Numerical solution of scattering from a hard surface in a medium with a linear profile

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The paper is concerned with the scattering of waves incident at grazing angles upon a rough surface on which Neumann boundary conditions apply, when the medium has a linearly varying refractive index. The scattering is governed by an integral equation, which uses a parabolic form of the Green's function. The numerical solution of this system requires careful analytical treatment of the Green's function, and the purpose of this paper is to describe the details both of the analysis and the numerical scheme. Some computational results are shown, and the accuracy of the inversion of the integral equation is tested by comparison with an analytical approximation.

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INTRODUCTION

The problem of acoustic scattering from rough surfaces has been widely studied in a variety of scattering regimes. A hard surface (Neumann condition) is needed for many applications in aero- and ocean acoustics. Although it is frequently assumed that the medium has a constant refractive index, in practice this is often not the case, and an important extension of the models is to allow variations of the refractive index. The parabolic form of the Green's function has recently been extended (Uscinski[1]) to include the effect of a linear refractive index profile; this is valid for a wave field incident at low grazing angles, when scattering is accurately described by the parabolic equation method.[2–4]

The governing equations take the form of an integral equation and an integral, which for a Neumann boundary condition, relate the field to the vertical derivative $H(\mathbf{r};\mathbf{r}')$ of the Green's function. (This approximates the normal derivative[1] that usually appears in integral equation approaches.) Numerical treatment requires the integral with respect to range of H. However H is not explicitly defined when its arguments coincide, and along a flat surface behaves like a delta function. The solution is complicated by the presence of weak singularities in the equations, as with the Dirichlet boundary condition. These difficulties are resolved by careful evaluation of H near the surface and extension of H to the surface by continuity. The main purpose of this paper is to describe this analytical treatment and the numerical solution that is then applied.

Some results and illustrations are given. In particular, for the case when the profile variation vanishes, a simple analytical approximation for the field at the surface is examined. This provides a test of the numerical solution for a given rough surface and allows the separation of the deterministic and stochastic components.

I. GOVERNING EQUATIONS

We consider the problem of a wave field at grazing incidence scattered from a one-dimensional rough surface. The Neumann boundary condition is assumed, that is, that the normal derivative of the field vanishes along the surface, and the refractive index in the medium itself is allowed to change linearly with depth. The coordinate axes are x and z, where x is the horizontal, $x \geqslant 0$, and z is the vertical, increasing upwards (i.e., z is directed out of the medium). Let $\mathbf{r} = (x,z)$. The source is centered about $\mathbf{r} = (0,0)$, and the mean surface level is at $z = z_0$. The rough surface itself is denoted $h_1(x)$, and $h \equiv h_1 - z_0$ so that h has mean zero. The surface is assumed here to have a bounded first derivative. In the numerical examples, h has Gaussian statistics and is stationary. Denote the rms of the surface h by ϕ and its correlation length by L_x.

The slowly varying part E of the wave field is defined by

$E(x,z) = p(x,z)e^{-ikx}$.

The parabolic form of the Green's function G when the medium has an acoustic refractive index $n(z) = n_0(1 + az)$, where n_0 is a constant reference value, is given by (see Uscinski[1])

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{2}\sqrt{\frac{i}{2\pi k(x-x')}} \exp \frac{ik}{2}\left(\frac{(z-z')^2}{x-x'} + a(z+z')(x-x') - \frac{a^2(x-x')^3}{12}\right), \quad (1)$$

when $x' \leqslant x$ and $G = 0$ otherwise, where $\mathbf{r}' = (x',z')$. Then the derivative $H = \partial G/\partial z'$ with respect to the second vertical coordinate becomes

$$H(\mathbf{r};\mathbf{r}') = \alpha(H_1 - H_2), \quad (2)$$

where

$$H_1 = \frac{z-z'}{(x-x')^{3/2}} \exp\left[\frac{ik}{2}\left(\frac{(z-z')^2}{x-x'} + a(z+z')(x-x') - \frac{a^2(x-x')^3}{12}\right)\right], \quad (3)$$

$$H_2 = \frac{a}{2}(x-x')^{1/2}\exp\left[\frac{ik}{2}\left(\frac{(z-z')^2}{x-x'} + a(z+z')(x-x') - \frac{a^2(x-x')^3}{12}\right)\right], \quad (4)$$

