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Numerical solution of the second moment equation for waves in an inhomogeneous waveguide

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Abstract. An accurate numerical method for the solution of the second moment equation is presented. This may be applied to scalar wavefields in a wide variety of scattering regimes and inhomogeneous waveguides, for many of which explicit analytical solutions cannot be obtained. The method is applied here to the solution of a waveguide similar to the acoustic near-surface ocean channel. For the case of a Gaussian beam in a parabolic channel the method yields the asymptotic behaviour of focusing with increasing scattering, expressed as functional laws.

1. Introduction

Waves propagating in waveguides with random variations of refractive index occur in many applications and have been studied extensively. Of principal interest are the moments of the wavefield, which describe average quantities such as the scintillation index, and are governed by differential equations. Analytical solutions are known for the second moment with no channelling, and in special cases such as a parabolic waveguide [1, 2]; approximate solutions are now well-established for the more difficult fourth moment in the absence of channelling (e.g. [3, 4]), although the numerical approach is still the more accurate over a wide range of scattering strengths [5]. However in many realistic applications in optics and acoustics explicit analytical solutions cannot be obtained even for the second moment; the problem becomes intractable for any refractive index profile which has no simple algebraic form, and is further complicated when the source intensity varies in the transverse direction.

The main purpose of this paper is to describe an accurate numerical solution of the second moment which can be applied in a great variety of realistic scattering regimes, including those in which the profile is known only from data. As an illustration the paper applies the method to a Gaussian beam propagating in a random medium whose average profile is similar to the near surface acoustic channel found in the ocean, and results of this study are presented. In addition scaling laws are obtained describing the behaviour of intensity peaks with scattering strength in a parabolic channel. The method is by means of operator splitting techniques, similar to those already widely used for the simulation of wave propagation [6], and more recently with great success for the solution of the fourth moment equation [5]. The second moment equation is written in terms of sum and difference coordinates, and operator splitting is applied to the transformed equation. The accuracy of the solution, which is quantified using existing analysis, is confirmed by application to cases for which exact theory is available. Although the numerical solution here is described for wave propagation in a channel or waveguide, it is easy to extend the scheme to more general regimes provided the incident wave is of finite spatial extent.

In Section 2 the second moment equation and other theoretical preliminaries are given. The formal and numerical solutions are described in Section 3, and in Section 4 numerical results are compared with existing theory, the scaling laws are found, and an illustrative example is presented for propagation in a more general waveguide.

2. Second moment equation

The treatment here is confined to a monochromatic scalar wave of wave-number k, propagating in a two-dimensional random medium which occupies the half-space $z \geqslant 0$ of the Cartesian plane (x, z). The refractive index $n(x, z)$ in the medium varies randomly as a function of both coordinates, and we write

$$n(x, z) = n_0 + n_s(x, z) + n_r(x, z), \tag{1}$$

where n_0 is a fixed reference value, and n_s and n_r are the systematic and stochastic variation of n about this value respectively. Thus n_r has mean zero, and

$$n_s = \langle n \rangle - n_0, \qquad n_r = n - \langle n \rangle,$$

where the angled brackets denote the ensemble average. It will be assumed here that n_s and n_r are small and vary slowly with range, so that they result in weak scattering. The constant n_0 will for convenience be set to unity. The r.m.s. $(\langle n_r^2 \rangle)^{1/2}$ of refractive index fluctuations will be denoted by μ, and the autocorrelation function $\langle n_r(x_1, z_1) n_r(x_2, z_2) \rangle / \mu^2$ by ρ_0, which is assumed to be a function of spatial separations $x_1 - x_2$ and $\zeta = z_1 - z_2$ only. The projected autocorrelation function f_{12} is defined by

$$f_{12} = \int_{-\infty}^{\infty} \rho_0(x_1, x_2, \zeta)\, d\zeta,$$

and we write $f_0 = f_{11}$.

We will consider a wave propagating mainly in the forward z direction in this medium, with slowly-varying component $E(x, z)$ which is well-described by the parabolic wave equation.

