Coherent field and specular reflection at grazing incidence on a rough surface

Mark Spivack
Department of Applied Mathematics and Theoretical Physics, The University of Cambridge, Cambridge CB3 9EW, England

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The coherent component of the field scattered at grazing angles from a slightly rough pressure release surface is found. This is valid for multiple scattering and is based upon the parabolic integral equation method. Also examined are the scattering of planes waves under the method and in particular, the effect of truncating the boundary integral. It is shown that the coherent field remains invariant when the source and receiver are displaced vertically by equal and opposite distances, as was found numerically in a previous paper. In general, this can be shown to hold because the coherent field due to any plane wave is specular; however, under the parabolic equation method reflection is *not* specular, and thus the result is of particular interest. Reflection coefficients are given in closed form for several surface statistics, valid asymptotically at large distances.

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INTRODUCTION

One of the main aims of the study of acoustic scattering by rough surfaces is to relate quantities such as the mean or coherent component of the field to the statistics of the surface.[1–4] A common approach to this is to consider the effect on incident plane waves. For a pressure release surface, the Helmholtz equations describe the field as a boundary integral in the normal derivative, which in turn is expressed as the solution of an integral equation. When the incident and scattered wave fields propagate at small angles to the surface, the full Helmholtz formulation may be replaced by the parabolic equation (PE) method.[5–8] In this forward-scattering approximation the Green's function is recast in a paraxial form, and the region of boundary integration is truncated to lie between the vertical planes of a source and the "receiver." In particular, the derivative of the field at the surface is treated as the result of scattering only from the direction of the source plane.

The PE method has proved extremely useful for forward scattering; it is highly efficient computationally and this formulation has allowed accurate inverse scattering solutions[9,10] to be developed. (For the inverse solutions at nongrazing incidence see, e.g., Ref. 11.) Nevertheless in several respects the method is not well understood. In Ref. 12 it was found numerically that for a Gaussian beam at grazing incidence, scattering under the PE method obeys an image property: The coherent field remains invariant when the source and receiver are displaced vertically by equal distances in opposite directions. Now as is shown below, this is a general property of rough surface scattering, because for arbitrary angles of incidence the coherent reflection of plane waves in specular. However, specular reflection under the PE method cannot easily be checked and it has remained unclear why the method should preserve this invariance.

In this paper we obtain the coherent scattered field at low grazing angles, and examine the scattering of planes waves under the parabolic equation method. In particular, we describe the effect of boundary truncation in the method on plane waves; we obtain the corresponding coherent component and show that it is not specular. This is applied to a full incident field, and is used to show that the coherent field obeys the image invariance property. This is an important test of the method, in view of the nonspecular reflection of plane waves. The expressions obtained are valid for multiple scattering from slightly rough surfaces and depend upon the autocorrelation function of the surface and its derivative. At large horizontal distances from the origin, however, specular reflection is recovered. Effective reflection coefficients for plane waves are then given, and in several cases are expressed in closed form. This calculation also provides a convenient measure of the dependence of accuracy of the PE method upon incident angle.

The results are obtained as follows: The scattered field is found along a plane close to the surface by an expansion in terms of the vertical derivative of the field at the surface. This derivative is solved using the governing integral equation under certain approximations, and further manipulation yields tractable expressions which can easily be averaged.

I. PARABOLIC EQUATION METHOD

We consider the problem of a scalar time-harmonic wave field p scattered from a one-dimensional rough surface $h(x)$, with a pressure release boundary condition. (For electromagnetic waves this corresponds to s or TE polarization, and perfect conductivity.) The wave field propagates with wave number k, and will be assumed to be incident and scattered at small grazing angles with respect to the surface. The field is governed by the wave equation $(\nabla^2+k^2)p=0$. The coordinate axes are x and z, where x is the horizontal $x \geqslant 0$, and z the vertical, directed out of the medium. It will be assumed that the surface is statistically stationary to second order, i.e., its autocorrelation function is translationally invariant. The mean surface level is taken at $z=0$, so that $h(x)$ has mean zero. The autocorrelation function $\langle h(x)h(x+\xi)\rangle$ is denoted by $\rho(\xi)$, and we assume that $\rho(\xi)\to 0$ at large separations ξ. (The angled brackets denote the ensemble average.) Then $\sigma^2=\rho(0)$ is the variance of surface height, so that the surface is of order $O(\sigma)$. We will denote by L the characteristic correlation length of the surface.

