Wave scattering in a rough elastic layer adjoining a fluid half-space

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Wave propagation and scattering are considered in a medium consisting of a slightly rough elastic layer adjoining a fluid half-space. The solution is obtained for the mean wave potentials in both media, due to multiple scattering within and reradiation from the layer. This is found by deriving effective transmission and reflection coefficients for each irregular boundary, and then showing that the problem is equivalent to the deterministic one for a plane-sided layer with these effective coefficients. The solution exhibits the dependence upon the variance and correlation length of each boundary explicitly.

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INTRODUCTION

Wave propagation in layered elastic/fluid media is a feature of many problems (see for example Brekhovskikh[1]), ranging from ice-covered oceans and seismology to nondestructive testing. In many applications the boundaries are to some extent irregular (Takenaka *et al.*[2]). However, in most theoretical treatments this is neglected, largely because of the intrinsic complications of multiple reflection and scattering. A natural approach to the problem is to treat the irregular surfaces as stochastic, and to seek the statistics of the scattered field. This is a common approach to half-space problems, in which it is frequently of interest to find the mean or coherent component of the scattered field (e.g., Dacol[3] and Bass and Fuks[4]). Such results are almost always restricted to slight roughness, with some notable exceptions which treat the opposite extreme (Talbot *et al.*[5]).

In this paper we obtain the mean field for wave scattering in an elastic fluid-loaded layer, in which one or both boundaries of the layer are slightly rough. The solution exhibits in a simple way the approximate dependence on the statistical characteristics of both surfaces. Furthermore, the method can in principle be extended to arbitrarily rough boundaries. The method is as follows: consider first the scattering of plane waves incident at a rough interface. The resulting mean field must obey a generalization of Snell's law, so that the average transmitted shear component, for example, is a plane wave propagating at the same angle as for a plane interface but with a modified transmission coefficient (see DeSanto and Brown,[6] Spivack[7]). These coefficients depend on the statistics of the irregular interface. Now for an irregular layer the total field may be treated as an infinite sum of components due to multiple reflection within and reradiation from the layer. Provided the layer depth is large compared with both the wavelength and the scale size of the surface, we can make the assumption that successive scatterings are independent, i.e., that after each interaction with a surface $H(x)$ and double propagation across the layer the features of the diffracted field are approximately uncorrelated with $H(x)$ itself. We then show that the mean total field is formally equivalent to that for a plane-sided layer in which the reflection and transmission coefficients for each boundary are replaced by the above modified ones. The solution we apply for the plane-layer problem follows the formulation of Sheard and Uscinski.[8] (See also Deschamps and Chengwei[9] who similarly treat an immersed layer.) Thus the problem is reduced to finding these coefficients; this is done to second order in surface height by finding the field near to the surface using a tangent plane approximation. The solution is then complete; it is straightforward to extend this to a rough elastic layer immersed in a fluid, to multilayered media, to a model in three dimensions and so on.

The tangent plane approximation developed here imposes fairly strong restrictions on the surface roughness, but is used because it leads to tractable results which exhibit the main features of the statistical dependence. The main limitations of the solution are associated with this approximation: the coefficients have cusps at certain angles associated with poles and branch points, which may give rise to surface waves. Near such cusps each coefficient varies rapidly with angle of incidence, and so the approximation becomes restricted to much smaller roughness. In addition the local scattering assumption is violated by surface modes. These problems, however, may effectively be removed by taking absorption into account. This is described in more detail below, where the solution is examined for parameters pertaining to an ice/ocean medium.

The plan of the paper is as follows: The preliminary equations are given in Sec. I. In Sec. II the effective coefficients are derived, the system is solved, and the limitations and their resolution are explained.

I. PRELIMINARIES AND BOUNDARY CONDITIONS

We consider wave propagation in a two-dimensional system, consisting of an elastic layer with irregular boundaries $H_1(x)$, $H_2(x)$ adjoining a fluid half-space. It will be

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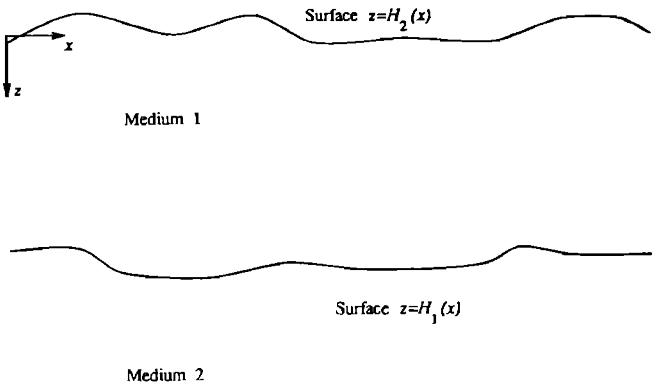

FIG. 1. Schematic view of the scattering geometry. Medium 1 denotes the elastic layer, and Medium 2 the fluid half-space. A plane wave is incident from the fluid. The x and z axes are horizontal and vertical, respectively, and H_1, H_2 denote the irregular upper and lower boundaries of the layer.

assumed that these rough boundaries are drawn from some ensemble obeying given statistics. In particular we write $H_i(x)=z_i+h_i(x)$ where we assume that h_1, h_2 are stationary to second order. Thus $\langle h_i(x)\rangle$ is zero and $\langle h_i(x)h_i(x+\xi)\rangle$ is a function of ξ only. (Here and elsewhere the angled brackets denote the ensemble average.) The axes are, respectively, parallel and perpendicular to the mean surfaces (see Fig. 1). The propagation of elastic waves in the two media are modeled by potentials ϕ_w in the fluid, and ϕ_i and ψ_i in the solid (e.g., Ewing[10]). (The subscripts here denote "water" and "ice" corresponding to the envisaged application.) Displacement and stress are related to derivatives of these potentials; the boundary conditions are given by the (local) continuity of stress and normal displacement across each interface. These give equations relating the derivatives of the potentials at the surface.

Consider for the moment solid and fluid half-spaces separated by a flat boundary. A plane acoustic wave impinging on the surface of the solid from the fluid (see Fig. 2) leads to a reflected wave in the fluid, and two transmitted waves (transverse and longitudinal) in the solid. The relative amplitudes and phases of these waves may be calculated by applying the boundary conditions. This gives rise to the various well-known reflection and transmission coefficients. Those required for the layer are listed in Appendix A; they are included for completeness since we need to describe the modification of these coefficients by surface irregularity.

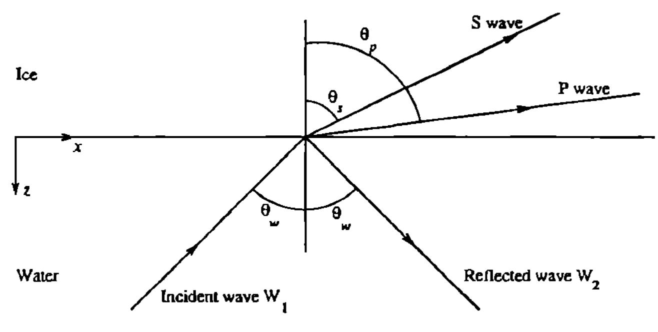

FIG. 2. Plane wave incident at angle θ_w on a flat interface, and the resulting reflected wave and two transmitted components.

We denote by θ_w, θ_p, and θ_s the angles between the normal to the surface and the wave vector of the longitudinal wave in the fluid, the longitudinal wave in the solid and the transverse wave in the solid, respectively. These are related by Snell's law for a flat surface

$$k_w \sin \theta_w = k_s \sin \theta_s = k_p \sin \theta_p, \tag{1}$$

where k_w, k_p, and k_s are the corresponding wave numbers.[1]

It is necessary to restate here the generalization of Snell's law which holds for an irregular boundary. This is well known (at any rate for perfectly conducting surfaces—see DeSanto and Brown,[6] Spivack[7]) and can be stated as follows: Given a plane wave incident on a stationary rough surface, the average of all resulting waves leaving the boundary are plane waves with wave numbers which obey Snell's law [Eq. (1)].

Briefly this can be shown in the following way: an incident plane wave is invariant in x in the sense that translation by ξ is equivalent to multiplication by $e^{ik_x\xi}$, where k_x is the x component of the wave number. For an irregular statistically stationary boundary, although waves are scattered into a spectrum of directions, the mean scattered field of each type obeys the same invariance by linearity of the governing equations. Thus the coherent field is composed of plane waves as for a flat boundary, with modified coefficients determined by the surface statistics. These will be referred to as *effective coefficients*.

