JOSA COMMUNICATIONS

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Accuracy of the moments from simulation of waves in random media

M. Spivack

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom

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Numerical simulation of wave propagation in random media is used principally to obtain the statistical moments of the wave field. It has been shown [J. Computat. Appl. Math. 27, 349 (1989)] that both local and cumulative accuracy of the second moment, as calculated from the solution, are greater than for the wave field itself. The purpose of this Communication is to extend these results to the higher moments and to demonstrate them computationally. The error is expressed as a function of operators acting on the wave field, and this permits the accuracy of the moments to be examined in terms of the error and its autocorrelation function. A model in which the random medium may be represented in arbitrarily fine detail is used to obtain the computational results.

1. INTRODUCTION

Numerical simulation is now widely used in the study of wave propagation in random media. A parabolic approximation to the wave equation is normally used and, in the solution, is approximated analytically by the split-step method[1,2] in which the medium is represented by a series of independent phase screens.

The main interest lies in finding the statistical moments of the wave field u and their relationship with those of the medium. (The problem of solving the moment equations directly has received much attention: see, for example, Refs. 3–5.) It is therefore essential to quantify the effect on these moments of numerical errors in the underlying wave field. In Ref. 6 it is shown that the accuracy produced in the second moment is of higher order than that of the field itself. This relationship holds both for local and cumulative accuracy. The purpose of this Communication is to show how this relationship extends to the higher moments and to demonstrate the results numerically. In our approach the error ϵ is expressed explicitly as a function of the operators acting on u, whose properties we can examine. ϵ is a continuous function that varies randomly in all directions. The error induced in the moments of the wave field can be expressed in terms of the correlation of ϵ with itself and with u. The generation of correlated sets of random phase screens enables us to represent a medium in arbitrarily fine detail, and this representation is used to demonstrate the results computationally.

In Section 2 the results from Ref. 6 are explained and extended, giving the local and global errors for the moments of the wave field. The computational scheme and numerical results are described in Section 3.

2. MATHEMATICAL FORMULATION AND ACCURACY OF THE SOLUTION

We consider a monochromatic plane-wave incident on a half plane $(x, z > 0)$ with a randomly varying refractive index n. The wavelength λ is assumed to be significantly shorter than the typical scale size of irregularities so that scattering takes place mainly in a forward z direction. The full elliptic wave equation can then be approximated by a parabolic form in which z becomes a timelike direction of propagation. In two-dimensional Cartesian coordinates (x, z), this equation is

$$u_z = -\frac{i}{2k} u_{xx} - i\frac{k}{2}(n^2 - 1)u, \tag{1}$$

where u is the complex wave field $k = 2\pi/\lambda$, and the subscript denotes differentiation. We can write the refractive index as $n = 1 + n_d(x, z) + \mu W(x, z)$, where n_d is a deterministic departure from 1 and μW varies randomly with zero mean and standard deviation μ. Here W is assumed to have Gaussian statistics but an arbitrary correlation function, with correlation lengths l and l_z transverse to and parallel to the direction of propagation, respectively. (Time dependence has been excluded since the wave speed is typically very fast relative to changes in n.) We employ the usual scaling and set $X = x/l$, $Z = z/kl^2$, and $L_Z = l_z/kl^2$. Since we assume n_d and μ to be small, Eq. (1) can be written, in terms of operators, as

$$\frac{\partial u}{\partial Z} = (A + B)u, \tag{2}$$

where $A = -i/2\ \partial^2/\partial X^2$ and $B = -ik^2L^2(n_d + \mu W)$ are the

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distance and scattering operators, respectively. The formal solution of Eq. (2) over a distance ΔZ is then approximately

$$u(Z + \Delta Z) \approx \exp\left[\int_Z^{Z+\Delta Z} (A + B)\, dZ\right] u(Z). \quad (3)$$

This approximation is shown in Ref. 6 to have a stepwise error that is of order $(\Delta Z)^3$ and is, to this order, a multiple of the commutator $[A, B] \equiv AB - BA$ of A and B. The split-step approximation, which was introduced to this problem by Tappert and Hardin[1] and applied by Macaskill and Ewart[2] to random media, is

$$u(Z + \Delta Z) \approx \exp\left(\int_Z^{Z+\Delta Z} A\, dZ\right) \exp\left(\int_Z^{Z+\Delta Z} B\, dZ\right) u(Z). \quad (4)$$

This iteratively applied solution has a stepwise error of order $(\Delta Z)^2$. [This error is sometimes improved to $(\Delta Z)^3$ with a slight modification of relation (4), but the cumulative accuracy is not affected.]

