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Recovery of a Variable Coefficient in a Coastal Evolution Equation

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A method is presented for reconstructing an unknown coefficient in a linear diffusion equation from measured data. This equation arises in the description of coastline evolution, and preliminary results are presented here. The unknown term may vary with both space and time, although time variation is assumed to be slow. Inversion is carried out by first expressing the solution of the direct problem formally in terms of the governing operators and making explicit approximations to these expressions. Using data at two time steps this then allows equations to be derived and solved to give explicit expressions for the required function.

1. INTRODUCTION

A central aim in many applications of wave propagation is to recover the scatterer or source terms from measured data. Problems of this kind range from acoustic tomography and medical imaging to coastline evolution, and the scattering function may for example represent boundary or refractive index variations, or sediment supply (see, for example, [2, 8, 9, 13, 14]). Particular difficulties arise when additional functions in the governing equations are unknown, and values of leading parameters or functions must often be estimated in order to circumvent this problem.

In this paper a method is presented for the recovery of a variable coefficient $K(x, t)$ in an equation of the form $y_t = (Ky_x)_x$, where the function K may vary both in space and time. This is widely used to describe coastline movement, for which initial results are presented based both on simulated and experimentally measured data. As in [8] it is assumed that data are available at discrete time steps t_i, and that the function $K(x, t)$ which is sought varies sufficiently slowly to be treated as constant over two successive time intervals. This does not necessarily imply that y itself varies slowly with t.

The method for the recovery of the coefficient term K is described. The solution of the direct problem over each time interval τ is first obtained formally in terms of the governing

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differential operator. A second order approximation in τ is then applied, giving the evolution of y explicitly in terms of $K(x, z)$. This expression is easily inverted to recover the function K.

Results are first obtained for simulated data, which can be chosen to be "exact." However, the envisaged application requires numerical differentiation of measured data. This procedure is ill-posed, so that small measurement errors may give rise to large changes in the derivative. This problem and regularization methods have been widely studied (for example, see [1, 3, 5, 6]). For the data which we intend to analyze, the spatial autocorrelation function is known to be "smooth," and if necessary this can to some extent be used to regularize the data. Preliminary results are given from application of the method to measurements relating to the United Kingdom coast. These give satisfactory solutions which are relatively insensitive to discretization lengths.

In Section 2 the governing equations are set out, and the solution of the direct problem by operator splitting is described. Section 3 gives the solution for the recovery of the unknown coefficient and presents numerical results. Here numerical noise is added in order to gain some measure of the robustness of the procedure. The preliminary results from measured data are then given in order to demonstrate the feasibility of the approach.

2. FORMULATION OF EQUATIONS AND SOLUTION FOR DIRECT PROBLEM

The paper will be concerned with the inversion of the equation

$$\frac{\partial y}{\partial t}(x, t) = Ly, \tag{1}$$

where

$$L = \frac{\partial}{\partial x}\left[K\frac{\partial}{\partial x}\right] \tag{2}$$

and $K(x, t)$ is unknown and will be assumed to vary slowly as a function of time. Boundary conditions must be specified in order to ensure uniqueness of the solution. In any practical application the solution will be carried out on a finite domain, and these may have to be modified or replaced by empirical or statistical conditions. Equation (1) will be inverted to obtain an approximation to K. (Note that the equation is sometimes modified to include a source term. The solution of this problem is feasible but requires further approximations and will not be tackled here.) We will refer to K as the coefficient. Denote by y_0 the initial value

$$y_0 = y(x, 0). \tag{3}$$

We consider in this section the direct initial value problem, i.e., the approximate solution for y from Eq. (1) when the term K is given.

The formal solution of (1) can be written *approximately* as

$$y(x, t) \cong \exp\left[\int_0^t L\, dt'\right] y_0, \tag{4}$$

where for any operator B, say, the exponential operator $\exp(B)$ is defined by the series

$$\exp(B) = 1 + B + \frac{B^2}{2!} + \cdots. \tag{5}$$

Note that when K is constant with respect to t, Eq. (4) is exact.

By analogy with results elsewhere [10, 11] the error in writing Eq. (4) can be shown to be a function of the quantity

$$\frac{t^3}{2}\left[L, \frac{\partial L}{\partial t}\right] \tag{6}$$

(where in general $[A, B]$ denotes the *commutator* $AB - BA$ of operators A and B). Over a small time step τ, Eq. (4) gives

$$y(x, t+\tau) \cong \exp\left[\int_t^{t+\tau} L(x, t')\, dt'\right](y(x, t)). \tag{7}$$

By Eq. (6) this has an error which to leading order is

$$\frac{\tau^3}{2}\left[L, \frac{\partial L}{\partial t}\right]. \tag{8}$$

Since K is assumed to vary slowly in t, Eq. (7) is therefore a good approximation when τ is reasonably small with respect to the scales of time variation. Consider now the term $\exp(\int L\, dt')$ in Eq. (7). As K is slowly varying we will treat K as constant over the interval $[t, t+\tau]$, and replace this exponential by

$$\exp\left[\int L\, dt'\right] \cong \exp\left[\tau L\left(x, t + \frac{\tau}{2}\right)\right]. \tag{9}$$

We now introduce a *rational approximation* for the operator $\exp(\tau L)$,

$$\exp(\tau L) \cong \left(1 - \frac{\tau L}{2}\right)^{-1}\left(1 + \frac{\tau L}{2}\right). \tag{10}$$

This is accurate to second order in τL. (We could instead write $\exp(\tau L) \cong 1 + \tau L + \tau^2 L^2$, but (10) will be more convenient for us because explicitly it involves only second as opposed to fourth order derivatives in x.) The above will be applied below to the recovery of coefficient K.