and

$$\alpha = -(i/2)\sqrt{ik/2\pi}.$$

The scattering and propagation is then governed by the following equations,[1] which relate the total field to the Green's function and the incident field along the rough surface:

$$E_{\rm inc}(x,z) = \int_0^x E(x',z')\frac{\partial G}{\partial z'}(x,z,x',z') \times dx'|_{z=h_1(x),\ z'=h_1(x')} + E(x,z), \quad (5)$$

which may be written in operator form as

$$E_{\rm inc} = TE,$$

and

$$E_s(x,z) = -\int_0^x E(x',z')\frac{\partial G}{\partial z'}(x,z,x',z')\bigg|_{z'=h_1(x')} dx'. \quad (6)$$

Equation (5) departs from the usual conventions for the analogous Helmholtz equations in the literature, and some discussion of this is required. The integral represents the limit of the integral in (6) as z approaches the surface. This differs from the common convention (for example, see Refs. 5 and 6) in which the limit is taken *first*, i.e., the integral is over the pointwise limit of the integrand in (6). The integration and limit operations fail to commute because as z approaches the surface $\partial G/\partial z'$ does not converge in the space L_1 of integrable functions. With the pointwise limit, for a flat surface the integrand vanishes almost everywhere, and the last term $E(x,z)$ acquires a 1/2 factor. The integral limit (5) is arguably more convenient here, since this equation arises as the limit of Eq. (6), and also because the numerical treatment of these expressions approximates point values by integrals over small intervals.

Now, the integral equation (5) must be inverted to give the total field E at the surface, which may then be substituted in (6) to find the field everywhere in the medium. In the derivation of these equations the usual normal derivative $\partial G/\partial z'$ of the Green's function has been replaced by the outward vertical derivative. This implies a further small slope assumption, which is consistent with the parabolic formulation, and is discussed further elsewhere.[1] These equations have weak (i.e., integrable) singularities as $x' \to x$. The main problem is to identify fully these singularities when $\mathbf{r}' \sim \mathbf{r}$, and to treat them numerically. The method of solution and the evaluation of the integral of H, will be given in Sec. II. An analytical solution of (5) as an infinite series when $a = 0$, and the approximation given by its truncation, will also be discussed.

The incident wave in the numerical examples here is a simple Gaussian beam of width w traveling at a small angle θ to the surface:

$$E_{\rm inc}(x,z) = \frac{w}{\sqrt{w^2 + 2ix/k}} \times \exp\left(-\frac{2z^2 + ikSw^2(Sx-z)}{2(w^2+2ix/k)}\right), \quad (7)$$

where $S = \sin(\theta)$. In the computational examples that follow we have taken $k = 1$ and $w = 8$, and have considered the forward-traveling case $\theta = 0$. The surface correlation length is of the order of a wavelength.

Some additional notation will be needed: Let E_0 denote the incident field $E_{\rm inc}(z_0)$ along z_0, and denote by T_0 the integral operator T [Eq. (5)] that would be due to a flat surface. The numerical scheme (described below) requires the region of integration $(0,x_N)$, say, to be discretized using a regular grid of N points $\{x_r\}$, where $x_r = r\Delta x$, and Δx is small compared with variation in the surface and in the field $E_{\rm inc}$ incident upon it.

II. SOLUTION

The approach adopted in solving Eqs. (5) and (6) can now be described. Although the numerical method is similar to that which has been applied to a pressure release surface,[4] careful analytical treatment is required, to deal with the singularities in the Green's function and in particular to determine the limiting behavior of $H(\mathbf{r};\mathbf{r}')$ as $\mathbf{r}' \to \mathbf{r}$. The integral of H over small intervals along the surface must therefore be evaluated explicitly. Although it is not possible to do this *exactly* for an arbitrary rough surface, the approximations that will be applied capture correctly the behavior near the singularities and are exact when the surface is flat. The integrals of the function H away from the singularities are also required for the numerical inversion of (5) and the evaluation of (6), in which the factor E in the integrands is treated as approximately constant over sufficiently small intervals. In the following equations z' denotes the value $h_1(x')$ at the surface, and the derivative dh/dx is written h'.