Denote by m the second moment

$$m(x_1, x_2, z) = \langle E(x_1, z) E^*(x_2, z) \rangle$$

of the complex field. Then the propagation of m and its dependence upon the medium and range is described by the partial differential equation

$$\frac{\partial m}{\partial z} = -\frac{i}{2k}\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) m - ik(n_{s1} - n_{s2}) m - \beta[1 - \rho(x_1, x_2)] m. \tag{2}$$

Here β is given by $k^2 \mu^2 f_0$ and $\rho(x_1, x_2) = f_{12}/f_0$. In what follows ρ is written as a function of separation only and is assumed to have a length scale L.

The equation above holds for any incident wavefield in which the energy propagates predominantly in the z direction. In this paper this is assumed to be a Gaussian beam, which may travel at a small angle θ to the horizontal:

$$E(x,0)=E_0\exp\left[-\left(\frac{x-x_0}{w}\right)^2-\mathrm{i}kS(x-x_0)\right],$$

where $S=\sin(\theta)$.

2.1. *Transformation to sum and difference coordinates*

In order to solve equation (2) it is convenient to write the equations in sum and difference coordinates, using the transformations:

$$\xi=\frac{x_1-x_2}{L},$$

$$X=\frac{x_1+x_2}{2L},$$

$$Z=z/kL^2. \tag{3}$$

Under this transformation the second moment may be written in operator form as

$$\frac{\partial m}{\partial Z}=(A+B)m(X,\xi,Z), \tag{4}$$

where

$$A=-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial X\,\partial\xi},$$

and

$$B=-\mathrm{i}k^2L^2(n_{s_1}-n_{s_2})-\Gamma[1-\rho(\xi)],$$

in which $\Gamma=\beta kL^2$ and all functions are understood to be in the new coordinates. The media in the examples which follow have either a Gaussian spectrum, whose autocorrelation function is $\rho(\xi)=\exp(-\xi^2)$, or a fourth-order power-law spectrum, with $\rho(\xi)=(1+|\xi|)\exp(-|\xi|)$. The initial condition must also be specified in these coordinates. With $\theta=0$ the function m at $z=0$ is:

$$m(X,\xi,0)\equiv\langle E(x_1,0)E^*(x_2,0)\rangle=E_0^2\exp\left[-\left(\frac{4X^2+\xi^2+4X_0-8X_0X}{2\alpha^2}\right)\right],$$

where $\alpha=w/L$ and $X_0=x_0/L$.

3. Numerical solution

3.1. *Formal solution*

If the mean variation n_s in the medium is independent of range, then the operators in (4) are also range-independent and over any distance ΔZ the equation has the formal solution:

$$m(Z+\Delta Z)=\exp\left[\int_Z^{Z+\Delta Z}(A+B)\,\mathrm{d}z\right]m(Z), \tag{5}$$

where $\int(A+B)\,\mathrm{d}z$ here is simply $\Delta Z(A+B)$. *If* n_s does vary with range then this is no longer an exact solution of (4), but it remains a good approximation provided n_s changes slowly over a distance ΔZ. The error in this case can be expressed as a function of the commutator $[A,B]=AB-BA$, and is of order $(\Delta Z)^3$ (see [7] for details). Now, analytical solutions of the expression (5) cannot be found in general. However when treated separately the terms involving the diffraction operator A and the scattering operator B may be solved explicitly. This leads to the approximation of (5) by operator splitting

$$m(Z+\Delta Z)=\exp(\Delta Z\,A)\exp\left(\int B\,\mathrm{d}z\right)m(Z). \tag{6 a}$$

Operator splitting has long been used for simulations in random wave problems, and more recently for the very effective solution of previously intractable fourth moment problems [5]. Equation (6 *a*) has an error by comparison with (5) which is of order $(\Delta Z)^2$ [7], and is again a multiple of the commutator $[A,B]$. This error is easily improved to $(\Delta Z)^3$ by writing

$$m(Z+\Delta Z)=\exp\left(\frac{\Delta Z}{2}A\right)\exp\left(\int B\,\mathrm{d}z\right)\exp\left(\frac{\Delta Z}{2}A\right)m(Z), \tag{6 b}$$

which is known as Strang's splitting. (Although it is easy to formulate higher-order schemes, there is little point in doing so because, firstly, refractive index profiles are in practice not known exactly, and secondly the equation (5) is itself approximate when the medium varies systematically with range.) Note that the commutator $[A,B]$ is in some sense small, and so the accuracy of equation (6) is better in practice than is suggested by the quantity $(\Delta Z)^2$ (see [8]).

