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The split-step solution in random wave propagation *

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Abstract: This paper considers the accuracy of the split-step solution for wave propagation in random media, both analytically and numerically. The main interest lies in finding the statistical moments of the wavefield, averaged over time or realisations of the random medium. The perturbation error is found explicitly as a function of operators acting on the wavefield. The accuracy of the wave and its moments is then examined, in terms of the error and its autocorrelation function. It is shown that both step-wise and cumulative accuracy of the moments is greater than it is for the wavefield itself. A model is also described in which the random medium may be represented, using a series of correlated phase-screens, in arbitrarily fine detail. This is used to examine the convergence of the split-step solution as the step-size is reduced. It is shown that the independent phase-screen model is valid for small scattering when the scale-size of the medium is less than the step-size in the direction of propagation.

Keywords: Operator splitting, stochastic differential equation, random wave propagation.

1. Introduction

Numerical simulation is now widely used in the study of wave propagation in random media, which arises in several branches of optics and acoustics. A parabolic approximation to the wave equation is normally used and the solution is approximated analytically by the split-step method [4,6]. A numerical scheme is then applied, in which the medium is represented by a series of independent phase-screens.

The equation is in the form of a linear Schrödinger equation perturbed by the addition of a multiplication operator which varies randomly and continuously in all coordinates. This function represents random fluctuations in the medium, which causes continuous scattering of the propagating wave. In practice, it may be characterised statistically, but its detailed features will not be known everywhere. The main interest thus lies in finding the statistical moments of the resultant wavefield u, averaged over time or realisations, and their relationship with those of the medium. These quantities include the autocorrelation function and that of intensity, which describe such effects as the scintillation of stars. The moments, which are range-dependent, are themselves governed by (higher-dimensional) differential equations and can be compared with theory and experiment.

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The accuracy with which the split-step solution approximates the analytical form is not well understood, and its effect on the moments has not previously been examined at all. The main purpose of this paper is to consider these problems both analytically and by use of new numerical techniques. In our approach the error ϵ is expressed explicitly in terms of a commutator of the operators acting on u, whose properties we examine. ϵ is itself a continuous function varying randomly in all directions in a way determined by the medium. The error induced in the autocorrelation function of the wavefield can be expressed in terms of the correlation functions of ϵ with itself and with u. It is shown that both step-wise and cumulative accuracy of the moments is of higher order than for the wavefield itself. The generation of correlated sets of random phase-screens enables us to represent a medium in arbitrarily fine detail, and this is used to conduct a numerical examination of the convergence of the solution. (This technique is also essential in modelling problems which involve strong Markov scattering, i.e., where successive scatters are correlated. Such applications of the model will be presented elsewhere.)

In Section 2 the mathematical preliminaries are given and the solution is considered analytically, giving the perturbation and global errors for the wavefield and its moments. The computational scheme, the generation of the medium, and numerical results are described in Section 3.

2. Mathematical formulation and solution

The situation we consider is that of a monochromatic plane wave incident on a half plane $(x,\ z>0)$ with randomly-varying refractive index n. The wavelength λ is assumed to be significantly shorter than the typical scale-size of irregularities so that scattering takes place mainly in a forward z-direction. The full wave equation can then be approximated by a parabolic form in which z becomes a time-like direction of propagation. This expression is, in 2-dimensional Cartesian coordinates $(x,\ z)$,

$$u_z = -\frac{\mathrm{i}}{2k} u_{xx} - \mathrm{i}\frac{k}{2}(n^2-1)u, \tag{2.1}$$

where u is the complex wavefield, $k=2\pi/\lambda$, and the subscript denotes differentiation. We can write the refractive index as follows:

$$n = 1 + n_{\mathrm{d}}(x,\ z) + \mu W(x,\ z), \tag{2.2}$$

where n_{d} is a deterministic departure from 1 and μW varies randomly with zero mean and standard deviation μ. Here W is assumed to have Gaussian statistics but arbitrary correlation function, with correlation lengths l and l_z transverse to and parallel to the direction of propagation respectively. (Time-dependence has been excluded since the wave-speed is typically very fast relative to changes in n.) We employ the usual scaling (e.g., [7]) and set $X=x/l$, $Z=z/kl^2$ and $L_Z=l_z/kl^2$. Since we assume n_{d} and μ to be small, (2.1) can be written, in terms of operators, as

$$\frac{\partial u}{\partial Z} = (A+B)u, \tag{2.3}$$

where $A = -\frac{1}{2}\mathrm{i}\ \partial^2/\partial X^2$ and $B = -\mathrm{i}k^2L^2(n_\mathrm{d} + \mu W)$ are the distance and scattering operators respectively. The formal solution of (2.3) over a distance ΔZ is then approximately

$$u(Z + \Delta Z) \approx \exp\left[\int_Z^{Z+\Delta Z} (A + B)\ \mathrm{d}Z\right] u(Z). \tag{2.4}$$

This is invariably taken as the basis of numerical approaches, and its accuracy is discussed below. The split-step approximation, introduced to this problem by Tappert and Hardin [6] and applied by Macaskill and Ewart [4] to random media, is

$$u(Z + \Delta Z) \approx \exp\left[\int A\ \mathrm{d}Z\right] \exp\left[\int B\ \mathrm{d}Z\right] u(Z). \tag{2.5}$$

The implementation of this iteratively applied solution will be described in Section 3. It has a step-wise error of order $(\Delta Z)^2$. (A slight modification to (2.5) yields Strang's splitting which has error of order $(\Delta Z)^3$.) However, more crucial are its dependence on the scattering strength and the propagating wavefield. In other words, accuracy is dominated by the behaviour of the operators A and B.