Now, $\int_Z^{Z+\Delta Z} A\, dZ = -i\Delta Z/2\, \partial^2/\partial X^2 = A'$, say. If ΔZ is sufficiently small so that B is almost constant in Z over that distance, say $B \approx B(X)$, then $\int_Z^{Z+\Delta Z} B\, dZ \approx \Delta Z \cdot B = B'$, say. Consider the error $\epsilon(u) = [\exp(A' + B') - \exp(A')\exp(B')]u$ between relations (3) and (4). To first order, $\epsilon(u)$ is equal to $\epsilon_1(u)/2$, where ϵ_1 is the commutator $[A', B']$ (e.g., Iserles and Sheng[7]).

Local Errors

Since the computed wave field is used principally to find statistical quantities, it is essential to find how these are corrupted by the statistics of the numerical error. We are mostly interested in the second moment (autocorrelation function) $\rho(\xi, Z) = \langle u(X, Z)u^*(X + \xi, Z)\rangle$ and the fourth moment, where angle brackets denote the ensemble average. Write $e = \epsilon_1(u)$, so that $e = -i(\Delta Z)^2[B_{XX}u + 2B_X u_X]$, and put $\phi = (\Delta Z)^{-2}e$. Note that u and B become independent, and $\langle u\rangle$ and $\langle B\rangle$ are zero. The derivatives $\partial^n B/\partial X^n$ also have zero mean, so the mean stepwise error $\langle e\rangle$ is zero. All these quantities are, by assumption, stationary and invariant under reflections about any point X.

Second Moment

We sketch the analysis given in Ref. 6. Let $\rho_e(\xi)$ be the correlation function $\langle e_1 e_2^*\rangle$ of the error, where e_i denotes $e(X_i)$ and $\xi = X_1 - X_2$. Then, if $U = u + e$ and $\rho_U(\xi) = \langle U_1 U_2^*\rangle$,

$$\rho_u(\xi) = \rho(\xi) + \rho_e(\xi) + \langle e_1 u_2^*\rangle + \langle u_1 e_2^*\rangle.$$

Since B is independent of u, the first cross term $\langle e_1 u_2^*\rangle$ is given by

$$-i(\Delta Z)^2[\langle B_{1XX}\rangle \rho(\xi) + 2\langle B_{1X}\rangle\langle u_{1X}u_2^*\rangle],$$

which vanishes since the derivatives of B have mean zero. The second cross term $\langle u_1 e_2^*\rangle$ similarly disappears, so the stepwise error in the correlation function is exactly $\rho_e(\xi)$. We now have $\langle \rho_e(\xi)\rangle = (\Delta Z)^4\langle \phi_1\phi_2^*\rangle$, and the error is therefore of order $(\Delta Z)^4$. The fact that this is more accurate than the wave field is due to the disappearance of the cross correlation of e with u.

Fourth and Higher Moments

Now consider the fourth moment $m = \langle u_1 u_2^* u_3 u_4^*\rangle$, where u_i denotes $u(X_i)$, and let m_U be the fourth moment of U. Then

$$m_U = \langle (u_1 + e_1)(u_2 + e_2)^*(u_3 + e_3)(u_4 + e_4)^*\rangle, \quad (5)$$

so that $m - m_U$ involves terms of first, second, third, and fourth orders in the error e. Now the terms in the expansion of Eq. (5) in which e appears once are of the form $\langle u_1 u_2^* u_3 e_4^*\rangle$ or similar, and these terms all disappear as they do in the second moment. The terms of second order in e are again of order $(\Delta Z)^4$. The remaining terms depend on higher powers of ΔZ so the error in the fourth moment is again $O[(\Delta Z)^4]$. The reasoning for the higher moments is identical, so the stepwise error in all cases is of this order.