Since the field propagates predominantly in one direction, we can define a slowly varying part ψ by

$$\psi(x,z)=p(x,z)\exp(-ikx).$$

Incident and scattered components ψ_i and ψ_s are defined similarly, so that $\psi=\psi_i+\psi_s$. We will sometimes denote by the ψ_s^0 the field which would be reflected from a flat surface, $\psi_s^0(x,z)=-\psi_i(x,-z)$. It may be assumed that $\psi_i(x,h(x))=0$ for $x\leqslant 0$, so that the area of surface illumination is restricted, as for example when the field is a directed Gaussian beam. The governing equations for the parabolic equation method[5,6] are then

$$\psi_i(\mathbf{r})=-\int_0^x G(\mathbf{r};\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial z}\,dx', \qquad (1)$$

where both $\mathbf{r}=(x,h(x))$, $\mathbf{r}'=(x',h(x'))$ lie on the surface; and

$$\psi_s(r)=\int_0^x G(\mathbf{r};\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial z}\,dx', \qquad (2)$$

where $\mathbf{r}'$ is again on the surface and $\mathbf{r}$ is now an arbitrary point in the medium. Here G is the parabolic form of the two-dimensional Green's function given by

$$G(x,z;x',z')\begin{cases} =\alpha\sqrt{\dfrac{1}{x-x'}}\exp\left[\dfrac{ik(z-z')^2}{2(x-x')}\right], & \text{for } x'<x,\\ =0, & \text{otherwise,}\end{cases} \qquad (3)$$

where $\alpha = 1/2\sqrt{i/2\pi k}$. [This form gives rise to the finite upper limit of integration in (1) and (2).] The Green's function is derived under the assumption of forward scattering, i.e., that the field obeys the parabolic wave equation,

$$\psi_x+2ik\psi_{zz}=0, \qquad (4)$$

which holds provided the angles of incidence and scattering are fairly small.

II. COHERENT FIELD AND IMAGE PROPERTY

In this section we will examine the parabolic equation method, first showing the characteristics of scattered plane waves, and then finding the coherent field for multiple scattering at grazing incidence. We first discuss the scattering of plane waves from a rough surface in general, and the specular description of the mean field due to an arbitrary source.

A. Specular reflection for arbitrary angles of incidence

In general, provided the rough surface is statistically stationary, the coherent scattered field due to an incident plane wave is again a plane wave. This is well known (e.g., DeSanto and Brown[1]), but it is useful to demonstrate it briefly here. The invariance which we require follows immediately from this.

Suppose that a plane wave $\exp(ik_xx+ik_zz)$ is incident with wave number k upon the statistically stationary rough surface $h(x)$, where $k_z=\sqrt{k^2-k_x^2}$. Denote the resulting scattered field by p_{scat}. We can choose $\langle h\rangle=0$.

Now, translation by ξ is equivalent to multiplication by $e^{ik_x\xi}$, that is,

$$\exp(ik_x[x+\xi]+ik_zz)=e^{ik_x\xi}\exp(ik_xx+ik_zz)$$

for all x and ξ. Then, since the equations are linear in the incident field, and the surface is stationary,

$$\langle p_{\text{scat}}(x+\xi,z)\rangle=e^{ik_x\xi}\langle p_{\text{scat}}(x,z)\rangle \qquad (5)$$

for z above the highest part of the surface. Since this holds for all x and ξ it follows that

$$\langle p_{\text{scat}}(x,z)\rangle=R'_{k_x}(z)e^{ik_xx}, \qquad (6)$$

where $R'_{k_x}(z)=\langle p_{\text{scat}}(0,z)\rangle$. Thus $\langle p_{\text{scat}}(x,z)\rangle$ is a plane wave as a function of x along any z plane. Now, away from the surface the coherent field is governed by the wave equation. Therefore

