Moments and angular spectrum for rough surface scattering at grazing incidence

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This paper considers the statistics of a plane wave scattered at low grazing angles from a one-dimensional rough surface, when the interaction with the surface is mainly in the forward direction. The mean intensity and autocorrelation of the scattered field and the corresponding angular spectrum, to second order in surface height is found. The diffuse component of the spectrum vanishes with the square of the grazing angle, while the specular part approaches unity linearly. The higher moments of the field at the mean surface are also obtained. It is shown that to first order in the grazing angle the moments of the scattered field change only by a phase factor with distance into the medium, so that in the small angle limit the moments in the medium can be derived from those at the surface. The results depend explicitly upon the autocorrelation function of the surface and its first derivative.

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INTRODUCTION

In the problem of wave scattering by a rough surface we usually seek a statistical description of the resulting field. The first two moments can be obtained by several methods such as the Kirchhoff approximation and perturbation theory,[1–4] but fewer results are known for the higher moments and little progress has been made on the fundamental question of the probability density function. For further discussion the reader is referred to the excellent review of DeSanto and Brown.[3]

At near-grazing incidence, the above approximations break down, although approaches such as the smoothing method[5] have been successfully applied in this régime. Further progress can be made, however, under the assumption that the field is predominantly due to forward scattering. The field can then be described by the parabolic equation approximation, and the usual Helmholtz equations may be replaced by the parabolic integral equation method.[6,7] This approach has proved very useful; for example it has allowed the development of accurate inverse scattering solutions,[8,9] and the inclusion of channeling in the medium due to a linear refractive index profile.[10,11] In a previous paper[12] the mean field was obtained to second order in surface height.

In this paper we study the second and higher moments at near-grazing incidence and consider the angular spectrum. We first find the moments of the field at the surface; the accuracy depends upon surface roughness and angle of incidence, and diverges for moments of high order. We then consider the second moment in the medium, which describes the mean intensity and autocorrelation of the field and its angular distribution. This is shown to be independent of distance from the surface, apart from evanescent components. It is found that the diffuse component of the angular spectrum vanishes with the square of the grazing angle, while the specular part linearly approaches unity. Although the approximation takes into account multiple scattering it cannot exhibit backscatter enhancement, because the restriction to forward interaction at the surface precludes reversible ray paths. Finally we show that to first order in the grazing angle the moments in the medium may be derived from those at the surface since they are unchanged apart from phase factors.

The results here are derived from expressions for the near-surface field given in Ref. 12. Powers and cross products of the field are formulated and averaged, and these are truncated at second order. This procedure is divergent for high powers, and this places a bound on the accuracy of the higher-order moments which is established below. It is assumed that multiple scattering at the surface takes place in the same direction as the forward-traveling incident field; scattering outward from the surface is not subject to this restriction.

The paper is organized as follows: The preliminary equations are set out in Sec. I. In Sec. II the moments at the mean surface plane are found and the accuracy of the binomial expansion is examined. The second moment in the medium is considered in Sec. III, where it is shown in effect to be independent of distance from the surface, and the corresponding angular spectrum is given. In Sec. IV the moments in the medium are given to first order in grazing angle, and are shown in this case to be related in a simple way to those on the surface.

I. PARABOLIC EQUATION METHOD AND PRELIMINARIES

We consider the problem of a scalar time-harmonic wave field p scattered from a one-dimensional rough surface $h(x)$, with a pressure release boundary condition. (For electromagnetic waves this corresponds to s or TE polarization, and perfect conductivity. It is applicable to corrugated surfaces provided polarization does not change under scattering.) The wave field propagates with wave number k, and is thus governed by the wave equation $(\nabla^2+k^2)p=0$. The field will be assumed to be incident and scattered at small grazing

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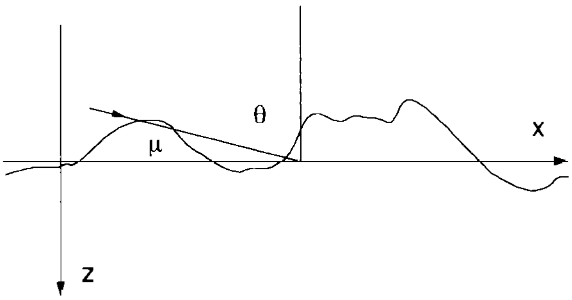

FIG. 1. Geometry of rough surface scattering, where the medium is in the upper-half-plane. The arrow indicates a wave incident at an angle θ from the normal, and μ denotes the grazing angle $\pi/2-\theta$.

angles with respect to the surface. The coordinate axes are x and z, where x is the horizontal $x \geqslant 0$, and z the vertical, directed out of the medium. This geometry is shown in Fig. 1. It will be assumed that the surface is statistically stationary to second order, i.e., its mean and autocorrelation function are translationally invariant. The mean surface level is taken at $z=0$, so that $h(x)$ has mean zero. The autocorrelation function $\langle h(x)h(x+\xi)\rangle$ is denoted by $\rho(\xi)$, and we assume that $\rho(\xi)\rightarrow 0$ at large separations ξ. (The angled brackets denote the ensemble average.) Then $\sigma^2=\rho(0)$ is the variance of surface height, so that the surface roughness is of order $O(\sigma)$. We will denote by L the characteristic correlation length of the surface, where applicable.

