Sound propagation in an irregular two-dimensional waveguide

Mark Spivack
Department of Applied Mathematics and Theoretical Physics, The University of Cambridge, CB3 9EW, England and Smith Institute, Surrey Research Park, Guildford, Surrey GU2 5YP, England

(Received 4 October 1995; revised 23 September 1996; accepted 31 October 1996)

A method is presented for the numerical calculation of a scalar wave propagating along a two-dimensional rough-sided or irregular waveguide. In this situation the wave becomes multiply scattered, with simultaneous interaction at the two boundaries. The field can be expressed in terms of a pair of coupled integral equations; these are derived and solved in an approach based on the parabolic integral equation method, which assumes that all energy is carried in a forward direction. An extended formulation encompassing backscatter is also derived, and a method given for its treatment. This paper serves in part to explain the computational results presented in B. J. Uscinski, ''High-frequency propagation in shallow water,'' J. Acoust. Soc. Am. **98**, 2702–2707 (1995).
 [S0001-4966(97)03703-X]

PACS numbers: 43.20.Fn, 43.20.Mv, 43.30.Hw [ANN]

INTRODUCTION

In many acoustic and electromagnetic applications, waves propagate along irregular or rough-sided waveguides,[1–5] becoming scattered progressively as they impinge upon the boundaries. This is an important mechanism, for example, for sound propagation in shallow water and ice, or electromagnetic waves in ducts and dielectric layers. The surfaces often vary randomly on a scale comparable with a wavelength; this gives rise to a high degree of multiple scattering, which is enhanced by the low angles of incidence. The field can be expressed exactly in terms of boundary integrals; these give rise to a pair of coupled integral equations whose treatment is difficult both numerically and analytically.

In the simpler case of a single boundary (i.e., a rough half-space) the analogous low grazing angle problem has been treated successfully using parabolic integral equation method.[6–8] Under the approximation that all energy is forward scattered, the full wave equation is replaced by the parabolic equation, an appropriate Green's function derived, and the exact boundary integral formulation replaced by the parabolic integral equations. (These equations allow for waves traveling only in the direction away from the source, but an extended description taking into account multiple backscatter has been derived.[9])

The same approach can be applied to the present problem,[4] giving rise to a set of coupled equations, again within the approximation that forward scatter predominates. The first purpose of this note is to explain the treatment of the coupled system, and to show how the parabolic form allows fast and efficient solution. The method is implemented here for the case in which both surfaces have pressure release boundary conditions. The system is first discretized with respect to range, giving a pair of coupled matrix equations in the unknown vertical derivatives of the field along the two surfaces. The ''one-way'' form of the equations allows the system to be solved progressively from the left, say. This requires only $O(N^2)$ operations, where N is the number of points used in the discretization. The advantages can be seen when the pair of matrix equations are rewritten formally in terms of a single ''matrix operator'' $\mathscr{M}$, whose entries are themselves 2×2 matrices; this matrix operator is lower triangular, and can therefore be inverted efficiently. Computational examples are given below for a Gaussian beam traveling at various angles with respect to the horizontal.

One drawback is that backscatter, the relevant quantity in applications such as detection and imaging, is precluded by this description as it stands. The second aim of the paper is to extend the method to include direct and indirect backscatter components. The one-way Green's function is replaced by an analogous two-way form; the resulting equations thereby include components of the field scattered back toward the source. In this form the equations do not directly yield a lower triangular system, so that the computational advantage is initially lost, but the solution can be expressed as a series in which each term is again lower triangular. It is explained below how this system may be treated in a way similar to that above.

The problem is formulated in Sec. I, and the implementation of the numerical solution given in Sec. II, with computational examples. In Sec. III an extended form of the equations is derived taking backscatter into account, and it is explained how these equations may be treated similarly.