Cumulative Errors

Cumulative Error in the Wave Field

We will here sketch the results for the field itself and show how those for its autocorrelation function extend to the higher moments. Suppose the total range of propagation is divided into n steps, so that n is of order $(\Delta Z)^{-1}$. Let U_k be the numerical solution of $u(Z_k)$. Denote by S_k the operator, which represents propagation from Z_k to Z_{k+1}, as determined by relation (3), and let S_k' be the composition of S_k, $S_{k+1}, \ldots, S_{n-1}$. Define the kth stepwise error by $e_k = S_k(u_k) - u_{k+1}$. Then we can write $U_n = u_n + e_{n-1} + S_{n-1}'e_{n-2} + \ldots + S_1'e_0$. Now, put $\phi_k = (\Delta Z)^{-2}e_k$ and $\psi_k = S_k'\phi_{k-1}$. If we define $\Psi = \psi_n + \psi_{n-1} + \ldots + \psi_1$ (where $\psi_1 = \phi_{n-1}$), then U_n can be written $U_n = u_n + (\Delta Z)^2\Psi$. Thus the cumulative error is $(\Delta Z)^2\Psi$, where Ψ consists of n terms. It can be shown that, for each realization, $\Delta Z\Psi$ is approximately equal to the integral with respect to Z of a fixed continuous function ψ, which is independent of step size, over the range (Z_1, Z_n). Thus the cumulative error in the wave field is approximately $\Delta Z \int \psi dZ$, which is of order ΔZ.

Cumulative Error in the Moments

As with the stepwise error, we can show that the moments are of higher accuracy than the field itself. Now, $U_n = u_n + \Delta Z \int \psi dZ$. By analogy with the analysis[6] for the second moment, let m_U be the fourth moment of U_n. Then we can express the error $m - m_U$ as a sum of terms of first, second, third, and fourth orders in $\Delta Z \int \psi dZ$. Just as in the case of the stepwise error, the first-order terms, such as $\langle u_1 u_2^* u_3 e_4^*\rangle$, vanish identically, although the calculation is now more complicated. Denote by G_k the set of functions of the form

$$g = B^r(Z_k)h[B(Z_{k+1}), \ldots, B(Z_n), u],$$

where the superscript denotes an X derivative of some degree, and h is any function. Let G_k^S denote the set of series (or sums) of elements of G_k. Then it is easy to show that G_k^S is invariant under the operator $S_l = \exp(\Delta Z A)\exp(\Delta Z B_l)$, for $l > k$. Thus, since each e_k is in G_k^S, it follows that ϕ_{n-1} and each function ψ_k is also in G_k^S. Now, for each of the functions g, $B^r(Z_k)$ is independent of h by our assumption that $\Delta Z \geq L_Z$ and since B and u are independent. It follows that

$$\langle g u_1{}^* u_2 u_3{}^* \rangle = \langle B^r(Z_k) \rangle \langle h u_1{}^* u_2 u_3{}^* \rangle = 0.$$

Therefore every term in the expansion of $\langle u_1{}^* u_2 u_3{}^* \psi_k \rangle$ is zero, and the cross correlations $\langle u_1 u_2{}^* u_3 \Psi^* \rangle$ and $\langle u_1{}^* u_2 u_3{}^* \Psi \rangle$ vanish as required.

Now the remaining terms, which are of second order or greater in $\Delta Z \int \psi \mathrm{d}Z$, are all $O[(\Delta Z)^2]$. This also holds for the higher moments, so that the cumulative error for all the moments is of order $(\Delta Z)^2$, which is compared with an error of $O(\Delta Z)$ for the wave field itself. Note that although the first-order-error terms disappear when averages are taken, they may still be present in individual realizations.

These results indicate that the moments of the wave field can be calculated very reliably from their numerical approximations, even when the scattering strength is large.

Other methods for solving relation (3) are in use (e.g., Lee and Papadakis[8]), and these methods are not discussed here. However, a crucial element of the split-step scheme applied to random media is the use of step size at least as large as L_Z. The generation of the random medium is at least as expensive computationally as the solution of relation (4), requiring $2n$ one-dimensional fast Fourier transforms, where n is the number of steps. Any reduction in ΔZ introduces correlations between the phase screens and immediately entails the use of two-dimensional fast Fourier transforms, which greatly increase the order of computation.