Since the field propagates predominantly in one direction, we can define a slowly varying part ψ by

$$\psi_{\text{tot}}(x,z)=p(x,z)\exp(-ikx).$$

Incident and scattered components ψ_i and ψ are defined similarly, so that $\psi_{\text{tot}}=\psi_i+\psi$. It may be assumed that $\psi_i[x,h(x)]=0$ for $x\leqslant 0$, so that the area of surface illumination is restricted, as for example when the field is a directed Gaussian beam. The governing equations for the parabolic equation method[6,7] are then

$$\psi_i(\mathbf{r})=-\int_0^x G(\mathbf{r};\mathbf{r}')\,\frac{\partial\psi(\mathbf{r}')}{\partial z}\,dx', \tag{1}$$

where both $\mathbf{r}=(x,h(x))$, $\mathbf{r}'=(x',h(x'))$ lie on the surface; and

$$\psi(\mathbf{r})=\int_0^x G(\mathbf{r};\mathbf{r}')\,\frac{\partial\psi(\mathbf{r}')}{\partial z}\,dx', \tag{2}$$

where $\mathbf{r}'$ is again on the surface and $\mathbf{r}$ is now an arbitrary point in the medium. Here G is the parabolic form of the Green's function in two dimensions given by

$$G(x,z;x',z')=\begin{cases}\alpha\sqrt{\dfrac{1}{x-x'}}\,\exp\left(\dfrac{ik(z-z')^2}{2(x-x')}\right), & \text{for } x'<x,\\ 0, & \text{otherwise,}\end{cases} \tag{3}$$

where $\alpha=\frac{1}{2}\sqrt{i/2\pi k}$. [This form gives rise to the finite upper limit of integration in (1) and (2).] The Green's function is derived under the assumption of forward scattering, i.e., that the field obeys the parabolic wave equation,

$$\psi_x+2ik\psi_{zz}=0, \tag{4}$$

which holds provided the angles of incidence and scattering are fairly small with respect to the x direction. (G can also be obtained directly from the full free-space Green's function by the appropriate low-angle approximation.) We will invert (1) to give the induced source $\partial\psi/\partial z$ at the surface, but the field in the medium will be related to this using the full wave equation rather than the integral (2). This allows scattering into all directions.

Now, Eqs. (1) and (2) do not apply to the scattering of plane waves at small or negative x because of the truncated lower limit of integration, equivalent to the restriction on surface illumination. Nevertheless, we can formally apply the integral equation to a plane wave, to obtain a solution which will be asymptotically accurate and physically meaningful at large values of x. This procedure is used in Ref. 12 to derive the mean field. In the remainder of the paper we will assume that x is sufficiently large for this to hold. (It is easier simply to set the lower integration limit to $-\infty$ but we retain this form for consistency.)

Consider an incident plane wave $\exp(ik[x\sin\theta+z\cos\theta])$, where θ is the angle with respect to the normal. The grazing angle is then denoted $\mu=\pi/2-\theta$ (see Fig. 1). This plane wave has slowly varying component $\exp(ik[Sx+z\cos\theta])$, where

$$S=\sin\theta-1, \tag{5}$$

which we refer to as the reduced plane wave. Denote the scattered field due to such a plane wave by $\psi^\theta(x,z)$. We introduce the following notation for field cross products

$$\psi_{r,s}(\mathbf{v};z)=\psi^{\theta_1}(x_1,z)\cdots\psi^{\theta_r}(x_r,z)\times\overline{\psi^{\theta_{r+1}}}(x_{r+1},z)\cdots\overline{\psi^{\theta_{r+s}}}(x_{r+s},z), \tag{6}$$

where the bar denotes complex conjugation and for convenience we define

$$\mathbf{v}=\{\theta_1,\ldots,\theta_{r+s},x_1,\ldots,x_{r+s}\}. \tag{7}$$

We then define the moments $m_{r,s}$ by

$$m_{r,s}(\mathbf{v};z)=\langle\psi_{r,s}(z)\rangle. \tag{8}$$

Among the most important of these are the one-point and intensity moments. We write

$$M_n(\theta,x;z)=\langle(\psi^\theta(x,z))^n\rangle \tag{9}$$

and the moments of intensity or "symmetric" moments

$$M_{n,n}(\theta,x;z)=m_{n,n}(\theta,\ldots\theta,x,\ldots,x;z)=\langle|\psi^\theta(x,z)|^{2n}\rangle. \tag{10}$$

For incident plane waves the quantities in (9) and (10) are statistically stationary and we may omit the variable x.

II. HIGHER MOMENTS

In this section we obtain the moments of the scattered field at the mean surface, for incident plane waves. We begin by summarizing the method and results of Ref. 12 for the scattered field.

It is assumed that the surface is only slightly rough, so that there is a horizontal plane at $z=z_1$ close to all points on the surface $h(x)$. For each x, the scattered field $\psi(x,z_1)$ may be expanded to second order in (z_1-h) about $h(x)$, and thus written in terms of the field itself and its first two derivatives at the surface. Of these, the field is given by the boundary condition, the first derivative is found by an analytical solution of Eq. (1) to second order in h, and the remaining term is solved similarly.

We thereby obtain the following [Eq. (23) of Ref. 12]:

$$\psi^\theta(x,z_1)$$
$$=-\left(1+ikh(x)\cos\theta-\frac{k^2}{2}\cos^2\theta[h^2(x)+(z_1-h)^2]\right)$$
$$\times e^{ikxS}-(z_1-h)\left[\frac{D_\theta(x)}{\pi}+\frac{1}{\pi}\frac{dI_\theta}{dx}+(ik\cos\theta\right.$$
$$\left.-k^2h(x)\cos^2\theta-i\frac{k^3}{2}h^2(x)\cos^3\theta)e^{ikxS}\right], \tag{11}$$

where

$$D_\theta(x)=\frac{d}{dx}\int_0^x\frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}\,dx', \tag{12}$$

and

$$I_\theta(x)=\int_0^x ikh(x')\cos\theta\,\frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}\,dx'. \tag{13}$$

If we take $z_1=0$ several terms cancel, and we can write (11) as

$$\psi^\theta(x,0)=-e^{ikSx}+\frac{h}{\pi}D_\theta+\frac{h}{\pi}\frac{dI_\theta}{dx}. \tag{14}$$

We may set z_1 to zero here for the following reason: The results which follow are obtained by taking polynomials in expression (11) and neglecting terms of order $O(\sigma^3)$ or higher. However, z_1 is of order σ, and so any remaining terms which contain z_1 are either linear in h and thus vanish on averaging, or deterministic. The latter components are those due to reflection from a flat surface and are therefore valid at all $z_1\geq 0$.

