A numerical approach to rough-surface scattering by the parabolic equation method

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The parabolic equation method is an effective approach when the acoustic wave field is incident at low grazing angles onto a rough surface. The method consists of an integral equation and an integral, the first of which yields the surface field derivative. The main part of this paper is concerned with an approximation to this equation, valid when wavenumber times surface height is up to order unity. The approximation has several advantages. First, it allows a decomposition of the equation into deterministic and stochastic components. The stochastic part depends only locally upon the surface in certain regimes, and this can give rise to a great reduction in computational expense. Some basic statistical moments of the stochastic component are also considered. These are nonstationary, but for the incident field a simple stationary transformation is found, which can therefore be compared with Monte Carlo simulations using far fewer realizations. These results are demonstrated computationally. The final part of the paper describes the numerical implementation of the full parabolic equation method. Both of the integrals contain singularities, and these are treated semianalytically.

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INTRODUCTION

The problem of acoustic scattering from rough surfaces has received considerable attention in recent years. Different theoretical methods are used for various scattering regimes, such as perturbation theory for small surface heights and the Kirchhoff approximation for small slope. When the field is incident at low grazing angles, neither of these remains accurate at moderately large surface heights (Thorsos[1]), and a successful approach is the parabolic equation method (e.g., Thorsos,[2] Dozier,[3] Tappert and Nghiem-Phu[4]). This consists of an integral equation and an integral, relating the vertical derivative $\partial\Psi/\partial z$ of the field Ψ at the surface to the incident and scattered fields, respectively. The first of these is a Volterra equation, which is inverted to give $\partial\Psi/\partial z$ at each range in terms of the incident field Ψ_{inc} at all previous ranges. In this way, the influence of Ψ_{inc} at one point upon $\partial\Psi/\partial z$ may persist for many correlation lengths along the surface.

The main purpose of this paper is to introduce and exploit an approximate form of this Volterra integral equation, which is shown to be accurate for surface heights times wavenumber up to order unity. The approximation has several advantages. It first allows us to decompose the problem naturally into two parts: a deterministic part plus a random part; these correspond to the field incident on a flat surface and its perturbation δ due to the surface variation. When the correlation length is comparable to a wavelength, the solution $\partial\delta/\partial z$ is far more localized than the corresponding full solution $\partial\Psi/\partial z$, depending only on nearby values of δ, and this then greatly reduces the computational expense. This field decomposition also enables us to examine some of the statistical moments, and we formulate some results for the autocorrelation of the stochastic component δ and for the vertical derivative. In particular, it is shown that δ can be written as the product of a stationary function $\bar{\delta}$ and a deterministic one. By transforming to $\bar{\delta}$, far fewer realizations are needed to form reliable statistics from Monte Carlo simulations. Accurate, nonstationary expressions are also given for the mean and autocorrelation of Ψ_{inc} for very rough surfaces. The basic equations for the method are given in Sec. I, and in Sec. II the results are described and demonstrated computationally.

Both integrals contain singularities, arising from zero denominators and an exponent tending to infinity. These are treated semianalytically, and the procedure and its numerical implementation are described in Sec. III. The equations are discretized with respect to range and product integration used for both integrals. Where the singularities are most acute, away from the surface, the equations are recast in terms of Fresnel integrals.

I. BASIC EQUATIONS

We set out here the equations to be solved, following Thorsos,[2] and make some preliminary definitions. We also give the form of the incident field and the surface statistics that were applied for the computational results presented below.

We consider in this paper the two-dimensional scattering problem, i.e., from a one-dimensional surface. We will be treating the case in which the field is incident at low grazing angles and is well described by the parabolic wave equation. (For the case of nearly normal incidence, an integral equation method can be found in Kachoyan and Macaskill.[5]) Coordinate axes x and z are taken as in Fig. 1, where x is horizontal, $x \geqslant 0$, and z is the vertical, decreasing downwards. Let $\bar{r} = (x,z)$. The source will be centered around $\bar{r}_0 = (0,0)$, at a distance $z_0 \approx 22.4$ below the surface. Denote by $h_1(x)$ the rough surface, and let $h \equiv h_1 - z_0$. Here, h will