$$\langle p_{\text{scat}}(x,z)\rangle=R_{k_x}e^{ik_xx-ik_zz}, \qquad (7)$$

where R_{k_x} is the effective reflection coefficient, $R_{k_x} = R'_{k_x}(z_1)e^{ik_zz_1}$, and this is the required result. Thus, since any incident field can be expressed as a distribution

$$\psi_i=\int A(k_x)e^{ik_xx+ik_zz}\,dk_x$$

of such plane components, the coherent scattered field is equivalent to the field due to the modified "image"

$$\langle\psi_s\rangle=\int A(k_x)R_{k_x}e^{ik_xx-ik_zz}\,dk_x.$$

This leads immediately to the following, which we refer to as the *image property*: The coherent field remains unchanged if the source and observation point are displaced vertically by equal distances in opposite directions.

This property can be viewed as the result of the commutativity between the operators S_{z_1} and $L_{\pm\zeta}$ acting on functions of x, where $L_{\pm\zeta}$ is free-space propagation through a distance ζ between two horizontal planes (the sign being chosen according to the direction of propagation), and S_{z_1} is an averaged scattering operator:

$$S_{z_1}[\psi_i(x,z_1)]=\langle\psi_s(x,z_1)\rangle.$$

Here, S_{z_1} and L_ζ commute because, as is shown by (7), they have in common the eigenvectors $e^{i\lambda x}$. Therefore for a rough surface at $z=0$, and z_1 near the surface, we can write the mean scattered field at any z' in terms of the incident field $\psi_{\rm inc}(x)$ at z:

$$\langle\psi_{\rm scat}(x,z')\rangle = L_{z_1-z'}S_{z_1}L_{z_1-z}(\psi_{\rm inc}(x))$$
$$= L_{z_1-z'}L_{z_1-z}S_{z_1}(\psi_{\rm inc}(x))$$
$$= L_{2z_1-(z'+z)}S_{z_1}(\psi_{\rm inc}(x)).$$

This depends only on the sum $z'+z$, which shows the required invariance. Analogous results clearly hold for three-dimensional wave propagation, and for elastic and layered media with irregular interfaces.

Approximate values for the effective reflection coefficients R_{k_x} have been provided by various methods, for example, perturbation analysis,[1] the Kirchhoff approximation,[1,2] and the smoothing method.[3]

B. Coherent field at grazing incidence

We now examine the scattering of plane waves under the parabolic equations set out in Sec. I. The coherent component will be obtained, and from this will be found the coherent field due to an arbitrary forward-going source.

The significance of plane-wave scattering in the PE method is to some extent formal rather than physical, since plane waves violate the assumption of restricted surface illumination. However, the results yield insight into the effect of boundary truncation and provide a tractable route to the calculation of the full coherent field.

In order to proceed we find the field along a plane, as follows: We assume slight roughness so that there is a plane at z_1 close to every point on the surface $h(x)$. By expanding ψ_s to second order in (z_1-h) about $h(x)$, we write ψ_s in terms of functions including $d\psi/dz$. The plane wave contributions to $d\psi/dz$ are found by applying a nonlocal approximation to Eq. (1). Note that, since the expansion of ψ_s is in functions which themselves depend on h, the h dependence of each term is not restricted to its coefficient. The results will be obtained to second order in h, i.e., to $O(\sigma^2)$. Since z_1 is of order σ, we may neglect terms such as h^2z_1 and hz_1^2.