Returning now to the irregular layer, the depth δ of the layer is defined as

$$\delta = z_2 - z_1. \tag{2}$$

We will require δ to be large for the validity of the assumption of independent scatter at successive interactions with a surface. This is quantified in Sec. II below.

II. SOLUTION

In this section we derive the main results. We will first examine the scattered field, along a plane near each rough surface. To do so we use a tangent plane approximation evaluated to second order in surface height. The expressions obtained in this way allow us to derive effective transmission and reflection coefficients for each boundary. Finally we apply these, using the summation method (see Refs. 8 and 9), to the deterministic problem of a plane-sided layer, and show that this problem is equivalent to that of finding the average over a statistical ensemble of irregular layers. This completes the solution for the mean field.

A. Scattered field near the surface

We derive an approximation to the (unaveraged) transmitted field just beyond a rough surface, due to a plane wave incident from the fluid. This determines the effective transmission coefficient. The calculations for the remaining wave modes and coefficients are straightforward and similar, and the corresponding results will be summarized in Appendix A.

The incident plane wave W_1 in the region $z>h_1(x)$ is given by

$$\phi_{\text{inc}}(x,z) = A e^{ik_w(\sin \theta_w x - \cos \theta_w z)}. \tag{3}$$

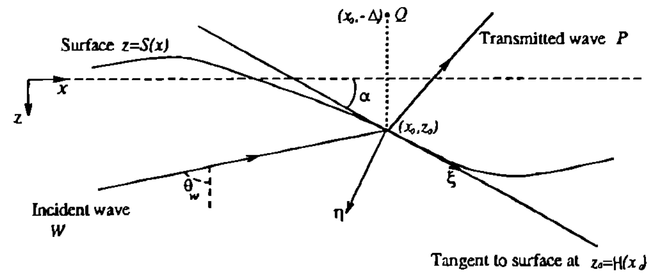

FIG. 3. Showing the first two sets of waves for the plane-sided layer.

Since the effective coefficients are independent of the mean surface level z_1, it is assumed in this derivation for simplicity that $z_1=0$.

The fluid/solid interface is written in terms of a size parameter ϵ_1 and an unscaled surface $\hbar_1(x)$; for brevity in this derivation we drop the subscript. Then

$$h(x)=\epsilon\hbar(x) \quad \text{and} \quad h'(x)=\epsilon\hbar'(x), \tag{4}$$

where the prime denotes differentiation with respect to x. The angle α between the tangent to the surface at (x_0,z_0) and the horizontal (see Fig. 3) is therefore given by

$$\tan\alpha=\epsilon\hbar'(x). \tag{5}$$

The surface is treated as being flat in the neighborhood of each point (x_0,z_0). This leads to a tangent plane approximation for the transmitted field in the vicinity of (x_0,z_0). We calculate the field at point Q with coordinates $(x_0,-\Delta)$.

For convenience we define a local coordinate system (ξ,η) aligned with the tangent and the normal at each point (x_0,z_0) on the surface (see Fig. 3). This is related to (x,z) by

$$\begin{aligned} x&=x_0+\xi\cos\alpha-\eta\sin\alpha,\\ z&=z_0+\xi\sin\alpha+\eta\cos\alpha, \end{aligned} \tag{6}$$

which thus defines η and ξ

$$\begin{aligned} \xi&=(x-x_0)\cos\alpha+(z-z_0)\sin\alpha,\\ \eta&=-(x-x_0)\sin\alpha+(z-z_0)\cos\alpha. \end{aligned} \tag{7}$$

With respect to these coordinates, the point Q is given by

$$\begin{aligned} \xi&=-(\Delta+z_0)\sin\alpha,\\ \eta&=-(\Delta+z_0)\cos\alpha, \end{aligned} \tag{8}$$

and by (6) the incident field ϕ_{inc} due to the wave W_1 [Eq. (3)] can be written as

$$\phi_{\text{inc}}(\xi,\eta) =A e^{ik_w\sin(\theta_w-\alpha)\xi}e^{-ik_w\cos(\theta_w-\alpha)\eta}e^{ik_w(\sin\theta_w x_0-\cos\theta_w z_0)}. \tag{9}$$

The solution is now formulated locally in the same way as for a horizontal boundary. Assuming the surface to be approximately flat near (x_0,z_0), Snell's law gives

$$k_w\sin\theta'_w=k_s\sin\theta'_s=k_p\sin\theta'_p, \tag{10}$$

where $\theta'_w=\theta_w-\alpha$ is the angle to the normal η at (x_0,z_0) of the incoming plane wave and θ'_p and θ'_s are are the angles to the wave normal of the transmitted P and S waves, respectively. In general, of course $\theta'_p\neq\theta_p-\alpha$ where θ_p is the angle to the normal of the transmitted plane wave for a horizontal flat surface. The angle θ_p is given by Snell's law for a horizontal surface, Eq. (1). The relationship between $\theta'_p(\theta'_s)$ and $\theta_p(\theta_s)$ will be derived later.

Applying Snell's law and the fluid/elastic P-wave transmission coefficient $T_p(\theta_w)$ to Eq. (3) gives the P field in the ice near (x_0,z_0)

$$\phi_1(\xi,\eta)=Ae^{ik_w(\sin\theta_w x_0-\cos\theta_w z_0)}e^{ik_p\sin\theta'_p\xi} \times e^{-ik_p\cos\theta'_p\eta}T_p(\theta_w-\alpha). \tag{11}$$

This is assumed to be valid at the point Q. Using Eq. (8) this gives

$$\phi_1(x_0,-\Delta)=Ae^{ik_w(\sin\theta_w x_0-\cos\theta_w z_0)}e^{-ik_w\sin\theta'_w\sin\alpha(\Delta+z_0)} \times e^{ik_p\cos\theta'_p\cos\alpha(\Delta+z_0)}T_p(\theta'_w), \tag{12}$$

where $k_p\sin\theta'_p$ has been replaced by $k_w\sin\theta'_w$ using Eq. (10).

Now, at Q the field ϕ_1° at Q for a flat surface at $z=0$ due to the same incident field [Eq. (3)] may be written as

$$\phi_1^\circ(x_0,-\Delta)=Ae^{ik_w\sin\theta_w x_0}e^{ik_p\cos\theta_p\Delta}T_p(\theta_w). \tag{13}$$

Our aim is to express the field ϕ_1 (12) in terms of the flat surface field ϕ_1° given by Eq. (13), in order to find an effective transmission coefficient T_p^r.

We must now expand the unknown terms in Eq. (12) in the small parameter ϵ. Only terms up to second order in ϵ will be needed explicitly. First, α can be written:

$$\alpha=\arctan(\epsilon\hbar')=\epsilon\hbar'+O(\epsilon^3). \tag{14}$$

Using Eq. (14), $\sin\theta'_w$ may be expanded about $\sin\theta_w$ to give

$$\sin(\theta_w-\alpha)=\sin\theta_w-\epsilon\hbar'\cos\theta_w-\epsilon^2\frac{\hbar'^2\sin\theta_w}{2} +O(\epsilon^3). \tag{15}$$

The expansion of the transmission coefficient T_p at $\theta'_w=\theta_w-\alpha$ is straightforward:

$$T_p(\theta'_w)=T_p(\theta_w)-\epsilon\hbar' T'_p(\theta_w)+\epsilon^2\frac{\hbar'^2}{2}T''_p(\theta_w) +O(\epsilon^3), \tag{16}$$

where $T'_p(\theta_w)$ is the first derivative of T_p with respect to its angular argument evaluated at $\theta=\theta_w$.