I. FORMULATION OF PROBLEM

Consider a two-dimensional waveguide with boundaries varying irregularly about the horizontal direction, as shown schematically in Fig. 1. We will as far as possible follow the conventions of previous papers.[6,8] The derivation of equations is similar to that for the case of a single surface (irregular half-space)[6,7] and details will be kept to a minimum here. Let x be the horizontal and z the vertical axis, and denote the upper and lower surfaces by $h_1(x)$, $h_2(x)$ respectively. We consider time-harmonic solutions p of the wave equation $(\nabla^2+k^2)p=0$, with wave number k. Then p can be considered as the sum

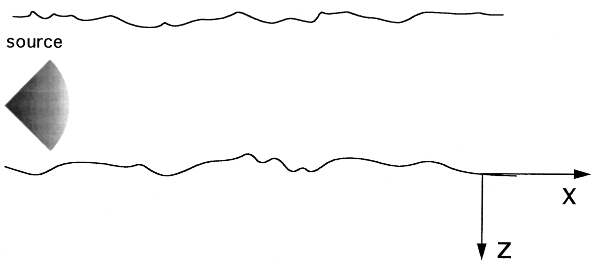

FIG. 1. Schematic view of the scattering geometry.

$$p=p_{\mathrm{inc}}+p_s$$

of an incident field p_{inc} and a scattered component p_s due to the presence of the surfaces. It will be assumed that angles of propagation and scatter are small with respect to the horizontal, so the field has a rapidly varying phase component $\exp(ikx)$. This can be factored out, and the reduced wave ψ introduced,

$$\psi=p\ \exp(-ikx).$$

Then ψ obeys the parabolic or one-way wave equation

$$\psi_x+2ik\psi_{zz}=0, \tag{1}$$

and we can write

$$\psi=\psi_{\mathrm{inc}}+\psi_s\,, \tag{2}$$

where the reduced forms ψ_{inc}, ψ_s of the incident and scattered fields are defined similarly. Now the parabolic form of the Green's function can be derived for wave equation (1). This is given by

$$G_p(x,z;x',z')=\begin{cases}\dfrac{1}{2}\sqrt{i/2\pi k}\sqrt{\dfrac{1}{x-x'}}\exp\left[\dfrac{ik(z-z')^2}{2(x-x')}\right] & \text{for } x'<x\\ 0, \quad \text{otherwise}\,. & \end{cases} \tag{3}$$

By analogy with the Helmholtz equations, and as for previous applications of the parabolic integral equation method to a rough half-space, the field at a point r inside the waveguide can be expressed as an integral across the bounding surfaces:

$$\psi_s(\mathbf{r})=\psi_{\mathrm{inc}}+\int_S\left[G(\mathbf{r};\mathbf{r}')\,\frac{\partial\psi}{\partial z}\,(\mathbf{r}')-\frac{\partial G}{\partial z}\,(\mathbf{r};\mathbf{r}')\,\psi(\mathbf{r}')\right]dx', \tag{4}$$

where $\mathbf{r}'=(x',S(x'))$, and S is the union of the surfaces h_1 and h_2.

For convenience we will henceforth specialize to the case of pressure release surfaces. The method is equally applicable when either or both surfaces obey Neumann boundary conditions. (See concluding remarks for discussion of more general cases.) In this case Eq. (4) becomes

$$\psi_s(\mathbf{r})=\int_{h_1+h_2}G(\mathbf{r};\mathbf{r}')\,\frac{\partial\psi}{\partial z}\,(\mathbf{r}')\,dx'. \tag{5}$$

Taking limits as $\mathbf{r}$ tends, respectively, to the upper and to the lower surfaces, and applying the boundary conditions, we obtain two coupled integral equations for the unknown values of the quantity $\partial\psi/\partial z$ along each of the two surfaces. It is convenient to regard these as separate functions of the single coordinate x, and accordingly we define

$$\begin{aligned}\psi^1(x)&=\frac{\partial\psi}{\partial z}\,(x,h_1(x)),\\ \psi^2(x)&=\frac{\partial\psi}{\partial z}\,(x,h_2(x)).\end{aligned} \tag{6}$$