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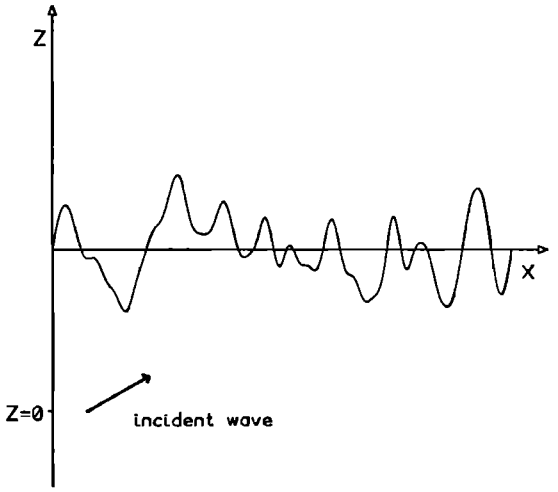

FIG. 1. Representation of typical rough surface, $z = h_1(x)$.

be taken to have Gaussian statistics and zero mean. In the treatment that follows, the derivative of h is assumed to be bounded. In the numerical work shown here, h is given a Gaussian spectrum. Denote the rms of the surface $h(x)$ by ϕ and its correlation length by L_x.

Here, $\Psi_{\rm inc}(\bar{r})$ denotes the (complex) incident wave field. It will be assumed that $\Psi_{\rm inc} = 0$ for $x \leqslant 0$ and along the bottom boundary, and that $h(x)$ is a pressure-release surface; $\Psi_s(\bar{r})$ denotes the wave field scattered from the surface and $\Psi(\bar{r})$ the total field, so that $\Psi = \Psi_s + \Psi_{\rm inc}$, and $\Psi = 0$ at the surface. The governing equations for the parabolic equation method are[2]

$$\Psi_{\rm inc}(\bar{r}) = -\int_0^x \left(G(\bar{r};\bar{r}') \frac{\partial \Psi}{\partial z'} \right)_{\bar{r}',\bar{r}' \text{ at surface}} dx', \tag{1}$$

and

$$\Psi_s(\bar{r}) = \int_0^x \left(G(\bar{r};\bar{r}') \frac{\partial \Psi}{\partial z'} \right)_{\bar{r}' \text{ at surface}} dx'. \tag{2}$$

Here, G is the Green's function given by

$$G(x,z;x',z') = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{i}{2\pi k(x-x')}} \exp\left(i\dfrac{k}{2}\dfrac{(z-z')^2}{x-x'} \right), \\ \quad \text{for } x > x', \\ 0 \quad \text{for } x < x'. \end{cases} \tag{3}$$

$\bar{r}' \equiv (x',z') \equiv (x',h_1(x'))$ along the surface, and k is the wavenumber.

Denote $\partial\Psi/\partial z$ along the surface by Ψ'. Then, in terms of linear operators we can write (1) as

$$\Psi_{\rm inc} = A\Psi', \tag{4}$$

and (2) as

$$\Psi_s = B\Psi'. \tag{5}$$

Although these operators are linear (for a given surface), the equations are not, since A and B vary with the surface, together with the vectors $\Psi_{\rm inc}$ and Ψ'. In other words, Ψ' and Ψ_s depend linearly upon $\Psi_{\rm inc}$, but nonlinearly upon the surface $h(x)$.

In the work shown here, a simple Gaussian-beamed source of width w is taken as the incident wave:

$$\Psi_{\rm inc}(x,z) = \frac{w}{\sqrt{w^2 + 2ix/k}} \exp -\left(\frac{z^2}{w^2 + 2ix/k} \right). \tag{6}$$

We will take $k = 1$ and $w = 8$. In most of our numerical results, we have taken $L_x = 8$, which is of the order of a wavelength, and this defines a situation of interest in many applications. Note that the large-scale variation of $\Psi_{\rm inc}$ takes place over many wavelengths.

We need some additional notation: Let $\Psi_0 = \Psi_{\rm inc}(z_0)$ be the incident field along z_0, and let Ψ_0', Ψ_{0s}, and A_0 denote the (deterministic) forms of Ψ', Ψ_s, and A, respectively, which would be due to a flat surface at z_0. We can then write

$$\delta = \Psi_{\rm inc} - \Psi_0,$$

$$\delta' = \Psi' - \Psi_0' \equiv A^{-1}\Psi_{\rm inc} - A_0^{-1}\Psi_0,$$

and

$$\delta_s = \Psi_s - \Psi_{0s} = B\delta'.$$

For any of these quantities f, say, $\langle f \rangle$ will denote its ensemble average, taken over realizations of the surface. The autocorrelation function of the surface (with mean removed) is given by

$$\rho(\xi) = \langle h(x)h(x+\xi) \rangle,$$

so that in this work $\rho(\xi) = \phi^2 \exp(-\xi^2/L_x^2)$.