Consider the slowly varying part $\psi_i = pe^{-ikx}$ of an incident field p which obeys the parabolic equation (4). We may assume that this field can be written as a distribution of plane-wave components

$$\psi_i(x,z) = \int_0^{2\pi} A(\theta)\psi_i^\theta(x,z)\,d\theta, \tag{8a}$$

where

$$\psi_i^\theta(x,z) = e^{-ikx}\exp(ik[x\sin\theta + z\cos\theta]). \tag{8b}$$

Define $S=\sin\theta-1$. When $kh(x)\cos\theta$ is small we can write

$$\psi_i^\theta(x,h) = \exp\{ik[xS+h(x)\cos\theta]\}$$
$$\cong e^{ikxS}[1+ikh(x)\cos\theta-(k^2/2)h^2(x)\cos^2\theta]. \tag{9}$$

For fixed wave number k this holds at low grazing angles and for moderate surface roughness, but of course is not uniform in k. Along the surface $h(x)$ the component (9) has z derivative

$$\frac{\partial\psi_i^\theta(x,h)}{\partial z} = ik\cos\theta\,\psi_i^\theta(x,h)$$
$$\cong e^{ikxS}[ik\cos\theta - k^2h(x)\cos^2\theta - (ik^3/2)h^2(x)\cos^3\theta]. \tag{10}$$

Consider now the scattered field at z_1. Writing this as

$$\psi_s(x,z_1) = \psi_s(x,h+[z_1-h])$$

and expanding about the surface h we obtain to second order in (z_1-h)

$$\psi_s(x,z_1) \cong \psi_s(x,h) + [z_1-h]\frac{\partial\psi_s(x,h)}{\partial z} + \frac{1}{2}[z_1-h]^2\frac{\partial^2\psi_s(x,h)}{\partial z^2}. \tag{11}$$

We will restrict attention to $x \gg L$, where L is the surface correlation length. This is not a significant limitation since, in this regime, ψ_s is negligible for small x. Now, the boundary condition gives $\psi_s(x,h) = -\psi_i(x,h)$. Also the zeroth-order (i.e., flat surface) solution for the scattered field is $\psi_s^0(x,z) = -\psi_i(x,-z)$. From this the second-order term in (11) may be written

$$\frac{\partial^2\psi_s(x,h)}{\partial z^2} = -\frac{\partial^2\psi_i(x,0)}{\partial z^2} + O(\sigma) \tag{12}$$

and the quadratic factor $(z_1-h)^2$ allows the $O(\sigma)$ term to be neglected:

$$\frac{\partial\psi_s(x,h)}{\partial z} = \frac{\partial\psi(x,h)}{\partial z} - \frac{\partial\psi_i(x,h)}{\partial z}. \tag{13}$$

The term $\partial\psi(x,z)/\partial z$ in (13) can be written as the solution of Eq. (1) in Sec. I. It was found in Ref. 12 that for moderately rough surfaces this is reasonably well approximated if G in the integrand is replaced by its flat surface form $G(x,z;x',0) = \alpha/\sqrt{x-x'}$. This introduces an error of order $O(\sigma^2)$, which may again be neglected because it occurs in (11) with an additional factor (z_1-h). Thus

$$\psi_i(x,h) \cong -\int_0^x \frac{\alpha}{\sqrt{x-x'}}\frac{\partial\psi(x',h)}{\partial z}\,dx'. \tag{14}$$

This form is known as a generalized Abel's equation, and has the solution (e.g., Ref. 13)

$$\frac{\partial\psi(x,h)}{\partial z} = -\frac{1}{\pi}\frac{d}{dx}\left[\int_0^x \frac{\psi_i(x',h(x'))}{\alpha\sqrt{x-x'}}\,dx'\right]. \tag{15}$$

Using (12), (13), and (15), we can rewrite (11) as

$$\psi_s(x,z_1) \cong -\psi_i(x,h) - (z_1,h) \times\left[\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\psi_i(x',h(x'))}{\alpha\sqrt{x-x'}}\,dx' + \frac{\partial\psi_i(x,h)}{\partial z}\right] - \frac{(z_1-h)^2}{2}\frac{\partial^2\psi_i(x,0)}{\partial z^2}. \quad (16)$$

