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Determination of a source term in the linear diffusion equation

D E Reeve† and M Spivack‡

† Sir William Halcrow and Partners Ltd, Burderop Park, Swindon SN4 0QD, UK
‡ Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK
and
Smith Institute, Research Park, Guildford, Surrey GU2 5YP, UK

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Abstract. A method is presented for the reconstruction of a source term in the linear diffusion equation. This is applied here to a problem of coastline evolution and sediment transport. The method is based upon inversion of the split-step solution, which is widely used to treat the related parabolic wave equation for propagation in inhomogeneous media. In this approach the effects of diffraction and scattering operators are treated separately, using the fact that their commutator is 'small' in an appropriate sense. It is explained how the method can also be applied to the recovery of the ocean sound-speed profile.

1. Introduction

In many areas of wave propagation and scattering a central problem is to recover the scatterer from measured data. Such applications range from acoustic ocean tomography and medical imaging to the prediction of coastline evolution. The scattering function may represent surface roughness, an inhomogeneous medium, or sediment supply (see for example Cannon and Ewing 1976; Wombell and DeSanto 1991; Spivack 1992; Flatté *et al* 1979; Weir 1989). In each physical situation the inverse problem will be posed according to the available data. Uniqueness of the solution cannot in general be assured, because dependence on the scatterer is highly nonlinear, and also one is typically attempting to reconstruct a continuous medium from finite information.

The general form of the governing equations involves temporal and spatial derivatives of the field plus a source/sink or forcing term. In underwater acoustics, for example, the source term describes the effects of wave refraction and scattering due to variations in the ocean refractive index. In the prediction of coastline evolution this term represents the supply or removal of sediment along the shoreline through cross-shore movement.

In this paper we present a method for the inversion of a linear diffusion equation with an unknown source term; this is applied here to a problem of prediction of coastline evolution. The equation is closely related to the parabolic equation for wave propagation in a randomly varying medium, and it is in this form that the problem is treated here. It is assumed that data is available at discrete range steps z_i, and from this the integral of the unknown function over each interval is found. The solution is based upon the 'split-step' method, which is widely used to treat the direct problem of finding the field when the medium is known.

The method is applied here to the problem of predicting coastline movement. By inverting a generalized form of the equation describing the transport of beach material along the shoreline, the magnitude and spatial variation of cross-shore sediment transport over time may be recovered. This type of information can be extremely important for planning the construction of beach and coastline protection works and also the maintenance of near-shore shipping channels.

The paper is organized as follows: in section 2 the governing equations are set out and the inverse problem is posed. The solution of the 'direct' problem is described and from this the inversion algorithm is derived. The method is applied in section 3 to the specific problem of coastline evolution, and computational examples are given.

2. Inverse problem and solution

In this section we describe the problem to be solved, which is in effect to invert a diffusion equation, and give the method by which we will carry out this inversion. The accuracy of the algorithm will also be discussed. Computational examples will be given in the next section to illustrate one of several potential applications.

2.1. Problem and equations

Consider the following two-dimensional linear differential equation

$$\frac{\partial y}{\partial z} = \alpha \frac{\partial^2 y}{\partial x^2} + G(x, z) y(x, z) \tag{2.1}$$

where α is some constant, and $G(x, z)$ is a continuous bounded function of both coordinates, which we will assume is slowly-varying in some sense to be discussed. The functions here may be either real or complex. The complex form of equation (2.1) arises, for example, in underwater acoustics or for electromagnetic propagation through a turbulent atmosphere. In such cases z denotes range, and α takes the value $\mathrm{i}/2k$ where k is a reference wavenumber. The term $\partial^2/\partial x^2$ is then the diffraction operator, and the function G is the scattering operator, which represents variation in the refractive index

$$G = \mathrm{i}k(n^2 - 1)/2.$$

A related diffusion equation involving real-valued functions occurs in describing thermal diffusion and also coastline movement. This equation can be written as

$$\frac{\partial y}{\partial z} = K \frac{\partial^2 y}{\partial x^2} + F(x, z) \tag{2.2}$$