II. APPROXIMATIONS AND RESULTS

A. Approximations

We will focus attention on the solution of Ψ' from integral equation (4). In the numerical treatment described in Sec. III, the incident field $\Psi_{\rm inc}$ and integrals are discretized, using N points, say. Here, A is represented by a lower-triangular matrix, the inversion of which requires $O(N^2)$ operations and relates the solution $\Psi'(x)$ to the values $\Psi_{\rm inc}(x')$ at all previous ranges $0 \leqslant x' \leqslant x$. In this way the influence of $\Psi_{\rm inc}$ can persist non-negligibly for many correlation lengths along the surface.

As noted in Sec. I, Eq. (4) gives Ψ' as a nonlinear function of surface roughness $h(x)$. We now approximate the solution by linearizing this dependence: The nonlinearity corresponds the stochastic part of A, which is in the exponential term

$$\exp\left[i\frac{k}{2}\left(\frac{(z-z')^2}{x-x'} \right) \right],$$

of the kernel G, and this becomes 1 when the surface is flat. We thus replace the operator A by its deterministic counterpart A_0:

$$\Psi_{\rm inc} \cong A_0\Psi'. \tag{7}$$

This approximation proves to be accurate for fairly high surface roughness, with $k\phi$ up to order unity, and turns out to be very convenient. For smaller values of $k\phi$, the approximation gives results almost identical with those from the full form.

To get a heuristic idea of the order of accuracy of (7), consider the point values of G. Let ϵ denote $x - x'$. The

above exponential tends to 1 as ϵ becomes large, and the exponent vanishes at $\epsilon = 0$ since h' is bounded. For ϵ greater than a correlation length L_x, the exponent is on average of order $O\{k[\phi^2/(x-x')]\}$. When $\epsilon < L_x$, we can show that it is of order $O(\epsilon k h'^2)$. Assuming a gentle surface slope, the exponent is small in both regions. The effect of integration over the kernel is to further reduce the departure of the operator A from its deterministic form A_0. (This comment also applies when there is a *large* surface slope because of the rapid phase oscillations which this induces.)

We can now show some numerical results, which indicate more directly the accuracy of (7). Figure 2 compares the amplitudes of Ψ' and its approximation $A_0^{-1}\Psi_{\text{inc}}$ for typical realizations, with (a) $k\phi = 0.5$ and (b) $k\phi = 1.0$. For values of $k\phi$ lower than about 0.5, the results are virtually indistinguishable. In these figures, $L_x = 8$.

B. Field decomposition

The approximation now allows us to deal separately with the deterministic and the purely stochastic effects: Since $\Psi_0' = A_0^{-1}\Psi_0$, $\Psi' \cong A_0^{-1}\Psi_{\text{inc}}$, and the operators are linear, we can write

$$\delta' \cong A_0^{-1}\delta. \tag{8}$$

When we calculate the full field Ψ, using the scattered field Ψ_s from (5), we have

$$\Psi = \Psi_s + \Psi_{\text{inc}} = B(\Psi_0' + \delta') + \Psi_{\text{inc}}$$
$$= \delta_s + (\Psi_{0s} + \Psi_{\text{inc}}).$$

The functions Ψ_{inc} and Ψ_{0s} are deterministic, and so for each realization it is sufficient to find δ'. Figure 3(a) shows an example of the real part of δ' for one realization with surface

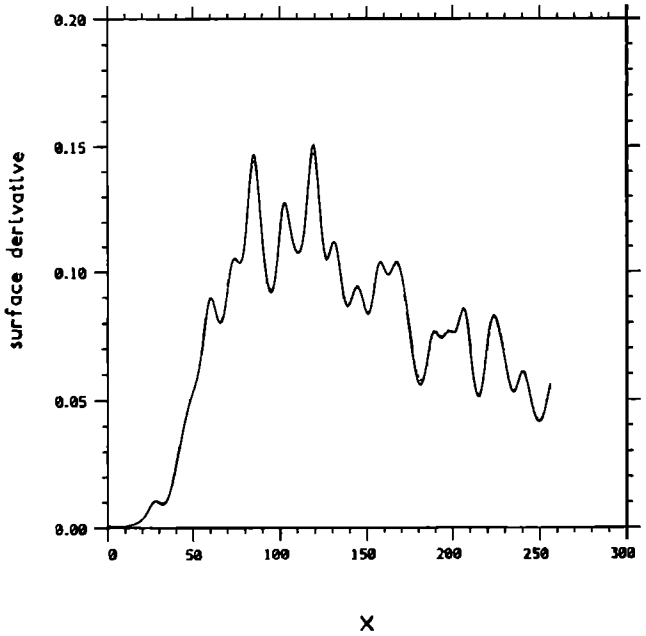

FIG. 2. Amplitude of $\partial\Psi/\partial z$ from Eq. (4) (full line) as a function of x compared with approximation (7) (dotted line) for (a) $k\phi = 0.5$ and (b) $k\phi = 1.0$. The correlation length is $L_x = 8.0$. The results are virtually identical for smaller values.