Now, $\int_Z^{Z+\Delta Z} A\ \mathrm{d}Z = -\frac{1}{2}\mathrm{i}\Delta Z\ \partial^2/\partial X^2 = A'$, say. If ΔZ is sufficiently small, such that B is almost constant in Z over that distance, say $B \approx B(X)$, then $\int_Z^{Z+\Delta Z} B\ \mathrm{d}Z \approx \Delta Z\ B = B'$, say. Consider the error, $\epsilon(u) = [\exp(A' + B') - \exp(A')\exp(B')]u$, between (2.4) and (2.5). $\epsilon(u)$ is strongly dependent on u, and therefore upon distance, and on the stochastic quantity B'. However $\epsilon(u)$ is, to first order, $\frac{1}{2}\epsilon_1(u)$ where ϵ_1 is the commutator $[A', B'] \equiv A'B' - B'A'$ (e.g., [1]) and when A' and B' "nearly" commute, the solution (2.5) approximates (2.4) very accurately.

Accuracy of equation (2.4)

First consider the accuracy of (2.4) itself. Let $C = A + B$ and let Y be the integral $\int_{Z_1}^{Z} C\ \mathrm{d}Z$. Suppose that U satisfies (2.4) exactly for all Z_1, Z, i.e., $U(Z) = \exp(C)\ U(Z_1)$. Then we may expand the exponential (by definition) as a Taylor's series, and take derivatives with respect to Z, to get

$$\frac{\partial U(Z)}{\partial Z} = [1 + C + \tfrac{1}{2}(CY + YC) + \cdots]U(Z_1).$$

This, to first order in $[C, Y]$, gives

$$\frac{\partial U}{\partial Z} = C \exp(Y)\ U(Z_1) - \tfrac{1}{2}[C, Y]U(Z_1).$$

Now, since A is constant in Z and B is just a multiplication operator, each commutes with its own integral. Thus, if $Z_1 = Z - \Delta Z$, $[C, Y]$ is just $[A, \int B - \Delta Z\ B]$. Finally, from the Taylor expansion of the function B, to second order in ΔZ this is simply $\frac{1}{2}\epsilon_1$, so that U satisfies an equation

$$\frac{\partial U}{\partial Z} = (A + B)U - \tfrac{1}{2}\epsilon_1(U(Z - \Delta Z)),$$

differing from (2.3) by a multiple of the same commutator ϵ_1. We can also consider directly the perturbation error resulting from the approximation (2.4). First note that derivatives of A, B, and C are well-defined and obey the following chain-rules and other properties: $A_Z = 0$, $(Au)_Z = Au_Z = ACu$, $(Bu)_Z = B_Z u + Bu_Z$, so that $C_Z = B_Z$ and

$$u_{ZZ} = (Cu)_Z = C_Z u + Cu_Z = B_Z u + Cu_Z,$$

where subscripts again denote differentiation. Now, we take the Taylor expansion of $u(Z + \Delta Z)$ about Z and apply these rules together with (2.3), to get

$$u(Z+\Delta Z) \approx u + \Delta Z\, Cu + \tfrac{1}{2}(\Delta Z)^2 (C^2 + B_Z)u + \tfrac{1}{6}(\Delta Z)^3 [C^3 + 2B_Z C + CB_Z + B_{ZZ}]u + \cdots. \tag{2.6}$$

By taking the Taylor expansion for B as before, we find that

$$\int_Z^{Z+\Delta Z} C\, \mathrm{d}Z \approx \Delta Z\, C + \tfrac{1}{2}(\Delta Z)^2 B_Z + \tfrac{1}{6}(\Delta Z)^3 B_{ZZ} + \cdots. \tag{2.7}$$

Expanding the exponential in (2.4) and using (2.7) we get

$$u(Z+\Delta Z) \approx u + \Delta Z\, Cu + \tfrac{1}{2}(\Delta Z)^2 (C^2 + B_Z)u + \tfrac{1}{6}(\Delta Z)^3 [C^3 + \tfrac{3}{2}B_Z C + \tfrac{3}{2}CB_Z + B_{ZZ}]u. \tag{2.8}$$

Subtracting (2.8) from (2.6) we find that the error in the formal solution (2.4) is, to lowest order, $\frac{1}{12}(\Delta Z)^3 [C, B_Z]$.

The significance of this is that no scheme based on the formal solution (2.4) can have a higher order of accuracy. Furthermore the behaviour of ϵ_1 is reflected in the accuracy not only of (2.5), but also of any other schemes based on (2.4).

Perturbation error

Perturbation error in the wavefield. Let $u(X, Z) = r(X, Y)\, \mathrm{e}^{\mathrm{i}\phi(X,Z)}$ where r is the amplitude and ϕ is the cumulative phase. Although the theory of phase-statistics is incomplete it is known that $\langle \phi^2 \rangle$ and $\langle |\phi| \rangle$ increase unboundedly with range, $\langle r^2 \rangle$, $\langle r \rangle$ remain bounded, and r, ϕ become independent. (The angled brackets denote ensemble averages. These may be thought of as very long time averages, or averages over realisations of the random process, and they correspond to integrals with respect to some unknown probability distribution functions. The average is thus linear, and if two random functions, f and g say, are *independent*, then the ensemble average has the property that $\langle fg \rangle = \langle f \rangle \langle g \rangle$.) Since

$$u_X = [r_X + \mathrm{i}\phi_X r] \exp(\mathrm{i}\phi),$$

and

$$u_{XX} = \left[2\mathrm{i} r_X \phi_X + r_{XX} + \mathrm{i}\phi_{XX} r - (\phi_X)^2 r\right] \exp(\mathrm{i}\phi),$$

it can easily be shown that $\langle |u_X| \rangle / \langle |u| \rangle$ and $\langle |u_{XX}| \rangle / \langle |u_X| \rangle$ increase unboundedly as Z increases.