A. Medium with constant profile

1. Evaluation of H(r; r') as r approaches the surface

Suppose that the refractive index in the medium is constant. Then $H = \alpha H_1$ and the exponent in H_1 simplifies. We consider the behavior of H when $\mathbf{r} - \mathbf{r}'$ is small. This is necessary, firstly, because the integral of H is not completely defined by (2) about the point $(x,z) = (x',z')$, and must be extended by continuity. Furthermore, in doing so care must be taken to avoid taking the "wrong" limit. As an illustration, consider H in the case of a flat surface. Then, for $x' \neq x$, putting $z' = z$ in Eq. (2) gives $H = 0$. This is consistent with (5) only if H behaves as a delta function at $x = x'$ (see below). If we wish to quantify this by integrating H along the surface in a neighborhood of x, we might try do so for a slightly rough surface and let the roughness tend to zero. For small β, expanding h about x this would give

$$\int_{x-\beta}^{x} H(x,z;x',z')dx' \cong \alpha h'(x)\int_{x-\beta}^{x}\frac{\exp[(ik/2)h'^2(x)(x-x')]}{\sqrt{x-x'}}dx';$$

this tends to zero with h', which is not valid. However if the integral of H is instead evaluated for z, at a small distance ϵ away from the surface, and ϵ is allowed to go to zero, the integral will approach the correct value in the limit.

Now, let $z - h_1(x) = \epsilon$, where the surface h_1 is again rough. (Since z increases upwards ϵ is negative.) Consider the integral

$$\int_{x-\beta}^{x} H(x,z;x',z')dx' ,$$

where β (which will be the numerical discretization length Δx) is a fixed small parameter. If we expand $z = h_1(x')$ about x, the term $z - z'$ becomes approximately $\epsilon + (x - x')h'(x)$ and the function H becomes

$$H \cong \alpha\left(\frac{\epsilon}{(x-x')^{3/2}} + \frac{h'(x)}{(x-x')^{1/2}}\right) \times \exp\left[\frac{ik}{2}\left(\frac{\epsilon^2}{x-x'} + 2\epsilon h'(x) + (x-x')h'^2(x)\right)\right].$$

The last term in the exponent is slowly varying and can be treated as zero over this interval of integration, since the form of the coefficients means that its main contribution comes as $x' \to x$. This equation can therefore be written

$$H \cong \alpha \exp[(ik/2)2\epsilon h'(x)](I_1 + I_2) , \qquad (8)$$

where

$$I_1(x,x') = \frac{\epsilon}{(x-x')^{3/2}} \exp\left(\frac{ik\epsilon^2}{2(x-x')}\right)$$

and

$$I_2(x,x') = \frac{h'(x)}{(x-x')^{1/2}} \exp\left(\frac{ik\epsilon^2}{2(x-x')}\right).$$

Now, under the substitution

$$y = \epsilon\sqrt{\frac{k}{2(x-x')}}$$

the integral of I_1 becomes

$$\int_{x-\beta}^{x} I_1(x,x')dx' = -2\sqrt{\frac{\pi}{k}} \int_{b}^{\infty} \exp\left(\frac{i\pi}{2}y^2\right)dy, \quad (9)$$

where $b = -\epsilon\sqrt{k/\pi\beta}$ (so that b is positive). This is a Fresnel integral, and in the limit $\epsilon \to 0$, b vanishes and the integral takes the value

$$\lim_{\epsilon\to 0} \int_{x-\beta}^{x} I_1(x,x')dx' = -\sqrt{\frac{\pi}{k}}(1+i) . \qquad (10)$$

The term I_2 can be treated similarly. With the substitution $y = \epsilon/\sqrt{x-x'}$ the integral becomes

$$\int_{x-\beta}^{x} I_2(x,x')dx' = -2\epsilon h'(x) \int_{c}^{\infty} \frac{e^{iky^2/2}}{y^2} dy , \qquad (11)$$

where $c = -\epsilon/\sqrt{\beta}$. Integration by parts gives

$$-2h'(x)\sqrt{\beta}\exp\left(\frac{ik\epsilon^2}{2\beta}\right) + 2\epsilon ikh'(x) \int_{B}^{\infty} \exp\left(\frac{ik}{2}y^2\right)dy . \qquad (12)$$