The evaluation of $\cos\theta'_p$ is slightly longer. Using Eq. (10),

$$\cos\theta'_p=\sqrt{1-\sin^2\theta'_p}=\left(1-\frac{k_w^2}{k_p^2}\sin^2\theta'_w\right)^{1/2} \tag{17}$$

into which we substitute Eq. (15) to obtain

$$\cos\theta_p'=\left(1-\frac{k_w^2}{k_p^2}\sin\theta_w^2+\epsilon\frac{k_w^2}{k_p^2}2\hbar'\sin\theta_w\cos\theta_w -\epsilon^2\frac{k_w^2}{k_p^2}(\cos^2\theta_w-\sin^2\theta_w)\hbar'^2+O(\epsilon^3)\right)^{1/2}. \tag{18}$$

Since $\cos^2\theta_p=1-(k_w^2/k_p^2)\sin\theta_w^2$ this simplifies and expands to

$$\cos\theta_p'=\cos\theta_p\left[1+\epsilon\frac{\hbar'}{2}\frac{k_w^2}{k_p^2}\frac{\sin 2\theta_w}{\cos^2\theta_p} -\epsilon^2\frac{\hbar'^2}{2\cos^2\theta_p}\frac{k_w^2}{k_p^2}\left(\cos 2\theta_w +\frac{1}{8}\frac{k_w^2}{k_p^2}\frac{\sin^2 2\theta_w}{\cos^2\theta_p}\right)+O(\epsilon^3)\right]. \tag{19}$$

For convenience this will be written

$$\cos\theta_p'=\cos\theta_p+\epsilon\gamma_1\hbar'+\epsilon^2\gamma_2\hbar'^2+O(\epsilon^3), \tag{20}$$

where γ_1,γ_2 are the coefficients in (19) of ϵ,ϵ^2, respectively. The exponentials in Eq. (12) may now be expanded. Consider first the term

$$\exp\{ik_p\cos\theta_p'\cos\alpha(\Delta+z_0)\}. \tag{21}$$

Using Eqs. (20) and (14) the exponent may be written

$$ik_p[\cos\theta_p+\epsilon\gamma_1\hbar'+\epsilon^2\gamma_2\hbar'^2+O(\epsilon^3)] \times\left(1-\epsilon^2\frac{\hbar'^2}{2}+O(\epsilon^4)\right)(\Delta+z_0). \tag{22}$$

The exponential in (21) thus becomes

$$\exp\{ik_p\cos\theta_p'\cos\alpha(\Delta+z_0)\} =\exp\left\{ik_p\left[\cos\theta_p\Delta+\epsilon(\cos\theta_p\hbar+\gamma_1\hbar'\Delta) +\epsilon^2\left(\gamma_2\hbar'^2\Delta-\frac{\cos\theta_p\hbar'^2}{2}+\gamma_1\hbar'\hbar\right)\right]+O(\epsilon^3)\right\}, \tag{23}$$

which is

$$\exp\{ik_p\cos\theta_p'\cos\alpha(\Delta+z_0)\} =\exp\{ik_p\cos\theta_p\Delta\}\exp\left\{ik_p\left[\epsilon(\cos\theta_p\hbar+\gamma_1\hbar'\Delta) +\epsilon^2\left(\gamma_2\hbar'^2\Delta-\frac{\cos\theta_p\hbar'^2}{2}+\gamma_1\hbar'\hbar\right)+O(\epsilon^3)\right]\right\}. \tag{24}$$

Expanding the second exponential on the right, we finally obtain

$$e^{ik_p\cos\theta_p'\cos\alpha(\Delta+z_0)}=e^{ik_p\cos\theta_p\Delta}[1+\epsilon\delta_1+\epsilon^2\delta_2+O(\epsilon^3)], \tag{25}$$

where δ_1 and δ_2 are given by

$$\delta_1=ik_p(\hbar\cos\theta_p+\gamma_1\hbar'\Delta),$$
$$\delta_2=ik_p\left(\hbar'^2\gamma_2\Delta+\gamma_1\hbar'\hbar-\frac{\Delta\cos\theta_p\hbar'^2}{2}\right) -k_p^2(\hbar^2\cos^2\theta_p+\gamma_1^2\hbar'^2\Delta^2+2\hbar\hbar'\gamma_1\cos\theta_p\Delta). \tag{26}$$

Similarly the second exponential in (12) can be written

$$\exp\{-ik_w\sin\theta_w'\sin\alpha(\Delta+z_0)\} =\exp\left\{-ik_w\left(\sin\theta_w-\epsilon\cos\theta_w\hbar'-\epsilon^2\frac{\hbar'^2\sin\theta_w}{2} +O(\epsilon^3)\right)[\epsilon\hbar'+O(\epsilon^3)](\Delta+z_0)\right\} \tag{27}$$

and we eventually obtain

$$e^{-ik_w\sin\theta_w'\sin\alpha(\Delta+z_0)}=1+\epsilon\beta_1+\epsilon^2\beta_2+O(\epsilon^3), \tag{28}$$

where β_1 and β_2 are defined by

$$\beta_1=-ik_w\hbar\sin\theta_w\Delta$$
$$\beta_2=ik_w\hbar'^2\cos\theta_w\Delta-k_w\sin\theta_w\hbar\hbar'-k_w^2\sin^2\theta_w\hbar'^2\Delta. \tag{29}$$

Finally from the first exponential in Eq. (12) we have

$$e^{-ik_w\cos\theta_wz_0}=1-\epsilon ik_w\cos\theta_w\hbar-\epsilon^2k_w^2\cos^2\theta_w\hbar^2+O(\epsilon^3). \tag{30}$$

Substituting expressions (16), (25), (28) and (30) into Eq. (12), we obtain

$$\phi_1(x_0,-\Delta)=A\,e^{ik_w\sin\theta_wx_0}e^{ik_p\cos\theta_p\Delta} \times\left(T_p(\theta_w)-\epsilon\hbar'T_p'(\theta_w) +\epsilon^2\frac{\hbar'^2}{2}T_p''(\theta_w)+O(\epsilon^3)\right)[1+\epsilon\delta_1 +\epsilon^2\delta_2+O(\epsilon^3)]e^{ik_w\sin\theta(\theta_w-\alpha)\sin\alpha(\Delta+z_0)} \times e^{-ik_w\cos\theta_wz_0}. \tag{31}$$

This expression is related in a transparent way to Eq. (13) for the flat surface form ϕ_1^0, so that ϕ_1 may be written as the flat surface field modified by terms involving ϵ:

$$\phi_1(x_0,-\Delta)=\frac{\phi_1^\circ(x_0,-\Delta)}{T_p(\theta_w)}[1+\epsilon\delta_1+\epsilon^2\delta_2+O(\epsilon^3)][1+\epsilon\beta_1+\epsilon^2\beta_2+O(\epsilon^3)]\left(T_p(\theta_w)-\epsilon\hbar' T_p'(\theta_w)+\epsilon^2\frac{\hbar'^2}{2}T_p''(\theta_w)\right.$$

$$\left.+O(\epsilon^3)\right)[1-\epsilon ik_w\cos\theta_w\hbar-\epsilon^2k_w^2\cos^2\theta_w\hbar^2+O(\epsilon^3)]. \tag{32}$$

This is multiplied out to give

$$\phi_1(x_0,-\Delta)=\frac{\phi_1^\circ(x_0,-\Delta)}{T_p(\theta_w)}\left(T_p(\theta_w)+\epsilon[T_p(\theta_w)(\delta_1-ik_w\hbar\cos\theta_w+\beta_1)-\hbar' T_p'(\theta_w)]+\epsilon^2\left\{T_p[\delta_1\beta_1-k_w^2\hbar^2\cos^2\theta_w\delta_2\right.\right.$$

$$\left.\left.+\beta_2-ik_w\hbar\cos\theta_w(\beta_1+\delta_1)]+\frac{\hbar'^2}{2}T_p''+ik_w\cos\theta_w\hbar\hbar' T_p'(\theta_w)-(\beta_1+\delta_1)\hbar' T_p'(\theta_w)\right\}+O(\epsilon^3)\right). \tag{33}$$

From this we can obtain directly a 'local' transmission coefficient $\hat{T}_p(x_0)$, which defines the transmitted field near to the surface:

$$\hat{T}_p(x_0)=T_p(\theta_w)+\epsilon(T_p(\theta_w)(\delta_1-ik_w\hbar\cos\theta_w+\beta_1)-\hbar' T_p'(\theta_w))+\epsilon^2\left\{T_p(\theta_w)(\delta_1\beta_1-k_w^2\hbar^2\cos^2\theta_w)+\frac{\hbar'^2}{2}T_p''(\theta_w)\right.$$

$$\left.-(\beta_1+\delta_1)\hbar' T_p'(\theta_w)+T_p(\theta_w)[\delta_2+\beta_2-ik_w\hbar\cos\theta_w(\beta_1+\delta_1)]+ik_w\cos\theta_w\hbar\hbar' T_p'(\theta_w)\right\}+O(\epsilon^3). \tag{34}$$

This is the expression we seek for the local form of the scattered field. The remaining reflected and transmitted components at each boundary follow similarly (see Appendix A).