3. COMPUTATIONAL SCHEME AND RESULTS

The implementation of the split-step scheme[2] and the generation of the random medium have been fully described elsewhere (see Refs. 2 and 6), and we will not give the details here. However, we will briefly outline the method of increasingly fine subdivision of the medium. Given the two-dimensional correlation function ρ of the medium W, we wish to construct a phase screen $\Phi(X)$ to represent the cumulative effect of W in the strip of medium $(Z, Z + \Delta Z)$. The transverse correlation function ρ_1 of Φ will be given by

$$\rho_1(\xi, \Delta Z) = \int \int_Z^{Z+\Delta Z} \rho(\xi, Z_1 - Z_2)\, \mathrm{d}Z_1\, \mathrm{d}Z_2.$$

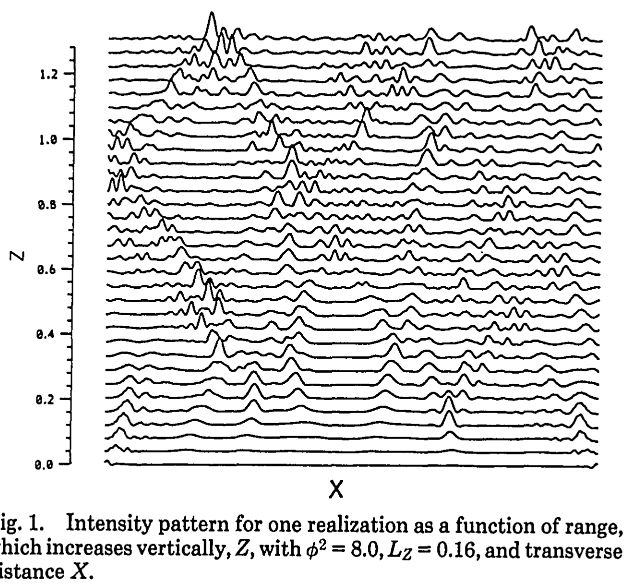

Fig. 1. Intensity pattern for one realization as a function of range, which increases vertically, Z, with $\phi^2 = 8.0$, $L_Z = 0.16$, and transverse distance X.

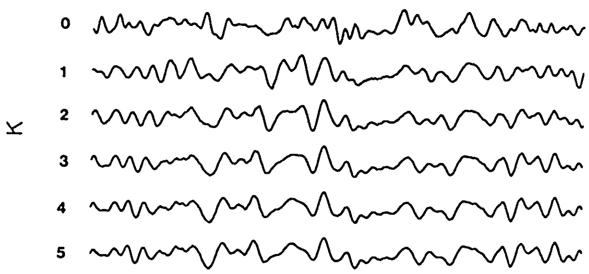

Fig. 2. Real parts of wave field at maximum range $Z = 1.28$ compared for step size 2^{-K}, as K is varied from 5 (bottom curve) to 0 (top curve), for same parameters as in Fig. 1.

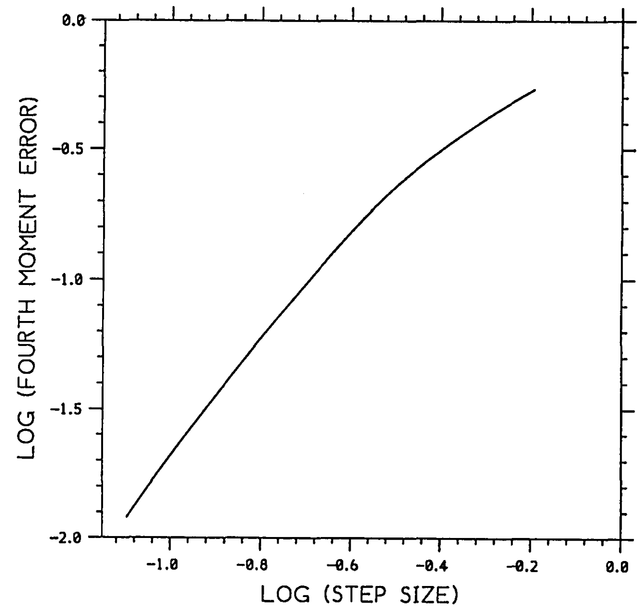

Fig. 3. Logarithmic plot of error e_k in the fourth moment of wave field as function of step size for $\phi^2 = 2.0$, $L_Z = 0.08$.