The change in m over a distance ΔZ can thus be found by successive application of the operators $\exp(\int A)$ and $\exp(\int B)$, and this procedure repeated to find m at any distance into the medium. The solution for each of these terms is described below.

3.2. *Numerical implementation of the solution*

The numerical treatment of the above solution is now straightforward. Provided the underlying wavefield is confined to a channel, as we assume, there is a corresponding rectangular region of the (x_1,x_2) plane beyond which the second moment is negligible for all z. If this is given, say, by x_1, $x_2 \in (x_{\min}, x_{\max})$ then in the coordinates (X,ξ) the second moment is again essentially confined to a fixed finite region, bounded by $X_{\min}=x_{\min}/L$, $X_{\max}=x_{\max}/L$, $\xi_{\min}=-2X_{\max}$, $\xi_{\max}=-\xi_{\min}$.

The diffraction and scattering terms in equation (6) have formal solutions which are easily implemented. The scattering term $\exp(\int B\,\mathrm{d}Z)$ is just a multiplication operator. Since B is constant or varies slowly over a distance ΔZ its integral may be accurately represented by the numerical integration over the interval $[Z, Z+\Delta Z]$. The solution for the diffraction term $\exp(\Delta Z A)$ is equivalent to solving (4) with the scattering component suppressed. If F denotes the two-dimensional Fourier transform with respect to X and ξ, this gives

$$\exp(\Delta Z A)\,m(Z+\Delta Z)=F^{-1}[\exp(\mathrm{i}v_x v_\xi Z)\,F(m(Z))].$$

In the numerical scheme F is replaced by the fast Fourier transform (FFT).

Although the results here are presented for a completely channelled field, it is easy to extend the method to a fully spreading beam by a simple adaptation of the grid, by allowing the grid-points to spread with the second moment. A more sophisticated 'adaptive grid' method could also be employed (see [9]) but would be significantly more complicated in this case because the solution lacks coordinate symmetry such as that of the standard fourth moment problem.

4. Computational results

The accuracy of the numerical solution can first be confirmed by comparison with cases for which analytical solutions have been found. For example the solution to equation (2) for $n_s = 0$ and plane wave initial conditions has the simple analytical form (e.g. [10])

$$m(\xi) = \exp\{-\Gamma Z[1-\rho(\xi)]\}. \tag{8}$$

Comparison between this and the full numerical solution of (2) showed extremely close agreement. In a typical case, for a Gaussian autocorrelation function with $\Gamma = 50$, differences were restricted to the fifth decimal place.

Analytical solutions in the presence of a waveguide exist [1, 2] for the case of a parabolic profile,

$$n_s = -bx^2. \tag{9}$$

Some features of these may be compared with the numerical solution, but they are not strictly comparable except at $\Gamma = 0$ because the analytical calculations are based