Write $e = \epsilon_1(u)$. Now, $e = -\mathrm{i}(\Delta Z)^2 [B_{XX} u + 2B_X u_X]$ and B is statistically independent of range. So as Z increases, for fixed ΔZ, the term $2B_X u_X$ dominates and $|e|$ increases. In other words, the accuracy of the split-step solution deteriorates locally with distance of propagation. This behaviour is demonstrated numerically in the following section, where the separate question of cumulative accuracy is also examined. However, as will also be shown, the accuracy of the split-step method is more than sufficient when used with fixed step-size and scattering strength within the limits adopted in previous work.

Perturbation error in the correlation function. Since the computed wavefield is used principally to calculate statistical quantities, such as its autocorrelation function $\rho(\xi, Z) = \langle u(X, Z)u^*(X+\xi, Z)\rangle$, it is essential to know how these are corrupted by the statistics of the error. This question has apparently not been examined elsewhere. First note that u and B become independent, and $\langle u\rangle$ and $\langle B\rangle$ are zero, and so the mean step-wise error $\langle e\rangle$ is zero. The derivatives $\partial^n B/\partial X^n$ also have zero mean, and these quantities are all assumed to be stationary and invariant under reflections about any point X. (This means that, for any such function f and any displacement ξ, $\langle f(X_1, X_2,\dots)\rangle = \langle f(-X_1, -X_2,\dots)\rangle = \langle f(X_1+\xi, X_2+\xi,\dots)\rangle$.) Let $\rho_e(\xi)$ be the correlation function $\langle e_1 e_2^*\rangle$ of the error, where e_i denotes $e(X_i)$ and $\xi = X_1 - X_2$. Then, if $U = u + e$ and $\rho_U(\xi) = \langle U_1 U_2^*\rangle$,

$$\rho_U(\xi) = \rho(\xi) + \rho_e(\xi) + \langle e_1 u_2^*\rangle + \langle u_1 e_2^*\rangle.$$

Since B is independent of u the first cross-term $\langle e_1 u_2^*\rangle$ is given by

$$-\mathrm{i}(\Delta Z)^2\left[\langle B_{1XX}\rangle\rho(\xi) + 2\langle B_{1X}\rangle\langle u_{1X}u_2^*\rangle\right],$$

which vanishes (even though e depends on u) since the mean derivatives of B are zero. Similarly, the second cross-term $\langle u_1 e_2^*\rangle$ disappears so the step-wise error in the correlation function is just $\rho_e(\xi)$. We now have

$$\begin{aligned}\langle\rho_e(\xi)\rangle &= (\Delta Z)^4\langle[B_{1XX}u_1 + 2B_{1X}u_{1X}][B_{2XX}u_2^* + 2B_{2X}u_{2X}^*]\rangle\\ &= (\Delta Z)^4\left[\langle B_{1XX}B_{2XX}\rangle\rho(\xi) + 4\langle B_{1X}B_{2XX}\rangle\langle u_{1X}u_2^*\rangle + 4\langle B_{1X}B_{2X}\rangle\langle u_{1X}u_{2X}^*\rangle\right].\end{aligned}$$

This expression becomes dominated by the last term, a product of the known correlation function of B_X and that of u_X, which is unknown. However, the bracketed sum is independent of step-size and the error is therefore of order $(\Delta Z)^4$. This fact hinges on the disappearance of the cross-correlation of e with u. The result indicates that the second moment of the wavefield can be calculated very reliably from its numerical approximation, and has great significance for work already carried out. It is easily shown that this analysis extends to the fourth moment.

Cumulative errors

Cumulative error in the wavefield. Suppose the total range of propagation is divided into n steps, so that n is of order $(\Delta Z)^{-1}$. Let U_k be the numerical solution of $u(Z_k)$. Denote the solution operator (2.5) from Z_k to Z_{k+1} by S_k, and the composition of $S_k, S_{k+1},\dots, S_{n-1}$ by S_k'. Define the kth perturbation error by $e_k = S_k(u_k) - u_{k+1}$. Then $U_1 = u_1 + e_0$, $U_2 = u_2 + e_1 + S_1 e_0$, etc., and we can write

$$U_n = u_n + e_{n-1} + S_{n-1}' e_{n-2} + \cdots + S_1' e_0.$$

Now, for $k < n$ define $\phi_k = (\Delta Z)^{-2} e_k$, and $\psi_k = S_k'\phi_{k-1}$, and put $\psi_n = (\Delta Z)^{-2} e_{n-1}$. Write $\Psi = \psi_n + \cdots + \psi_1$. Then U_n can be written

$$U_n = u_n + (\Delta Z)^2\Psi. \tag{2.9}$$

Thus the cumulative error is $(\Delta Z)^2 \Psi$, where Ψ consists of n terms. Some care should be taken in examining Ψ and its dependence on step-size.

Now, ϕ_{k-1} is simply the well-defined continuous function $\phi = B_{XX}u + B_X u_X$ evaluated at Z_{k-1}. The composition operator S_k' approximately represents propagation from Z_k to Z_n governed by equation (2.4), and ψ_k is S_k' applied to $\phi(Z_{k-1})$. Thus, with an error of order ΔZ resulting from the difference $\phi(Z_k) - \phi(Z_{k-1})$, ψ_k is the value at Z_k of a fixed continuous function $\psi(Z)$. We can therefore write

$$(\Delta Z)^2 \Psi = \Delta Z(\Delta Z\ \Psi) \approx \Delta Z \int_{Z_1}^{Z_n} \psi(Z)\, \mathrm{d}Z + \mathrm{O}\big((\Delta Z)^2\big).$$

Thus, from (2.9), the cumulative error in the solution U is of order ΔZ.