Substituting Eq. (9) into (7), using approximation (10), and applying the operator $(1 - \tau L/2)$ to both sides, we obtain

$$\left(1 - \frac{\tau L}{2}\right) y(x, t+\tau) \cong \left(1 + \frac{\tau L}{2}\right) y(x, t). \tag{11}$$

This gives an approximate analytical formula relating the evolution of y to K over each time-step τ.

3. INVERSE PROBLEM

3.1. *Recovery of Unknown Coefficient*

We now consider the approximate recovery of the function $K(x, t)$ from Eq. (1). (The extension to include a known, non-zero source term is possible but this will not be discussed here.) Suppose we are given accurately measured values of the function y at two time-steps t_1, t_2, where $\tau = t_2 - t_1$ is "small' in the sense that the solution is well-approximated by

Eq. (10). Denote

$$y^{(1)}(x) = y(x, t_1), \tag{12}$$

$$y^{(2)}(x) = y(x, t_2). \tag{13}$$

We further assume (here and throughout) that y can be measured with sufficient spatial resolution to recover the second derivative $\partial^2 y/\partial x^2$. (This clearly imposes quite stringent data requirements; however, it is reasonable for the intended applications.) From Eq. (11) we obtain

$$\frac{y^{(2)} - y^{(1)}}{\tau/2} = \frac{\partial}{\partial x}\left[K\left(\frac{\partial y^{(1)}}{\partial x} + \frac{\partial y^{(2)}}{\partial x}\right)\right], \tag{14}$$

where K is evaluated at $t + \tau/2$ in view of Eq. (9).

In this equation, both the left hand side and the term in round brackets on the right are known or can be obtained since $y^{(1)}$ and $y^{(2)}$ are given as data. It is possible to retrieve K by expanding the right hand side, to get an equation in terms of K and K_x. However, it is more convenient to treat Eq. (14) directly, as we now describe.

For convenience denote the known function $y^{(2)} - y^{(1)}$ and the spatial derivative $y_x^{(2)} + y_x^{(1)}$ by F and F', respectively. Assuming that data at consecutive times are given at n equally spaced x-values, $x_1, \ldots, x_n$ say, we can regard these functions as vectors of length n, so that

$$F_j \equiv F(x_j) = y^{(2)} - y^{(1)}, \tag{15}$$

$$F'_j \equiv F'(x_j) = \frac{\partial y^{(2)}}{\partial x} + \frac{\partial y^{(1)}}{\partial x}. \tag{16}$$

Numerical differentiation is unstable and can introduce significant errors and must be applied with caution. Many authors have considered the effect of perturbations of data and examined regularization methods (in particular see [1, 3, 5]). In the case of coastline measurements, empirical information may be available (in the form of a spatial autocorrelation function) which allows smoothing to be applied to remove spurious high-frequency components. As discussed below, however, this was not found to be necessary for the measurements examined here.

Denote by ξ the constant spacing in x. The coefficient K can similarly be replaced by discretized form, $K_j = K(x_j)$ for $j = 1, \ldots, n$. The outer derivative in Eq. (14) can now be approximated by a central finite difference, giving rise to a set of equations

$$\frac{2}{\tau}F_j = \frac{1}{2\xi}\left(K_{j+1}F'_{j+1} - K_{j-1}F'_{j-1}\right) \tag{17}$$

for $j = 2, \ldots, n-1$ and (using one-sided differences at the end values)

$$\frac{2}{\tau}F_1 = \frac{1}{\xi}\left(K_2F'_2 - K_1F'_1\right), \tag{18}$$

$$\frac{2}{\tau}F_n = \frac{1}{\xi}\left(K_nF'_n - K_{n-1}F'_{n-1}\right). \tag{19}$$

$$\tag{20}$$

Note that the difference scheme at the end-points is accurate only to first order in ξ, and we therefore lose some of the advantage of using central differences elsewhere. (We can improve upon this by specifying boundary conditions y periodic on $[x_1, x_n]$ and $\partial y/\partial x = 0$ at the edges of the domain, provided the second derivative of y is non-vanishing. In that case we can make use of fictitious data points outside the domain in order to formulate second order conditions. For simplicity this is not done here.) This is equivalent to a matrix equation for K

$$\mathbf{F} = A\mathbf{K}, \tag{21}$$

where A is the tridiagonal matrix with elements

$$\begin{aligned} A_{i,i} &= 0, \\ A_{i,i\pm 1} &= \pm\frac{\tau}{4\xi}F'_{i\pm 1} \end{aligned} \tag{22}$$

for $i = 2, \ldots, n-1$, and in the first and last rows

$$A_{1,1} = -\frac{\tau}{2\xi}F'_1, \qquad A_{1,2} = \frac{\tau}{2\xi}F'_2, \tag{23}$$

$$A_{n,n-1} = -\frac{\tau}{2\xi}F'_{n-1}, \qquad A_{n,n} = \frac{\tau}{2\xi}F'_n. \tag{24}$$

This is easily and rapidly inverted by back-substitution.

