \hat{y}(\nu, t+\tau) >= e^{-\nu^2 \ t \ <\varepsilon>} \ \exp\left(-\frac{\nu^4}{2} \left\langle \int\limits_t^{t+\tau} \int\limits_t^{t+\tau} \delta(t')\delta(t'')dt' \ dt'' \right\rangle \right) \ \hat{y}(\nu, t). \quad (7)$$

The average can now be taken under the integral signs, to obtain

$$< \hat{y}(\nu, t+\tau) >= e^{-\nu^2 \ t \ <\varepsilon>} \ \exp\left(-\frac{\nu^4}{2} \int\limits_t^{t+\tau} \int\limits_t^{t+\tau} \rho(t'-t'')dt' \ dt'' \right) \ \hat{y}(\nu, t). \qquad (8)$$

The term ρ here is simply the given autocorrelation function of δ,

$$\rho(t'-t'') = \langle \delta(t')\delta(t'') \rangle \, .$$

Solution in specific cases

For many cases of practical interest the integral in eq. (8) can be written in closed form. We consider first the case when the correlation function for d(t) is Gaussian or exponential. These are given respectively by

$$\rho(\eta) = e^{-\left(\frac{\eta}{T}\right)^2} \qquad (9)$$

and

$$\rho(\eta) = e^{-\left|\frac{\eta}{T}\right|} \qquad (10)$$

The extent to which values of the coefficient are similar is governed by the 'correlation time', T, while the statistics of $\delta(t)$ obey a Gaussian distribution. A diffusion coefficient with variations obeying a Gaussian correlation function will exhibit irregularities that are very closely grouped about the single scale, T. In contrast, the exponential correlation function falls off to zero more slowly and so considerable contributions to the fluctuations in diffusion coefficient come from a wide range of temporal scales.

As mentioned above, if the initial coastline configuration depends on alongshore position then the statistics of $y(x, t)$ will not be stationary. Equation (8) is valid for an arbitrary initial coastline configuration. For comparisons with previous analytical solutions for the instantaneous coastline position we will consider the case of a rectangular beach recharge scheme on a straight beach, extending a distance a either side of the origin on the x-axis and protruding a distance V from the rest of the shoreline. The initial condition is therefore given by

$$y(x,0)=\begin{cases} V & for \quad |x|<a \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The Fourier transform of this function is well-known to be

$$\tilde{y}(\nu,0)=V\sqrt{\frac{2}{\pi}}\frac{\sin(\nu a)}{\nu}, \qquad \nu\neq 0 \tag{12}$$

For $\nu=0$ we obtain $y(0,0)=a\sqrt{2/p}$. For the two forms of the autocorrelation function above the second exponential term on the right hand side of equation (8) may be written respectively as:

$$\exp\left\{-\frac{\nu^4\sigma^2}{2}(\tau\sqrt{\pi}V\mathrm{erf}(\tau/T)+2T^2\left[\exp\left\{-\left(\frac{\tau}{T}\right)^2\right\}-1\right])\right\} \tag{13}$$

$$\exp\left\{-\frac{\nu^4\sigma^2 T}{2}\left(2\exp\left\{-\frac{\tau}{T}\right\}(\tau+2T)+2(\tau-2T)\right)\right\} \tag{14}$$

Inserting (12) and either of (13) or (14) into equation (8) yields an expression for the Fourier transform of the ensemble average shoreline position after time t. The resulting expression is generally not amenable to analytical methods and the inverse Fourier transform must be evaluated numerically. Here, we have used a discrete Fourier transform for this purpose.

Computations have been performed for the two correlation functions for a range of temporal correlation scales. In order to provide some tangible measure of the impact of the presence of temporal variations in diffusion coefficient, comparisons are made against the analytical solution for the instantaneous shoreline position for the same initial condition with the diffusion coefficient set equal to its ensemble average value. For ease of reference, we include the analytical solution to equation (2) subject to initial condition (11):

$$y(x,\tau)=\frac{V}{2}\left[\mathrm{erf}\left(\frac{a+x}{2\sqrt{<\varepsilon>\tau}}\right)+\mathrm{erf}\left(\frac{a-x}{2\sqrt{<\varepsilon>\tau}}\right)\right] \tag{15}$$

This solution is well-known in heat conduction problems, and the details can be found in textbooks on the subject, (see eg Carslaw & Jaeger 1959). Fig. 1 shows this comparison for the rectangular beach profile above. The beach is straight apart from a 'top-hat' protrusion of 20 m seawards, over a length of 1000 m. The results are taken over a time of 0.3 years, and ε has a value of 10^6 m^2/year The temporal correlation function is Gaussian, with a correLation time of 0.1 years.

Here, the full line is the shoreline position assuming no fluctuations, and the dashed line is the solution when fluctuations are taken into account. This demonstrates, in this case, that neglecting the fluctuations results in an underestimate of the rate at which 'beach nourishment' is spread along the shoreline.

Second moment equation and the variance

While equations such as (1) are very useful as a means of simulating changes in beach profiles and examining qualitatively the dependence on ε, the first moment above allows us to look at long-term evolution of the mean profile, examine underlying persistent effects due to features such as a breakwater, and quantify the dependence on the statistics of ε. It does not by itself, however, reveal anything either about the *typical* spatial variation (for example how rapidly the profile will vary with x), or the variation about this mean. For this we require the second moment equation.

We can define the *second moment* (i.e. autocorrelation function) m_2 of y,

$$m_2(x_1, x_2, t) =< y(x_1, t)y(x_2, t) > \tag{16}$$

where x_1, x_2 are any two points. (Note that in statistically stationary problems, more usual in ocean acoustics, the second moment simplifies to a function of time and spatial separation $\xi = x_2 - x_1$ only.) From the equations above we can form an exact evolution equation for the second moment. This will be discussed in a separate paper.

Our aim is to develop a set of equations to examine the mean spatial pattern and more importantly to quantify the departure of the typical profile from the mean. This is expressed as the variance of $y - m_1$, which can be expanded and expressed in terms of the second moment (16) as follows:

$$\begin{aligned} < (y - m_1)^2 > \; &= \; < y^2 > \; - \; 2 < y(x,t) > m_1 \; + \; m_1^2 \\ &= \; < y^2 > \; - \; 2m_1^2 \; + \; m_1^2. \end{aligned} \tag{17}$$

The first term on the right-hand-side is the second moment, so that this becomes

$$< (y(x,t) - m_1(x,t))^2 > \; = \; m_2(x,x,t) - m_1^2(x,t). \tag{18}$$

CONCLUSIONS

Moment equations provide rigorous basis on which to examine long term shoreline evolution from a stochastic perspective. Numerical solution of the equations for more complicated situations is the subject of ongoing work.

The first and second moments, and therefore the variance (18), can be obtained either analytically or numerically, for a given value of $< \varepsilon >(t)$ and initial and boundary conditions. The boundary conditions for m_1, m_2 must be obtained from those of the underlying profile y.

It should be noted that, since ε is treated as constant in x, any spatial variation in the mean arises entirely from the initial and boundary conditions. (This contrasts with the moment equations in ocean acoustics and random media, where the medium depends on x but the simplification is commonly used that the initial conditions do not, and the resulting solutions are statistically stationary.)

This will allow treatment of important cases such as a breakwater, which leads to specific form for the boundary conditions; and an arbitrary initial beach profile $y(x, 0)$.

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