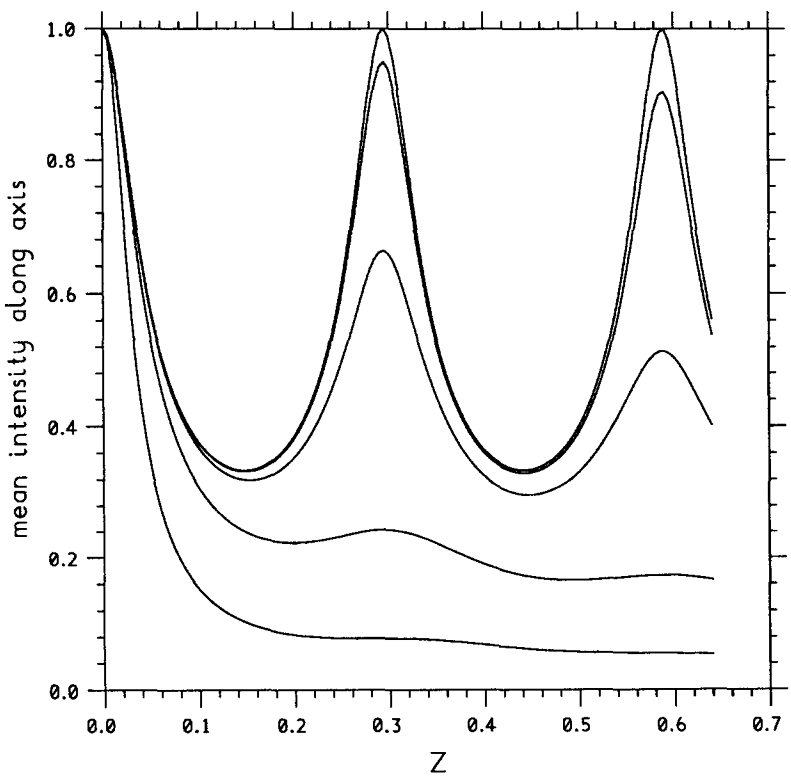


Figure 1. Mean intensity as function of scaled range on axis of parabolic channel, for $\Gamma = 0$ to 1000 in powers of 10. The amplitude decreases with increasing Γ.