B. Effective transmission and reflection coefficients

The expression (34) is now in a form which is easily averaged. As mentioned in Sec. I we assume that the surfaces are statistically stationary. We will again suppress subscripts in $h_{1,2}$ since no confusion arises.

The first order terms in ϵ are all linear in $\hbar$ and $\hbar'$, which both have mean zero, and thus the $O(\epsilon)$ term vanishes.

The normalized autocorrelation function $\rho(\xi)$ of the surface $h(x)$ is defined as

$$\rho(\xi)=\frac{\langle h(x)h(x+\xi)\rangle}{\langle h(x)h(x)\rangle}=\langle\hbar(x)\hbar(x+\xi)\rangle, \tag{35}$$

so that $\rho(0)=1$.

We require the quantities $\langle\hbar^2\rangle$, $\langle\hbar\hbar'\rangle$, and $\langle\hbar'^2\rangle$. Since the surface is stationary, these may be given in terms of the correlation function $\rho(\xi)$ (Papoulis)[11] as follows:

$$\langle\hbar\hbar'\rangle=\left.\frac{d\rho(\xi)}{d\xi}\right|_{\xi=0},\quad \langle\hbar'^2\rangle=-\left.\frac{d^2\rho(\xi)}{d\xi^2}\right|_{\xi=0}. \tag{36}$$

In many cases $\langle\hbar\hbar'\rangle$ will vanish, i.e., the autocorrelation function has continuous derivative at zero. This is an indication of smoothness of the surface. For example, a Gaussian autocorrelation function with characteristic scale length L

$$\rho(\xi)=\exp\left\{-\left(\frac{\xi}{L}\right)^2\right\} \tag{37}$$

gives

$$\langle\hbar\hbar'\rangle=0,\quad \langle\hbar'^2\rangle=\frac{1}{2L^2}, \tag{38}$$

which leads to a relatively simple form for the modified coefficients. However, surfaces with, for example, fractal autocorrelation function

$$\rho(\xi)=\exp\{-(\xi/L)\}, \tag{39}$$

have features of arbitrarily small scale, and these cross correlations are then given by

$$\langle\hbar\hbar'\rangle=-\frac{1}{L},\quad \langle\hbar'^2\rangle=-\frac{1}{L^2}. \tag{40}$$

We have considered the field point Q with coordinates $(x_0,-\Delta)$. Henceforth, we will set Δ to zero, to obtain the averaged field at the mean surface level. The justification for this is as follows: We can suppose that Δ is at most of order $O(\epsilon)$. In the analysis which follows we discard terms of order $O(\epsilon^3)$ and higher. Therefore, any remaining terms which contain Δ are either linear in the surface, and thus vanish on averaging, or deterministic. The latter components must be due to a flat surface reflection/transmission and therefore valid for any value of Δ.

Setting $\Delta=0$ in Eq. (33) and averaging gives the effective transmission coefficient T_p^r as

$$T_p^r=T_p+\epsilon^2\left\{T_p(k_wk_p\cos\theta_w\cos\theta_p-k_p^2\cos^2\theta_p\right.$$

$$-k_w^2\cos^2\theta_w)+\langle\hbar\hbar'\rangle\left(iT_p\frac{k_w^2}{2k_p}\frac{\sin 2\theta_w}{\cos\theta_p}\right.$$

$$\left.+iT_p'(k_w\cos\theta_w-k_p\cos\theta_p)-T_pk_w\sin\theta_w\right)$$

$$\left.+\langle\hbar'^2\rangle\frac{T_p''}{2}\right\}+O(\epsilon^3). \tag{41}$$

For the particular case of a surface with a Gaussian autocorrelation function, this becomes

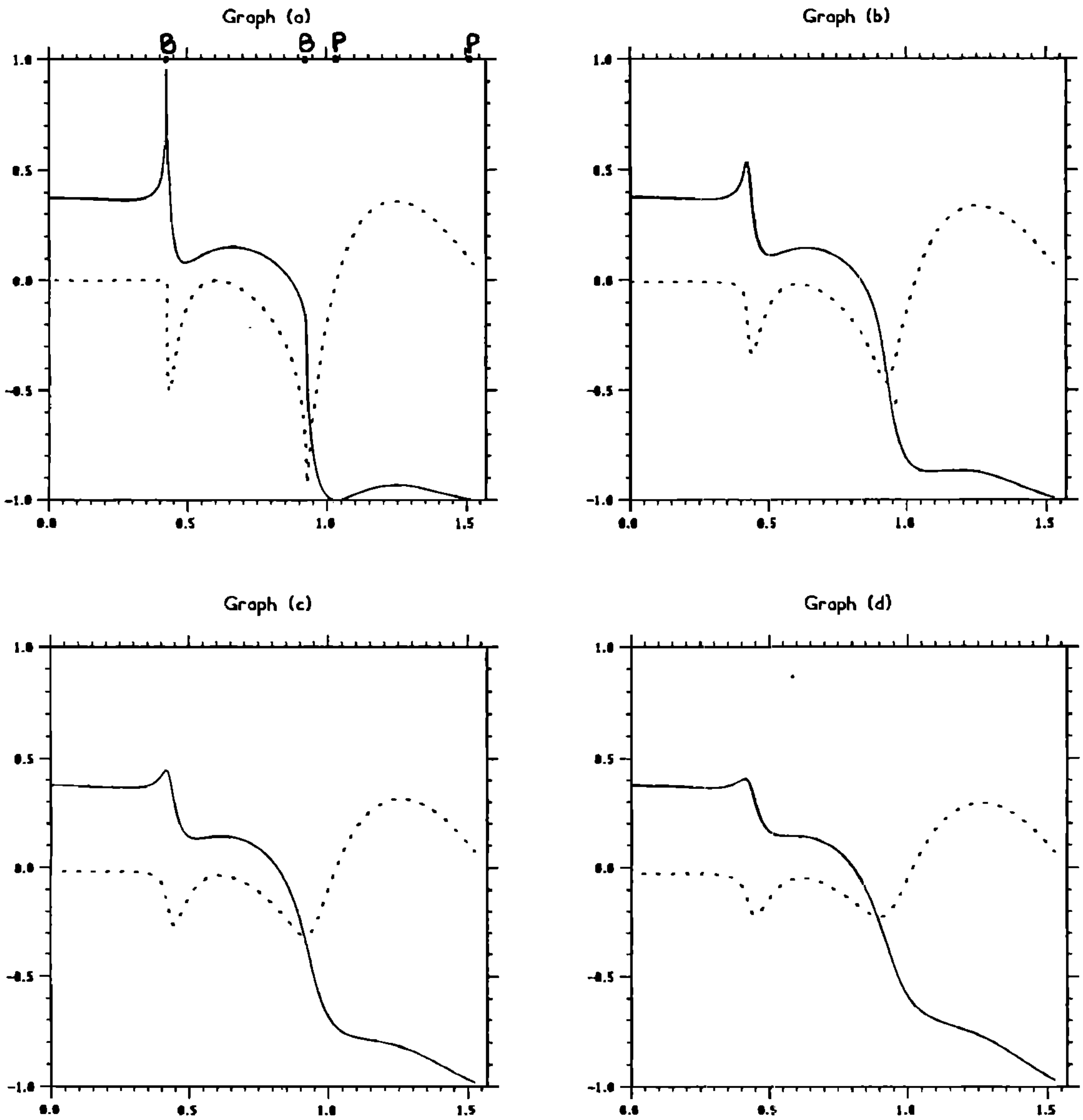

FIG. 4. Real (full line) and imaginary (dotted line) components of the reflection coefficient due to a wave incident on a flat interface from a fluid onto an elastic half-space, as functions of incident angle. Increasing degrees of absorption in the solid are taken into account by giving the wave numbers an imaginary part $k_p \to k_p(1+id)$ and $k_s \to k_s(1+id)$ where d is as follows: (a) no absorption; (b) $d=0.02$; (c) $d=0.04$; (d) $d=0.06$. The parameters of the media correspond to water and ice: speeds of propagation $c_p=3500\ \mathrm{ms}^{-1}$, $c_s=1800\ \mathrm{ms}^{-1}$, and $c_w=1440\ \mathrm{ms}^{-1}$, and densities $\rho_i=910\ \mathrm{kgm}^{-3}$ and $\rho_w=1000\ \mathrm{kgm}^{-3}$. The locations of nearby singularities due to poles and branch cuts are indicated by P and B, respectively.

$$T_p^r = T_p + \epsilon^2 \left\{ T_p (k_w k_p \cos\theta_w \cos\theta_p - k_p^2 \cos^2\theta_p - k_w^2 \cos^2\theta_w) + \frac{1}{2L^2}\frac{T_p''}{2} \right\} + O(\epsilon^3). \tag{42}$$

C. Absorption and properties of the coefficients

The approximations for the effective coefficients [Eqs. (41) and (A4)] depend upon the first and second derivatives of the flat surface coefficients with respect to angle of incidence. At certain critical angles of incidence although they remain finite these derivatives (especially the second) may become very large. Near such angles the validity of the approximation becomes weaker. Physically, this can be understood as follows: if, for a given incident angle θ, we consider the coefficient T say as a function of the relative angle α, then for θ near a critical angle this coefficient varies much more rapidly along the surface. Thus in this sense the surface appears "rougher" to a wave at these angles.