where z now denotes time, and the function F on the RHS is a forcing function. The derivation of this is discussed in section 3 below. We will make the assumption that F is continuous and that F/y remains finite everywhere. Thus y is bounded away from zero except where F itself vanishes. We can therefore define a function G (to be determined) by

$$G(x, z) = \frac{F(x, z)}{y(x, z)}. \tag{2.3}$$

If y vanishes at any point (x', z'), $G(x', z')$ must be determined numerically from (2.3) as a limit. (Since the governing equation is linear, the solution y remains continuous everywhere provided the initial condition $y(x, 0)$ is continuous.) It is clear that (2.2) can be written in the form (2.1), with $K = \alpha$, i.e.

$$\frac{\partial y}{\partial z} = K\frac{\partial^2 y}{\partial x^2} + G(x, z)y(x, z). \tag{2.4}$$

In what follows we therefore restrict attention to equations of the form (2.1).

Now suppose we are given a partial solution for the function $y(x, z)$. The problem we consider here is that of recovering the scattering function $G(x, z)$ as far as possible from the measured values of y.

2.2. *Split-step method*

We first consider the direct problem, i.e. the approximate solution of (2.1) to find y when the function G is given. For this we describe the method of operator splitting commonly used to treat wave propagation problems in random media.

Write (2.1) in the form

$$\frac{\partial y}{\partial z} = (A + G)y \tag{2.5}$$

where $A = \alpha\partial^2/\partial x^2$ and for each z the range-dependent function G is considered as a multiplication operator $G = G(x, z)$. The formal solution for this equation over a small step ζ can be written approximately as

$$y(z + \zeta) \cong \exp\left[\int_z^{z+\zeta} (A + G)\,\mathrm{d}z\right]\big(y(z)\big). \tag{2.6}$$

This approximation becomes exact if the range-dependent operator $(A + G)$ everywhere commutes with its integral. (This may be seen (Spivack and Uscinski 1989) by expanding the exponential operator as a Taylor series, taking derivatives of both sides of (2.6) with respect to z, and comparing the result with (2.5).) This condition is clearly satisfied if G is constant with respect to z, and is a good approximation provided G changes slowly over the interval $[z, z + \zeta]$. To lowest order the perturbation (i.e. step-wise) error can be shown to be (Spivack and Uscinski 1989)

$$\tfrac{1}{2}\zeta^3[A + G, \partial G/\partial z]$$

where $[C, D]$ denotes the commutator $CD - DC$ of any operators C and D. The form of (2.6) immediately suggests the use of the well-known split-step solution, which was introduced to underwater acoustics applications by Tappert and Hardin (1974), and is in widespread use. This solution has the form

$$y(x, z + \zeta) \cong \exp\left[\int_z^{z+\zeta} G(x, z)\,\mathrm{d}z\right]\exp(\zeta A)\big(y(x, z)\big). \tag{2.7}$$

This is exact when A and G commute. Specifically, the error ϵ is a function of the commutator $[A, G]$ and to leading order becomes

$$\epsilon = \frac{\zeta}{2}\left[A, \int_z^{z+\zeta} G(x, z)\,\mathrm{d}z\right]$$

which is of order ζ^2. This error dominates that of the exponential solution (2.6), which will therefore be neglected. The advantage of the above solution is that the individual exponential terms can be treated analytically; the function $\exp(\int G\,dz)$ is already in explicit form, and $\exp(\zeta A)$ can be written in terms of Fourier transforms

$$e^{\zeta A} y(x, z) = F^{-1}\big(\exp(i\nu^2\zeta)F(y)\big) \tag{2.8}$$

where F denotes the Fourier transform with respect to x and ν is the transform variable. In numerical evaluation this is efficiently implemented by use of the fast Fourier transform. Note that a more accurate form of operator splitting is *Strang's splitting*, in which the evolution operator is split into three exponential factors. This is of order $O(\zeta^3)$; for our purposes, however, (2.7) is more convenient.