A. Truncation of binomial expansions

In order to form the higher moments of the wave field we will take powers and polynomials of expressions such as $[1+a\epsilon(x)+b\epsilon^2(x)]$, where ϵ is a small random function, and then average, and truncate at second order. This procedure is divergent for polynomials of high order, and we need to quantify the accuracy for given order in terms of ϵ. Consider for simplicity the expression

$$P_n=(1+[\epsilon_1+\epsilon_2])^n, \tag{15}$$

where ϵ_j is of order ϵ^j for $j=1,2$. We will suppose that ϵ_j is random and that all odd-order terms (e.g., ϵ_1 and $\epsilon_1\epsilon_2$) have mean zero. The binomial expansion of (15) is

$$P_n=1+n(\epsilon_1+\epsilon_2)+C_{n,2}(\epsilon_1+\epsilon_2)^2+\cdots+(\epsilon_1+\epsilon_2)^n, \tag{16}$$

where

$$C_{n,j}=\frac{n!}{j!(n-j)!}.$$

Expanding further and averaging this yields

$$\langle P_n\rangle=1+n\langle\epsilon_2\rangle+C_{n,2}\langle\epsilon_1^2\rangle+\sum_{j=4}^{n}C_{n,j}\langle(\epsilon_1+\epsilon_2)^j\rangle$$
$$+[C_{n,2}+C_{n,3}]B_4, \tag{17}$$

where B_4 is a term of order $O(\epsilon^4)$. If we truncate (17) at second order in ϵ, we retain only the first three terms. The error thus incurred clearly diverges for large n since the coefficients $C_{n,j}$ grow exponentially with n. We may restrict attention to the sum since the coefficient of B_4 is small compared with the coefficient $C_{n,4}$ of the $O(\epsilon^4)$ term in the sum. Now $C_{n,j}$ is at most of order $n^{j/2}$ in n. Thus each term in the sum is at most proportional to $n^{j/2}\epsilon^j$. The sum is therefore at most of order $O(n^2\epsilon^4)$, and so Eq. (17) is reasonably approximated by the first three terms provided

$$n^2\epsilon^4\ll 1. \tag{18}$$

B. Moments at mean surface plane

Consider now the terms in Eq. (14). We are interested in the asymptotic forms of I_θ and D_θ with respect to x as discussed in Sec. I. At large x the term in D_θ has the behavior[12]

$$\frac{h(x)}{\pi}D_\theta(x)\sim h(x)f(\theta)e^{ikSx}, \tag{19}$$

where

$$f(\theta)=-2ik\sqrt{2-2\sin\theta}. \tag{20}$$

We will also require the mean $\langle h\,dI_\theta/dx\rangle$, which was obtained in Ref. 12. We briefly sketch the derivation: The chain rule is applied, to write

$$\left\langle h\frac{dI_\theta}{dx}\right\rangle=\left\langle\frac{d(hI_\theta)}{dx}\right\rangle-\left\langle I_\theta\frac{dh}{dx}\right\rangle.$$

In each term on the right, h appears, or can be brought, inside the integral I_θ [Eq. (13)], and the upper limit of integration is then allowed to go to infinity. We thus obtain

$$\frac{1}{\pi}\left\langle h\frac{dI_\theta}{dx}\right\rangle=-T_\theta e^{ikSx}, \tag{21}$$

where the coefficient T_θ is given by

$$T_\theta=\frac{k^2}{\pi\alpha}\cos\theta\left[\int_0^\infty\frac{e^{-ikS\xi}}{\sqrt{\xi}}\left(\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}\right)d\xi\right]. \tag{22}$$

The coefficient T_θ can be written explicitly in closed form for several surface correlation functions of interest. Three cases are given in the Appendix. (T_θ is related to the effective admittance obtained in various regimes in Ref. 1.)

Using (19) we write (14)

$$\psi^\theta(x,0)\cong-e^{ikSx}+h(x)f(\theta)e^{ikSx}+\frac{h(x)}{\pi}\frac{dI_\theta}{dx}. \tag{23}$$

The terms on the right are respectively zero, first and second order in surface height σ. We will therefore write

$$\psi^{\theta}(x,0)\cong a_0+a_1+a_2,$$

where

$$a_0(\theta,x)=-e^{ikSx},$$

$$a_1(\theta,x)=h(x)f(\theta)e^{ikSx}, \tag{24}$$

$$a_2(\theta,x)=\frac{h(x)}{\pi}\frac{dI_\theta}{dx}.$$

We have the following

$$\overline{a_0}=a_0^{-1}, \quad |a_0|=1,$$

$$\langle a_1(\theta,x)\overline{a_1(\theta',x')}\rangle=\rho(x-x')f(\theta)\overline{f(\theta')}a_0(\theta x)\overline{a_0(\theta',x')}, \tag{25}$$

and

$$\langle a_2(\theta,x)\rangle=T_\theta a_0.$$

Note that in these expressions [see (20)]

$$f(\theta)\overline{f(\theta')}=-8k^2\sqrt{SS'}, \tag{26}$$

where as before $S=\sin(\theta)-1$, and $S'=\sin(\theta')-1$.