3.2. *Numerical Implementation*

In order to illustrate the application of the method, results are first given based on a numerical simulation of the system of equations (1), (2). The simulation was carried out for various time steps. Figure 1 shows an input function $y^{(1)} = y(x, t_0)$ at an initial time $t = 0$. (This was obtained by solving the underlying equation for a smoothly varying function $K(x, t)$ using the methods described in Section 3.) After a further small time-step $\tau = 0.0155$ this takes the form $y^{(2)}$. Figure 2 shows the difference $y^{(2)} - y^{(1)}$. The coefficient $K(x, t)$ was taken as constant in t over the interval $[t_0, t_0 + \tau]$, with the form $K(x)$ say. The solution, which we can denote $\tilde{K}$, was calculated according to Eq. (21), and the two functions are compared in Fig. 3. As can be seen the functions agree closely, apart from a small difference which is constant across the x axis. When this is subtracted the curves become indistinguishable. Instead of this, however, we can make the assumption that the mean of the coefficient across a sufficiently large distance is zero. We therefore subtracted the spatial average from the solution obtained and compared it with the original coefficient. As seen in Fig. 4 this gives extremely close agreement.

In order to check the robustness of the approach, further simulations were carried out, but with white noise added to the initial data. Figure 5 shows the difference function $y^{(2)} - y^{(1)}$ with white noise added. This is fairly severe in places and completely distorts the difference function. The resulting solution exhibits similar delta-correlated noise, shown in Fig. 6. This can be smoothed, however, using an simple algorithm which effectively replaces each value by an average over a small window (in this case using 10 points). This removes the spurious noise from the solution, as seen in Fig. 7. The original function and solution with mean removed are compared directly in Fig. 8, and again show extremely close agreement.

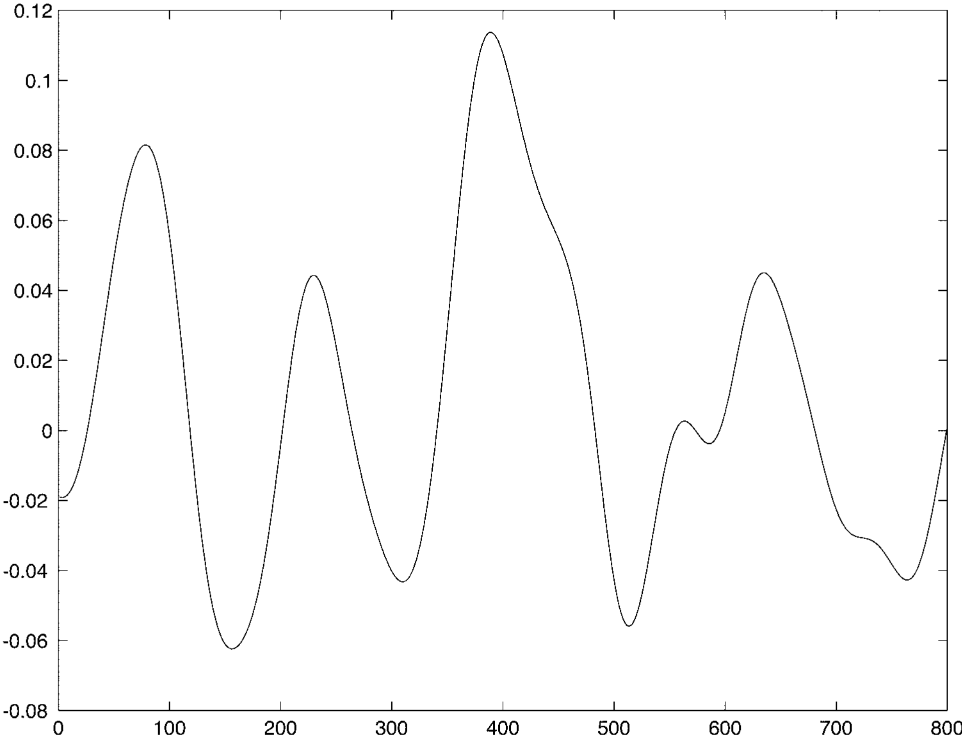

FIG. 1. Graph of input data at initial time step.