2.3. Solution of the inverse problem

We now return to the problem of inverting equation (2.1) when G is unknown, and derive the inversion algorithm. Suppose, for simplicity, that we are given the values of $y(x_i, z_j)$ on a rectangular grid $\{x_i, z_j\}$, where x_i, z_j are evenly spaced. We assume that the values of y are known at 'sufficiently many' points x_i, z_j for the split-step approximation to be accurate. This requires that the transverse (x) resolution of y is sufficient to ensure accuracy of the fast Fourier transform, used in evaluation of (2.8). However, this restriction does not necessarily apply to derivatives in the range direction. This will be discussed below.

Let $\zeta = z_{j+1} - z_j$, which we have assumed to be constant with respect to j. Denote the spatially-integrated form of G over each range step ζ by

$$C(x, z_j) = \int_{z_j}^{z_{j+1}} G(x, z)\,dz. \tag{2.9}$$

Given the function $y(x, z_j)$ *we can evaluate the diffraction term* $\alpha\partial^2/\partial x^2$ using the Fourier transform as discussed above. Denote this 'incomplete' solution by Y, so that

$$Y(x, z_j) = e^{\zeta A} y(x, z_j). \tag{2.10}$$

We can now apply the split-step solution (2.7) directly and, substituting (2.9) and (2.10) into (2.7), write

$$\exp[C(x, z_j)] = \frac{y(x, z_{j+1})}{Y(x, z_j)}. \tag{2.11}$$

This is well-defined provided $y(x, z)$ (and therefore Y) is bounded away from zero. We will assume for convenience that this holds everywhere. As remarked earlier, the method can easily be extended by continuity to allow for y vanishing at isolated points.

The cases in which the functions G and y are real or complex must now be treated separately.

(a) Real case. Suppose first that all functions are real-valued; this is the situation in the diffusion equation (2.2). Since y is assumed to be bounded away from zero (and also remains finite since G is bounded), we can write

$$C(x, z_j) = \ln[y(x, z_{j+1})] - \ln[Y(x, z_j)]. \tag{2.12}$$

This yields the integrated form of G, and is the solution which is sought. We cannot resolve the details of G more finely than the points at which y is known, although if G changes reasonably smoothly then we can interpolate, for example linearly, to approximate the values of $G(x, z)$ at any z

$$G(x, z) = G(x, z_j) + \frac{z - z_j}{\zeta}\big[C(x, z_{j+1}) - C(x, z_j)\big]. \tag{2.13}$$

(b) Complex case. Now suppose that the quantities α, y, and G are complex, as in acoustic wave propagation. In this case equation (2.12) again holds, but is ambiguous, since the complex logarithm is multi-valued. In order to choose a solution $C(x, z)$ uniquely for each z we may do the following: fix a value of x, say $x = x_0$, and then choose the principal value

$$C(x_0, z_j) = \mathrm{Ln}\left[\frac{y(x_0, z_{j+1})}{Y(x_0, z_j)}\right] \tag{2.14}$$

so that $\mathrm{Im}(C)$ lies in $[0, 2\pi]$. This choice is of course arbitrary. However, provided the resolution in the transverse x direction is sufficient, (2.14) fixes $C(x_k, z_j)$ for all other x_k by continuity of y and its first derivative. This 2π phase ambiguity is implicit in (2.12) but is unimportant because a constant phase shift along a z plane has no effect on propagation in the z direction. However, it can be resolved, for each x, if the spatial average $\langle G(x, z)\rangle_x$ with respect to the vertical is known; furthermore, if G varies slowly with respect to x then continuity can again be used to determine $G(x_i, z)$, say, in terms of $G(x_{i-1}, z)$.