In the limit, the second term tends to zero with ϵ and the first simplifies to give

$$\lim_{\epsilon\to 0} \int_{x-\beta}^{x} I_2(x,x')dx' = -2h'(x)\sqrt{\beta} . \qquad (13)$$

Thus from (10) and (13) we obtain

$$\lim_{\epsilon\to 0} \int_{x-\beta}^{x} H(x,z;x',z')dx' \cong \left(\sqrt{\frac{ik\beta}{2\pi}}\, h'(x) - \frac{1}{2}\right). \qquad (14)$$

For a flat surface this is exact, reducing to $-1/2$, so that $H(x,z;x',z')$ is the delta function, $-\delta(x-x')/2$.

We also require the integral of H over intervals of the surface in which x' does not approach x, but these are well behaved and are obtained easily from the more general case below.

2. Evaluation of H(r; r') for r in the medium

Retaining the assumption that the profile is constant, consider the subintegral

$$\int_{x_r}^{x_{r+1}} H(x,z;x',h(x'))dx'$$

over a small interval (x_r,x_{r+1}), where $x_{r+1} \neq x$, when (x,z) is not on the rough surface. Now $h(x_r)$ can be expanded about x_r, and omitting the factor $ik/2$ the exponent can be written

$$\frac{[z-h(x_r)]^2}{x-x'} \cong \frac{B}{x-x'} + 2h'(x_r)[z-h(x_r)] ,$$

where

$$B(x,z,x_r) = [z-h(x_r)] \times [z-h(x_r) - 2h'(x_r)(x-x_r)] . \qquad (15)$$

Similarly the coefficient becomes

$$\alpha\frac{z-z'}{(x-x')^{3/2}} \cong \alpha\left(\frac{A}{(x-x')^{3/2}} + \frac{h'(x_r)}{(x-x')^{1/2}}\right),$$

where

$$A(x,z,x_r) = z - h(x_r) - h'(x_r)(x-x_r) .$$

Thus H can be written

$$H(x,z;x',z') \cong \alpha\left(A + \frac{h'(x_r)}{(x-x')^{1/2}}\right)\exp\{ikh'(x_r) \times [z-h(x_r)]\}\exp\left(\frac{ikB}{2(x-x')}\right).$$

The separate components of H due to the coefficients A and h' can be approached much as before. Exponential factors that vary slowly are treated as constant over (x_r,x_{r+1}). The integral of H over (x_r,x_{r+1}) becomes

$$\int_{x_r}^{x_{r+1}} H\,dx' \cong \alpha A \exp\{ikh'(x_r)[z-h(x_r)]\} \times \int_{x_r}^{x_{r+1}} \frac{\exp[ikB/2(x-x')]}{(x-x')^{3/2}} dx'$$

$$+ \alpha h'(x_r)\exp\{ikh'(x_r)[z-h(x_r)]\}$$
$$\times\int_{x_r}^{x_{r+1}} \frac{\exp[ikB/2(x-x')]}{(x-x')^{1/2}}\,dx' . \qquad (16)$$

With the change of variable $y\rightarrow\sqrt{kB/\pi(x-x')}$ the first integral is a Fresnel integral,

$$2\sqrt{\frac{\pi}{kB}}\,[C+iS]_{A_2}^{B_2} , \qquad (17a)$$

where S and C are the sine and cosine Fresnel integral, respectively, and

$$A_2=\sqrt{\frac{kB}{\pi(x-x_r)}} \quad\text{and}\quad B_2=\sqrt{\frac{kB}{\pi(x-x_{r+1})}} .$$

With the substitution $y=kB/2(x-x')$, after an integration by parts and further change of variables, the second integral becomes

$$\sqrt{\frac{kB}{2}}\left[\frac{e^{iy}}{\sqrt{y}}\right]_{(\pi/2)A_2^2}^{(\pi/2)B_2^2} - i\sqrt{kB\pi}\,[C+iS]_{A_2}^{B_2} . \qquad (17b)$$

The case $r+1=n$ is similar.