We now consider the restriction of the right-hand side of Eq. (16) to the plane-wave components of ψ_i. Accordingly, define

$$\psi_s^\theta(x,z_1) \equiv -\psi_i^\theta(x,h) - (z_1-h) \times\left[\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\psi_i^\theta(x',h(x'))}{\alpha\sqrt{x-x'}}\,dx' + \frac{\partial\psi_i^\theta(x,h)}{\partial z}\right] - \frac{(z_1-h)^2}{2}\frac{\partial^2\psi_i^\theta(x,0)}{\partial z^2}. \quad (17)$$

By (9) the integral term in (17) can be written

$$\frac{d}{dx}\int_0^x \frac{\psi_i^\theta(x',h(x'))}{\alpha\sqrt{x-x'}}\,dx' \cong D_\theta(x) + I_2 + \frac{dI}{dx}, \quad (18)$$

where

$$D_\theta(x) = \frac{d}{dx}\int_0^x \frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}\,dx', \quad (19)$$

$$I_2(x) = -\frac{d}{dx}\int_0^x k^2h^2(x')\cos^2\theta\,\frac{e^{ikSx'}}{2\alpha\sqrt{x-x'}}\,dx', \quad (20)$$

and

$$I(x) = \int_0^x ikh(x')\cos\theta\,\frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}\,dx'. \quad (21)$$

Since I_2 is $O(\sigma^2)$ and occurs in (17) with a factor $O(\sigma)$, we may neglect it. Note that I is linear in h. Now the deterministic (i.e., surface independent) term D_θ can be expressed in the form

$$D_\theta(x) = \frac{d}{dx}\left[\frac{\sqrt{2\pi}}{\alpha\sqrt{-kS}}\{C_1(-kSx) + iS_1(-kSx)\}e^{ikSx}\right], \quad (22)$$

where C_1, S_1 are the Fresnel integrals[14]

$$C_1(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{\cos t}{\sqrt{t}}\,dt, \quad S_1(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{\sin t}{\sqrt{t}}\,dt.$$

For the moment, we will retain the notation D_θ. So by (9), (10), and (18) the expression (17) becomes

$$\psi_s^\theta(x,z_1) = -\left(1 + ikh(x)\cos\theta - \frac{k^2}{2}\cos^2\theta \times[h^2(x) + (z_1-h)^2]\right)e^{ikxS} - (z_1-h)\left[\frac{D_\theta(x)}{\pi} + \frac{1}{\pi}\frac{dI}{dx} + \left(ik\cos\theta - k^2h(x)\cos^2\theta - i\frac{k^3}{2}h^2(x)\cos^3\theta\right)e^{ikxS}\right]. \quad (23)$$

We must now find the mean of ψ_s^θ. On averaging, all terms which are linear in h vanish, and (23) gives

$$\langle\psi_s^\theta(x,z_1)\rangle = -\frac{z_1}{\pi}D_\theta(x) - \left(1 + ikz_1\cos\theta - \frac{k^2}{2}z_1^2\cos^2\theta\right)e^{ikSx} + \frac{1}{\pi}\left\langle h(x)\frac{dI}{dx}\right\rangle, \quad (24)$$

where σ^2 is the variance of the surface height. Here we have neglected an $O(\sigma^3)$ term $ik^3z_1\sigma^2\cos^3\theta/2$, which is also of third order in the grazing angle. Three other terms in σ^2 have cancelled.

It remains to evaluate the term involving the integral I. By the chain rule we can write

$$\left\langle h\frac{dI}{dx}\right\rangle = \left\langle\frac{d(hI)}{dx}\right\rangle - \left\langle I\frac{dh}{dx}\right\rangle. \quad (25)$$

Consider first the mean of $d(hI)/dx$. The average can be taken under the integral sign in I, so from (21) this term becomes

$$\left\langle\frac{d(hI)}{dx}\right\rangle = \frac{d}{dx}\left[\int_0^x \frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}\,ik\rho(x-x')\cos\theta\,dx'\right],$$

where $\rho(x-x')$ is the autocorrelation function $\langle h(x)h(x')\rangle$. We can put $\xi = x-x'$ and write this as

$$\left\langle\frac{d(hI)}{dx}\right\rangle = \frac{d}{dx}\left[\frac{e^{ikxS}}{\alpha}\int_0^x \frac{e^{-ikS\xi}}{\sqrt{\xi}}\,ik\rho(\xi)\cos\theta\,d\xi\right].$$

