# Solution of the inverse-scattering problem for grazing incidence upon a rough surface

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The inverse-scattering problem for a scalar wave field incident at grazing angles upon a one-dimensional moderately rough surface is considered. The problem is solved directly by treating the scattering integral as an integral equation in the unknown field derivative at the surface and coupling this to a simple equation relating the derivative to the surface itself. An algorithm is described for the solution of this system, and results in which complicated rough surfaces are accurately reconstructed are presented.

## 1. INTRODUCTION

In the study of rough surface scattering the solution of inverse problems, and in particular of the reconstruction of surfaces from scattered data, has many potential applications and is one of the main aims. Treatment is usually by iterative methods, and their success in many problems for complicated scattering surfaces seems to have been limited (see, e.g., Refs. 1–5). For scalar wave fields at arbitrary angles of incidence the scattered field is described by the Helmholtz integral formula.[6,7] When the field is incident at small angles to the surface, forward scattering predominates, and the Helmholtz equations may be replaced by the parabolic equation method.[8,9] This consists of an integral equation in the transverse derivative $\partial\Psi/\partial z$ of the field at the surface and a scattering integral. Despite the simplifications offered by the parabolic regime, multiple scattering occurs even for slight roughness, and the dependence of the scattered field on the surface remains highly nonlinear.

In this paper the inverse problem for grazing incidence upon a one-dimensional pressure-release surface $h(x)$ is formulated as an integral equation in $\partial\Psi/\partial z$. This is coupled to a simple expression relating $h$ to $\partial\Psi/\partial z$, and the system is solved directly by numerical inversion. The second equation is valid for moderate rms surface variation and is needed because the kernel depends on the initially unknown surface. The Volterra form of the integral equation allows us to find the surface progressively and to substitute the values back into the kernel. Thus although the method may be regarded as a direct solution it cannot be extended as such to the full Helmholtz formula. It is assumed that the incident wave field is known, together with scattered data along a line parallel to the mean surface level; these data permit the scattering integral to be treated as an integral equation. (A previous method[5] for this regime requires twice this amount of data, and the corresponding simultaneous inversion of two integral equations, and presents numerical difficulties that have not yet been fully overcome. A recent paper[4] also presents a successful direct solution, for nongrazing incidence, based on the Kirchhoff approximation.) The solution is implemented numerically, and results in which rapidly varying surfaces are accurately reconstructed are presented.

In Section 2 the parabolic equation method for surface scattering is given. The equations for the inverse problem are formulated in Section 3, and the numerical solution is described, together with the computational results.

## 2. MATHEMATICAL FORMULATION

We consider the problem of a scalar time-harmonic wave field $p$ scattered from a one-dimensional rough surface. The field is at grazing incidence. The Dirichlet boundary condition is assumed, i.e., $s$ or TE polarization and perfect conductivity. (The scalar treatment is applicable provided that we restrict attention to surfaces of one-dimensional variation so that polarization does not change under scattering.) The coordinate axes are $x$ and $z$, where $x$ is the horizontal $x \geq 0$ and $z$ is the vertical. For convenience it is assumed that the mean surface level is at $z = 0$. The source is centered about $\mathbf{r} = (0, z_0)$, with wave number $k$. The rough surface itself is denoted $h(x)$, so that $h$ has mean zero. In the numerical examples $h$ is drawn from an ensemble of normally distributed and statistically stationary processes, with rms surface height denoted by $\phi$. (This statistical description is used for convenience, but the stochastic nature of the surface is not central to the treatment here.)