2.4. Accuracy of algorithm and data requirements

It has been assumed above that values of $y(x, z)$ are known on a rectangular grid $\{x_i, z_j\}$ which is fine enough for the split-step approximation to hold. Specifically the resolution in x must be sufficient to ensure accurate numerical representation of the Fourier transform. This implies that $\partial^2 y/\partial x^2$ may also be well-approximated numerically. In the context of acoustic wave propagation, a widely-used rule-of-thumb in implementations of the split-step method (Spivack and Uscinski 1989), is

$$k\mu^2 L_{\mathrm{p}} \frac{\Delta z}{L^2} < \tfrac{1}{4}$$

where μ is the estimated standard deviation of wave-speed irregularities, L is the transverse correlation length of irregularities, and

$$L_{\mathrm{p}} = \int_{-\infty}^{\infty} \frac{\rho(0, \zeta)}{\rho(0, 0)}\, \mathrm{d}\zeta$$

which is a measure of longitudinal correlation length, where $\rho(\xi, \zeta)$ is the wave-speed autocorrelation function with respect to spatial separations ξ, ζ in the x, z directions respectively. An analogous condition for the linear diffusion equation is easily derived. This restriction requires far less resolution in range than in the transverse direction, and in particular does not assume that enough information is available to approximate the derivative $\partial y/\partial z$ numerically. (Indeed once the derivative is obtained the problem becomes trivial, since the unknown term in equation (2.1) can then be found by simple substitution.) This can be seen as follows.

Over the given interval $[z, z + \zeta]$, the diffraction effect may give rise to appreciable changes in amplitude, due to already large variations (in phase and amplitude) of the function $y(x, z)$. Thus the amplitude of the function $Y = \exp(\zeta A)y$ may be significantly different from that of y, and in that case $\partial y/\partial z$ cannot be accurately recovered from the data. On the other hand, provided the scattering term G is small, it may not in itself be converted (through diffraction) into a large change in y. The scattering and diffraction effects over this step ζ then remain roughly separated since the commutator of these operators is small, and the inverse method presented above will be accurate.

3. Application and computational example

3.1. Application to shoreline movement

Methods of controlling coastal erosion have traditionally consisted of constructing static structures between the sea and the shore (e.g. concrete sea walls). These methods for 'holding the shoreline' have met with mixed success and are now justifiably being questioned, particularly in the light of their impact on neighbouring lengths of coastline. An equation of the form (2.1) arises in the prediction of coastline erosion and accretion. Inversion of this equation allows the distribution of the long-term sediment supply to and removal from the coast to be retrieved from historical records of the coastline position, thus providing a new insight into the long-term spatial and temporal variations in coastline alignment. In this case the real dependent variable is the position of the coastline from a datum line, and the independent variables are distance along the datum line and time.

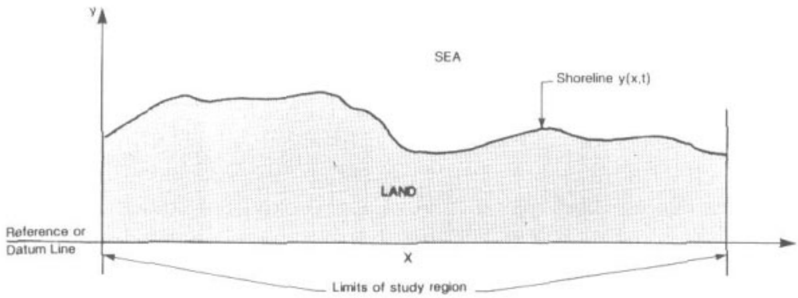

Figure 1. Schematic diagram showing the definition of variables.

Consider a length of coastline as shown in figure 1. The coastline position is defined as a function of distance x along the reference (datum) line—in this case the x axis (see figure 1). Two simplifying assumptions are now made. The first is that the time scale of changes in the beach cross section normal to the shore is very different to the time scale of changes along the shore. The second is that similar sediment transport processes occur along the length of the shoreline under study. These assumptions restrict consideration to the case where depth contours are orthogonal to the shore normals. It is further assumed that changes in beach cross section are bounded and lie within a fixed vertical range d, which is constant along coastline (see figure 2(a)).