Since $x \gg L$ and the autocorrelation function falls to zero at large separations, we can replace the upper limit of integration by ∞, so that the integral becomes independent of x and we obtain

$$\left\langle\frac{d(hI)}{dx}\right\rangle = -\frac{k^2S}{\alpha}\cos\theta\,e^{ikxS}\int_0^\infty \frac{e^{-ikS\xi}}{\sqrt{\xi}}\,\rho(\xi)\,d\xi. \quad (26)$$

Consider finally the remaining term $\langle I\,dh/dx\rangle$ in Eq. (25). The derivative dh/dx can be taken inside the integral, so that

$$\left\langle I\frac{dh}{dx}\right\rangle = \int_0^x \frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}\,ik\left\langle h(x')\frac{dh(x)}{dx}\right\rangle\cos\theta\,dx'. \quad (27)$$

Since the correlation function ρ is stationary, we can write (see Papoulis;[15] p. 317)

$$\left\langle h(x')\frac{dh(x)}{dx}\right\rangle=\frac{d\rho(\xi)}{d\xi}, \tag{28}$$

where $\xi=x-x'$. So, reasoning as before, (27) becomes

$$\left\langle I\frac{dh}{dx}\right\rangle=\frac{ik}{\alpha}\cos\theta\, e^{ikSx}\int_0^\infty \frac{e^{-ikS\xi}}{\sqrt{\xi}}\frac{d\rho(\xi)}{d\xi}\,d\xi. \tag{29}$$

Collecting terms, and using (25), (26), and (29), Eq. (24) can now be written

$$\langle\psi_s^\theta(x,z_1)\rangle\cong-\frac{z_1}{\pi}D_\theta(x)-\left(1+ikz_1\cos\theta-\frac{k^2}{2}z_1^2\cos^2\theta\right)e^{ikSx}-T_\theta\, e^{ikSx}, \tag{30}$$

where the coefficient T_θ is given by

$$T_\theta=\frac{k^2}{\pi\alpha}\cos\theta\left\{\int_0^\infty \frac{e^{-ikS\xi}}{\sqrt{\xi}}\left[\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}\right]d\xi\right\}. \tag{31}$$

The integral T_θ is calculated in closed form for certain cases below. It is convenient to reverse the approximation (9) and write (30) as

$$\langle\psi_s^\theta(x,z_1)\rangle\cong-\frac{z_1}{\pi}D_\theta(x)-e^{ik(Sx+z_1\cos\theta)}-T_\theta\, e^{ikSx}. \tag{32}$$

This is the first of the results required; it represents the mean field near the surface due to a plane incident wave *under the parabolic equation method*, using approximations (9)–(12) and (14) and is valid for $x\gg L$.

We obtain reflection coefficients for forward-scattering at grazing incidence by letting $z_1\to 0$ (see remarks later) or simply to write the deterministic part by appealing to the exact solution for a flat surface:

$$\langle\psi_s^\theta(x,z_1)\rangle\cong-e^{ikSx}(1+T_\theta). \tag{33}$$

Suppose now that ψ_i is an incident field written as in (8a). From (33)

$$\langle\psi_s(x,z)\rangle\cong-\psi_i(x,-z)-\int_0^{2\pi}A(\theta)T_\theta\, e^{ik(Sx-z\cos\theta)}. \tag{34}$$

As mentioned above, although this is restricted to $x\gg L$, it is not a significant limitation because we can assume ψ_s to be negligible at smaller values of x. The Appendix gives T_θ in closed form for three cases: Surfaces with Gaussian, exponential (fractal), and modified exponential (subfractal) autocorrelation functions.