Since the wave field propagates predominantly in one direction, it has a slowly varying part $\Psi$ defined by

$$\Psi(x, z) = p(x, z)\exp(-ikx).$$

Incident and scattered components $\Psi_{\text{inc}}$ and $\Psi_s$ are defined similarly, so that $\Psi = \Psi_{\text{inc}} + \Psi_s$. It is assumed that $\Psi_{\text{inc}}[x, h(x)] = 0$ for $x \leq 0$, so that the area of surface illumination is restricted as, for example, when the field is a Gaussian beam (see below). The governing equations for the parabolic equation method[8,9] are then

$$\Psi_{\text{inc}}(\mathbf{r}) = -\int_0^x G(\mathbf{r}; \mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial z}\mathrm{d}x', \tag{2.1}$$

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where both $\mathbf{r} = [x, h(x)]$, $\mathbf{r}' = [x', h(x')]$ lie on the surface and

$$\Psi_s(\mathbf{r}) = \int_0^x G(\mathbf{r}; \mathbf{r}') \frac{\partial \Psi(\mathbf{r}')}{\partial z} \mathrm{d}x', \tag{2.2}$$

where $\mathbf{r}'$ is again on the surface and $\mathbf{r}$ is now an arbitrary point in the medium. Here $G$ is the parabolic form of the Green's function given by

$$G(x, z; x', z') = \frac{1}{2}\left[\frac{i}{2\pi k(x - x')}\right]^{1/2} \exp\left[\frac{ik(z - z')^2}{2(x - x')}\right] \tag{2.3}$$

when $x' < x$ and $G = 0$ otherwise. [This gives rise to the finite upper limit of integration in Eqs. (2.1) and (2.2).] This Green's function is derived under the assumption of forward scattering, i.e., that the field obeys the parabolic wave equation

$$\Psi_x + 2ik\Psi_{xx} = 0,$$

which holds provided that the angles of incidence and scattering are fairly small. The accuracy has been examined by Thorsos.[8] These results are of course inapplicable to situations in which backscattering is measured.

Since $\partial\Psi/\partial z$ is considered only at points along the surface, it may be considered here as a function just of $x$ and denoted $\Psi'$. Let $\alpha = \frac{1}{2}(i/2\pi k)^{1/2}$. The incident field used in the examples below is a Gaussian beam of initial width $w$, assumed for simplicity to be traveling horizontally:

$$\Psi_{\mathrm{inc}}(x, z) = \frac{w}{(w^2 + 2ix/k)^{1/2}} \exp\left[-\frac{(z - z_0)^2}{w^2 + 2ix/k}\right]. \tag{2.4}$$

This field impinges upon the surface as it spreads; the pattern of illumination along a flat surface rises from zero to a peak and decays with $1/\sqrt{x}$. We refer to this field as being at zero grazing angle, although it is composed of a spectrum of angles; results similar to those shown here may be obtained for incidence at small nonzero angles.

## 3. SOLUTION OF INVERSE PROBLEM AND RESULTS

### A. Formulation and Numerical Solution

Suppose that the scattered field is known along an interval $[0, X]$ at some distance $z$ from the surface. Once the kernel $G$ is known, Eq. (2.2) can be regarded as a Volterra integral equation in $\partial\Psi/\partial z$. Although the surface is initially unknown, it has the simple integral relationship (3.3) (below) to $\partial\Psi/\partial z$. Thus, since $G$ is here a function of $h$, Eq. (2.2) and relation (3.3) can be treated as coupled equations. Now because scattering is mainly around one direction, these equations can be solved from the left to yield values at successive points along the interval; as the solution for $\partial\Psi/\partial z$ progresses, surface values can be found simultaneously and substituted back into the equation. This is only a slight modification of the standard procedure for inverting such integral equations by discretization and Gaussian elimination.

We first express the surface in terms of $\partial\Psi/\partial z$: When the surface is moderately rough (i.e., the rms variation $\phi$ is not too large), $\partial\Psi/\partial z$ varies approximately linearly with $\Psi_{\mathrm{inc}}[x, h(x)]$, which is in turn almost linear in $h$. Explicitly, for small $h$ the incident field [Eq. (2.4)] along the surface can be written as

$$\Psi_{\mathrm{inc}}[x, h(x)] \cong \Psi_{\mathrm{inc}}(x, 0)\left[1 + \frac{2z_0 h(x)}{w^2 + 2ix/k}\right], \tag{3.1}$$

and in Eq. (2.1) the exponent can be neglected[10] so that

$$\Psi_{\mathrm{inc}}[x, h(x)] \cong -\alpha \int_0^x \frac{1}{(x - x')^{1/2}} \Psi'(x') \mathrm{d}x', \tag{3.2}$$

where $\Psi' = \partial\Psi/\partial z$. From relations (3.1) and (3.2) we obtain

$$h(x) \cong \left[\frac{-\alpha \int_0^x (x - x')^{1/2} \Psi'(x') \mathrm{d}x'}{\Psi_{\mathrm{inc}}(x, 0)} - 1\right]\left(\frac{w^2 + 2ix/k}{2z_0}\right). \tag{3.3}$$