Figures 4(a)–6(a) show the standard reflection and transmission coefficients [Eqs. (A3)] for the case of a compressional wave incident from water onto an ice half-space. Two of the critical points present in these figures are at branch points of the coefficients (marked B), representing the places where $\sin\theta_p$ and $\sin\theta_s$ become unity. The waves produced at these angles are a type of surface wave referred to as lateral waves (Überall).[12] The presence of the branch points causes rapid phase variation of the coefficients, giving rise to large derivatives.

Two other critical points occur (indicated by P); these are the poles corresponding respectively to the generalized Rayleigh wave and the Scholte wave. Such poles are not on

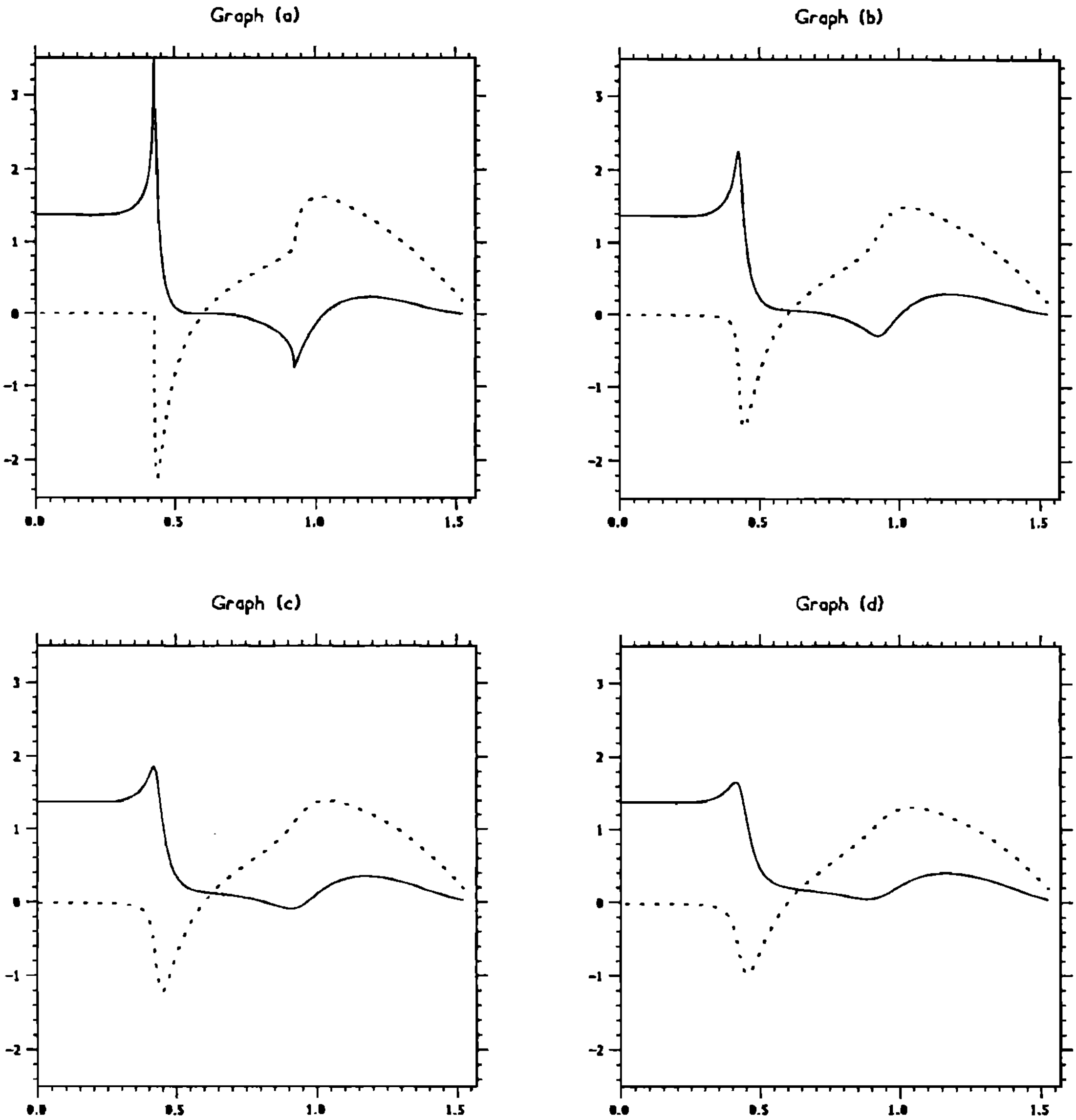

FIG. 5. Components of the transmitted compressional wave, for increasing degrees of absorption. All parameters are as in Fig. 4.

the real axis in the θ_w plane, and hence the coefficients do not strictly diverge. In the ice/water case shown here, the poles are sufficiently far off axis that their effect is not apparent. The magnitude of the coefficients varies rapidly near a pole, and so the derivatives would again become large.

These considerations limit the approximation to certain ranges of incident angles, to an extent determined by the surface roughness. However, the problem may be circumvented if absorption in the media is taken into account. The presence of absorption dramatically smooths out the rapid variations with angle. This is introduced into the model as nonzero imaginary components of the wave numbers. In the figures shown here this has been done for the solid only. The effect of increasing absorption on the flat surface coefficients is shown in graphs (b) to (d) in each of Figs. 4–6. The imaginary component of the wave numbers gives rise to imaginary angles in the solid for a real angle of incidence in the fluid. This effectively moves the branch points away from the real axis in the complex θ_w plane, and largely smooths out the cusps in the coefficients. Similarly the poles corresponding to the lateral and Rayleigh waves are moved further away from the real axis of θ_w. Although these measures are physically reasonable, values must be chosen appropriate to each situation to which these results are to be applied.

It should also be noted that the breakdown of the tangent plane approximation at the Rayleigh angle has another interpretation. The approximation (33) implicitly assumes single scatter at each surface interaction. However, the Rayleigh and lateral waves couple with the surface, and so this assumption is violated. The method therefore gives incorrect results near to the Rayleigh angle. Such an error arises with the Kirchhoff approximation, and this is quantified and discussed by Dacol.[3]

The effective reflection and transmission coefficients were calculated for the same amount of absorption as in Figs. 4, 5, and 6(d). The corresponding results are shown in Figs. 7, 8, and 9, respectively. Here the scale of roughness is given by $k_w^2\epsilon^2=0.1$, and the surface irregularities are assumed to have a Gaussian autocorrelation function. It is interesting to