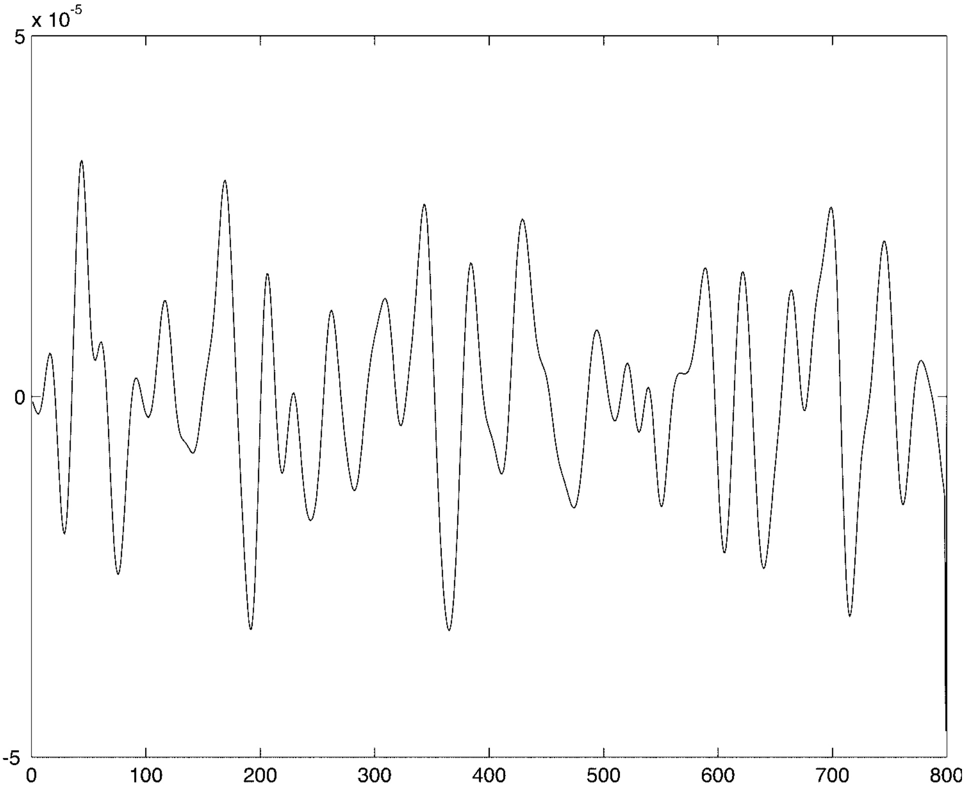

FIG. 2. Difference between input functions at successive time steps.

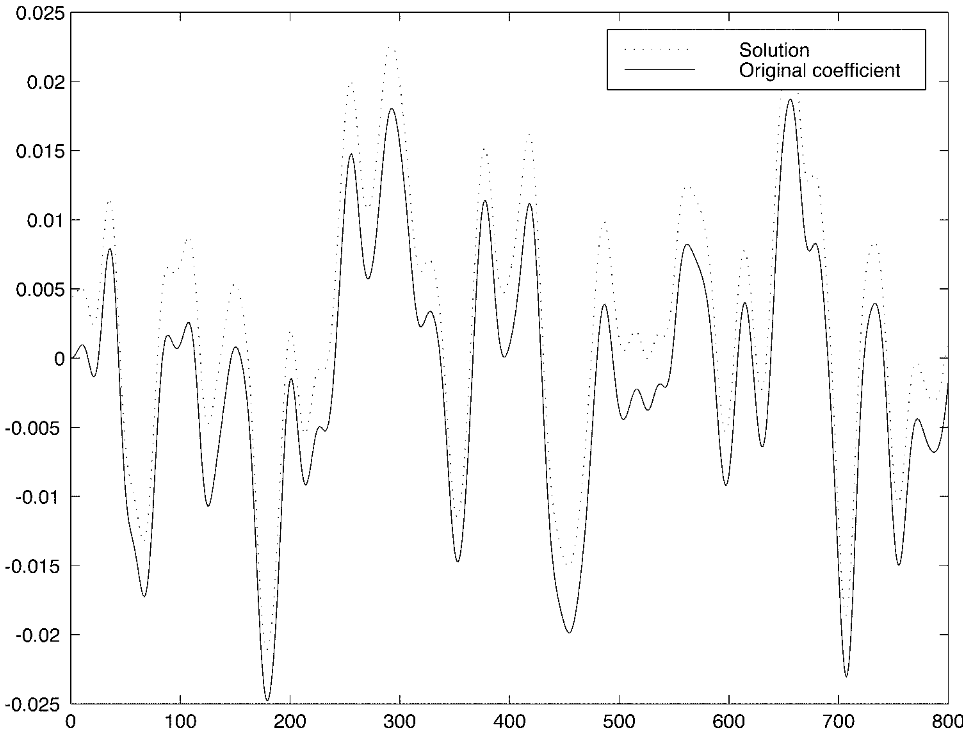

FIG. 3. Comparison between reconstructed function and exact solution (i.e., original coefficient).