B. Medium with linear profile

The treatment when there is a linearly varying profile in the medium is similar. The terms in the exponents in (3) and (4), which are due to the profile, vary slowly near the singularities and can be treated as constant over sufficiently small sub-integrals. Denote this slowly varying part as

$$F(x,x_r)=\exp\left[\frac{ik}{2}\left(a[z+h(X_r)](x-X_r) - \frac{a^2}{12}(x-X_r)^3\right)\right] , \qquad (18)$$

where X_r may be chosen to be the midpoint of the interval (x_r,x_{r+1}). [Note that although the terms $(x-X_r)$, $(x-X_r)^3$ in the exponent may become large, they do not give rise to rapid variation of F over distances considered here; in typical applications the profile parameter a is less than 10^{-2}. However it is easy to deal analytically with larger distances because F begins to vary quickly only when the other exponential factor is itself changing slowly.] The integral of the term H_1 can then be found immediately from the approximations for the constant-profile case. The extra term H_2 introduced by the profile remains to be dealt with.

1. Integral of $H_2(r;r')$ for r on the surface

The integral of H_2 over intervals (x_r,x_{r+1}) widely separated from x dominates that of H_1, and is non-negligible even for a flat surface. In effect it is the main contributor to the repeated "bouncing" of the field against the surface. Apart from the slowly varying factors H_2 becomes

$$\int_{x_r}^{x_{r+1}} \exp\left(\frac{ik}{2}\frac{\epsilon^2}{x-x'}\right)\sqrt{x-x'}\,dx' .$$

Substituting y (positive), where $\pi y^2=\epsilon^2k/(x-x')$, the integrand takes the form $e^{i\pi y^2}/y^4$, which integration by parts again yields in terms of Fresnel integrals. The contribution from H_2 is then

$$\int_{x_r}^{x_{r+1}} H_2(x,z;x',z')dx'$$
$$\cong -a\frac{\alpha}{3}[(x-x_{r+1})^{3/2}-(x-x_r)^{3/2}]$$
$$\times F(x,x_r)\exp\left(\frac{ik\Delta x}{4}h'^2(x)\right) . \qquad (19)$$

2. Integral of $H_2(r;r')$ for r in the medium

When (x,z) is a point inside the medium the integral is treated similarly. After a change of variables and integration by parts we finally obtain

$$\int_{x_r}^{x_{r+1}} H_2(x,z;x',z')dx'$$
$$\cong a\alpha F(x,x_r)B^{3/2}\exp\{ikh'(x_r)[z-h(x_r)]\}$$
$$\times\left\{\left[\frac{e^{iy^2}}{3y}\left(\frac{1}{y^2}+2i\right)\right]_{A_2\sqrt{\pi/2}}^{B_2\sqrt{\pi/2}} + \frac{4}{3}\sqrt{\frac{\pi}{2}}\,[C+iS]_{A_2}^{B_2}\right\} , \qquad (20)$$

where C and S are again the cosine and sine Fresnel integrals, with A_2, B_2 as in Eq. (17), and $B=B(x,z,x_r)$ as in (15).

C. Numerical treatment of equations

With the calculations above, the numerical solution of Eqs. (5) and (6) is now straightforward, and is similar to that adopted for a pressure release surface.[4] The discretization $\{x_r\}$, which may depend on the rms surface height ϕ, gives rise to a discretization of the integral equation (5) and the integral (6). For example, for each n (5) becomes

$$E_{\rm inc}(x_n,z)=\sum_{r=1}^{n}\int_{x_{r-1}}^{x_r} E(x',z') \times\frac{\partial G}{\partial z'}(x_n,z,x',z')\bigg|_{z=h_1(x_n),\ z'=h_1(x')} dx' + E(x_n,z) . \qquad (21)$$

Provided Δx is sufficiently small the slowly varying terms may be treated as constant over each subintegral and (21) may be written

$$E_{\rm inc}(x_n,z)\cong\hat{G}\hat{E} , \qquad (22)$$

where $\hat{E}$ denotes the vector $E_m\equiv E(x_m,h(x_m))$, $z=h(x_n)$, and $\hat{G}$ is the matrix:

$$\hat{G}_{n,r}(z)=\int_{x_{r-1}}^{x_r} H(x,z;x',z')dx', \quad \text{for } r\neq n ,$$

$$\hat{G}_{n,n}=1+\int_{x_{n-1}}^{x_n} H(x,z;x',z')dx' . \qquad (23)$$

The constant 1 here is due to the term $E(x,z)$ on the right-hand side of (5). The integrals are approximated by Eqs. (14) and (19) given earlier and this matrix equation is inverted to solve for the field at the surface. The integral (6) is