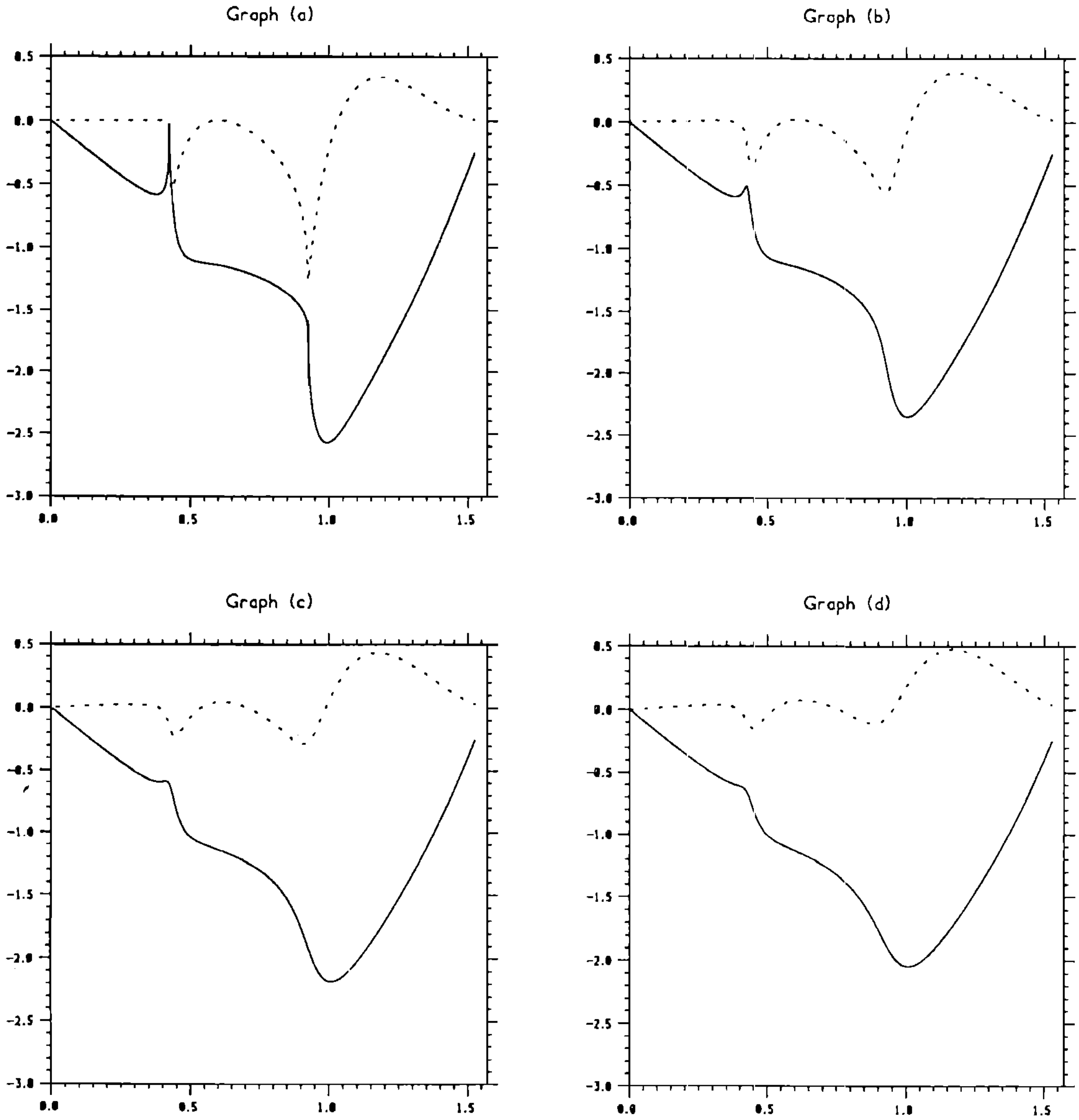

FIG. 6. Components of the transmitted shear wave, for increasing degrees of absorption, as in Fig. 4.

note the significant qualitative differences between the flat surface coefficients and the corresponding effective rough surface coefficients.

D. Solution for the total mean field in the layered medium

Having obtained all relevant effective reflection and transmission coefficients we can proceed as for a plane-sided layer to find the total mean field. The reasoning that this analogy can be made is given below. First we summarize the method for the plane layer. This has been described elsewhere[8,9] and will not be repeated in full detail here.

Consider first a field incident on a plane layer from the fluid. We may think of the resulting solution as composed of an infinite series of transmitted and reflected waves; thus the first component is the transmitted wave defined by the appropriate elastic/fluid coefficients. The second component is its reflection from the top boundary, and so on. These will be referred to as the first and second components, etc. The sum of these waves is convergent, and satisfies the boundary conditions. The series is therefore the solution for the layered system. (We need not distinguish at this point between wave modes.) The solution due to a source in the water consisting of a spectrum of plane waves may thus be obtained by superposition of the corresponding solutions as described above. This solution, following the notation in Sheard,[13] is summarized in Appendix B.

We now return to the problem of an irregular layer. We make the crucial assumption that successive scatters are independent at either surface. More specifically, consider the field scattered (either by reflection or transmission) at one surface h_1, say. We will assume that, after propagation across the layer, scattering at h_2, and propagation back to h_1, the variation in the resulting field is approximately statistically independent of h_1. This "decoupling" of the field from the scatterer results from the fact that the features imposed on the wave at each interaction change markedly with diffraction. (The independent scattering at h_2 is of course a further cause of statistical decoupling.) The effect is well known in the study of wave propagation beyond a random phase screen, and the analogous assumption is used in the formulation of the moment equations for propagation in random

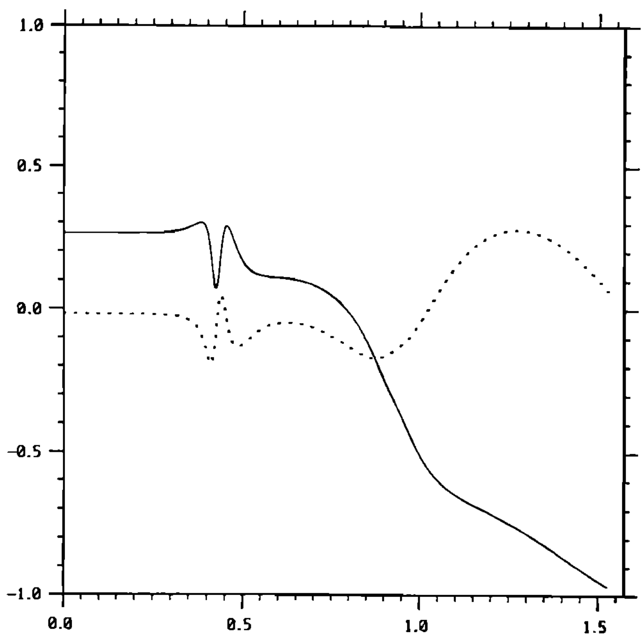

FIG. 7. Real (full line) and imaginary (dotted line) components of the effective reflection coefficient due to a rough interface, with $k_w^2\epsilon^2=0.1$. The amount of absorption here is $d=0.06$, corresponding to Fig. 4(d).

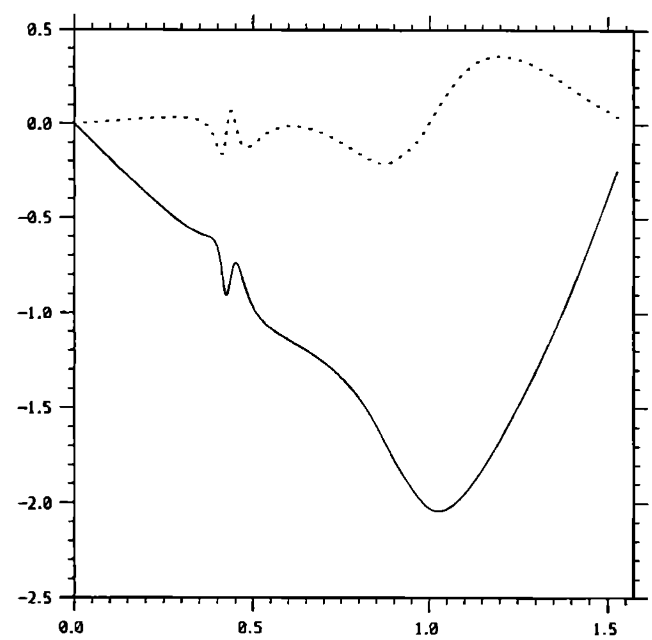

FIG. 9. Components of the effective transmitted shear wave, for roughness and absorption as in Fig. 7, corresponding to Fig. 6(d).

media (see Uscinski[14]). The assumption clearly breaks down for sufficiently shallow layers. The characteristic length scale for propagation beyond a phase screen is kL^2,[14] and so we require that the depth δ is much greater than this quantity, i.e.,

$$\delta \gg kL^2. \tag{43}$$

Now, the total field whose average we will eventually obtain can be thought of as the infinite sum of the scattered field components arising at and propagating from each surface interaction; the total mean field is thus given by the sum