The coupled integral equations for ψ^1, ψ^2 can then be written

$$\begin{aligned}\psi_{\mathrm{inc}}(\mathbf{r}_2)&=\int_0^x[G(\mathbf{r}_2;\mathbf{r}_2')\,\psi^2(x')-G(\mathbf{r}_2;\mathbf{r}_1')\,\psi^1(x')]\,dx',\\ \psi_{\mathrm{inc}}(\mathbf{r}_1)&=\int_0^x[G(\mathbf{r}_1;\mathbf{r}_2')\,\psi^2(x')-G(\mathbf{r}_1;\mathbf{r}_1')\,\psi^1(x')]\,dx',\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathbf{r}_1&=(x,h_1(x)),\quad \mathbf{r}_1'=(x',h_1(x')),\\ \mathbf{r}_2&=(x,h_2(x)),\quad \mathbf{r}_2'=(x',h_2(x')).\end{aligned} \tag{8}$$

The main task is to invert this set of coupled equations to find the field derivatives ψ^1, ψ^2 along the surfaces. These may then be substituted into Eq. (5) to yield the value of the field in the waveguide.

For computational purposes we will take as a source a Gaussian beam, centered at a point $(0,z_0)$, say, whose principal direction is at some small angle θ to the horizontal. This gives rise to an incident field

$$E_{\mathrm{inc}}(x,z)=\frac{w}{\sqrt{w^2+2ix/k}}\exp\left[-\frac{2z^2+ikSw^2(Sx-z)}{2(w^2+2ix/k)}\right], \tag{9}$$

where $S=\sin(\theta)$, and w is the width of the beam.

II. SOLUTION

In this section the numerical solution of the integral equations will be explained, and computational examples given.

A. Numerical implementation

The numerical treatment is similar in many respects to that of the corresponding rough half-space problem (described elsewhere[6]), so we concentrate here on the additional complications introduced by the coupling of the equations. Some care is required in treating the integrands, which contain weak (i.e., integrable) singularities.

Following previous treatments we discretize Eqs. (7) with respect to range x introducing, say, N equally spaced points $\{x_n\}$, $n=1,\ldots,N$. The first of Eqs. (7), for example, is written as a sum of subintervals

$$\psi_{\mathrm{inc}}(\mathbf{r}_2)=\sum_{j=1}^{n-1}\int_{x_j}^{x_{j+1}}[G(\mathbf{r}_2;\mathbf{r}_2')\,\psi^2(x')-G(\mathbf{r}_2;\mathbf{r}_1')\,\psi^1(x')]\,dx', \tag{10}$$

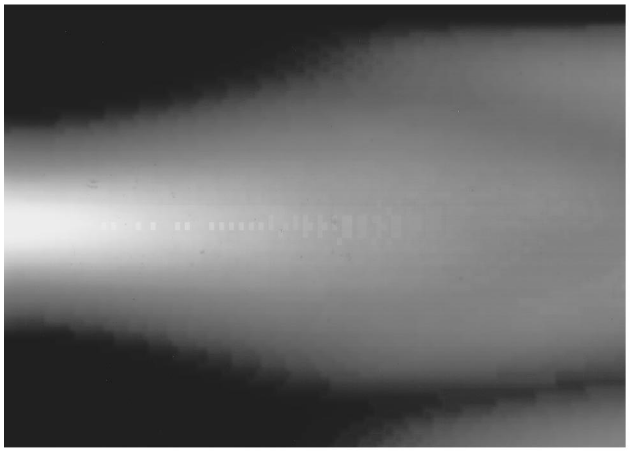

FIG. 2. Amplitude of the wave due to a horizontally traveling Gaussian beam, in a regular flat-sided waveguide.