Now, from (23) and (24), for any r, s the cross product $\psi_{r,s}$ at $z=0$ [Eq. (6)] can be expressed as

$$\prod_{j=1}^{r}[a_0(\theta_j,x_j)+a_1(\theta_j,x_j)+a_2(\theta_j,x_j)]\prod_{k=r+1}^{r+s}[\overline{a_0(\theta_k,x_k)}+\overline{a_1(\theta_k,x_k)}+\overline{a_2(\theta_k,x_k)}]. \tag{27}$$

It is straightforward to evaluate this to second order in h, and thus obtain any of the moments at $z=0$. We will do this explicitly only for the most important cases. From (20), (21), (24), and (25) we can treat the terms a_1 and a_2 as first and second order in the small parameter $\epsilon=k\sigma$. Therefore by (18) all moments up to the nth, obtained by truncating and averaging (27), will be accurate provided $n^2\ll 1/k^4\sigma^4$. For example, if $k\sigma=1/4$, second-order truncation will yield moments up to the eighth with reasonable accuracy. In addition the terms $\langle a_1^2\rangle$ and $\langle a_2\rangle$ depend on another small parameter, the grazing angle $\mu=\pi/2-\theta$, and they vanish in the limit $\mu\to 0$. This will be discussed below.

Consider first the one-point moments M_n (9) and intensity moments $M_{r,r}$ (10), i.e., the moments of the field at a point due to a single incident plane wave. Expanding (27) and retaining terms to order σ^2, and averaging so that the linear terms vanish, we use (17) and write

$$M_n(\theta,x;0)=\langle(\psi^{\theta}(x,0))^n\rangle=a_0^n+na_0^{n-1}\langle a_2\rangle+\frac{n(n-1)}{2}a_0^{n-2}\langle a_1^2\rangle.$$

From Eqs. (24) and (25) this becomes

$$M_n(\theta,x;,0)=(-1)^n e^{iknSx}\times\left\{1+nT_\theta-\frac{n(n-1)}{2}\sigma^2 f^2(\theta)\right\}, \tag{28}$$

where f is given by (26). The intensity moments $M_{n,n}$ become, after some simple manipulation,

$$M_{n,n}(\theta,x;0)=|a_0|^{2n}+n[\langle a_2\rangle\overline{a_0}|a_0|^{2n-2}+\langle\overline{a_2}\rangle a_0|a_0|^{2n-2}]+\frac{n(n-1)}{2}[\langle a_1^2\rangle\overline{a_0^2}|a_0|^{2n-4}+\langle\overline{a_1^2}\rangle a_0^2|a_0|^{2n-4}]+n^2\langle|a_1|^2\rangle|a_0^{2n-2}|. \tag{29}$$

By (24) and (25) this reduces to

$$M_{n,n}(\theta,x;0)=1+2n\ \mathrm{Re}(T_\theta)+\frac{n(n-1)}{2}\sigma^2(f^2(\theta)+\overline{f^2(\theta)})+n^2\sigma^2|f(\theta)|^2. \tag{30}$$

We also consider the two-point moments. The most useful of these are the second and fourth. To second order in σ, the second moment (i.e., the cross correlation between components due to two plane waves as a function of spatial separation) at $z=0$ is

$$m_{1,1}(\theta,\theta',x,x';0)\cong a_0\overline{a_0'}+[a_0\langle\overline{a_2'}\rangle+\langle a_2\rangle\overline{a_0'}]+\langle a_1\overline{a_1'}\rangle,$$

where the prime denotes that the arguments of the function are θ', x'. From (24)–(26) this can be written

$$m_{1,1}(\theta,\theta',x,x';0)\cong e^{ik(Sx-S'x')}[1+T_\theta+\overline{T_{\theta'}}-8\rho(\xi)k^2\sqrt{SS'}], \tag{31}$$

where $\xi=x-x'$ and again $S'=\sin(\theta')-1$. At $\theta=\theta'$ this reduces to the correlation function of the scattered component due to a plane wave

$$m_{1,1}(\theta,\theta,x,x';0)\cong e^{ikS(x-x')}[1+2\ \mathrm{Re}(T_\theta)-8\rho(\xi)k^2S]. \tag{32}$$

Before impinging on the surface the reduced plane wave has deterministic autocorrelation function given by the phase term $\exp[ikS(x-x')]$; this becomes modified by the autocorrelation function of the surface as described by Eq. (32). As the grazing angle approaches zero this modification becomes vanishingly small, i.e., the rough surface appears as a perfect reflector, as is expected from other considerations (e.g., Refs. 2, 13). We will quantify this more precisely in the next section, where the diffuse component is shown to vanish more rapidly than the specular term 2 $\mathrm{Re}(T_\theta)$.

For the case of a single angle θ the cross correlation of intensity (fourth moment of the field) can similarly be written

$$m_{2,2}(\theta,\ldots,\theta,x,x',x,x';0)$$
$$=\langle |a_0|^2|a_0'|^2+[a_2\overline{a_0}|a_0'|^2+\overline{a_2}a_0|a_0'|^2+a_2'\overline{a_0'}|a_0|^2$$
$$+\overline{a_2'}|a_0|^2a_0']+[|a_1|^2|a_0'|^2+a_1a_1'\overline{a_0a_0'}+a_1\overline{a_1'}\overline{a_0}a_0'$$
$$+\overline{a_1}a_1'a_0\overline{a_0'}+\overline{a_1a_1'}a_0a_0'+|a_1'|^2|a_0|^2]\rangle. \quad (33)$$

All the exponents in a_0, a_1 cancel, and from (24) and (33) we obtain

$$m_{2,2}(\theta,\ldots,\theta,x,x',x,x';0)=1+4\ \mathrm{Re}(T_\theta)+2\sigma^2|f|^2+\rho(\xi)(2\ \mathrm{Re}[f^2+|f|^2]), \quad (34)$$

where $f(\theta)$ is again given by (20).