Cumulative error in the correlation function. As with the step-wise error, we can show that the correlation function is of higher accuracy than the field itself. Now, $U_n = u_n + (\Delta Z)^2 \Psi$, so that if $\rho_n(\xi)$ is the autocorrelation function $\langle U_n(X) U_n^*(X+\xi)\rangle$ of U_n, and $\rho_e(\xi)$ is that of Ψ, then

$$\rho_n(\xi) = \rho(\xi, Z_n) + (\Delta Z)^4 \rho_e(\xi) + (\Delta Z)^2 \big[\langle u_n(X) \Psi^*(X+\xi)\rangle + \langle u_n^*(X) \Psi(X+\xi)\rangle\big].$$

The second term on the right-hand side can be written, as before, in terms of integrals,

$$(\Delta Z)^4 \rho_e(\xi) = (\Delta Z)^2 \left\langle \int \psi(X, Z)\, \mathrm{d}Z \left[\int \psi(X+\xi, Z)\, \mathrm{d}Z\right]^* \right\rangle,$$

which is of second order in ΔZ. The final term appears to be, by the same reasoning, of first order in ΔZ, but as with the step-wise cross-correlation we can show that it actually vanishes. In fact each term $\langle u_n \psi_k^* \rangle$ arising from the sum Ψ of functions ψ_k is identically zero: Denote by G_k the set of functions of the form

$$g = B^r(Z_k) h(B(Z_{k+1}), \dots, B(Z_n), u),$$

where the superscript denotes an X-derivative of some degree and h is any function. If we act on g with $\exp(\Delta Z\, B(Z_l))$ for $l > k$, we get another function in G_k. If we apply any power of the differential operator A we get a sum of such functions. Applying the operator $S_l = \exp(\Delta Z\, A)$ $\exp(\Delta Z\, B_l)$ for $l > k$, to a sum (or series) of functions in G_k, we thus obtain another series of functions in G_k. Then, since each e_k is a sum of functions of this type, it follows that each ψ_k can be expressed as a series of functions in G_k. Now, for each of these functions g, $B^r(Z_k)$ is independent of h, by our assumption that $\Delta Z \geqslant L_Z$ and since B and u are independent. It follows that

$$\langle g u^* \rangle = \langle B^r(Z_k) \rangle \langle h u^* \rangle = 0.$$

Therefore every term in the expansion of $\langle u \psi_k^* \rangle$ is zero, and the cross-correlations $\langle u \Psi^* \rangle$, $\langle u^* \Psi \rangle$ vanish, as required. The cumulative error $\rho_n(\xi) - \rho(\xi)$ in the second moment is thus of order $(\Delta Z)^2$.

Remarks. (1) The above calculation was simplified by the assumption, valid in most applications, that ΔZ remains greater than L_Z and that successive phase-screens $\exp(B)$ are therefore independent. The result is still true, however, for $\Delta Z \leqslant L_Z$, and can be shown by replacing ψ_k in the above argument with a sum of such functions over a correlation length.

(2) Higher-order splitting methods, such as Strang's splitting, are available, but the correlation function errors introduced by (2.5) are already well within our requirements, and in practice represent greater accuracy than can be achieved in physical measurements of the correlation function and of the medium.

(3) Other methods for solving (2.4) are in use (e.g., [3]), and discussion of these is beyond the scope of this paper. It should be noted, however, that a crucial element of the split-step scheme is the use of step-size at least as large as L_Z. The generation of the random medium is at least as expensive computationally as the solution of (2.5) itself, requiring $2n$ 1-dimensional Fast Fourier Transforms, where n is the number of steps. Even the slightest reduction in ΔZ introduces correlation between the phase-screens, and immediately entails the use of 2-dimensional FFTs. These remarks apply specifically to *random* media. The simpler problem of wave-scattering by deterministic refractive index fluctuations also arises, and requires different techniques. Operator-splitting has now also been applied to the higher-dimensional non-stochastic problem of the moments of random wave propagation (see [5]). In this case the accuracy is greater because the commutator is small on the "solution space" in a well-defined sense.

3. Numerical scheme, its application and results

We will describe first the numerical implementation of the split-step solution (2.5), and the representation on the computer of the random medium. Many of the details can be found in [4]. Some computational results will then be given.

In what follows the deterministic component n_d of variation is neglected. It is a simple matter to include such effects.

Implementation of (2.5)

The scattering effect $\exp(B')$ simply imposes a phase-change on the field. If we express this as $e^{i\psi(X)}$, then $\psi(X)$ represents the total scattering undergone in a strip of medium of width ΔZ. The medium is thus represented as a series of phase-screens ψ. The construction of these screens is described below.

The solution of the diffraction operator $\exp(A')$ is equivalent to solving (2.1) with the scattering component ignored, that is

$$u_Z = -\tfrac{1}{2}\mathrm{i}u_{xx}. \tag{3.1}$$

In most applications of the split-step solution to random media two methods are used to solve (3.1).

(a) The first is an implicit finite-difference scheme derived as follows: Equation (3.1) is discretised and the derivatives approximated by finite differences, using a Crank–Nicolson scheme. This yields an implicit system of linear equations of the form $SU_{j+1} = TU_j$ where S and T are tridiagonal matrices, and the vector U_j corresponds to $u(X)$ at range Z_j. S and T are constant with Z since the range dependence is included in the phase-screen. This system can then be solved by use of an efficient tridiagonal matrix inversion algorithm. Explicit boundary conditions are treated in the definitions of S and T.

(b) The solution of (3.1) can also be written as

$$u(Z+\Delta Z) = F^{-1}\left[e^{i\nu^2 \Delta Z/2} F(u(Z))\right], \tag{3.2}$$

where F denotes the Fourier transform with respect to X and ν is the transform variable. This suggests use of the Fast Fourier Transform. The technique was applied by Tappert and Hardin [6]. It is computationally fast and eliminates the need to deal explicitly with boundary conditions. Furthermore the accuracy of this step depends only on the discretisation with respect to X, and not on ΔZ.