where $\mathbf{r}_2=(x_n,h_2(x_n))$, $\mathbf{r}_1'=(x',h_1(x'))$, and $\mathbf{r}_2' = (x',h_2(x'))$. We may assume that the unknown functions ψ^1, ψ^2 vary sufficiently slowly to be treated as constant over each of the subintervals (x_j,x_{j+1}), and can therefore be taken outside the integral. Writing $X_n=(x_{n+1}+x_n)/2$, we then replace ψ^1, ψ^2, and the incident field along the surfaces by vectors:

$$\begin{aligned} a_n&=\psi_{\mathrm{inc}}[x_n,h_1(x_n)],\\ b_n&=\psi_{\mathrm{inc}}[x_n,h_2(x_n)],\\ c_n&=\psi^1(X_n),\quad d_n=\psi^2(X_n). \end{aligned} \tag{11}$$

Equations (7) then become

$$\begin{aligned} a_n&=\sum_{j=1}^{n-1}[S_{j,n}c_j+T_{j,n}d_j],\\ b_n&=\sum_{j=1}^{n-1}[S_{j,n}'c_j+T_{j,n}'d_j], \end{aligned} \tag{12}$$

where

$$\begin{aligned} S_{j,n}&=\int_{x_{j-1}}^{x_j}(\mathbf{r}_1;\mathbf{r}_1')dx',\\ T_{j,n}&=-\int_{x_{j-1}}^{x_j}G(\mathbf{r}_1;\mathbf{r}_2')dx',\\ S_{j,n}'&=\int_{x_{j-1}}^{x_j}G(\mathbf{r}_2;\mathbf{r}_1')dx',\\ T_{j,n}'&=-\int_{x_{j-1}}^{x_j}G(\mathbf{r}_2;\mathbf{r}_2')dx', \end{aligned} \tag{13}$$

and

$$\begin{aligned} \mathbf{r}_1&=(x_n,h_1(x_n)),\quad \mathbf{r}_1'=(x',h_1(x')),\\ \mathbf{r}_2&=(x_n,h_2(x_n)),\quad \mathbf{r}_2'=(x',h_2(x')). \end{aligned} \tag{14}$$

Note that, by expanding the integrand in each interval about the endpoint x_j, the integrals (13) can be carried out analytically and expressed in terms of Fresnel integrals. This becomes particularly important when the x values of the arguments of the Green's function are close (especially at large vertical separation), because $G(x,z;x',z')$ has a weak singularity as $x'\rightarrow x$. Further details can be found in Ref. 6.

The equations are now in a form which can be solved by induction, or progressively from the left. At each stage the calculation reduces to a pair of simultaneous linear equations for the values of c_n,d_n at a single range step x_n, say.

Step 1:

For $n=1$, Eqs. (12) give:

$$\begin{aligned} a_1&=S_{11}c_1+T_{11}d_1,\\ b_1&=S_{11}'c_1+T_{11}'d_1. \end{aligned} \tag{15}$$

Multiplying the first by S_{11}', the second by S_{11}, and subtracting, this gives the solutions for the field derivative at the initial range step

$$\begin{aligned} d_1&=\frac{S_{11}b_1-S_{11}'a_1}{S_{11}T_{11}'-S_{11}'T_{11}},\\ c_1&=(a_1-T_{11}d_1)/S_{11}. \end{aligned} \tag{16}$$

Step 2:

Assume that c_j,d_j are known for $j=1,\ldots,n-1$. Then Eqs. (12) can be written in the form

$$\begin{aligned} a_n-\sum_{j=1}^{n-1}[S_{jn}c_j+T_{jn}d_j]&=S_{nn}c_n+T_{nn}d_n,\\ b_n-\sum_{j=1}^{n-1}[S_{jn}'c_j+T_{jn}'d_j]&=S_{nn}'c_n+T_{nn}'d_n. \end{aligned} \tag{17}$$

This is solved to find c_n,d_n, exactly as for $n=1$: Write

$$\begin{aligned} X_n&=a_n-\sum_{J=1}^{n-1}[S_{jn}c_j+T_{jn}d_j],\\ Y_n&=b_n-\sum_{J=1}^{n-1}[S_{jn}'c_j+T_{jn}'d_J]. \end{aligned} \tag{18}$$

Then we obtain

$$\begin{aligned} d_n&=\frac{S_{n,n}Y_n-S_{n,n}'X_n}{S_{nn}T_{nn}'-S_{nn}'T_{nn}},\\ c_n&=\frac{X_n-T_{nn}d_n}{S_{nn}} \end{aligned} \tag{19}$$

as required. This is carried out progressively up to the maximum range x_N.