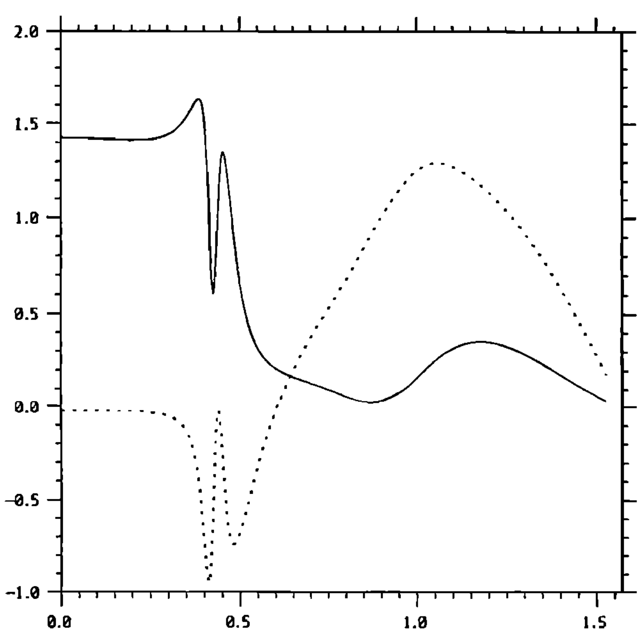

FIG. 8. Components of the effective transmitted compressional wave, for roughness as in Fig. 7, with absorption similarly corresponding to Fig. 5(d).

of the mean of these individual components, by analogy with the plane layer case. Consider therefore the nth component P_n, say. We can write this as

$$P_n = \langle P_n \rangle + P_n', \tag{44}$$

where the random component P_n' has mean zero. Suppose first that the mean $\langle P_n \rangle$ is a plane wave. Now P_n is the "incident" or driving field for the $(n+1)$th component P_{n+1}, and can write symbolically

$$P_{n+1} = \hat{S}(\langle P_n \rangle) + \hat{S}(P_n'), \tag{45}$$

where the linear operator $\hat{S}$ represents scattering due to the interaction with the surface. After averaging, terms $\langle \hat{S}(P_n') \rangle$ resulting from the component P_n' must therefore vanish, since this quantity arises from previous scatterings which can be assumed to be independent of the current scattering. The remaining term $\hat{S}(\langle P_n \rangle)$ then gives, after averaging, another plane wave, with amplitude determined by the effective coefficients as discussed previously for unscattered plane waves. Thus it follows that the mean

$$\langle P_{n+1} \rangle = \langle \hat{S}(\langle P_n \rangle) \rangle \tag{46}$$

is again a plane wave. Finally, since the incident field P_1 is a plane wave it follows by induction that $\langle P_n \rangle$ is indeed a plane wave, as assumed. Therefore at each scattering the mean field is modified by the effective coefficients, and so the mean total field is obtained as for a plane-sided layer with the relevant coefficients simply replaced by the effective coefficients. The complete solution is summarized in Appendix B, using these effective coefficients which are given in Appendix A. This mean field description is qualitatively valid for any degree of surface roughness, since only the coefficients are approximated.

III. CONCLUSIONS

The mean field has been found due to a wave impinging on an elastic layer with adjoining fluid half-space, in which the boundaries of the layer are slightly rough. The solution has been considered explicitly for the case of an ice/ocean medium. In this method effective coefficients were first found for the elastic/vacuum and elastic/fluid half space problems using a tangent plane approximation. It was then shown that under the assumption that successive scatterings are uncorrelated, the mean field is exactly equivalent to the solution of a plane-layer problem with the effective coefficients applied.

The approximation of the effective coefficients is nonuniform, in the sense that it becomes relatively poor at angles of incidence which are near to poles and branch points, and breaks down where these angles give rise to surface waves. This problem is circumvented, however, when absorption is taken into account. The existence and precise location of these poles depends upon the parameters of the medium in each case.

This method of solution can in principle be extended to a multilayered system, or to propagation in a three dimensional medium. The situation for the higher moments, however, is significantly more complicated, since they are not linear in the field.

ACKNOWLEDGMENTS

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APPENDIX A: SUMMARY OF EFFECTIVE COEFFICIENTS

This appendix sets out the effective coefficients which are obtained. Some of the flat surface forms of these coefficients are also included, but most can be found elsewhere and are omitted. Each expression is truncated after the term quadratic in the surface height. We introduce the following notation for surface statistics:

$$\langle \hbar_1^2 \rangle = 1, \quad \langle \hbar_1 \hbar_1' \rangle = A_1, \quad \langle \hbar_1'^2 \rangle = B_1,$$
$$\langle \hbar_2^2 \rangle = 1, \quad \langle \hbar_2 \hbar_2' \rangle = A_2, \quad \langle \hbar_2'^2 \rangle = B_2. \tag{A1}$$

Each effective coefficient will be a function of derivatives of the corresponding flat surface one with respect to the angle of incidence of the incident plane wave. These are simply denoted R_w', etc.

The effective impedances of the media to waves in the fluid and P and S waves in the solid are denoted by Z_w, Z_p, and Z_s, respectively, and are defined by

$$Z_w = \frac{c_w \rho_1}{\cos \theta_w}, \quad Z_p = \frac{c_p \rho_2}{\cos \theta_p}, \quad Z_s = \frac{c_s \rho_2}{\cos \theta_s}. \tag{A2}$$

Here, ρ_1 and ρ_2 are the densities of the fluid and solid, respectively, and c_w, c_p, and c_s are the speeds of propagation of, respectively, the transverse waves in the fluid, and the transverse and shear waves in the solid.

1. Fluid/solid coefficients

The reflection and transmission coefficients for a *plane* wave incident from a fluid onto a plane solid surface are then

$$R_w = \frac{Z_p \cos^2 2\theta_s + Z_s \sin^2 2\theta_s - Z_w}{Z_p \cos^2 2\theta_s + Z_s \sin^2 2\theta_s + Z_w},$$
$$T_{wp} = \frac{\rho_w}{\rho_i} \frac{2Z_p \cos 2\theta_s}{Z_p \cos^2 2\theta_s + Z_s \sin^2 2\theta_s + Z_w}, \tag{A3}$$
$$T_{ws} = -\frac{\rho_w}{\rho_i} \frac{2Z_s \sin 2\theta_s}{Z_p \cos^2 2\theta_s + Z_s \sin^2 2\theta_s + Z_w}.$$

The effective coefficients in the case of a rough surface are

$$R_w^r = R_w + \epsilon_1^2 \Big\{ -3R_w k_w^2 \cos^2 \theta_w + A_1 [2iR_w' k_w \cos \theta_w - R_w k_w \sin \theta_w (1+i)] + B_1 \frac{R_w''}{2} \Big\},$$

$$T_{wp}^r = T_{wp} + \epsilon_1^2 \Big\{ T_{wp}(k_w k_p \cos \theta_w \cos \theta_p - k_p^2 \cos^2 \theta_p - k_w^2 \cos^2 \theta_w) + A_1 \Big(iT_{wp} \frac{k_w^2}{2k_p} \frac{\sin 2\theta_w}{\cos \theta_p} + iT_{wp}'(k_w \cos \theta_w - k_p \cos \theta_p) - T_{wp} k_w \sin \theta_w \Big) + B_1 \frac{T_{wp}''}{2} \Big\}, \tag{A4}$$

$$T_{ws}^r = T_{ws} + \epsilon_1^2 \Big\{ T_{ws}(k_w k_s \cos \theta_w \cos \theta_s - k_s^2 \cos^2 \theta_s - k_w^2 \cos^2 \theta_w) + A_1 \Big(iT_{ws} \frac{k_w^2}{2k_s} \frac{\sin 2\theta_w}{\cos \theta_s} + iT_{ws}'(k_w \cos \theta_w - k_s \cos \theta_s) - T_{ws} k_w \sin \theta_w \Big) + B_1 \frac{T_{ws}''}{2} \Big\}.$$