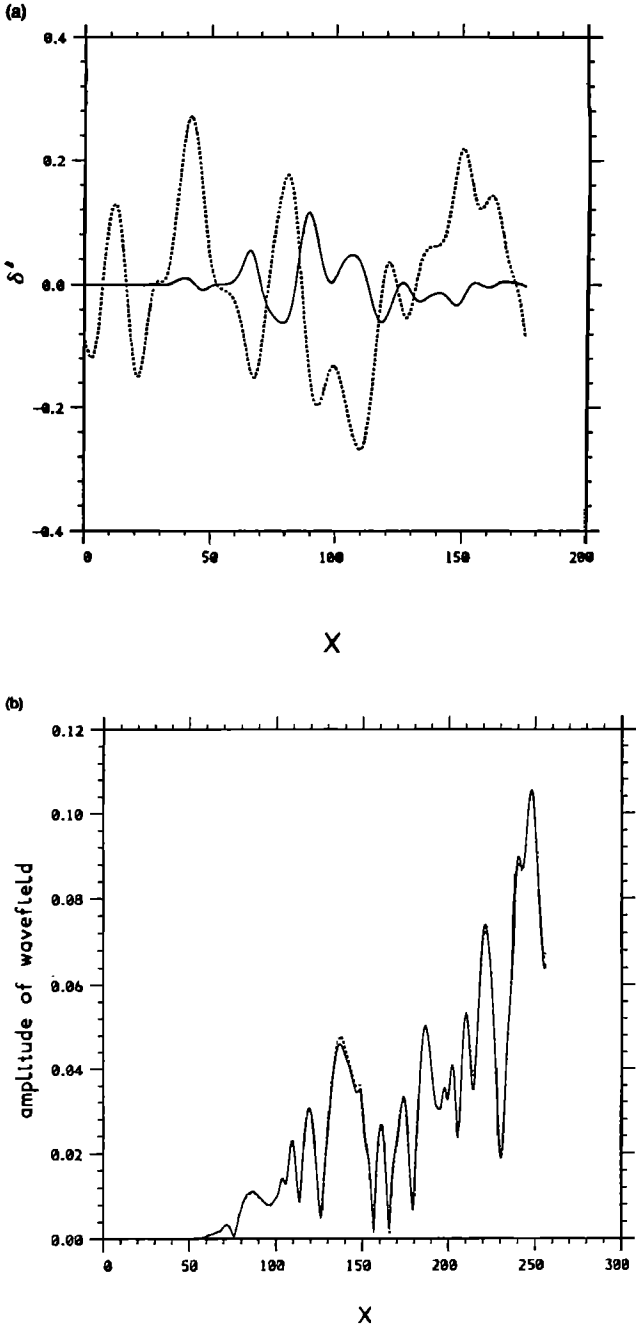

FIG. 3. (a) Real part of δ' (full line) as a function of x for one realization with $k\phi = 0.8$, against the surface itself (rescaled, dotted line). (b) Comparison between full field Ψ and its approximation as functions of range, at a distance 50.0 below a surface with roughness $k\phi = 0.8$, showing close agreement.

roughness $k\phi = 0.8$. The surface itself is shown for comparison. From δ' is obtained the stochastic component δ_s of the scattered field Ψ_s at any depth, from which we can find the full field Ψ. In Fig. 3(b) this approximation for Ψ is compared with the full solution at a depth $z = 50$ below the surface and shows very close agreement.

The field decomposition has a further advantage in the scattering regime L_x of the order of a wavelength, $k\phi$ up to 1. While Ψ_0' is determined purely by the source, the function δ' depends mainly on the surface roughness and its correlation

length. Thus δ' depends only on a small neighborhood of values of δ. Since $\Psi' = \Psi_0' + \delta'$, this allows a significant computational reduction for Ψ', from $O(N^2)$ to $O(N)$, assuming a fixed number of grid points per correlation length. (This, in turn, enables us to calculate the field out to arbitrarily large range.) In Fig. 4(a) the full form Ψ' was compared with an approximation using both (7) and a restricted "window" of about four correlation lengths to find the stochastic component δ'. The two curves again agree well. For comparison, another approximation to Ψ' was then calculated by applying the same restricted window to the full Eq. (4), without separating the stochastic part δ' from Ψ_0'. When compared with the full form of Ψ', there is a complete lack of agreement, as shown in Fig. 4(b), even though no approximation such as (7) has been applied. This illustrates that, through the integral equation, the range of dependence of Ψ_0' upon Ψ_0 is far greater than that of δ' on δ. [It should be stressed, however, that *physically* it is precisely the random variation which introduces multiple scattering and therefore nonlocal surface effects. The sense in which Ψ_0' depends nonlocally upon Ψ_0 is not that of surface interaction but arises, through (1), from the parabolic equation. In our situation δ is a fairly small, fast-varying perturbation of Ψ_0, and its surface effects are only local in comparison with this long-range dependence of Ψ_0' on Ψ_0.]