The relationship between these and the moments elsewhere in the medium will be discussed in the following sections. It should be remembered that these results pertain to the reduced (slowly varying) field $\psi(x,z)=\exp(-ikx)\times p(x,z)$ say; the moments of the full field contain additional phase factors $\exp(ikx)$.

III. AUTOCORRELATION OF FIELD AND INTENSITY

We now consider the moments in the medium. The most useful of these are perhaps $m_{1,1}$ and $m_{2,2}$ which give mean intensity $\langle I\rangle$ and mean-square intensity $\langle I^2\rangle$, as well as the cross correlation of the field on horizontal planes. We will consider here only $m_{1,1}$.

We show first that the second moment of any scattered field $\psi(x,z)$ does not change under propagation away from a fixed plane, if evanescent components of the field there are neglected. This is well known for three-dimensional electromagnetic propagation,[14] and is easily shown for propagation under the parabolic wave equation (where the principal direction of propagation is normal to the surface). The author is not aware of references for the present case, so a proof is given here. This does not rely explicitly on the Green's function and is easily generalized to electromagnetic and elastic waves.

Define the Fourier transform $\hat{\psi}$ of $\psi(x,0)$

$$\hat{\psi}(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(x,0)e^{-i\nu x}\,dx, \quad (35a)$$

so that

$$\psi(x,0)=\int_{-\infty}^{\infty}\hat{\psi}(\nu)e^{i\nu x}\,d\nu. \quad (35b)$$

Then the field at z is

$$\psi(x,z)=\int_{-\infty}^{\infty}\hat{\psi}(\nu)e^{i\nu x}e^{isz}\,d\nu, \quad (36)$$

where $s=\sqrt{k^2-\nu^2}$. We will exclude evanescent contributions to the field at $z=0$, and so we may assume that s is real. Consider the second moment $m_{1,1}(x,x',\theta,\theta;z)$ of the field at z for a single incident plane wave at angle θ, which we may denote $m_{1,1}(x-x';z)$. From (36) this can be written

$$m_{1,1}(x-x_2;z)=\int\int_{-\infty}^{\infty}\langle\hat{\psi}(\nu)\overline{\hat{\psi}(\nu')}\rangle e^{i(\nu x-\nu' x_2)}e^{i(s-s')z}\,d\nu\,d\nu', \quad (37)$$

where $s'=\sqrt{k^2-\nu'^2}$. Now by (35a) and the definition (9) the second moment of $\hat{\psi}$ is given by

$$\langle\hat{\psi}(\nu)\overline{\hat{\psi}(\nu')}\rangle=\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}m_{1,1}(x-x';0)\times e^{-i\nu x+i\nu' x'}\,dx\,dx'. \quad (38)$$

If we introduce the new variables

$$\xi=(x-x')/2,\quad X=(x+x')/2,$$

Eq. (38) becomes

$$\langle\hat{\psi}(\nu)\overline{\hat{\psi}(\nu')}\rangle=\frac{1}{4\pi^2}\int\int_{-\infty}^{\infty}m_{1,1}(2\xi;0)e^{i(\nu+\nu')\xi}\times e^{-i(\nu-\nu')X}\,dX\,d\xi. \quad (39)$$

Carrying out the integration with respect to X we obtain a delta function and (39) can be written

$$\langle\hat{\psi}(\nu)\bar{\hat{\psi}}(\nu')\rangle=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}2\pi\delta(\nu-\nu')\times m_{1,1}(2\xi;0)e^{i(\nu+\nu')\xi}\,d\xi. \quad (40)$$

When this is substituted in (37) the delta function removes the contributions from $\nu\neq\nu'$, so that the exponential $e^{i(s-s')}z$ can be set to unity. Thus the dependence upon z vanishes, i.e., the second moment for a single incident plane wave does not evolve with height above the surface and thus by Eq. (32)

$$m_{1,1}(\theta,\theta,x,x';z)\cong e^{ikS(x-x')}[1+2\ \mathrm{Re}(T_\theta)-8\rho(\xi)k^2S]. \quad (41)$$

We remark that in general, if the field is given on the plane $z=0$, the field elsewhere may be obtained from it in either of two equivalent ways: One is to invoke the Helmholtz–Kirchhoff equations using the free-space Green's function; the second is to Fourier transform the field to find the spatial Fourier components, include the relevant propagation factor for each, and transform back. In this way any of the moments at a given z plane can be found as a multiple integral in terms of the corresponding moment at $z=0$. Another approach to the problem is to seek linear differential equations for the moments and solve these directly. An advantage of this approach is that such equations can be formed even when the medium itself has a randomly varying refractive index.