B. Computational examples

The scheme described above can be applied both to regular and irregular waveguides. In the following examples, the vertical scale is exaggerated, approximately by a factor of 10, so that on a true scale the figures would be stretched horizontally. Figure 2 shows the amplitude of the total field in a regular flat-sided waveguide, due to a Gaussian beam traveling horizontally i.e., with principal direction parallel to the mean surface direction. The source is at the left, and the field vanishes on both surfaces. (This is visible at the lower surface which corresponds to the lower edge of the plot; the

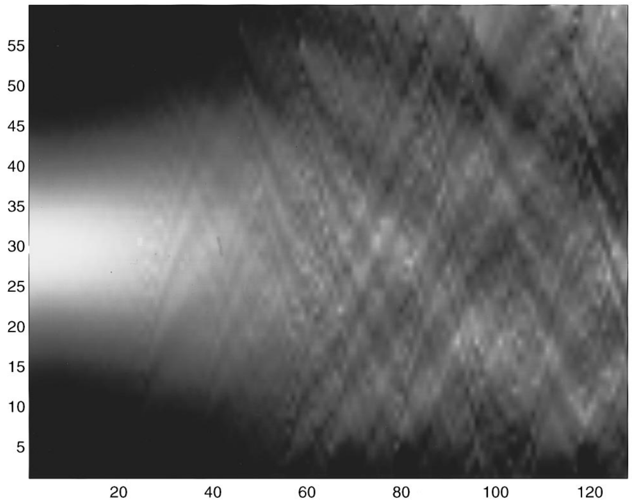

FIG. 3. Amplitude of the total (i.e., incident plus scattered) wave, for the same incident field as in Fig. 2 where the surfaces are now rough. The horizontal scale size L of each surface is approximately $\lambda/4$ where λ is the wavelength.

top of the graph extends up to slightly below the upper waveguide surface.) An example is shown in Fig. 3 of the field resulting when the surfaces are rough; here the scattering has largely destroyed the deterministic interference pattern. The surface correlation length, L, is about a quarter of a wavelength. Figure 4 shows a calculation for surfaces with the same rms height, but with scale sizes L about four times as large, i.e., of the order of a wavelength.

Figures 5 and 6 show similar configurations, but here the beam is at a nonzero angle θ to the horizontal. Again the disruption of the flat waveguide pattern is clear.

The parabolic equation method which we apply requires low angles of propagation and scatter, i.e., less than around 15° to the x direction. It is difficult to translate this into a precise limitation on the form of the rough surfaces, because a general solution for the scattered field is unavailable. However, the method should yield reasonably accurate results provided that the maximum average angles of slope relative to the direction of incidence are less than 15°.

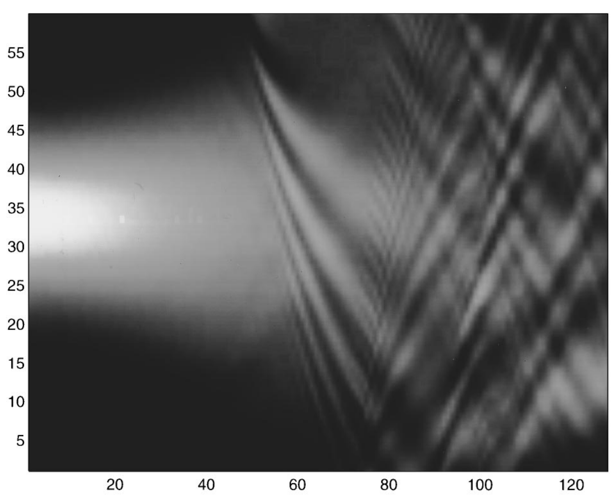

FIG. 4. Similar plot to Fig. 3, buth with scale sizes L four times as large.