The continuity equation for the sediment transport S may then be written as

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + d\frac{\partial y}{\partial t} = 0 \tag{3.1}$$

where sediment transport is taken to be primarily induced by wave-driven longshore currents. An important factor governing wave-driven longshore transport is the angle α_b between the wavecrest at the position where wave-breaking occurs and the local depth contour (see figure 2(b)). This angle may be related to the datum line via the relationship

$$\alpha_b = \tfrac{1}{2}\pi - \alpha - \frac{\partial y}{\partial x}. \tag{3.2}$$

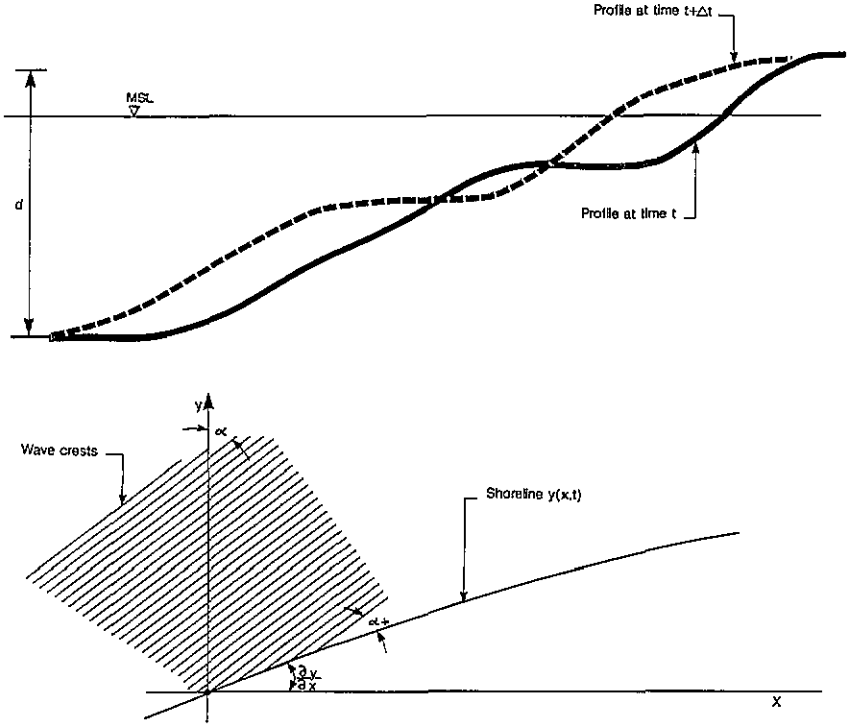

Figure 2. (a) Definition of the active beach profile depth d. (b) Schematic diagram illustrating the definition of wave-climate shoreline orientation.

For α and $\partial y/\partial x$ sufficiently small the longshore transport S is well approximated by its longshore component ($S_y \ll S_x = S$). Noting that with the chain rule $\partial S_x/\partial x$ may be expanded to

$$\frac{\partial S_x}{\partial \alpha_b}\frac{\partial \alpha_b}{\partial x}$$

it follows from equations (3.1) and (3.2) that

$$\frac{\partial y}{\partial t} = K\frac{\partial^2 y}{\partial x^2} \tag{3.3}$$

where

$$K = \frac{1}{d}\frac{\partial S}{\partial \alpha_b}.$$

Equation (3.3) was derived in the pioneering paper of Pelnard-Considere (1956), who used it to predict the impact of breakwaters and sand replenishment on a coastline. The quantity K plays the role of a diffusion coefficient and is usually taken to be a constant in applications (see, e.g. Tilmans 1991). Here the equation is written as

$$\frac{\partial y}{\partial t} = K\frac{\partial^2 y}{\partial x^2} + F(x,t) \tag{3.4}$$

where $F(x,t)$ represents the sediment source and sink distribution resulting from all processes other than longshore transport, and is considered to be separable as

$$F(x,t) = y(x,t)G(x,t)$$

as in equations (2.2) and (2.3).