Extremely close agreement between these two methods has been consistently obtained in practice, and demonstrates the accuracy of both as solutions of (2.5). (The results below were calculated using (b).) Higher-order methods may be applied, but as remarked previously physical constraints and the accuracy of (2.4) limit their usefulness, and the schemes described above are the most widely used.

Generation of random medium

We denote by $\rho(\xi, \zeta)$ the normalised correlation function of refractive index fluctuations W (2.2). Thus $\rho(\xi, \zeta) = \langle W(X, Z) W(X+\xi, Z+\zeta)\rangle$. Given ρ we wish to construct a phase-screen $\psi(X)$ to represent the cumulative effect of W in the strip of medium $(Z, Z+\Delta Z)$. The (unnormalised) transverse correlation function ρ_1 of ψ in this strip is therefore

$$\rho_1(\xi, \Delta Z) = \int\int_Z^{Z+\Delta Z} \rho(\xi, \zeta)\, dZ_1\, dZ_2, \tag{3.3}$$

where $\zeta = Z_1 - Z_2$. In previous work ΔZ has been taken equal to the horizontal scale-size L_Z.

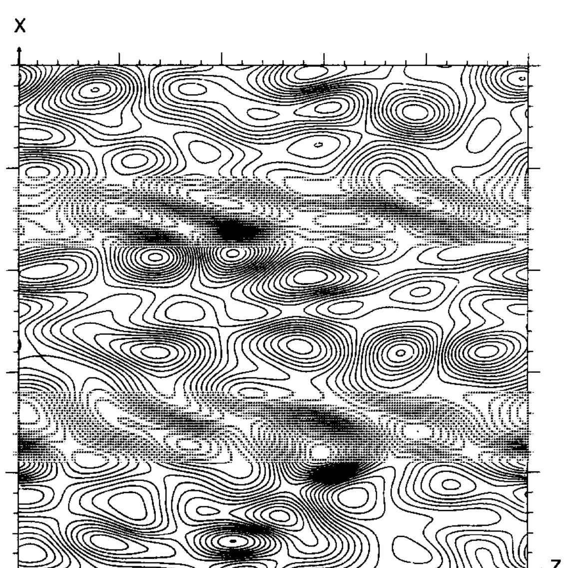

Fig. 1. Contour map of a typical computer realisation of $W(x, z)$, the 2-dimensional randomly varying part of the refractive index n.

Equation (3.3) then simplifies and the individual screens $\psi(X)$ are generated independently. Many important situations, however, are not adequately modelled in this way. Situations in which successive scatters are correlated, for example strong Markov scattering, require the use of phase-screens with correlation in the direction of propagation. Results of these simulations in comparison with new theoretical work will be given elsewhere. Our present use for the technique is to model the medium in arbitrarily fine detail, whilst keeping both L_z and the total scattering effect fixed. (This has proved invaluable in testing the validity of our computer models of specific ocean acoustics experiments.) The details of the generation of phase-screens are given in [4], and follow a method described in [2]. The screens are generated in the frequency domain, using a random number generator. White noise is filtered by the spectral function corresponding to ρ_1 and the whole is transformed back to the spatial domain using Fast Fourier Transforms to give