We can denote this linear transformation by $L$, that is, $h \cong L\Psi'$. The integral here is easy to evaluate numerically. Approximation (3.3) (which includes multiple-scattering effects via the integral) remains accurate for $k\phi$ up to $\sim 1$, which represents a fairly rough surface especially at grazing incidence. Note that more accurate, nonlinear, expressions can be used in place of relation (3.1).

The numerical solution is implemented as follows: The interval $[0, X]$ is discretized by evenly spaced points $\{x_r\}$, say, where $r = 0, \ldots, N$ and the intervals $\Delta x = x_r - x_{r-1}$ are small. It is supposed that scattered data are available at the points $x_r$. For $x_n \in [0, X]$ the integral [Eq. (2.2)] is written as a sum of subintegrals over the corresponding intervals:

$$\Psi_s(x_n, z) = \sum_{r=1}^{n} \int_{x_{r-1}}^{x_r} G[x_n, z; x', h(x')] \Psi'(x') \mathrm{d}x'.$$

The function $\partial\Psi/\partial z$ varies slowly compared with $G$ and may be treated as constant over each subinterval, and the equations may therefore be written as

$$\Psi_s(x_n, z) \cong \sum_{r=1}^{n} \Psi'(x_r) \gamma_{n,r}(z), \tag{3.4}$$

where

$$\gamma_{n,r}(z) = \alpha \int_{x_{r-1}}^{x_r} G[x_n, z; x', h(x')] \mathrm{d}x'. \tag{3.5}$$

When $h$ is known the coefficients $\gamma_{n,r}$ may be approximated as follows: For $x' \in [x_{r-1}, x_r]$, take the Taylor expansion of $h$ about $x_{r-1}$,

$$h(x') \cong h(x_{r-1}) + (x' - x_{r-1}) \frac{\mathrm{d}h(x')}{\mathrm{d}x}.$$

$\gamma$ can then be written in terms of Fresnel integrals after a change of variable followed by integration by parts.[9] (This analytical treatment is necessary to deal with the singularity as $x' \to x_n$.)

Therefore if $h$ has been obtained up to $x_n$ then all the coefficients $\gamma_{n,r}$ can be found. From relation (3.4) we

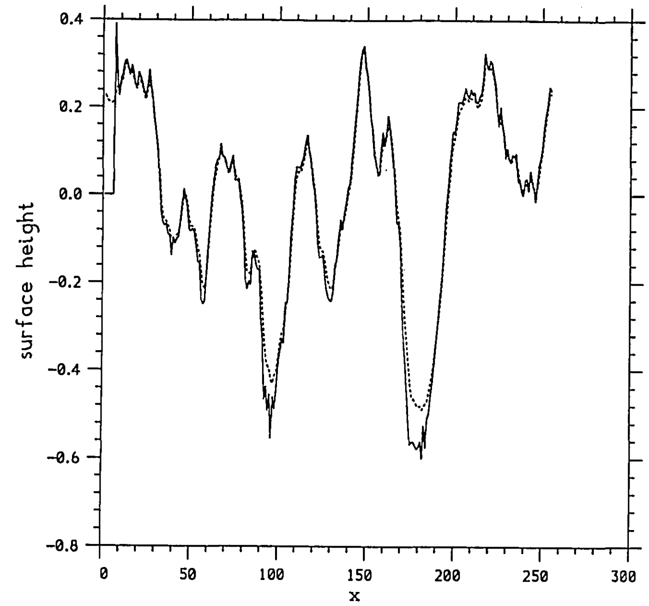


Fig. 1. Comparison between the surface $h$ (dotted curve) and its reconstruction $\bar{h}$ (solid curve).