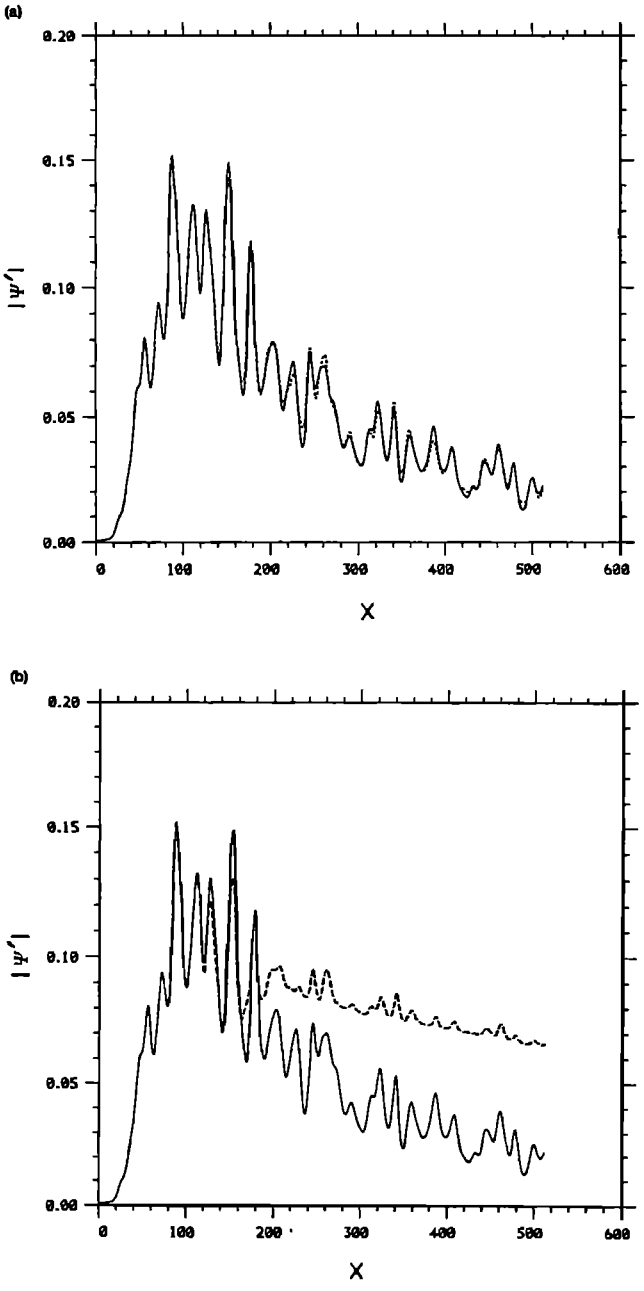

FIG. 4. (a) Amplitude $|\Psi'|$ from Eq. (4) (full line) compared with approximation (dotted line) using Eq. (7) and restricted window of four correlation lengths to find δ' and then applying $\Psi' = \Psi_0' + \delta'$. Here, $k\phi = 0.5$. (b) $|\Psi'|$ as in (a) (full line) compared with another approximation (dashed line) calculated by applying the same restricted window directly to Eq. (4), which gives complete lack of agreement.

C. Moments of Ψ_{inc} and Ψ'

One of the eventual aims is to find the moments of the scattered field Ψ_s, such as $\langle \Psi_s(x_1)\Psi_s^*(x_2)\rangle$, and their dependence on the statistics of the surface. Much progress has been made on this for volume scattering, e.g., Refs. 6 and 7, where the moments are governed by differential equations. For small surface heights the moments can be calculated by perturbation methods,[8,9] but for very rough surfaces the problem remains open (see, for example De Santo and Brown[8]). Equations (7) and (8) provide a convenient framework for one approach to the moments, and we can give some preliminary results for Ψ_{inc} and Ψ'.