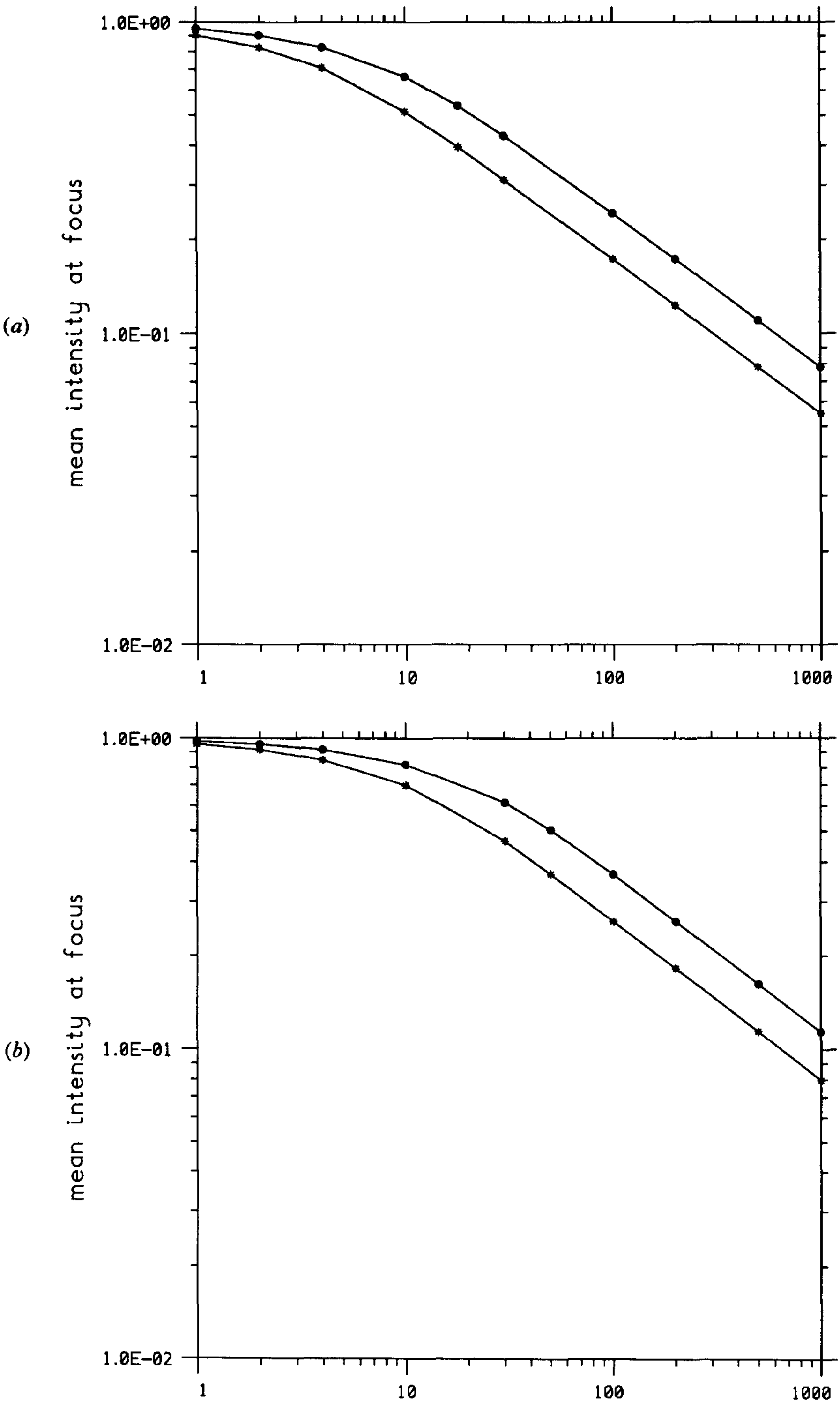


Figure 2. The peak intensity at the first (●) and second (*) foci as a function of Γ, for (*a*) a Gaussian spectrum and (*b*) a fourth order power law spectrum. Both axes are logarithmic.

on approximate forms of the transverse correlation function. One simple test of the numerical method is its ability to capture the repeated focusing which occurs in a homogeneous waveguide. The analytical solution for the waveguide (9) yields the spacing between consecutive foci as

$$\frac{\pi}{(2b)^{1/2}}. \tag{10}$$

Figure 1 shows the mean intensity along the axis of a parabolic channel with $b=5{\cdot}7\times10^{-7}\,\mathrm{m}^{-2}$ from the numerical solution of (4). Results are given for a homogeneous waveguide ($\Gamma=0$) and for various degrees of inhomogeneity (up to $\Gamma=1000$). The initial half-width of the Gaussian beam is a quarter of a correlation length. The spacing of consecutive foci is 2942 m in unscaled coordinates, assuming $k=1$ and $L=100\,\mathrm{m}$, in very close agreement with (10).

A useful measure of the scattered wavefield is the peak intensity $I_j(\Gamma)$ at the jth focus. (This is taken relative to the peak intensity at zero range.) Figure 2 (*a*) shows $I_j(\Gamma)$ at the first and second focus ($j=1$ and 2) as a function of Γ for a medium with a Gaussian spectrum. The corresponding result for a medium with a fourth-order power-law spectrum in shown in figure 2 (*b*). In all cases the source was centred on the axis of the waveguide. The behaviour clearly scales with Γ, asymptotically approaching the behaviour