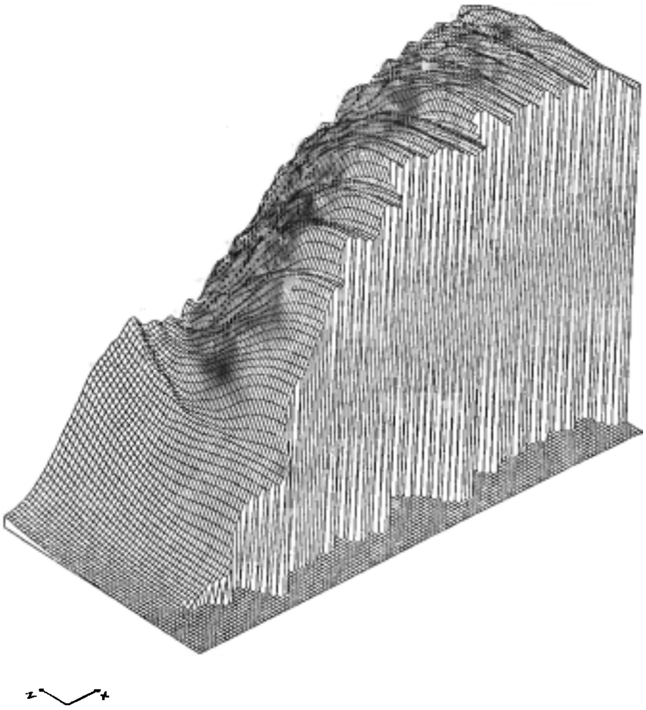

FIG. 1. Amplitude of the field as a function of range and distance z from the surface. The source is located (beyond the region shown) along the axis, which is to the left; the foreground extends slightly beyond the boundary of the real medium, as indicated where the field vanishes.

discretized in the same way; with the values now found for field at the surface, and the sub-integrals of H from (16), (17), and (20), the field everywhere can be computed.

Although no analytical solution for an arbitrary rough surface is available, the accuracy of this method in practice can be examined using a number of obvious tests. The error is of order $O(\Delta x)$; this cannot be improved upon without introducing substantially greater complexity in the analysis which is needed to evaluate $\hat{G}$. Provided Δx is chosen to be small compared with variation in the surface itself and with $E_{\text{inc}}(x',z')$, the accuracy will be satisfactory.

The first check is that the boundary condition is satisfied, i.e., that the normal derivative of the field E vanishes along the surface. This is clear from Fig. 1, which shows the amplitude of the field as a function of range x and the vertical z in a region around the surface (whose outline is marked by the discontinuity in E). In the case of a nonconstant profile a and a flat surface the "bounce" length x_1, i.e., the distance to the first intensity peak along the surface, is given by $x_1 = (2z_0/a)^{1/2}$. Provided that successively scattered peaks are separated, it is easily shown from the ray paths that they should occur at intervals of $2x_1$. Figure 2 shows the amplitude of the field scattered from a flat surface as a function of range and depth, in a medium with a fairly strong linear profile. The interference and channeling effects due to the profile are clearly seen. Further results and illustrations are given elsewhere[1] and these will not be reproduced here.

When the profile variation a is zero, an analytical check can be used for moderately rough surfaces, and this is described below, with some additional comparisons.

D. Further results

When there is a linear variation of the refractive index in the medium analytical treatment becomes intractable, particularly for a rough surface since the profile introduces multiple scattering against even a slightly rough surface. Suppose however that the profile is constant, and consider Eq. (5), $E_{\text{inc}} = TE$. The flat surface form T_0 is given in this case simply by 1/2. Recall that $E_0(x) = E_{\text{inc}}(x,z_0)$, i.e., the incident field along the line z_0 in the absence of the surface. We can write $T = T_0 + \Delta_T$, and $E_{\text{inc}} = E_0 + \Delta_E$. Then Δ_T is just the integral operator defined by