We can now consider the angular distribution of the scattered intensity. For this we remove the specular component $1+2\ \mathrm{Re}(T_\theta)$, and replace the rapidly varying phase factors $\exp(ikx)$, to obtain the second moment $m_{1,1}^d$ for the diffuse component of the full field

$$m_{1,1}^d(\theta,\theta,x,x';0)\cong-8e^{2ik\sin\theta\xi}\rho(\xi)k^2S. \quad (42)$$

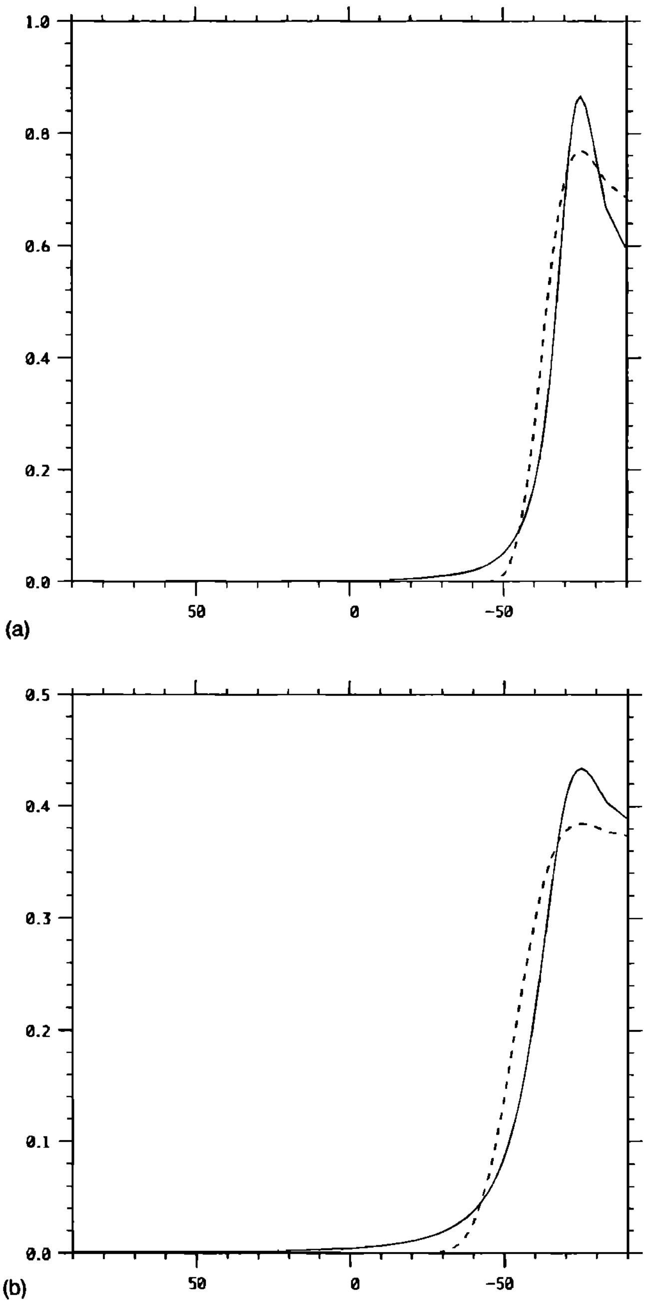

FIG. 2. Angular distribution of the diffuse component for μ=15° with (a) $L=10$ and (b) $L=5$, where L is the correlation length of the surface roughness. The angle θ is shown in degrees, measured from the normal. The distribution is shown for fractal (full line) and Gaussian (dashed line) autocorrelation functions.

By analogy with (40), we then define the angular distribution as

$$m(\nu)=\frac{1}{2\pi}\int_{-\infty}^{\infty} m_{1,1}^{d}(2\xi;0)e^{2i\nu\xi}\,d\xi. \qquad (43)$$

The above argument shows that this, as we would expect, also remains constant with distance from the surface. The quantity (42) is equivalent to that studied in Ref. 15, and is appropriate when the incident field is a plane wave. (The bistatic cross section, more commonly studied for incident beams which have a finite area of illumination, is not applicable here.)

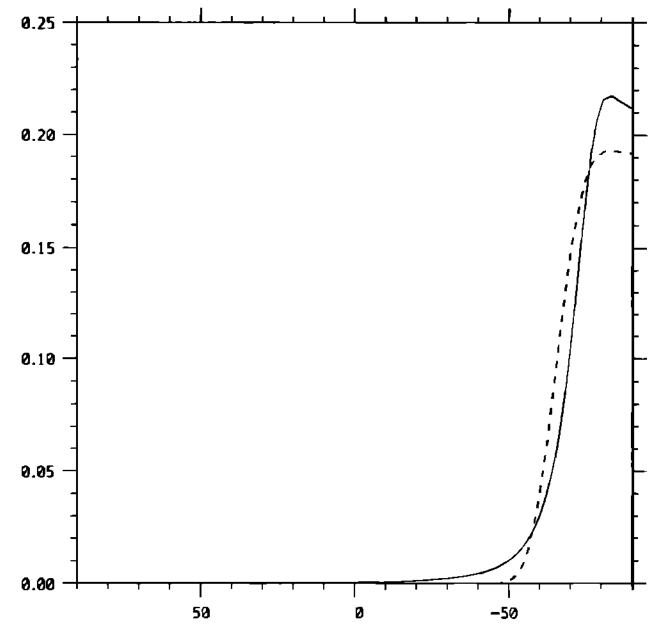

FIG. 3. Angular distribution as in Fig. 2, for μ=7.5° and $L=10$.

Figure 2(a) shows the angular distribution (43) due to a wave incident at a grazing angle of 15°, i.e., θ=75°, for surfaces with correlation length $L=10$. The full line corresponds to a surface with "fractal" autocorrelation function $\exp(-\xi/L)$, and the dashed line corresponds to a Gaussian autocorrelation function. The vertical scale on all figures here has been normalized by $k^2\sigma^2$. Figure 2(b) shows the same quantities for $L=5$. As would be expected the peak around the specular direction becomes broader as the correlation length decreases. A very small amount of backscattered energy (i.e., at negative angles) is visible in Fig. 2(b), but no backscatter enhancement can be observed because, as discussed above, the formulation does not take account of reversible paths. Note that in each case the small scale structure in the fractal surface leads to greater scatter away from the specular direction compared with that of a Gaussian surface. Finally Fig. 3 shows the angular spectrum for a grazing angle of 7.5°, i.e., half that of Fig. 2, again with correlation length $L=10$ and other parameters as in Fig. 2(a).