C. Spatial and asymptotic dependence

Although we have discarded certain terms in (32), the expression gives an opportunity to examine the angular dependence of the accuracy of the PE method. It can be seen that the coherent field (32) is nonplanar as a function of x, but having neglected the term in $z_1\sigma^2\cos^3\theta$ the departure from plane-wave behavior is contained completely in the deterministic part D_θ [see (22)]. Consider a flat surface, so that $T_\theta=0$ in (32). For sufficiently large kxS the coefficients C_1,S_1 approach 1/2, D_θ approaches a plane wave, and specular reflection is recovered:

$$\frac{z_1}{\pi}D_\theta(x)\sim\frac{z_1}{2\pi}\frac{\sqrt{2\pi kS}}{\alpha}(1+i)e^{ikSx}$$
$$=-2ikz_1\sqrt{-2S}e^{ikSx}. \tag{35}$$

(Some care is needed in choosing the sign of the square root correctly.) This is an asymptotic limit for large x, but is not uniform in angle in incidence. On the other hand, for low grazing angle $\pi/2-\theta$, the coefficients C_1,S_1 vary slowly with respect to x so again D_θ is plane wave e^{ikSx} modified by a slowly varying envelope, and in fact vanishes in the limit. Now, in the large x asymptotic limit (35), Eq. (32) remains inaccurate for a flat surface: The exact solution at z_1 is $e^{ik(xS-z_1\cos\theta)}$. However from (32) and (35) the reflection at z_1 is given as

$$\langle\psi_s^\theta(x,z_1)\rangle\cong-e^{ikSx}[1-ikz_1(2\sqrt{-2S}-\cos\theta)]$$
$$\cong-\exp[ikSx-ikz_1(2\sqrt{2-2\sin\theta}-\cos\theta)]. \tag{36}$$

This implies a phase error of

$$2kz(\cos\theta-\sqrt{2-2\sin\theta}), \tag{37}$$

which, to second order in the grazing angle $\theta'=\pi/2-\theta$, becomes

$$(kz/4)\theta'^3+O(\theta'^4). \tag{38}$$

This error arises in applying the integral Eq. (1) to relate the surface derivative $d\psi/dz$ and the incident field, and is therefore inherent in the PE method.

We also wish to show that the PE method obeys the image property. As we have discussed in Sec. II A, this follows immediately if the mean reflection due to plane waves is specular. The preceding comments make clear that this holds at low grazing angles, and at all other angles in the limit of large x. This is the case despite the above phase error (38).

Some remarks should be made concerning the relationship between the component $\psi_s^\theta(x,z)$ at $z=z_1$ and at $z=0$. First, we were able to set z_1 to zero in (33) because the only z_1-dependent terms remaining after averaging must be negligible or deterministic, and in the latter case this step is clearly valid. Thus the coherent field could have been obtained by setting $z_1=0$ from the start. However, for z_1 chosen above all points on the surface the (unaveraged) scattered field has a straightforward physical interpretation.

Second, it may at first seem inconsistent with the wave equation that a nonplane component z_1D_θ appears only for $z_1\neq 0$. Since this part is purely deterministic, we can suppose here that the surface is flat. Recall that D_θ (22) is the θ component of the exact solution (15) for the derivative $d\psi/dz$ due to the decomposition of ψ_i. The term z_1D_θ occurs in expanding the field at z_1 (11) about $h\equiv 0$. Thus the fields at $z=z_1$ and at $z=0$ are consistent with the parabolic wave equation, the additional x dependence arising purely from the boundary truncation at $x=0$. (As discussed earlier the individual θ components of the solution are to be interpreted formally rather than physically in the region of small x.) When the plane wave components are recombined, from (15) and (19) we get

$$\int_0^{2\pi} A(\theta)\,D_\theta\,d\theta=-\pi\,\frac{\partial\psi_0(x,0)}{\partial z}=-2\pi\,\frac{\partial\psi_i(x,0)}{\partial z}.$$

Thus for a flat surface we obtain from (32):

$$\psi_s(x,z_1)\cong 2z_1\,\frac{\partial\psi_i(x,0)}{\partial z}-\psi_i(x,z_1)$$

$$=-\psi_i(x,-z_1)+O(z_1^2),$$

where the second equation can be seen by taking Taylor expansions of ψ_i and $d\psi_i/dz$. This is correct to first order in z_1, so that the additional spatial dependence in the components D_θ does not cause inconsistencies in the reflected field.