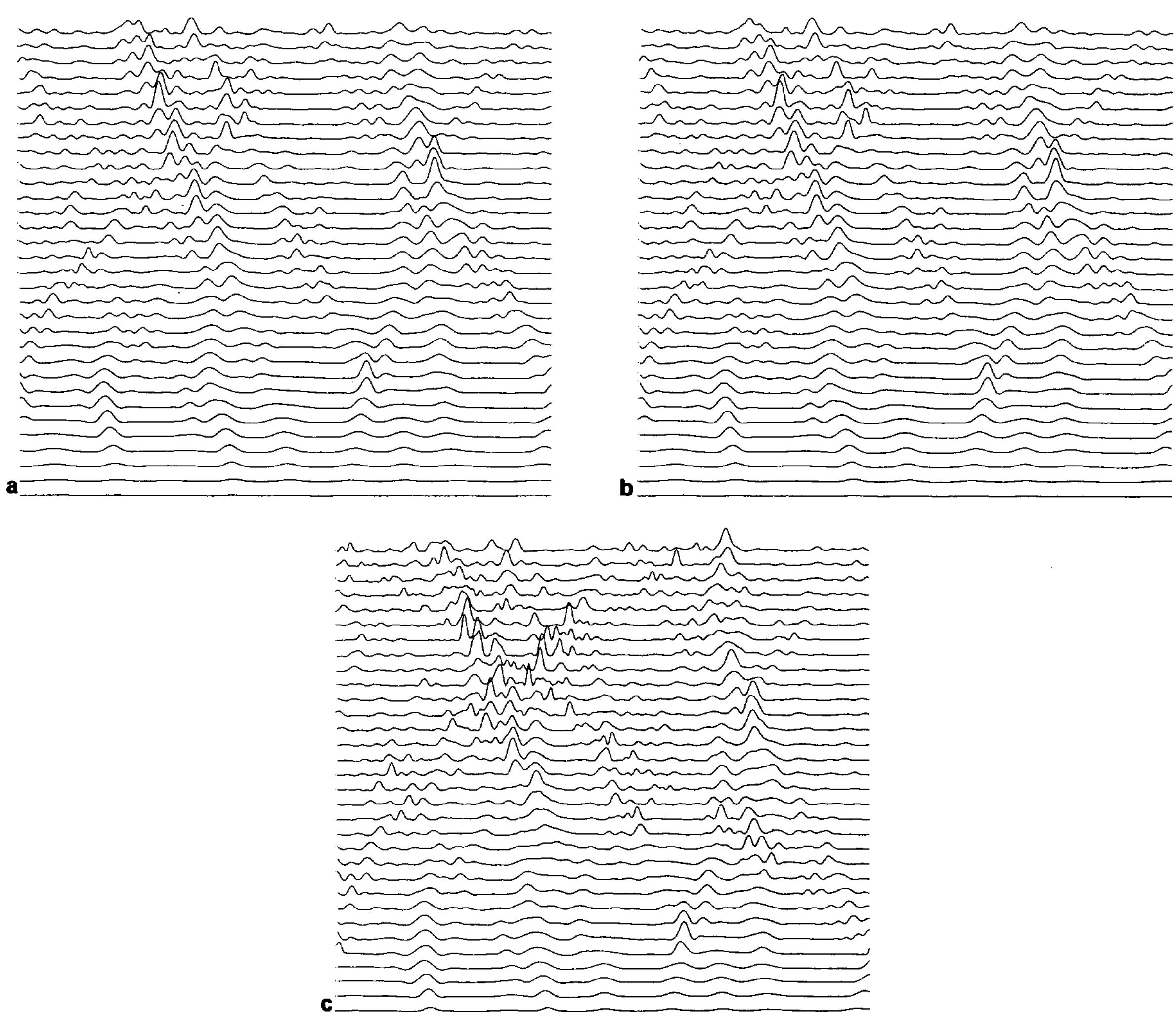

Fig. 2. Intensity pattern as function of Z in a medium with $L_Z = 0.16$, $\phi^2 = 8.0$, and (a) 32 steps/irregularity ($\Delta Z = L_Z/32$); (b) 8 steps/irregularity ($\Delta Z = L_Z/8$); (c) 1 step/irregularity ($\Delta Z = L_Z$).

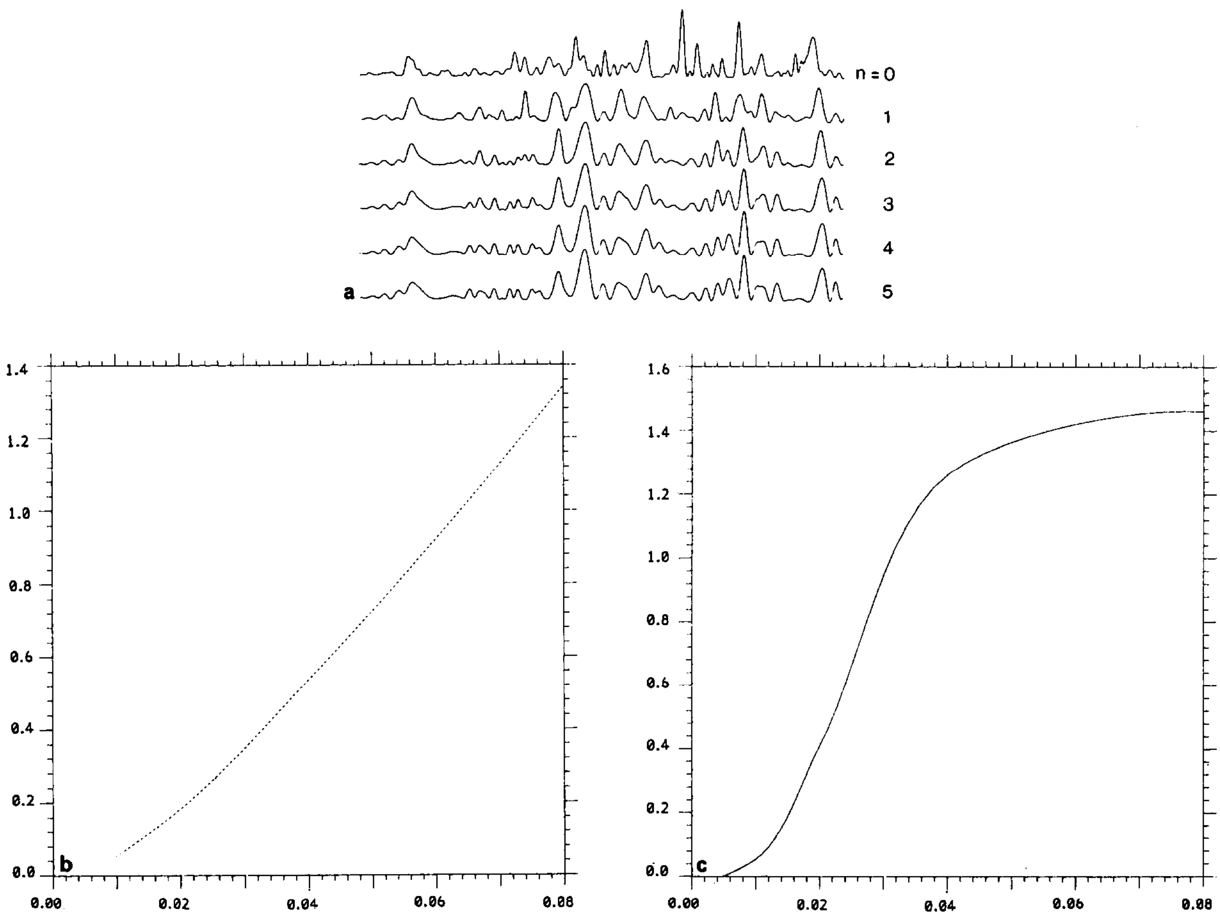

Fig. 3. (a) Comparison of intensity patterns $I_Z^n(X)$ at $Z=1.28$ for $L_Z=0.08$ and $\phi^2=8.0$. Here n determines how finely the medium is subdivided in the direction of propagation, with step-length $\Delta Z = L_Z/2^n$; (b) L_2-norm $\langle (I_Z^{n+1} - I_Z^n)^2 \rangle$ as a function of ΔZ; (c) L_2-norm $\langle (I_Z^N - I_Z^n)^2 \rangle$ as a function of ΔZ.