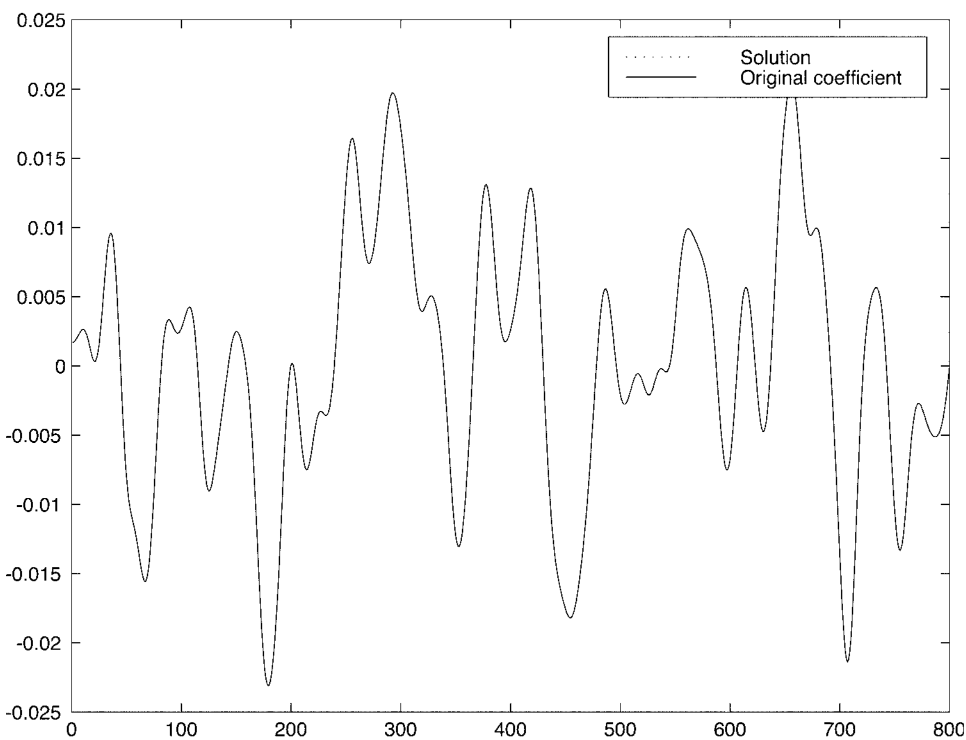

FIG. 4. Comparison as in Fig. 3 after subtraction of the spatial mean of the reconstructed function.

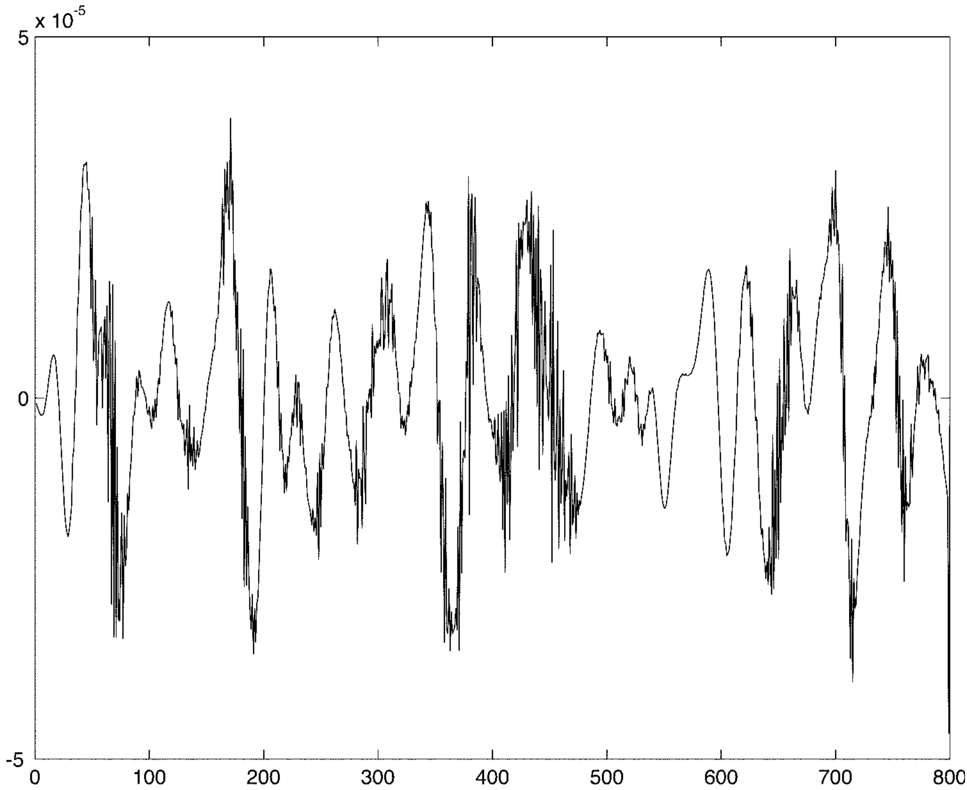

FIG. 5. Difference between functions at successive time steps, when white noise has been added.