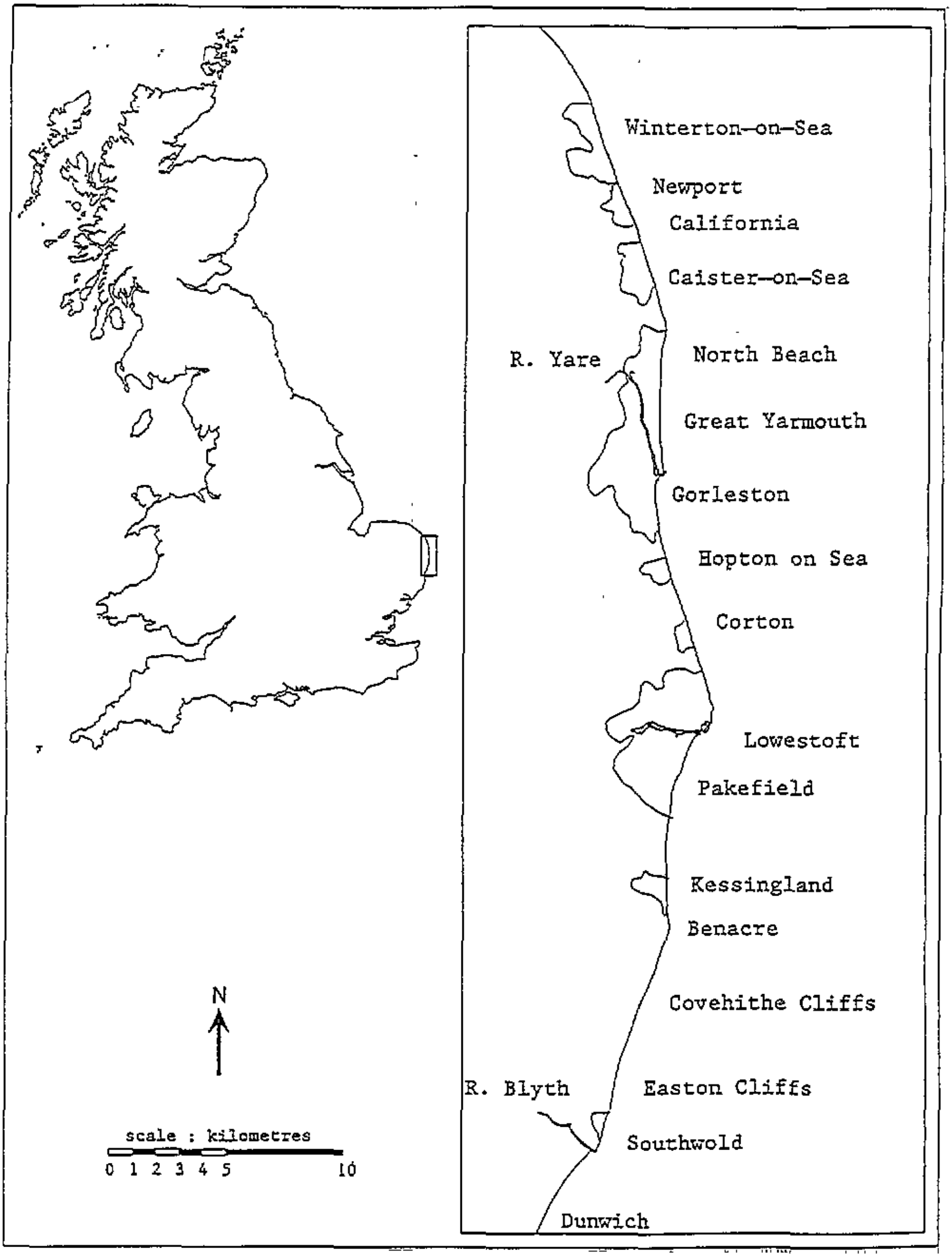

Figure 3. Map of study region.

3.2. Computational results

The shoreline of the East Anglian coast contains a wide range of beach types and provides a formidable problem for predicting long-term morphological changes. The study area for this case runs south from Happisburgh on the North Norfolk coast to Felixstowe. Beaches along the coast typically comprise sand or sand and gravel. The shoreline is generally retreating and beaches steepening. The exception is at the location of nesses (coastline bulges or gentle headland promontories). These features are known to migrate along the coast over a period of decades (Robinson 1966).