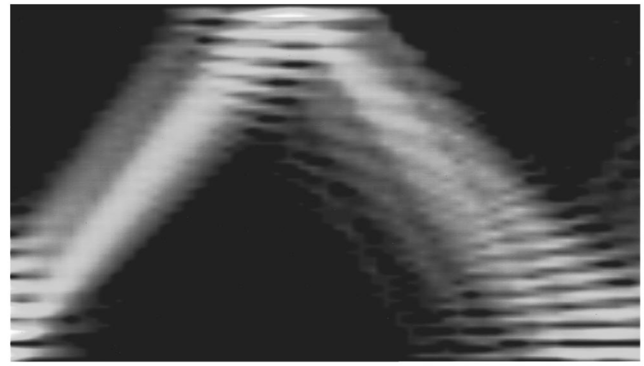

FIG. 5. Amplitude of the wave in a flat-sided waveguide, due to a Gaussian beam traveling at an angle θ=0.2.

III. MULTIPLE BACKSCATTER

In many circumstances, particularly in detection and imaging, the principal quantity required is power scattered back towards the source, and the above purely forward scatter calculation is not sufficient. In this section we formulate a method which takes account of backscatter, although numerical implementation will not be carried out here.

The above discretized solution is first expressed in more formal terms, in order to simplify the eventual backscatter calculation.

Define vectors $\mathbf{A}=(A_1,\ldots,A_N)$, $\mathbf{C}=(C_1,\ldots,C_N)$ of length N, whose elements are themselves two-dimensional vectors,

$$A_n=(a_n,b_n), \quad C_n=(c_n,d_n), \tag{20}$$

where a_n, b_n, c_n, d_n are given by (11). Thus, in effect $\mathbf{A}$, $\mathbf{C}$ are functions of x. Define the 2×2 matrices M_{jk} by

$$M_{jk}=\begin{pmatrix} S_{jk} & T_{jk} \\ S'_{jk} & T'_{jk} \end{pmatrix}. \tag{21}$$

Then Eq. (15) becomes

$$A_1=M_{11}C_n,$$

Eq. (17) is

$$A_n-\sum_{j=1}^{n-1}[M_{jn}C_j]=M_{nn}C_n,$$

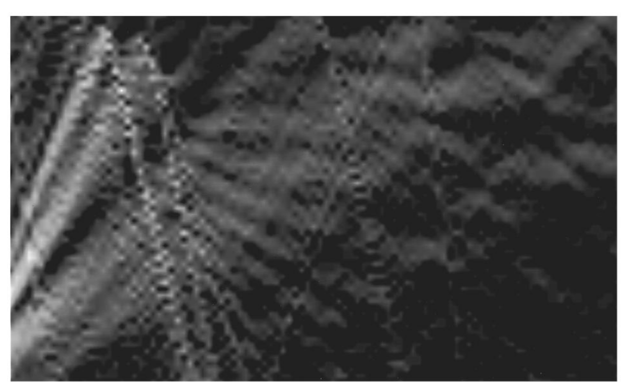

FIG. 6. Amplitude of the wave in a rough-sided waveguide, due to a Gaussian beam traveling at an angle θ=0.2.

and so on. As is easily checked, we thus obtain the matrix operator equation

$$\mathbf{A} = \mathscr{M}\mathbf{C}, \tag{22}$$

where $\mathscr{M}$ is a lower-triangular matrix operator whose entries are themselves the 2×2 matrices M_{jk}, for $j \leq k$. This is simply a convenient way of writing the coupled system of Eqs. (12).

The solution of (22) can formally be written

$$\mathbf{C} = \mathscr{M}^{-1}\mathbf{A}. \tag{23}$$

This expresses the discretized forms of the unknown functions ψ^1, ψ^2 in terms of the known incident field, and Eqs. (12)–(19) show how the inversion of $\mathscr{M}$ may be carried out in practice.