2. Solid/fluid coefficients—Incident *P* wave

A plane P wave incident from a solid onto a plane fluid interface gives rise to coefficients which we denote R_{pp} (P-wave reflected component), R_{ps} (S wave), and T_{pw} (transmitted wave in the fluid). For brevity these are omitted. The effective rough surface forms are

$$R_{pp}^r = R_{pp} + \epsilon_1^2 \Big\{ -3R_{pp} k_p^2 \cos^2 \theta_p + A_1 [R_{pp} k_p \times \sin \theta_p (1+i) - 2iR_{pp}' k_p \cos \theta_p] + B_1 \frac{R_{pp}''}{2} \Big\},$$

$$R'_{ps}=R_{ps}+\epsilon_1^2\left\{-R_{ps}(k_pk_s\cos\theta_p\cos\theta_s+k_s^2\cos^2\theta_s+k_p^2\cos^2\theta_p)+A_1\left(iR_{ps}\frac{k_p^2}{2k_s}\frac{\sin 2\theta_p}{\cos\theta_s}-iR'_{ps}(k_p\cos\theta_p+k_s\cos\theta_s)-R_{ps}k_p\sin\theta_p\right)+B_1\frac{R''_{ps}}{2}\right\}, \quad \text{(A5)}$$

$$T'_{pw}=T_{pw}+\epsilon_1^2\left\{T_{pw}(k_wk_p\cos\theta_w\cos\theta_p-k_p^2\cos^2\theta_p-k_w^2\cos^2\theta_w)+A_1\left(T_{pw}k_p\sin\theta_p-iT_{pw}\frac{k_p^2}{2k_w}\frac{\sin 2\theta_p}{\cos\theta_w}+iT'_{pw}(k_w\cos\theta_w-k_p\cos\theta_p)\right)+B_1\frac{T''_{pw}}{2}\right\}.$$

3. Solid/fluid—Incident *S* wave

For a plane S wave incident from a solid onto a plane fluid interface the coefficients are denoted R_{sp}, R_{ss}, and T_{sw}.

For a rough surface these become

$$R'_{ss}=R_{ss}+\epsilon_1^2\left\{-3R_{ss}k_s^2\cos^2\theta_s+A_1(R_{ss}k_s\sin\theta_s(1+i)-2iR'_{ss}k_s\cos\theta_s)+B_1\frac{R''_{ss}}{2}\right\},$$

$$R'_{sp}=R_{sp}+\epsilon_1^2\left\{-R_{sp}(k_pk_s\cos\theta_p\cos\theta_s+k_s^2\cos^2\theta_s+k_p^2\cos^2\theta_p)+A_1\left(iR_{sp}\frac{k_s^2}{2k_p}\frac{\sin 2\theta_s}{\cos\theta_p}-iR'_{sp}(k_p\cos\theta_p+k_s\cos\theta_s)-R_{sp}k_s\sin\theta_s\right)+B_1\frac{R''_{sp}}{2}\right\}, \quad \text{(A6)}$$

$$T'_{sw}=T_{sw}+\epsilon_1^2\left\{T_{sw}(k_wk_s\cos\theta_w\cos\theta_s-k_s^2\cos^2\theta_s-k_w^2\cos^2\theta_w)+A_1\left(T_{sw}k_p\sin\theta_s-iT_{sw}\frac{k_s^2}{2k_w}\frac{\sin 2\theta_s}{\cos\theta_w}+iT'_{sw}(k_w\cos\theta_w-k_p\cos\theta_s)\right)+B_1\frac{T''_{sw}}{2}\right\}.$$

4. Solid/vacuum—Incident *P* wave

In this case the flat-surface coefficients are denoted by R_{pp} and R_{ps}. For a rough surface these become

$$R'_{pp}=R_{pp}+\epsilon_2^2\left\{-3R_{pp}k_p^2\cos^2\theta_p+A_2[2iR'_{pp}k_p\cos\theta_p-R_{pp}k_p\sin\theta_p(1+i)]+B_2\frac{R''_{pp}}{2}\right\}, \quad \text{(A7)}$$

$$R'_{ps}=R_{ps}+\epsilon_2^2\left\{-R_{ps}(k_pk_s\cos\theta_p\cos\theta_s+k_s^2\cos^2\theta_s+k_p^2\cos^2\theta_p)+A_2\left(R_{ps}k_p\sin\theta_p-iR_{ps}\frac{k_p^2}{2k_s}\frac{\sin 2\theta_p}{\cos\theta_s}+iR'_{ps}(k_p\cos\theta_p+k_s\cos\theta_s)\right)+B_2\frac{R''_{ps}}{2}\right\}.$$

5. Solid/vacuum—Incident *S* wave

Finally, for a plane S wave incident on a flat solid/vacuum boundary the coefficients are R_{sp} and R_{ss}.

When the boundary is rough the effective coefficients are given by

$$R'_{ss}=R_{ss}+\epsilon_2^2\left\{-3R_{ss}k_s^2\cos^2\theta_s+A_2[2iR'_{ss}k_s\cos\theta_s-R_{ss}k_s\sin\theta_s(1+i)]+B_2\frac{R''_{ss}}{2}\right\}, \quad \text{(A8)}$$

$$R'_{sp}=R_{sp}+\epsilon_2^2\left\{-R_{sp}(k_pk_s\cos\theta_p\cos\theta_s+k_s^2\cos^2\theta_s+k_p^2\cos^2\theta_p)+A_2\left(R_{sp}k_s\sin\theta_s-iR_{sp}\frac{k_s^2}{2k_p}\frac{\sin 2\theta_s}{\cos\theta_p}+iR'_{sp}(k_p\cos\theta_p+k_s\cos\theta_s)\right)+B_2\frac{R''_{sp}}{2}\right\}.$$

APPENDIX B: SUMMARY OF SOLUTION FOR IRREGULAR LAYER

In the following, the notation used to distinguish coefficients is as in Appendix A, but an additional superscript denotes whether the coefficient refers to the solid/fluid boundary (superscript D) or the solid/vacuum boundary (superscript S). For convenience it will be assumed that the mean upper boundary z_2 is given by $z_2=0$. As before, δ denotes the depth of the layer.

Consider a plane wave $\phi_{\rm inc}$ incident on the solid layer from the fluid half-space,

$$\phi_{inc}=Ae^{ik_w \sin\theta_w(x-x_0)}e^{-ik_w\cos\theta_w z}$$
$$=Ae^{-ik_w\cos\theta_w\delta}e^{-ik_w\cos\theta_w(z-\delta)}. \quad \text{(B1)}$$

In the layered system the solutions for the fluid potential ϕ_w, and P and S potentials ϕ_i, ψ_i in the solid take the form:

$$\begin{aligned}\phi_w(x,z)=&Ae^{ik_w\sin\theta_w(x-x_0)}(e^{-ik_w\cos\theta_w z}\\&+R_{ww}e^{ik_w\cos\theta_w(z-2\delta)}\\&+T_{ws}e^{-ik_w\cos\theta_w\delta}e^{ik_s\cos\theta_s\delta}\{R_{ss}^S\mathbf{M}_6+R_{sp}^S\mathbf{M}_5\}\\&+T_{wp}e^{-ik_w\cos\theta_w\delta}e^{ik_p\cos\theta_p\delta}\{R_{ps}^S\mathbf{M}_6\\&+R_{pp}^S\mathbf{M}_5\}),\\\phi_i(x,z)=&Ae^{ik_p\sin\theta_p(x-x_0)}\\&\times(T_{wp}e^{-ik_w\cos\theta_w\delta}e^{ik_p\cos\theta_p(\delta-z)}\\&+T_{ws}e^{-ik_w\cos\theta_w\delta}e^{ik_s\cos\theta_s\delta}\{R_{ss}^S\mathbf{M}_2+R_{sp}^S\mathbf{M}_1\}\\&+T_{wp}e^{-ik_w\cos\theta_w\delta}e^{ik_p\cos\theta_p\delta}\{R_{ps}^S\mathbf{M}_2\\&+R_{pp}^S\mathbf{M}_1\}),\qquad\text{(B2)}\\\psi_i(x,z)=&Ae^{ik_s\sin\theta_s(x-x_0)}(T_{ws}e^{-ik_w\cos\theta_w\delta}e^{ik_s\cos\theta_s(\delta-z)}\\&+T_{ws}e^{-ik_w\cos\theta_w\delta}e^{ik_s\cos\theta_s\delta}\{R_{ss}^S\mathbf{M}_3+R_{sp}^S\mathbf{M}_4\}\\&+T_{wp}e^{-ik_w\cos\theta_w\delta}e^{ik_p\cos\theta_p\delta}\{R_{ps}^S\mathbf{M}_3\\&+R_{pp}^S\mathbf{M}_4\}).\end{aligned}$$

In these expressions the six quantities $\mathbf{M}_i$ arise as coefficients in mode-coupling matrices. These are described in Ref. 13 and are somewhat lengthy to reproduce in full, and are therefore omitted here.

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