In what follows, the inversion technique given in section 2.3 is used to investigate shoreline changes described by the position of the mean low water line (MLW). A local x axis was taken as the line running northwards from 649000E, 273250N (relative to the UK national grid) as shown by the left-hand edge of the rectangular study region in figure 3. The position of MLW was derived from historical maps for the years 1884, 1904, 1947, 1971 and 1980.

A value of $K = 10^3\,\mathrm{m}^2\,\mathrm{year}^{-1}$ was used. Typical values suggested by standard formulae for full beaches (e.g. Walton and Chiu 1979) range from 10^3 to 10^8; the choice here reflects the fact that in this case there is a limited volume of sand on the shoreline.

Results are shown as the approximate time-integrated forcing, calculated as

$$\int_t^{t+\Delta t} F(x,t)\,\mathrm{d}t \cong \int_t^{t+\Delta t} G(x,t)\,\mathrm{d}t \left(\frac{y(t)+y(t+\Delta t)}{2}\right). \tag{3.6}$$

Figure 4 shows the time-integrated forcing distribution for the periods 1884–1904 and 1884–1971 plotted along the datum line. This shows how the cumulative forcing distribution has developed over the course of the last century.

The computed distributions of the forcing show no definite underlying spatial trend, suggesting that systematic errors associated with the assumption of uniform longshore transport conditions are not significant.

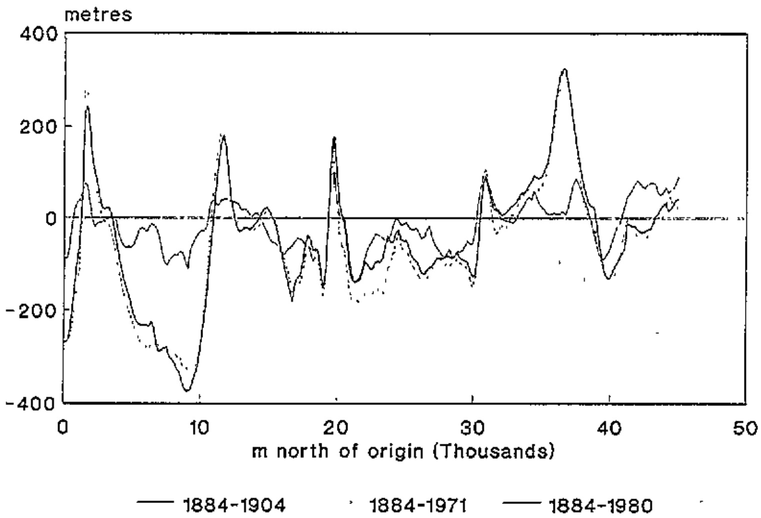

Figure 4. The computed distribution of time-averaged sediment supply for the periods 1884–1904, 1884–1971 and 1884–1980. Southwold, Benacre, Lowestoft, Gorleston, and North Beach correspond approximately to positions 2 km, 11 km, 19 km, 30 km and 37 km north of the origin.

Currently there are no means of verifying the calculated forcing functions directly as the historical measurements of sediment movement required for verification do not exist. However, the computed distributions show well-defined structure, which can be identified with known local coastal behaviour. In particular, maxima in the forcing (corresponding to sediment supply to the shore) coincide with Winterton, Caister and Benacre nesses.

This is in agreement with the findings of Robinson (1966, 1980) who argued that nesses are locations at which onshore transport is a dominant process. The advance of Great Yarmouth North Beach by 300 m since the 1930's has been reported by Clayton *et al* (1983). This is mirrored by a corresponding positive peak in the calculated forcing distributions. Minima in the forcing occur from Benacre to Southwold where there has been rapid retreat of the MLW (in excess of 2 m per year).

4. Conclusions

A method has been described for the inversion of the two-dimensional linear diffusion equation, based upon an analogy with the split-step solution used for the direct problem. The region of accuracy depends on the spacing of the data in the range direction; roughly speaking, the method is accurate provided diffraction and scattering effects are approximately separable on the scale of each interval. The method has been applied here to a problem of coastline evolution, in which all functions are real valued. It can be extended, however, to the related complex-valued equation for acoustic propagation in an inhomogeneous medium such as the ocean.

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