III. SUMMARY

We have considered scattering at grazing incidence from rough surfaces under the parabolic equation method. The coherent component of the field for slightly rough surfaces has been found (34), and expressed in terms of the autocorrelation function of the surface. Effective reflection coefficients (33) have been given for scattering due to incident plane waves. We have examined the effect of truncating the boundary integral upon plane waves; the field at a near-surface plane was obtained (32). Plane-wave reflection is not specular, but at low grazing angles or in the limit of large x it becomes specular. However a phase error remains (38) even in the large x limit. Nevertheless, this is sufficient to verify preservation of the image property. Extension of the results to the higher moments, from Eq. (23), should be straightforward.

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APPENDIX: EXPLICIT EVALUATION FOR SPECIFIC SURFACES

We return now to the effective coefficients (33). The integral in coefficient T_θ [Eq. (31)] is a Laplace transform. In several cases of interest we can express this integral in closed form, or in terms of standard functions. We give three examples. In each case the parameter L determines the correlation length of the irregular surface. The first two functions are characteristic of surface features which are jagged compared with the third, the Gaussian autocorrelation function.

(1) Consider the "fractal" autocorrelation function

$$\rho(\xi)=\sigma^2\exp(-\xi/L).$$

Then

$$\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}=\left(S+\frac{1}{ikL}\right)\sigma^2\exp\left(\frac{-\xi}{L}\right).$$

The coefficient T_θ takes the value[14]

$$T_\theta=\frac{k^2}{\pi\alpha}\cos\theta\int_0^\infty\sigma^2\left(S+\frac{1}{ikL}\right)\frac{e^{-\xi(ikS+1/L)}}{\sqrt{\xi}}\,d\xi$$

$$=\frac{k^2}{\sqrt{\pi}\alpha}\,\sigma^2\cos\theta\,\frac{S+1/ikL}{\sqrt{ikS+1/L}}. \qquad \text{(A1)}$$

(2) The "subfractal" autocorrelation function is

$$\rho(\xi)=\sigma^2(1+\xi/L)\exp(-\xi/L).$$

In this case

$$\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}=\sigma^2\exp\left(\frac{-\xi}{L}\right)\left[S+\xi\left(\frac{S}{L}+\frac{1}{ikL^2}\right)\right].$$

The coefficient is given by[14]

$$T_\theta=\frac{k^2}{\sqrt{\pi}\alpha}\,\sigma^2\cos\theta\left[\frac{S+1/2ikL}{\sqrt{ikS+1/L}}\right]. \qquad \text{(A2)}$$

(3) Finally, consider a Gaussian autocorrelation function,

$$\rho(\xi)=\sigma^2\exp(-\xi^2/L^2).$$

We can again find the integral exactly, with coefficients in terms of standard functions. Here,

$$\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}=\sigma^2\exp\left(\frac{-\xi^2}{L^2}\right)\left[S+\frac{2\xi}{ikL^2}\right].$$

The integral in T_θ is the sum of two Laplace transforms, and can be written (see p. 146 of Ref. 16)

$$T_\theta=(k^2/\pi\alpha)\,\sigma^2\cos\theta\exp(-(kLS)^2/8)$$

$$\times\left\{\frac{LS}{2}\,(ikS)^{1/2}K_{1/4}\left[\frac{(kLS)^2}{8}\right]-\frac{2}{ikL^2}\left(\frac{L}{\sqrt{2}}\right)^{3/2}\Gamma\left(\frac{3}{2}\right)D_{-(3/2)}\left(\frac{ikSL}{\sqrt{2}}\right)\right\}. \qquad \text{(A3)}$$

Here, Γ is the Gamma function, K is the modified Bessel function of the third kind, and D is the parabolic cylinder function.

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