The next step is to derive extended governing equations, making use of the *two-way* parabolic form of the Green's function $\mathscr{G}$, given by[9]

$$\mathscr{G}(x,z;x',z') = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{i}{2\pi k}}\sqrt{\dfrac{1}{x-x'}}\exp\left[\dfrac{ik(z-z')^2}{2(x-x')}\right], & x'<x \\ \dfrac{1}{2}\sqrt{\dfrac{i}{2\pi k}}\sqrt{\dfrac{1}{x'-x}}\exp\left[\dfrac{ik(z-z')^2}{2(x'-x)}\right] \\ \quad\times\exp[2ik(x'-x)], & x\geq x. \end{cases}$$

The factor $\exp[2ik(x'-x)]$ arises for $x' \geq x$ because we are solving for the reduced wave ψ. Using the new Green's function we can then derive extended forms of Eqs. (4) and (7):

$$\psi(\mathbf{r}) = \int_S \left[\mathscr{G}(\mathbf{r};\mathbf{r}')\,\frac{\partial\psi}{\partial z}(\mathbf{r}') - \frac{\partial\mathscr{G}}{\partial z}(\mathbf{r};\mathbf{r}')\,\psi(\mathbf{r}')\right]dx', \tag{24}$$

where again $\mathbf{r}' = [x', S(x')]$, and S is the union of the surfaces h_1 and h_2. Taking the limit as r approaches the surface gives rise as before to coupled integral equations:

$$\begin{aligned} \psi_{\text{inc}}(\mathbf{r}_2) &= \int_0^\infty [\mathscr{G}(\mathbf{r}_2;\mathbf{r}_2')\,\psi^2(x') - \mathscr{G}(\mathbf{r}_2;\mathbf{r}_1')\,\psi^1(x')]dx', \\ \psi_{\text{inc}}(\mathbf{r}_1) &= \int_0^\infty [\mathscr{G}(\mathbf{r}_1;\mathbf{r}_2')\,\psi^2(x') - \mathscr{G}(\mathbf{r}_1;\mathbf{r}_1')\,\psi^1(x')]dx'. \end{aligned} \tag{25}$$

The key difference in these equations is that, since $\mathscr{G}$ no longer vanishes for $x' > x$, integration is across the whole surface and therefore allows for left-traveling wave components. These equations may be discretized exactly as before. Omitting the details, this eventually results in a matrix operator equation, analogous to (22):

$$\mathbf{A} = \mathscr{P}\mathbf{C}, \tag{26}$$

with formal solution

$$\mathbf{C} = \mathscr{P}^{-1}\mathbf{A}, \tag{27}$$

where $\mathscr{P}$ is now a full matrix, whose entries are 2×2 matrices, which is simply the sum of $\mathscr{M}$ with an upper-triangular part $\mathscr{Q}$:

$$\mathscr{P} = \mathscr{M} + \mathscr{Q}. \tag{28}$$

It can be shown that, in the regime here in which most energy is scattered to the right, $\mathscr{Q}$ has the effect of a small perturbation. The inverse of $\mathscr{P}$ can thus be expanded in a series about $\mathscr{M}^{-1}$:

$$\mathscr{P}^{-1} = [1 - \mathscr{M}^{-1}\mathscr{Q} + (\mathscr{M}^{-1}\mathscr{Q})^2 - \cdots]\mathscr{M}^{-1} \tag{29}$$

and solution (27) can be approximated by truncating this and substituting in (27):

$$\mathbf{C} \cong [1 - \mathscr{M}^{-1}\mathscr{Q}]\mathscr{M}^{-1}\mathbf{A}. \tag{30}$$

This expresses the solution of the extended system as the forward-going result (27) plus a correction accounting for backscatter via the action of the operator $\mathscr{Q}$. Here $\mathscr{Q}$ simply acts by matrix multiplication, and so (30) requires the inversion only of the same ''lower-triangular'' coupled system represented by $\mathscr{M}$, which has been described earlier.