Since the incident field is range dependent, its statistics are nonstationary with respect to x. If we assume that $z \gg \phi$ [which is implicit in the condition $\Psi_{inc}(0) \simeq 0$], then from (6) Ψ_{inc} at the surface can be written

$$\Psi_{inc}(x) \simeq \Psi_0(x) e^{g(x)h(x)}, \tag{9}$$

where

$$g(x) = \frac{2z_0}{w^2 + 2ix/k}.$$

Since g is deterministic, we may calculate the moments of Ψ_{inc} using (9) and the known surface probability distribution p, say. The mean of any function of Ψ_{inc} (e.g., Papoulis[10]) is just the integral of its product with p over all possible values. The mean of Ψ_{inc} is thus found to be

$$\langle \Psi_{inc}(x)\rangle \simeq \Psi_0(x)\exp\left(\frac{g^2(x)\phi^2}{2}\right), \tag{10}$$

and if we denote the autocorrelation function $\langle \Psi_{inc}(x)\Psi_{inc}^*(x')\rangle$ by $\bar{\rho}(x,x')$ and put $\xi = x - x'$, then

$$\bar{\rho}(x,x') \simeq \Psi_0(x)\Psi_0^*(x')\exp\left(\frac{\sigma^2(x,x')}{2}\right), \tag{11}$$

where σ^2 denotes the variance of $g(x)h(x) + g^*(x')h(x')$ and is given by

$$\sigma^2(x,x') = \phi^2[g^2(x) + g^{*2}(x')] + 2g(x)g^*(x')\rho(\xi).$$

These expressions hold for any ϕ. One realization of Ψ_{inc} is compared with its approximation (9) for $k\phi = 10$ in Fig. 5(a) and agrees closely.

When ϕ is small, the situation simplifies. To first order in h, (6) becomes

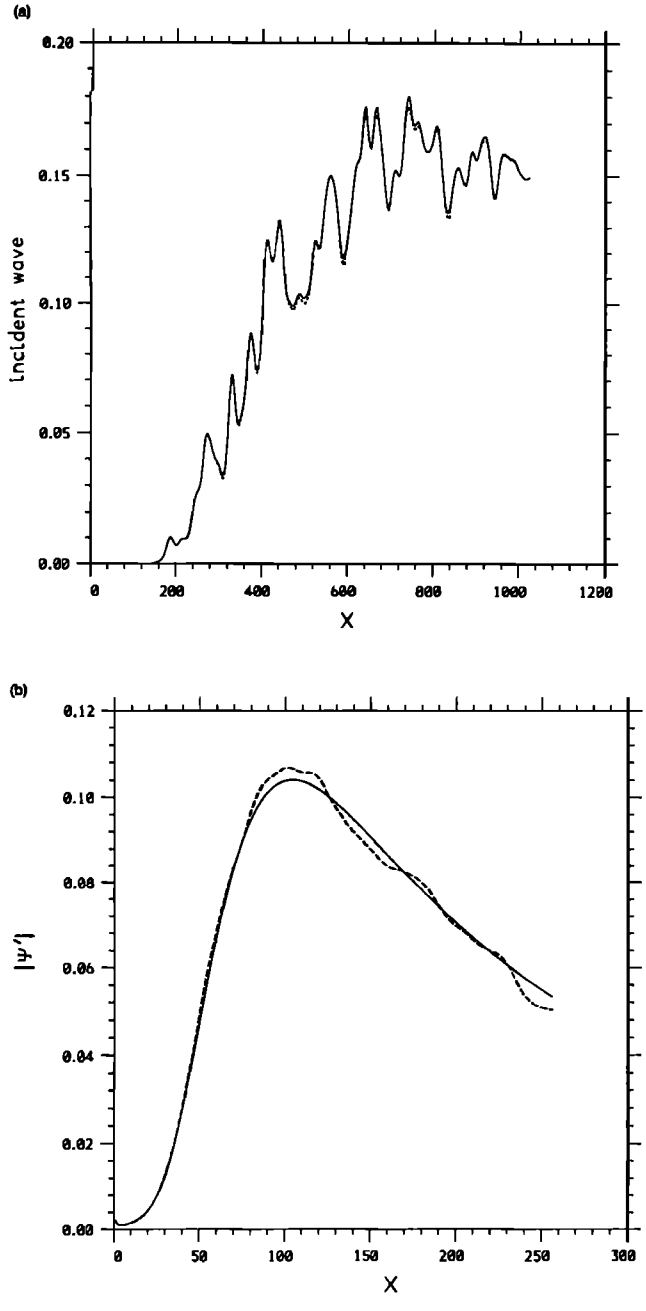

FIG. 5. (a) Comparison between Ψ_{inc} (dashed line) and approximation (9) (full line), along a typical surface with $\phi = 10$, when the source is at a depth of 100. (b) Predicted value of the amplitude of the mean of Ψ' (full line) as a function of x compared with simulations (dotted line), taking the average over 500 realizations, for $k\phi = 2.0$.