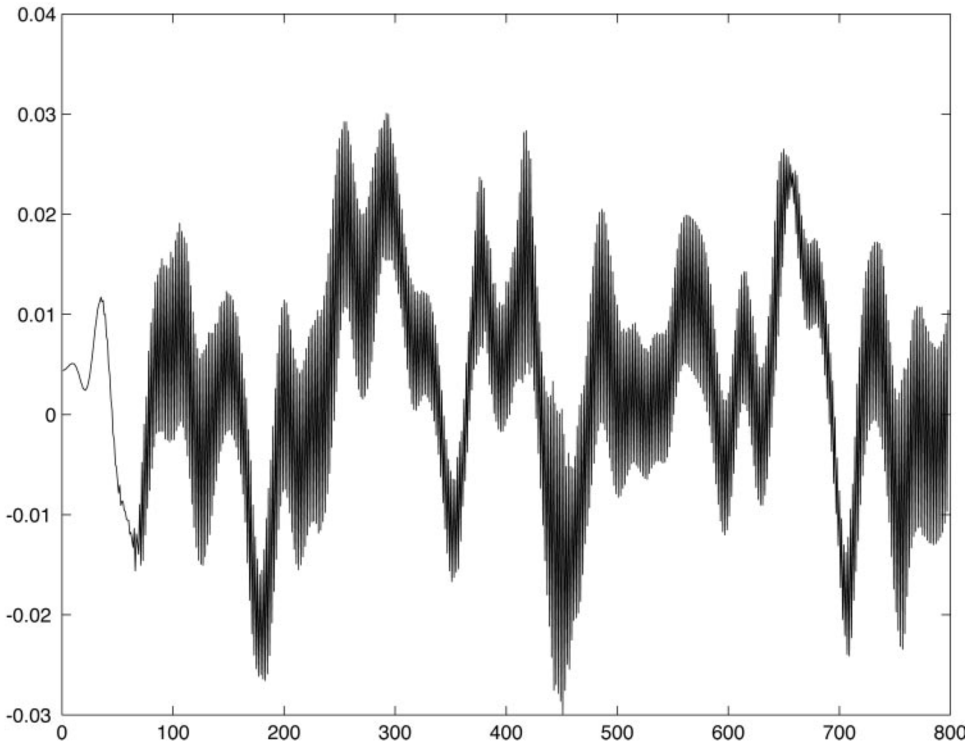

FIG. 6. Reconstruction of function from noisy data shown in Fig. 4.

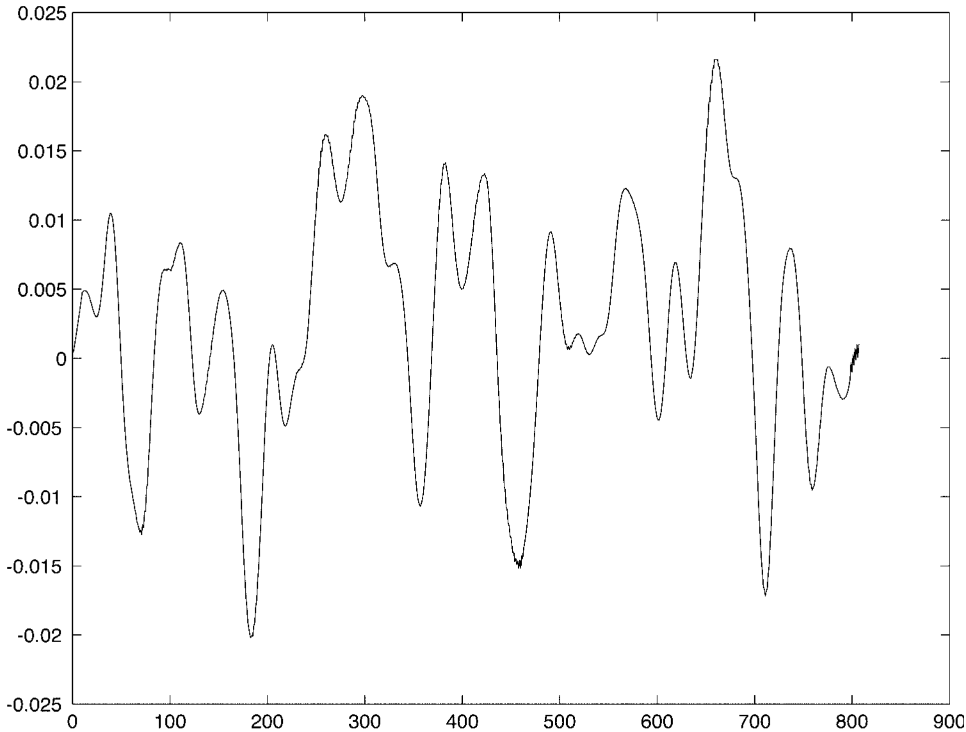

FIG. 7. Reconstruction as in Fig. 6, with simple smoothing applied.