IV. DISCUSSION

The paper has described the efficient solution for a scalar wave propagating along a rough-sided waveguide, a problem requiring the evaluation of a pair of coupled integral equations. Results have been presented for the purely forward scattered component, and the extension to backscatter of the governing equations and their numerical treatment have been explained.

This approach is computationally convenient and makes possible the numerical investigation of other questions which have not been discussed here, such as field statistics when the surfaces are randomly rough (see Refs. 5 and 11). One such question concerns the mean scattered field $\langle\psi_s\rangle$: It can be shown that the effect of surface roughness upon the mean field is equivalent to the solution for a *flat* waveguide, in which the reflection coefficient of each surface is replaced by an effective one, R_e say, depending on the surface statistics and the depth δ of the waveguide. However, R_e in general is not known, and some studies approximate it by the coefficient for the *isolated* rough surface, as in the rough half-space solution. It can be argued[10] that this holds in the limit of large δ (specifically when $\delta \gg kL^2$, due to the evolution of the wave on propagation across the waveguide). This is clear from Eq. (7); when δ becomes large the cross terms $G(\mathbf{r}_2;\mathbf{r}_1')$, $G(\mathbf{r}_1;\mathbf{r}_2')$ oscillate rapidly, and the two equations approximately decouple. This issue is also discussed by Voronovich[11] in terms of the ''skip distance,'' which depends upon the horizontal wavenumber and is in some sense a more refined estimate. It appears that these two measures in fact reflect separate mechanisms for decorrelation.

The method can be extended in principle to waveguides with penetrable surfaces, where the boundary conditions can no longer be expressed in terms of reflection coefficients independent of frequency. In the related vector wave case, for example in an elastic layer adjoining a fluid half-space, additional mechanisms come into play such as mode conversion ($P-S$, etc.) at the interfaces, and leakage of energy outside the layer. Such complications are less easy to treat with this approach because the angles of scattering due to wave conversion may easily exceed the limits imposed by the parabolic wave equation.

ACKNOWLEDGMENTS

The author holds the Smith Institute Research Fellowship in the Department of Applied Mathematics and Theoretical Physics in the University Cambridge, and is grateful to B. J. Uscinski and D. Berman for numerous discussions related to this work and to the referee for helpful comments.

[1] L. M. Brekhovskikh, *Elastic Waves in Layered Media,* Applied Mathematics and Mechanics, Vol. 16 (Academic, New York, 1980).

[2] T. W. Dawson and J. A. Fawcett, ''A boundary integral equation method for acoustic scattering in a waveguide with nonplanar surfaces,'' J. Acoust. Soc. Am. **87**, 1110–1125 (1990).

[3] A. S. Starkov, ''Normal mode propagation in a waveguide with an irregular section,'' Sov. Phys. Acoust. **36**, 419–423 (1990).

[4] B. J. Uscinski, ''High-frequency propagation in shallow water,'' J. Acoust. Soc. Am. **98**, 2702–2707 (1995).

[5] D. H. Berman, ''The mean acoustic field in layered media with rough interfaces,'' J. Acoust. Soc. Am. **98**, 542–551 (1995).

[6] M. Spivack, ''A numerical approach to rough surface scattering by the parabolic equation method,'' J. Acoust. Soc. Am. **87**, 1999–2004 (1990).

[7] E. Thorsos, ''Rough surface scattering using the parabolic wave equation,'' J. Acoust. Soc. Am. Suppl. 1 **82**, S103 (1987).

[8] M. Spivack, ''Moments and angular spectrum for rough surface scattering at grazing incidence,'' J. Acoust. Soc. Am. **97**, 745–753 (1995).

[9] M. Spivack, ''Backscatter at low grazing angles from rough surfaces,'' J. Acoust. Soc. Am. (submitted).

[10] J. D. Sheard and M. Spivack, ''Wave scattering in a rough elastic layer adjoining a fluid half-space,'' J. Acoust. Soc. Am. **97**, 72–83 (1995).

[11] A. G. Voronovich, *Wave Scattering from Rough Surfaces* (Springer-Verlag, Berlin, 1994).