The change in incident angle has little effect on the breadth of the peak, but as the grazing angle approaches zero the energy in the diffuse component rapidly decreases. This is in accord with the fact that the rough surface acts as a "perfect reflector" in the limit as $\mu\to 0$.[2,13] From (42) and (43) the diffuse component vanishes with the square μ^2 of the grazing angle, since

$$S=\cos\mu-1=-\mu^2/2+O(\mu^4). \qquad (44)$$

The specular component $1+2\,\mathrm{Re}(T_\theta)$ (32) also approaches unity, but does so more slowly. From (22) this is linear in μ:

$$1+2\ \mathrm{Re}(T_\theta)=1+2\mu\frac{k^2}{\pi}\mathrm{Re}\left[\int_0^\infty\frac{e^{-ikS\xi}}{\alpha\sqrt{\xi}}\left(\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}\right)d\xi\right]+O(\mu^3). \tag{45}$$

We note that these expressions hold independently of the form of the surface autocorrelation function.

IV. VOLUME MOMENTS IN THE SMALL ANGLE LIMIT

Although we cannot easily obtain the general volume moments from those at the surface, the problem turns out to be trivial if the grazing angle is sufficiently small that its square can be neglected. In this limit the dependence of the moments upon distance from the surface is given purely by deterministic phase terms.

Consider the scattered field in the medium in relation to its value at $z=0$ (23). The evolution of each of the three terms a_i [Eq. (24)] is governed by the wave equation and by the radiation condition at infinity. We denote by L_z the linear evolution operator acting on functions of x, under which each function propagates outwards through a distance z, so that $L_z[\psi(x,z_1)]=\psi(x,z_1+z)$. Thus

$$\psi(x,z)=L_z\{a_0(\theta,x)+a_1(\theta,x)+a_2(\theta,x)\}. \tag{46}$$

As we have seen earlier, when the moments are formed the term a_2 appears only to first order and a_1 only to second order. Now, if we again denote by μ the grazing angle

$$\mu=\frac{\pi}{2}-\theta,$$

then we have

$$\sqrt{2-2\sin\theta}=\sqrt{2-2\cos\mu}=\mu+O(\mu^3)$$

and

$$\cos\theta=\sin\mu=\mu+O(\mu^3).$$

Thus from (24) and (20)

$$a_1=-2ikh(x)\mu+O(\mu^3) \tag{47}$$

and similarly from (13) and (24),

$$a_2=\mu\frac{h(x)}{\pi}\frac{d}{dx}\left\{\int_0^x ikh(x')\frac{e^{ikSx'}}{\alpha\sqrt{x-x'}}dx'\right\}+O(\mu^3). \tag{48}$$

Therefore at small μ, a_2 varies linearly with μ while the quadratic terms in a_1 vary like μ^2 and we will neglect them. This approximation may be considered valid for grazing angles from zero up to about 5°. With this restriction the scattered field becomes

$$\psi(x,z)\cong L_z\{a_0+a_2\}. \tag{49}$$

Consider now the moments of the field based on this reduced equation. Reasoning as before, by the binomial expansion (16) the one-point nth moment is

$$M_n(\theta,x;z)=\langle[\psi^\theta(x,z)]^n\rangle\cong(L_za_0)^n+na_0^{n-1}\langle L_za_2\rangle. \tag{50}$$

From the wave equation,

$$L_za_0(\theta,x)=-L_ze^{ikSx}=-e^{ik(Sx-z\cos\theta)}, \tag{51}$$

and by (25), since L is deterministic,

$$\langle L_za_2(\theta,x)\rangle=L_z\langle a_2(\theta,x)\rangle=T_\theta L_za_0(\theta,x) \tag{52}$$

so that from (24) and (50)–(52)

$$M_n(\theta,x;z)=-e^{ikn(Sx-z\cos\theta)}(1+nT_\theta). \tag{53}$$

Thus to first order in μ the one-point moments can be written [see (28)]

$$M_n(\theta,x;z)=e^{iknz\cos\theta}M_n(\theta,x;0).$$

In a similar way, the symmetric moments $M_{n,n}$ [Eq. (10)] can be written

$$M_{n,n}(\theta,x;z)=|L_za_0|^{2n}+n|L_za_0|^{2n-2}[\langle L_za_2\rangle\overline{L_za_0}+\langle\overline{L_za_2}\rangle L_za_0] \tag{54}$$

and by (30), (51), and (52) this gives

$$M_{n,n}(\theta,x;z)=(1+n[T_\theta+\overline{T_\theta}])=M_{n,n}(\theta,x;0). \tag{55}$$

Thus the intensity moments are unchanged to first order in the grazing angle as the field is scattered into the medium. It is easy to see that these results generalize to all the moments, which in this limit depend on distance from the surface only through some deterministic propagation factors $\exp(iknz\cos\theta)$.