$$\Psi_{inc}(x) = \Psi_0(x)[1 - g(x)h(x)],$$

and therefore $\delta = \Psi_0 g(x)h(x)$. We can thus transform δ to the quantity $\bar{\delta} = \delta/\Psi_0 g$, so that $\bar{\delta} \cong h$. Then, $\bar{\delta}$ is approximately stationary, with mean zero and autocorrelation function ρ. A major advantage of transforming to a stationary function is that far fewer realizations are needed when calculating statistics from numerical simulations.

Consider again Ψ'. Since A_0 is linear and deterministic, it follows that $\langle A_0^{-1}\Psi_{inc}\rangle = A_0^{-1}\langle\Psi_{inc}\rangle$. So from (7), $\langle\Psi'\rangle \cong \langle A_0^{-1}\Psi_{inc}\rangle$, where $\langle\Psi_{inc}\rangle$ is known from (10). This is again confirmed by numerical simulations and despite the approximation for A holds well for $k\phi$ up to around 2, as shown in Fig. 5(b).

In a similar way one can express the second moment of Ψ' approximately as a double sum (or integral) over the function $\langle\Psi_{inc}(x)\Psi^*_{inc}(x+\xi)\rangle$. Using the semianalytical forms given in Sec. III below, it is then possible to write the moments of Ψ_s in terms of those of Ψ_{inc}. This will be the subject of another paper. However, the accuracy of the approximation for A rapidly deteriorates as $k\phi$ increases, and there is a need to develop a more consistent approach.

III. NUMERICAL SOLUTION

We give a brief description of the numerical approach used for the inversion of integral equation (4), and the forward integration (5).

A. Solution of Ψ' from (1)

The region of integration $(0,x_N)$, say, is discretized using a regular grid of N points $\{x_l\}$. Sufficient resolution is required for both $h(x)$ and $\Psi_{inc}(x)$. This gives rise to discretization of the Volterra integral equation (1). Each subintegral

$$-\int_{x_{l-1}}^{x_l}\left(G(\bar{r};\bar{r}')\frac{\partial\Psi}{\partial z'}\right)_{\bar{r},\bar{r}' \text{ at surface}} dx'$$

is treated by a simple combination of product integration and midpoint rules, to be approximated by

$$-\frac{1}{2}\sqrt{\frac{i}{2\pi k}}\exp\left(i\frac{k}{2}\frac{(z_n - Z_l)^2}{x_n - X_l}\right)\Psi'(X_l) \times\int_{x_{l-1}}^{x_l}\frac{dx'}{\sqrt{x_n - x'}},$$

where $\bar{r} = (x_n, h_1(x_n))$, $X_l = \frac{1}{2}(x_l + x_{l-1})$, and $Z_l = h_1(X_l)$. (Product integration is suitable here because both Ψ' and the exponential, and their derivatives, are effectively bounded.) This yields a set of linear equations, so that (4) becomes a matrix equation in which A is lower trangular and easily inverted.

It is easy to test the numerical solution for a flat surface, since the integral can then be carried out analytically when Ψ' is any polynomial in x.

B. Solution of Ψ_s from (2)

Although this is a forward integration, the nature of the singularity in G makes it more difficult to solve than (1). This difficulty increases with depth and integration cannot be dealt with by simple quadrature.

Consider again a subintegral

$$\int_{x_r}^{x_{r+1}}\left(G(\bar{r};\bar{r}')\frac{\partial\Psi}{\partial z'}\right)_{\bar{r}' \text{ at surface}} dx', \tag{12}$$

where $\bar{r} = (x_n, z)$. The exponent occuring in G in (3) can be written

$$i(k/2)\{[z - h_1(x')]^2/(x_n - x')\}.$$

Expanding h_1 as a Taylor's series about x_r, ignoring terms in

$(x_r - x')$ of second order or higher, and writing $x_r - x' = (x_n - x') - (x_n - x_r)$, this becomes

$$\frac{ik}{2}\left(\frac{z - h_1(x_r)}{x_n - x'}[z - h_1(x_r) - 2h'(x_r)(x_n - x_r)] + 2h'(x_r)[z - h_1(x_r)]\right). \quad (13)$$