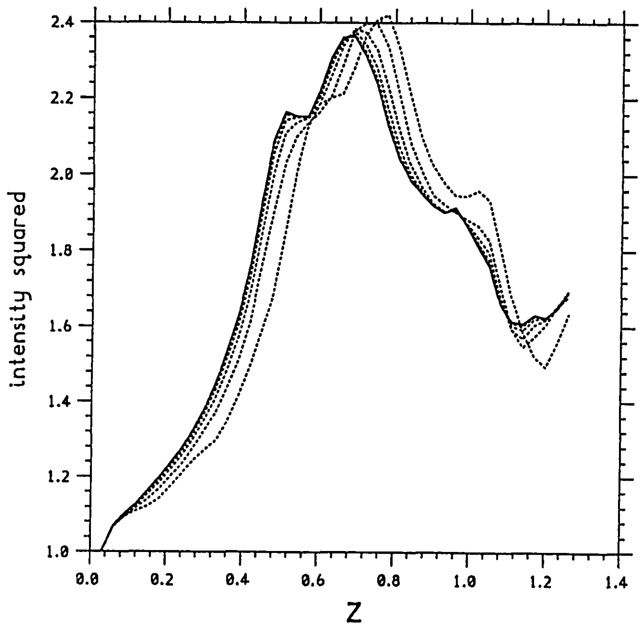

Fig. 4. Fourth moment (average of intensity squared) as a function of range Z for one realization, for $\phi^2 = 0.4$ and $L_Z = 0.08$.

Using this correlation function, we can generate a series of phase screens at a distance ΔZ apart, and each screen can be given the appropriate variance $k^2\mu^2\rho_1(0, \Delta Z)$. We denote by ϕ^2 the mean-square fluctuation $k^2\mu^2\rho_1(0, L_Z)$ that is imposed by each irregularity. When $\Delta Z \geq L_Z$ the phase screens may be modeled independently, but if ΔZ becomes less than a correlation length, the appropriate degree of correlation between the screens must be introduced. By using this procedure we can build up sets of random phase screens that represent the medium in arbitrarily fine detail, and we can propagate the wave numerically, in effect keeping the medium fixed while varying the step size between the screens.

In the results that we show here, the step size was successively halved until the solution had effectively converged. Several realizations were run, and the results were averaged over these. Thus, for example, for each realization, we obtained the error produced by a given step size and then averaged the results over a number of realizations. More precisely, denote by Δ_0 the smallest step size used. Let a be the exact solution (for whatever quantity is being calculated), and let a_k be the approximate solution obtained with step size $2^k\Delta_0$. We measure the errors e_k relative to a_0, since a is unknown, so that $e_k = a_k - a_0$ for $k = 1, 2, \ldots$ The order of the error should then be reflected in the ratios $r_k = e_{k+1}/e_k$. If we assume that the solution is of order α, then $a_k \approx a + \lambda(2^k\Delta_0)^\alpha$, where λ is some constant that will vary with realizations. Thus $e_k \approx \lambda(2^{\alpha k} - 1)\Delta_0{}^\alpha$, and so for k greater than 0 (and $\alpha \geq 1$), $r_k \approx 2^\alpha$. However for first-order error $r_0 \approx 3$, and $r_0 \approx 5$ when the error is of second order.

The scattering strength and maximum step size in the results shown here were much greater than the usual limits, in order to illustrate the behavior of the accuracy of the moments when errors for the wave field itself are large. Figure 1 shows the intensity pattern for one realization as a function of range, up to $Z_n = 1.28$. Here, the scattering per irregularity is $\phi^2 = 8.0$, and the correlation length is $L_Z = 0.16$. The real parts of the functions u at Z_n as the step size is successively reduced are compared in Fig. 2. The detailed features that arise when the maximum step size $\Delta Z = L_Z$ is used are very different from those that arise at the lower limit, $\Delta Z = 2^{-5}L_Z$.

The error $|e_k|$ in the fourth moment m_U (intensity squared) was measured as described above for $\phi^2 = 2.0$, $L_Z = 0.08$, and $Z = 1.28$ and averaged over 16 realizations. In Fig. 3 the logarithmic plot of this quantity as a function of ΔZ shows the second-order dependence on step size. Note that the curve starts to tail off as the maximum step size is approached. This is probably due to the fact that, in the limit of large step size, the numerical solution becomes completely uncorrelated with the exact solution, and the error is then determined by the variance of the ensemble of the functions $\langle u_1u_2{}^*u_3u_4{}^*\rangle$.

In Fig. 4 the fourth moment is shown as a function of range for a single realization with $\phi^2 = 0.4$ and $L_Z = 0.08$. The finest subdivision is represented by the solid curve and coarser subdivisions by the dotted curves. The order in which these subdivisions occur is obvious from the figure; in this case the dependence on ΔZ is of first order as would be expected in a single realization.

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