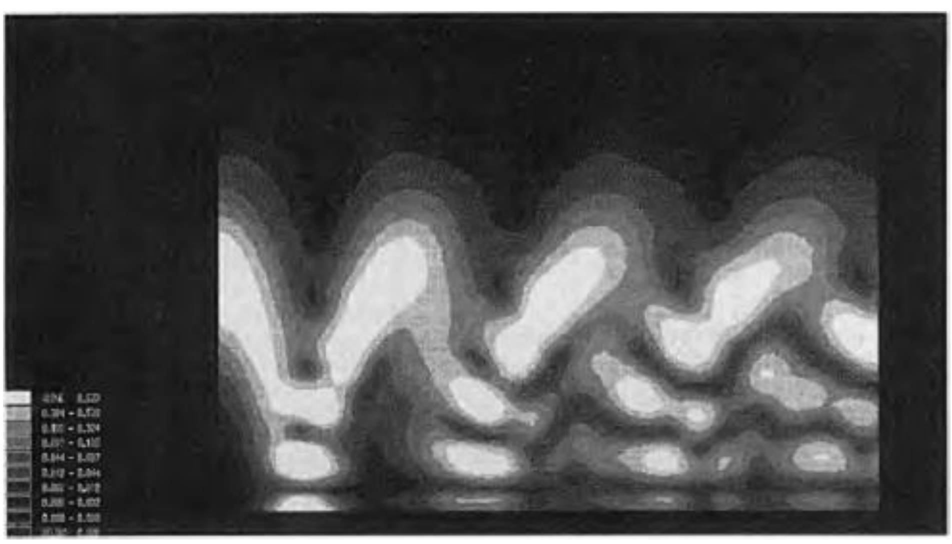

FIG. 2. Contour plot of the amplitude of the field in a medium with a profile $a = 0.015$. Range and depth are represented by the horizontal and vertical axes, respectively, with the source at the left, and the surface along at the bottom edge. The source is at 22 m, and the graph shows the field over a distance of 520 m.