The derivative $h'(x_r)$ here can be approximated by finite differences. Substituting (13) into (12), making a change of variables, and again using product integration, the subintegral takes the approximate form

$$e^{-i(k/2)a}\Psi'(X_{r+1})\int_{b_1}^{b_2}\frac{e^{iby}}{y^{3/2}}\,dy. \quad (14)$$

Here,

$$y = (x_n - x')^{-1/2}, \quad a = -2h'(x_r)[z - h_1(x_r)],$$
$$b_1 = (x_n - x_r)^{-1}, \quad b_2 = (x_n - x_{r+1})^{-1}$$

and

$$b = \tfrac{k}{2}[z - h_1(x_r)][z - h_1(x_r) - 2h'(x_r)(x_n - x_r)].$$

Equation (14) can be integrated by parts and put in terms of Fresnel integrals. These subintegrals can then be summed over r to give the scattered field $\Psi_s(x_n, z)$ for any depth z.

Numerical simulations using the above scheme can be tested both against full analytical solutions, in special cases, and more generally against "exact" numerical solutions.[2] Figure 6(a) shows the amplitude $|\Psi|$ of the full field as a function of range and depth for a flat surface. The amplitude of Ψ for one realization of a rough surface with $k\phi = 0.8$ is given in Fig. 6(b).

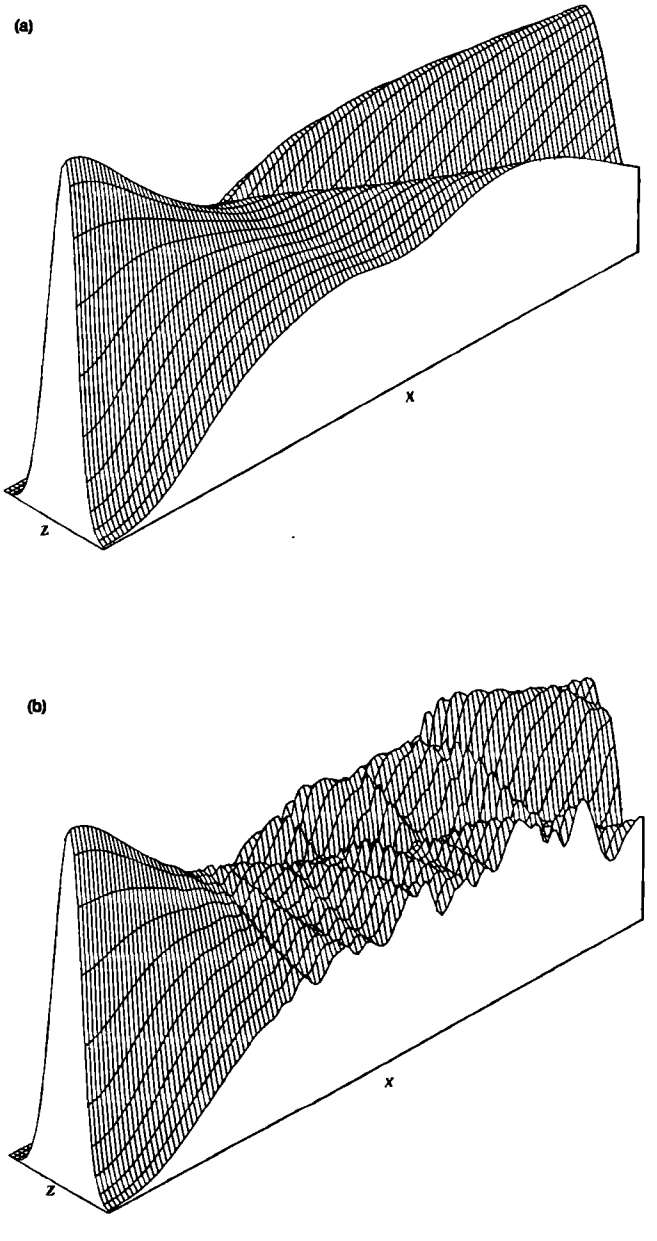

FIG. 6. Amplitude $|\Psi|$ of the full field as a function of range and depth (a) for a flat surface and (b) for one realization of a rough surface with $k\phi = 0.8$. Here, the source is at the left and the surface along the far side of the figure.

IV. CONCLUSIONS

The parabolic equation method[2] is an effective one for a wave incident at low grazing angles on a rough surface. Our numerical approach has been described, and an approximation to the method has been introduced, which is valid for $k\phi$ up to order unity. The approximation has several consequences. It allows the wave field to be split into a sum of deterministic and stochastic components. This, in turn, greatly reduces the computational expense because the stochastic part has a localized behavior. The decomposition also provides a framework for examination of some of the statistical moments. These results and the numerical scheme have been demonstrated computationally.

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