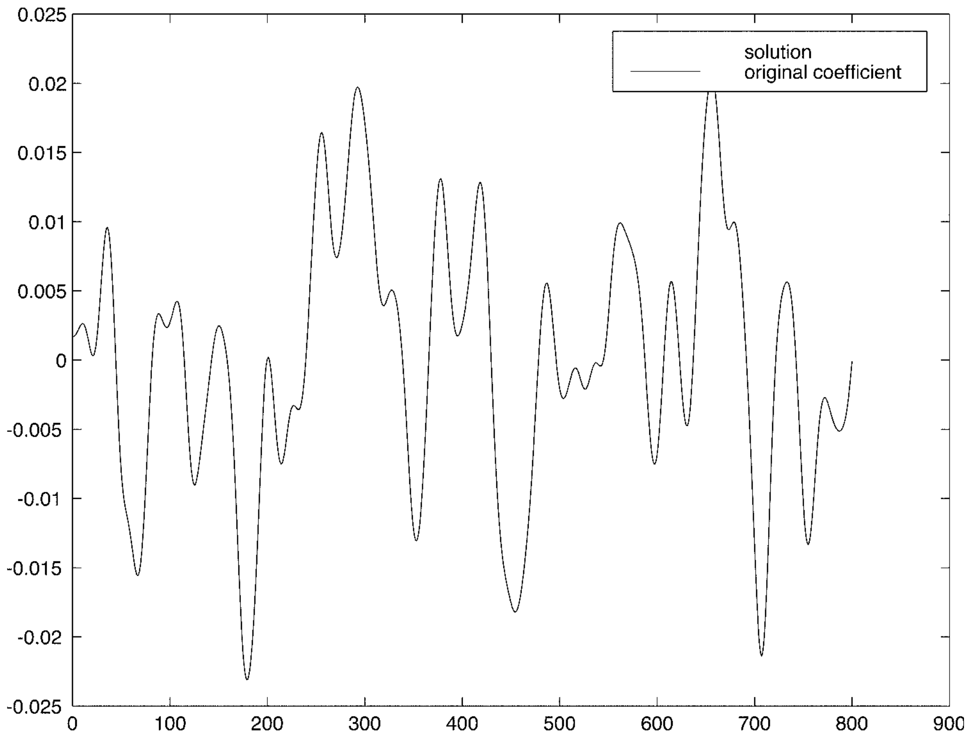

FIG. 8. Comparison between reconstructed function, with smoothing applied as in Fig. 7, and exact solution.

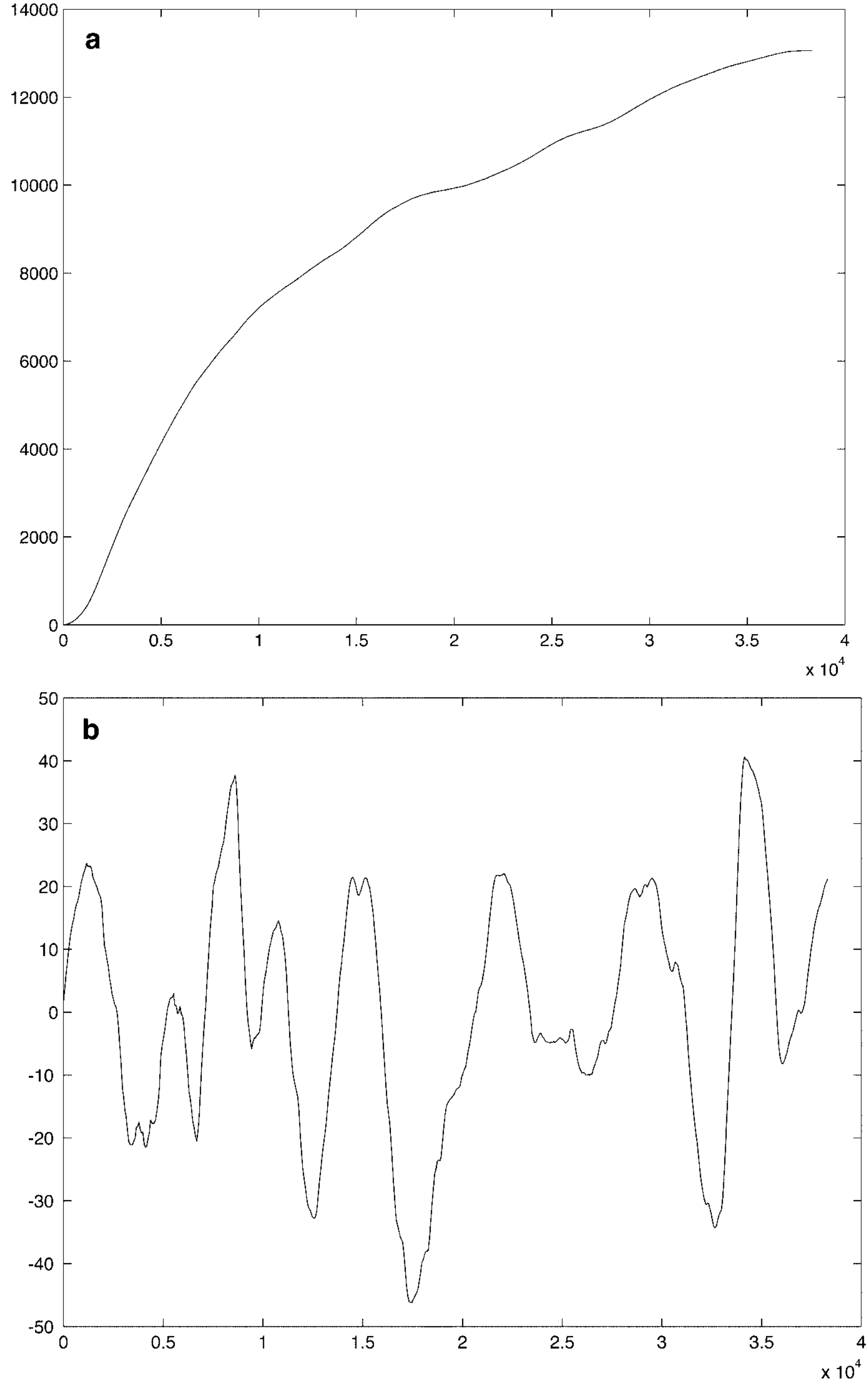

FIG. 9. (a) Profile of a 25 kilometre stretch of the East Anglian coastline, measured in 1992. (b) Change in coastline of (a) over the period from 1992 to 1994.