A. Probability density

We briefly discuss the question of obtaining the probability density function (p.d.f.) of the scattered field. All statistics of the scattered wave field can be found from this function and it is therefore one of the ultimate goals of such work (see DeSanto and Brown[3]). If, for example, we are given the density function $f(X)$ of the wave field $\psi(x,z)$ (which will depend upon z), the mean of any function G is given by

$$\langle G(\psi)\rangle=\int_{-\infty}^{\infty}G(X)f(X)\,dX. \tag{56}$$

One way to calculate the probability density function is in terms of the moments, via the moment theorem (see Papoulis[16]) or characteristic function. The characteristic function is defined as the Fourier transform of the p.d.f.:

$$\Phi(\omega)=\int_{-\infty}^{\infty}e^{i\omega\Omega}f(\Omega)d\Omega. \tag{57}$$

Expanding the exponential and using (56) this can be written

$$\Phi(\omega)=1+\sum_{j=1}^{\infty}\frac{\langle(i\omega\psi^\theta(x))^j\rangle}{j!}.$$

If we have "enough" of the moments it is thus in principle a simple matter to form an approximation to the characteristic function:

$$\Phi(\omega)\cong 1+\sum_{j=1}^{n}\frac{\langle(i\omega\psi^{\theta}(x))^{j}\rangle}{j!}=\sum_{j=0}^{n}(i\omega)^{j}M_{j}(\theta,x;z), \tag{58}$$

where the moments M_j are given by (53) and the upper limit $n=n(\sigma)$ is determined according to Eq. (18) with $k\sigma=\epsilon$. Therefore in the limit of low angle (i.e., to first order in the angle and to second order in surface height) we may take the inverse transform of (58), by (57), to obtain an approximation to the probability density function as required.

The above argument, which applies to the one-point statistics, can be extended trivially to the two-point and higher-order statistics. If we drop the restriction to first order in the grazing angle, the same calculation holds for the statistics at the mean surface using the results of Sec. II, but in general no simple method exists to obtain from these the moments in the medium.

V. CONCLUSIONS

We have considered the scattering of a plane-wave incident upon a slightly rough surface at a low grazing angle μ. The second moment and angular distribution of intensity have been found to second order in surface height, and their dependence upon incident angle and the surface autocorrelation function has been shown. It has been found that as $\mu\rightarrow 0$ the diffuse component vanishes with μ^2, and the specular part approaches unity linearly in μ. The higher moments have been found at the mean surface, and to first order in μ these yield the corresponding quantities everywhere in the medium.

These results do not exhibit enhancement of backscattering in the antispecular direction; this effect[15,18–20] is largely due to coherent addition of reversible ray paths, which are precluded by the restriction to forward scattering at the surface. Any backscatter enhancement would probably be very small in this case. It is nevertheless important to quantify this, and the above results raise the question of whether backscatter enhancement vanishes at the same rate as the diffuse or the specular component. Another important effect which we have not considered here is that of depolarization due to scattering of electromagnetic waves, about which few theoretical results are known for this régime.

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APPENDIX: EFFECTIVE REFLECTION COEFFICIENTS

The integral in coefficient T_θ [Eq. (22)] is a Laplace transform. In several cases of interest we can express this integral in closed form, or in terms of standard functions. We give three examples, taken from Ref. 12. In each case the parameter L determines the correlation length of the irregular surface.

(1) Consider the "fractal" autocorrelation function

$$\rho(\xi)=\sigma^2\exp(-\xi/L).$$

Then

$$\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}=\left(S+\frac{1}{ikL}\right)\sigma^2\exp\left(-\frac{\xi}{L}\right).$$

The coefficient T_θ takes the value[17]

$$T_\theta=\frac{k^2}{\pi\alpha}\cos\theta\int_0^\infty\sigma^2\left(S+\frac{1}{ikL}\right)\frac{e^{-\xi(ikS+1/L)}}{\sqrt{\xi}}\,d\xi$$

$$=\frac{k^2}{\sqrt{\pi\alpha}}\sigma^2\cos\theta\frac{S+1/ikL}{\sqrt{ikS+1/L}}. \tag{A1}$$

(2) The "subfractal" autocorrelation function is

$$\rho(\xi)=\sigma^2(1+\xi/L)\exp(-\xi/L).$$

In this case

$$\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}=\sigma^2\exp(-\xi/L)\left[S+\xi\left(\frac{S}{L}+\frac{1}{ikL^2}\right)\right].$$

The coefficient is given by[17]

$$T_\theta=\frac{k^2}{\sqrt{\pi\alpha}}\sigma^2\cos\theta\left[\frac{S+1/2ikL}{\sqrt{ikS+1/L}}\right]. \tag{A2}$$

(3) Finally, consider a Gaussian autocorrelation function,

$$\rho(\xi)=\sigma^2\exp(-\xi^2/L^2).$$

In this case we can express the coefficients exactly in terms of standard functions. Here

$$\rho(\xi)S-\frac{1}{ik}\frac{d\rho(\xi)}{d\xi}=\sigma^2\exp\left(-\frac{\xi^2}{L^2}\right)\left[S+\frac{2\xi}{ikL^2}\right].$$

The integral in T_θ is the sum of two Laplace transforms, and can be written (see p. 146 of Ref. 17)

$$T_\theta=\frac{k^2}{\pi\alpha}\sigma^2\cos\theta\exp(-(kLS)^2/8)$$

$$\times\left[\frac{LS}{2}(ikS)^{1/2}K_{1/4}\left(\frac{(kLS)^2}{8}\right)\right.$$

$$\left.-\frac{2}{ikL^2}\left(\frac{L}{\sqrt{2}}\right)^{3/2}\Gamma\left(\frac{3}{2}\right)D_{-3/2}\left(\frac{ikSL}{\sqrt{2}}\right)\right]. \tag{A3}$$

Here Γ is the Gamma function, K is the modified Bessel function of the third kind, and D is the parabolic cylinder function.

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