$$I_j(\Gamma)\approx a_j\Gamma^{-1/2}. \tag{11}$$

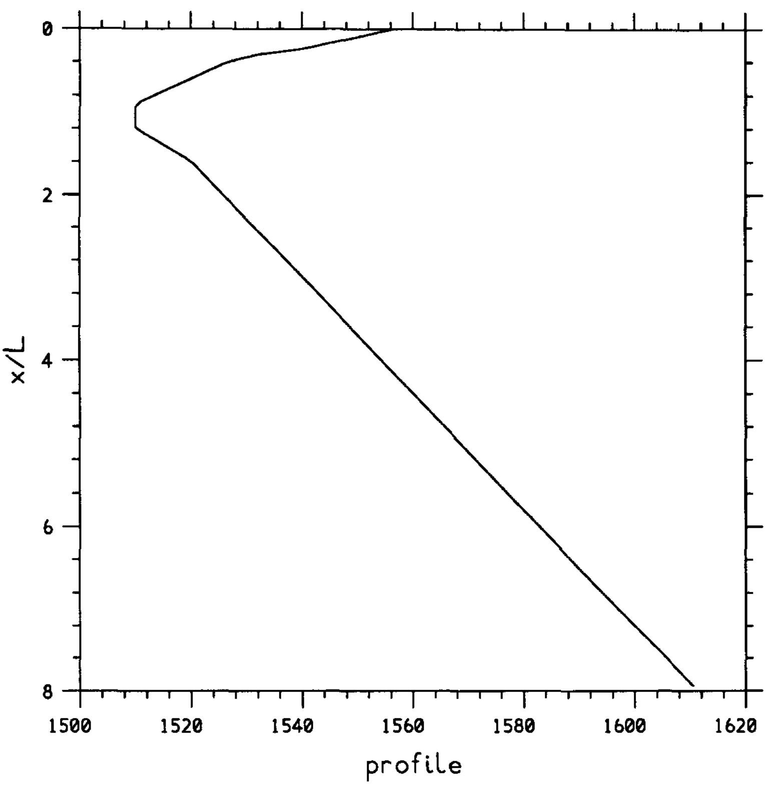
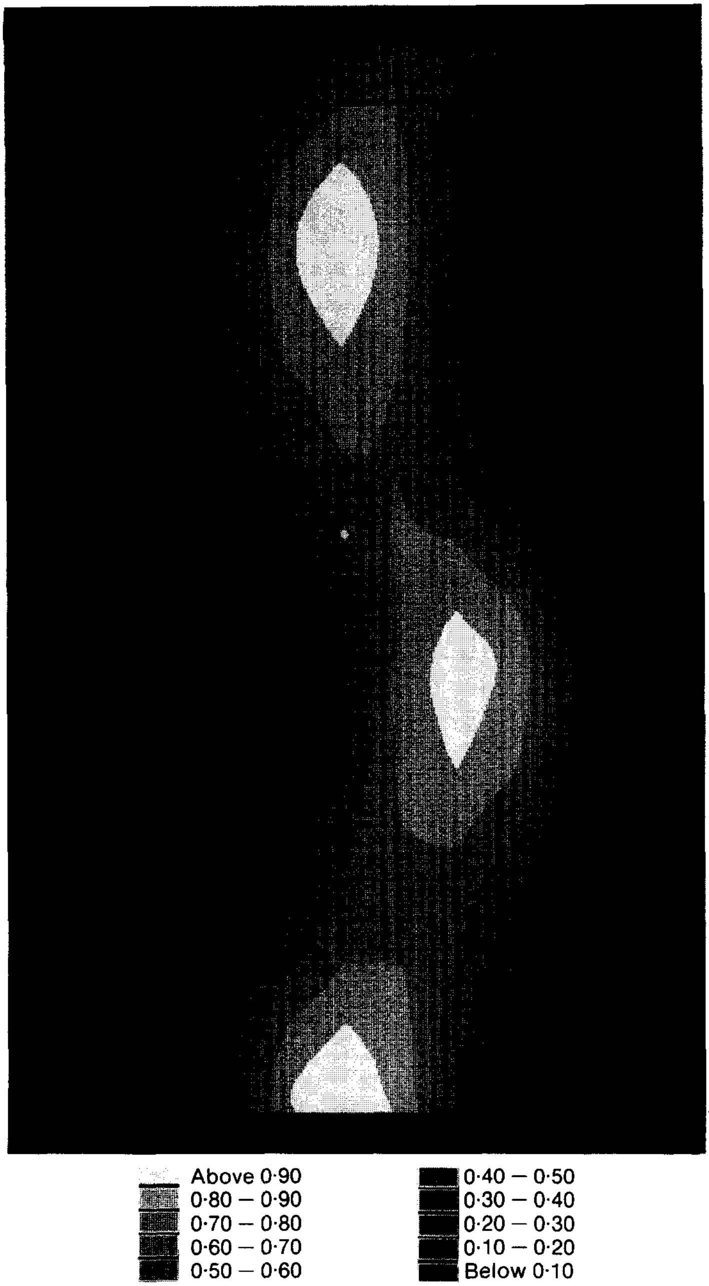
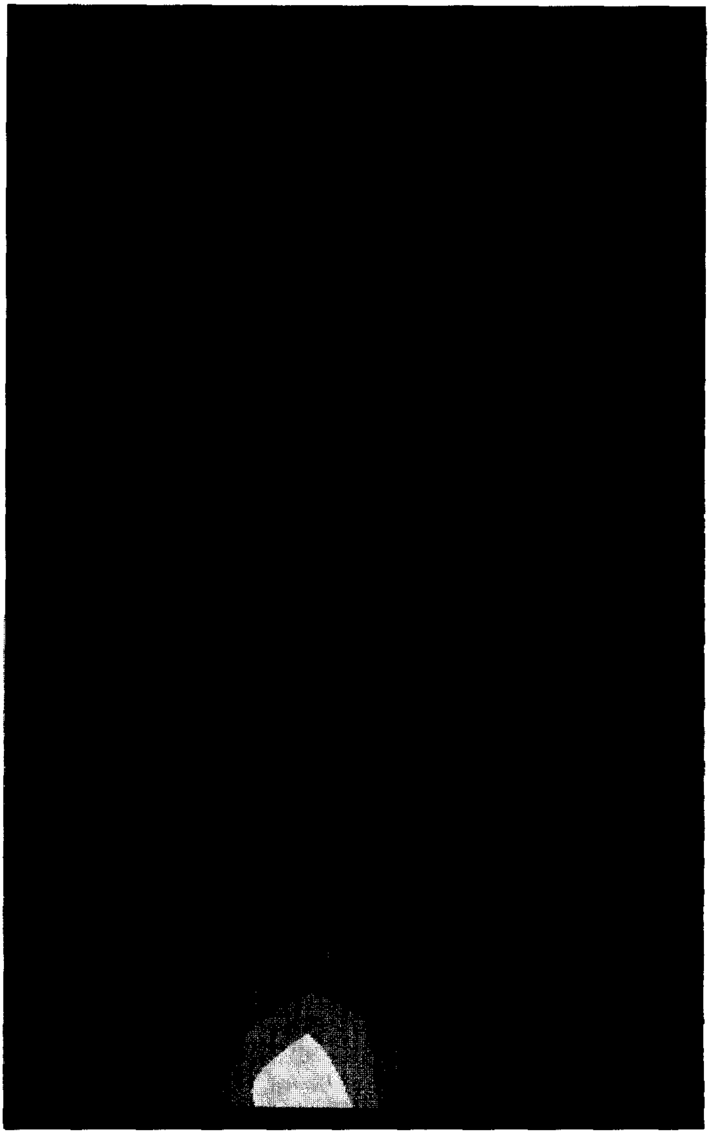