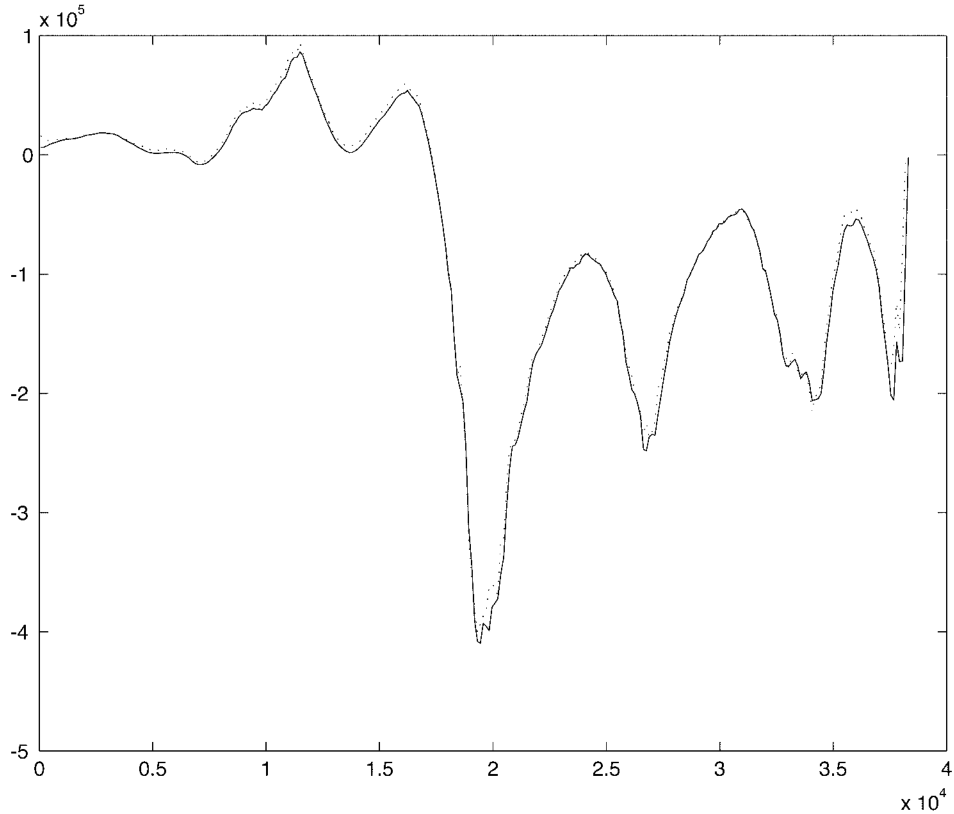

FIG. 10. The solution obtained from the data shown in Fig. 9. The full line is obtained using a spatial resolution of 64.17 metres and the dotted line is obtained at half-resolution, i.e., a step-size of 128.34 metres.

The method was then applied to data measured along a stretch of the United Kingdom coast in East Anglia. Measurements are available at various intervals since 1883, on a regular grid. The data used here are along a 25 kilometre stretch, measured in steps of 64.17 metres, in 1992 and 1994. The values represent perpendicular distances from a fixed line parallel to the coast. Figure 9a shows the coastline itself, and Fig. 9b gives the change over this period. Note that the change over time is around 100 metres, compared with variation along the coast of many kilometres. The reconstruction itself (with no smoothing applied) is given in Fig. 10 (full line). As a further rudimentary check the spatial resolution was halved, so that mid-points were discarded. This resulted in almost no change in the solution, as can be seen from this figure (in which the dotted line represents the lower resolution result). This gives some confidence that the results are not significantly affected by the instability arising from numerical differentiation.

4. CONCLUSIONS

A method has been described for the direct inversion of a diffusion equation describing coastal evolution, in which a variable coefficient term is unknown. The approximate solution of the forward problem described above forms the basis for the inversion scheme. Measured data at two successive time steps was used to formulate and solve simultaneous equations for the unknown coefficient. The approximations which are used impose fairly strong limitations on the behaviour of the solutions, and in particular the coefficient must vary slowly compared with the time intervals between available data. Nevertheless the method gives a prescription for the treatment of a difficult inverse problem, avoiding the use of iterative schemes. It is expected that the method will be also applicable in underwater acoustics in certain problems of image reconstruction and tomography. The problem for which this is immediately intended arises in the prediction of long term coastline evolution [7], which can be formulated in terms of a diffusion equation whose source term and diffusion coefficient may both vary. Using historical coastline position data this equation can be inverted to provide valuable information on the gross effects of the underlying physical processes responsible for coastal movement. Preliminary numerical results have been presented and have shown good agreement with the coefficient which is sought.

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