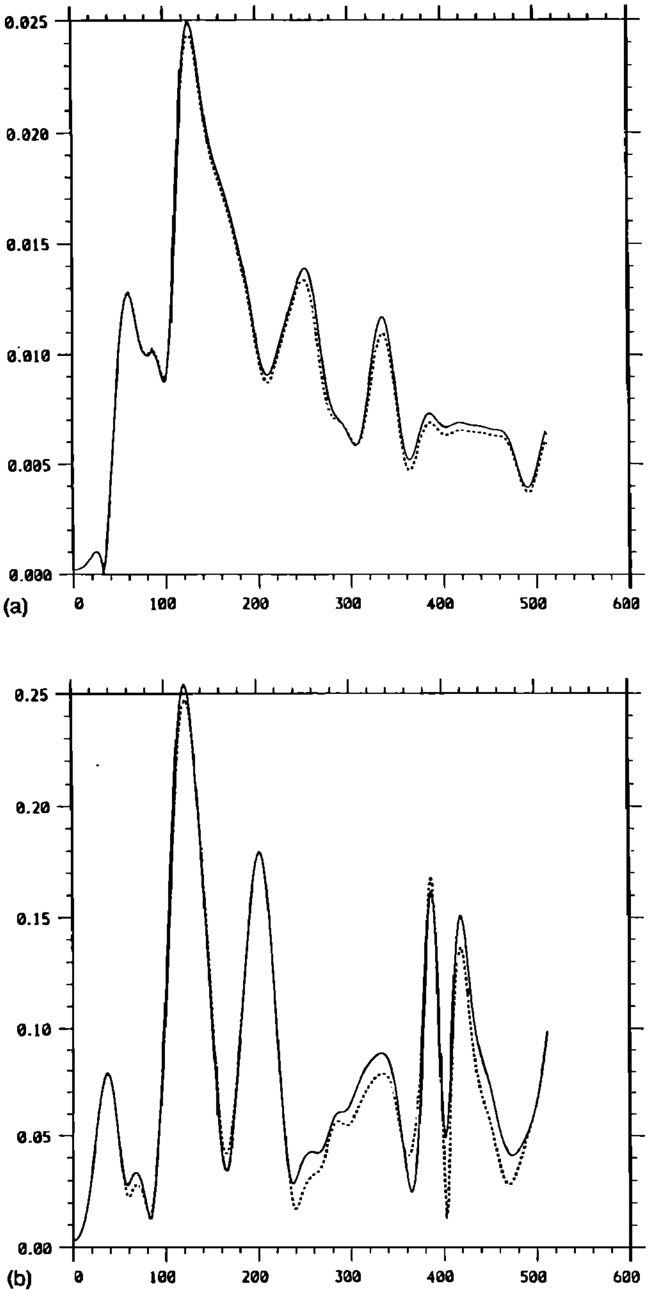

FIG. 3. Amplitude of the field with the deterministic component removed from the numerical solution (dashed line) and the approximation equation (25) (dotted line), for a rough surface with $k\phi = 0.2$. (b) Comparison as in (a) for a different surface with $k\phi = 2.5$, with an extra term from Eq. (24) included in the approximation.

$$\Delta_T(f)(x) = \int_0^x H(\mathbf{r};\mathbf{r}') f(x')dx' - f(x)/2 ,$$

where $\mathbf{r},\mathbf{r}'$ lie on the surface. We want the solution $E = T^{-1}E_{\text{inc}}$, and since T_0 is constant (and therefore commutes with all other operators) we can write

$$T^{-1} = T_0^{-1}(1 - T_0^{-1}\Delta_T + T_0^{-2}\Delta_T^2 - \cdots) \qquad (24)$$

so that

$$E = (2 - 4\Delta_T + 8\Delta_T^2 - \cdots)E_{\text{inc}}.$$

The terms Δ_T^m are easily evaluated by repeated application of Δ_T. Truncation then gives the approximate solution

$$E(x,h(x)) \approx 2E_0 + 2\Delta_E - 4\Delta_T E_0 , \qquad (25)$$

where we have neglected products or powers of Δ_T, Δ_E. These expressions are quite convenient in several ways. Most usefully here, (25) provides a simple analytical test of the numerical solution for slightly rough surfaces. Figure 3(a) compares Eq. (25) with the full numerical solution for the amplitude of the field, after the deterministic component $E_{\text{inc}}(\mathbf{r})$ has been subtracted, for $k\phi = 0.2$. It is easy and computationally inexpensive to take further terms in the series (24). Figure 3(b) shows the comparison for a different surface with $k\phi = 2.5$, when one extra term from the series (24) has been included.

The first two terms on the right of Eq. (25) are purely "local," and although the last term may be regarded as once-scattered, it gives rise to the leading order multiple scattering component anywhere away from the surface. (Multiple scattering is usually considered to arise where the field results from more than one surface integration.[5]) The expression can also be examined (by analogy with the pressure release surface[4]) to quantify the distance over which the details of the surface appreciably affect the scattered field. The stochastic part of E along the surface is given by the last two terms, and from these an approximate form for the mean field along the surface is easily obtained.

When there is a linear profile in the medium, the expansion (24) (for the appropriate operators T_0 and Δ_T) does not hold because T_0 no longer commutes with Δ_T. It is also clear that no "local" approximation can be valid in that case since the wave field is repeatedly scattered at the surface, and analytical treatment both for hard and soft surfaces becomes difficult.

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[1] B. J. Uscinski, "Sound propagation with a linear sound-speed profile over a rough surface" (submitted).
[2] E. Thorsos, "Rough surface scattering using the parabolic wave equation," J. Acoust. Soc. Am. Suppl. 1 82, S103 (1987).
[3] E. Thorsos, "The numerical approach to understanding rough surface scattering," J. Acoust. Soc. Am. Suppl. 1 83, S9 (1988).
[4] M. Spivack, "A numerical approach to rough surface scattering by the parabolic equation method," J. Acoust. Soc. Am. 87, 1999–2004 (1990).
[5] H. A. DeSanto and G. S. Brown, "Analytical techniques for multiple scattering for rough surfaces," in *Progress in Optics XXIII*, edited by E. Wolf (North-Holland, Amsterdam, 1986), p. 2-62.
[6] W. C. Meecham, "On the use of the Kirchhoff approximation for the solution of reflection problems," J. Rational Mech. Anal. 5, 323–33 (1956).