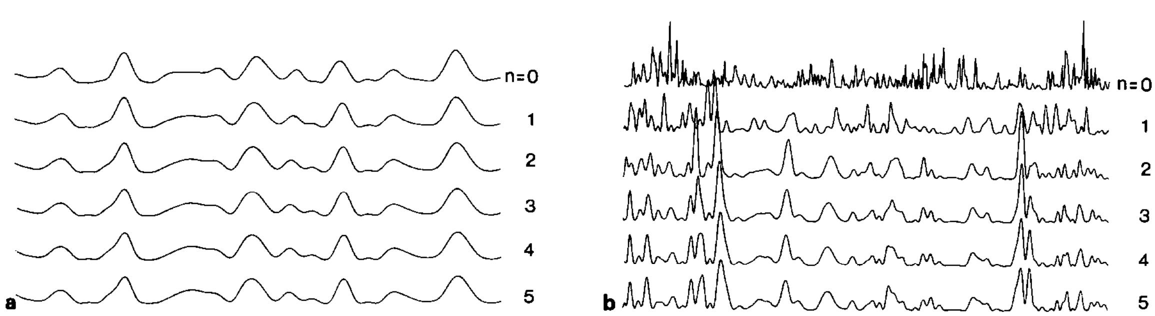

Fig. 4. As for Fig. 3(a), with (a) $\phi^2 = 0.25$; (b) $\phi^2 = 64.0$.

the required screens $\psi(X_i, Z_j)$. ψ is a given variance $k^2\mu^2\rho_1(0, \Delta Z)$ for each screen, representing the phase modulation in a strip $(Z, Z + \Delta Z)$. We denote by ϕ^2 the variance $k^2\mu^2\rho_1(0, L_Z)$. ϕ^2 can be thought of as the mean-square phase fluctuation imposed per irregularity. Figure 1 shows a contour map of a typical set of correlated screens.

Computational results

We now describe the simulations and show some typical results. Some figures are also given to illustrate the deterioration of accuracy with range, and the good overall accuracy which is obtained. All the results shown here are for a medium with a Gaussian correlation function.

Values are chosen for L_Z and ϕ^2 and the medium is generated numerically. A simulation is performed for a fixed distance ΔZ between phase-screens. ΔZ is halved and the computation repeated, and so on. In this way a realistic predetermined medium can be progressively subdivided into increasingly fine screens and the effect on wave propagation examined. Figure 2 compares the intensity patterns as functions of range in one such series of simulations. Note that such a comparison cannot be made using independent screens, since the degree of correlation between the screens very significantly affects the intensity pattern and its statistics. The intensity patterns $I_Z^n(X)$ at chosen range Z are then compared for different n, where n defines the step-size $\Delta Z = 2^{-n} L_Z$ (Fig. 3(a)). Figure 3(b) shows the L_2-norm $\langle (I_Z^{n+1}(X) - I_Z^n(X))^2 \rangle$ as a function of ΔZ, taken over X and averaged over several realisations of the medium. Figure 3(c) shows $\langle (I_Z^N - I_Z^n)^2 \rangle$ where N is the largest index used. Since the numerical scheme is highly accurate, the effects as ΔZ changes almost entirely reflect the change in accuracy of the split-step solution (2.5). The strength of scattering ϕ^2 here is many times greater than is normally used in actual applications of the method (see [4]).

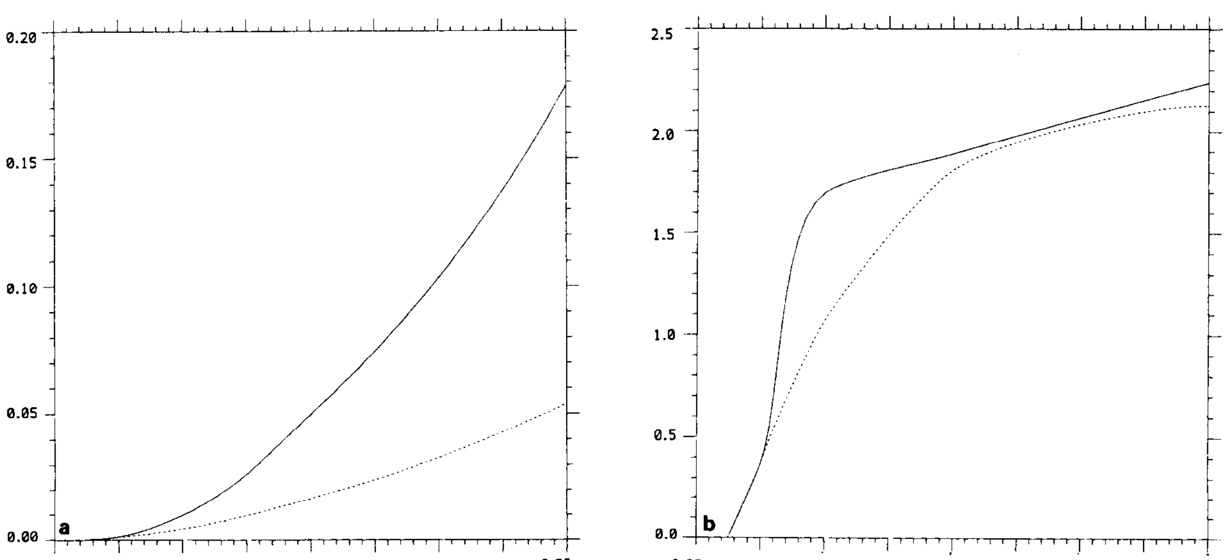

Fig. 5. As for Fig. 3(b) and (c) combined, with $L_Z = 0.08$, and (a) $\phi^2 = 0.25$; (b) $\phi^2 = 64.0$.