Figure 3. Mean wave-speed profile as a function of the transverse coordinate x/L where L is the correlation length, with $x=0$ at the surface.


Above 0·90
0·80 – 0·90
0·70 – 0·80
0·60 – 0·70
0·50 – 0·60
0·40 – 0·50
0·30 – 0·40
0·20 – 0·30
0·10 – 0·20
Below 0·10

(*a*)



(*b*)

Figure 4. Contour plot of mean intensity due to propagation in waveguide with mean profile as in figure 3, as a function of depth (horizontal axis, increasing left to right) and range Z (vertical, increasing upwards), for (*a*) $\Gamma = 0$, and (*b*) $\Gamma = 100$. (The slight irregularity of some contours is an artifice of the plotting routine.)

The values of a_j depend upon j, the correlation function of irregularities, and the exact form of the source. It is interesting that, as these results show, the asymptotic behaviour (11) holds even for relatively low values of Γ. For the parabolic waveguide the behaviour described by equation (11) may be inferred from approximate analytical solutions (e.g. [1, 2]), for media with either Gaussian or power law spectra.

As an illustration of a realistic application the numerical scheme has been used to calculate the mean acoustic intensity in a waveguide typical of those found in the ocean. Figure 3 shows the profile of sound speed with depth, scaled by the correlation length L. This is a fairly sharp axially asymmetric waveguide, incresing almost linearly at larger depths. Such behaviour is well-known and has been repeatedly observed experimentally (e.g. [11]). Figure 4 shows the mean intensity field as a function of longitudinal and transverse coordinates, for (*a*) $\Gamma=0$ and (*b*) $\Gamma=100$. The graph represents the axis $\xi=0$ of the full second moment as a function of Z. Propagation is upwards to a scaled distance of $Z=0{\cdot}64$, with a transverse extent of about two correlation lengths. The mean pattern of intensity weaves back and forth due partly to the asymmetric profile, and attenuates because of the decreased coherence of the wavefield caused by repeated scattering.

5. Summary

An accurate and efficient numerical scheme for solving the second moment equation has been described. This allows mean spectra and intensities to be found for propagation within inhomogeneous waveguides in a wide variety of scattering regimes; such models exhibit behaviour due to both refraction and diffraction. In addition functional laws governing the intensity peaks in a parabolic channel have been obtained, for two different media. Analytical solutions are obtainable only for a restricted class of problems, and the present numerical method thus considerably increases the range of applications which can be studied.

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