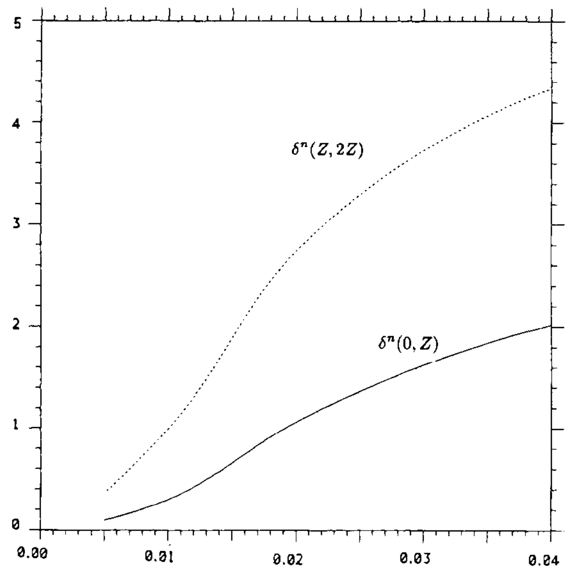

Fig. 6. The quantities $\langle (I^n_{0,Z} - I^{n-1}_{0,Z})^2 \rangle$ (———) and $\langle (I^n_{Z,2Z} - I^{n-1}_{Z,2Z})^2 \rangle$ (------) as functions of $\Delta Z = L_Z/2^n$, for $L_Z = 0.04$, $\phi^2 = 16.0$ and $Z = 0.64$.

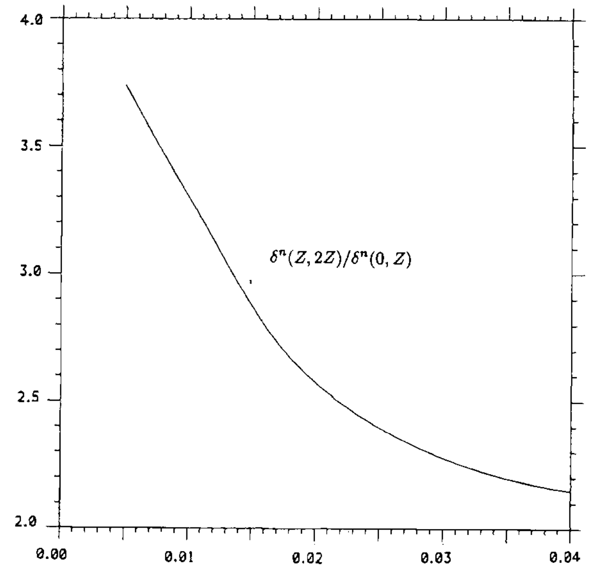

Fig. 7. The ratio r of $\langle (I^n_{Z,2Z} - I^{n-1}_{Z,2Z})^2 \rangle$ to $\langle (I^n_{0,Z} - I^{n-1}_{0,Z})^2 \rangle$ as a function of ΔZ. The increase in r as ΔZ decreases demonstrates that the accuracy of the split-step method decreases as the wave propagates.

The whole procedure was repeated for various values of ϕ^2 and L_z and some of these results are shown in Figs. 4 and 5. The value $\phi^2 = 0.25$ treated in Figs. 4(a) and 5(a) represents the maximum strength that would normally be used in simulations [4]. (ϕ^2 is usually chosen such that $\phi^2 \ll 1$, so that the scattering per screen is small.) All the curves indicate satisfactory convergence as ΔZ is reduced. Viewed together with the direct comparison of intensities they give a clear idea of the good qualitative behaviour of the numerical simulations. Figures 4(a) and 5(a) suggest that for $\phi^2 = 0.25$ one screen per irregularity is more than sufficient and gives well over 95% accuracy.

In Section 2 the analysis indicated that local accuracy decreases with range, for fixed ΔZ. In order to confirm this, changes in intensity over intervals $(0, Z_1)$ and $(Z_1, 2Z_1)$ were calculated and compared as ΔZ was reduced. If we put $I^n_{Z,Z'}(X) = I^n_{Z'}(X) - I^n_Z(X)$, then $I^n_{0,Z} - I^{n-1}_{0,Z} = I^n_Z - I^{n-1}_Z$ since $I^n_0 \equiv 1$. Figure 6 shows the quantities $\delta^n(0, Z) = \langle (I^n_{0,Z} - I^{n-1}_{0,Z})^2 \rangle$ and $\delta^n(Z, 2Z) = \langle (I^n_{Z,2Z} - I^{n-1}_{Z,2Z})^2 \rangle$. This example, typical of the results obtained, clearly shows the expected behaviour. The intensity change itself is greater over the interval $(0, Z)$ than it is over $(Z, 2Z)$ and the curve $\delta^n(Z, 2Z)$ is therefore higher than $\delta^n(0, Z)$. However the ratio between the curves increases as ΔZ decreases (Fig. 7) and this indicates that $\delta^n(0, Z)$ is converging faster, and that the scheme is more accurate over $(0, Z)$ at a fixed value of ΔZ than over $(Z, 2Z)$.

4. Summary

The accuracy of the split-step solution for wave propagation in random media has been considered, both analytically and numerically. The error has been established for the wavefield and for its statistical moments, and it has been shown that these moments are given with greater accuracy. It has been described how the random medium may be represented, by use of correlated phase-screens, in arbitrarily fine detail. This model has been used to examine the convergence of the split-step solution as the step-size ΔZ is reduced. The numerical model has been employed to confirm that, for constant step-size, accuracy decreases locally as the wave propagates. It has nevertheless been demonstrated that the solution is very accurate for small scattering per step and that the independent phase-screen model is valid for small scattering when the longitudinal scale-size is less than ΔZ.

This numerical technique can be very useful to gain insight into the accuracy of simulations, for given parameter values, with a single realisation of the random medium. It has proved to be a relatively simple test of validity in our modelling of certain large-scale ocean acoustics experiments.

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