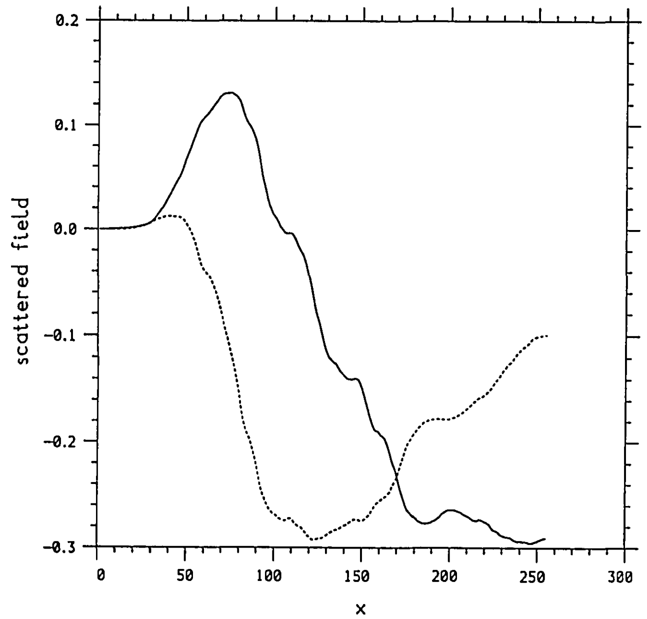


Fig. 2. Real (solid curve) and imaginary (dotted curve) components of scattered data along a line at distance 0.7 from the mean surface level.

then have

$$\Psi'(x_n) \cong \left[\Psi_s(x_n, z) - \sum_{r=1}^{n-1} \Psi'(x_r)\gamma_{n,r}(z)\right] \Big/ \gamma_{n,n}(z). \quad (3.6)$$

From this we obtain the surface height at the point $x_n$,

$$h(x_n) \cong L\Psi'(x_n), \quad (3.7)$$

where $L$ is as defined above. To begin this solution an initial value is required either for $\Psi'(0)$ or for $h(0)$ (or $\gamma_{1,1}$). We make the simplest assumption, that the surface around $x = 0$ is flat, setting $h(0) = 0$. (The results overall are insensitive to this single guess; this is essentially because the field is not illuminated around $x = 0$.)

This procedure yields values for $\Psi'$ and the surface $h$ itself throughout the interval $[0, X]$. The assumption that the scattering is predominantly about one direction permits the solution to be progressively substituted back into the scheme. Consequently this method could be extended to the full Helmholtz formula only as an iterative solution.

## B. Results

The above procedure was followed to obtain a solution for $\Psi'$ when the surface and the scattered wave field had been produced by simulations using the full form of the parabolic equation method. (Details of this and the generation of random surfaces are given in Ref. 9.)

In the first example a surface was generated with $\phi = 0.3$, and scattering was calculated for an incident field that is due to a source with wave number $k = 1$ and initial width $w = 8$ and located at a distance $z_0 = 22.4$. The random surface was fairly jagged, with a subfractal autocorrelation function of the form

$$\langle h(x)h(x + \xi)\rangle = \left(1 + \frac{|\xi|}{l}\right)\exp\left(-\frac{|\xi|}{l}\right),$$

where $l = 8$, slightly more than a wavelength. In Fig. 1 the surface $h$ is compared with its solution $\bar{h}$, say. The solution recaptures the detailed features of $h(x)$. Since $\bar{h}(0) = 0$ whereas $h(0)$ is in general nonzero, this also illustrates the relative lack of sensitivity of the solution to the guess of the first surface height. (The solution in and beyond a deep shadow zone would behave similarly; the shadow zone itself could not of course be reconstructed.) In the same way it was found that the results are almost unaffected by an arbitrary error of phase in measurement of the complex amplitude $\Psi_s$. For this surface Fig. 2 shows the real and the imaginary components of the scattered field at $z_1 = 0.7$, which represents the given data. Another example is given in Fig. 3, this time for a

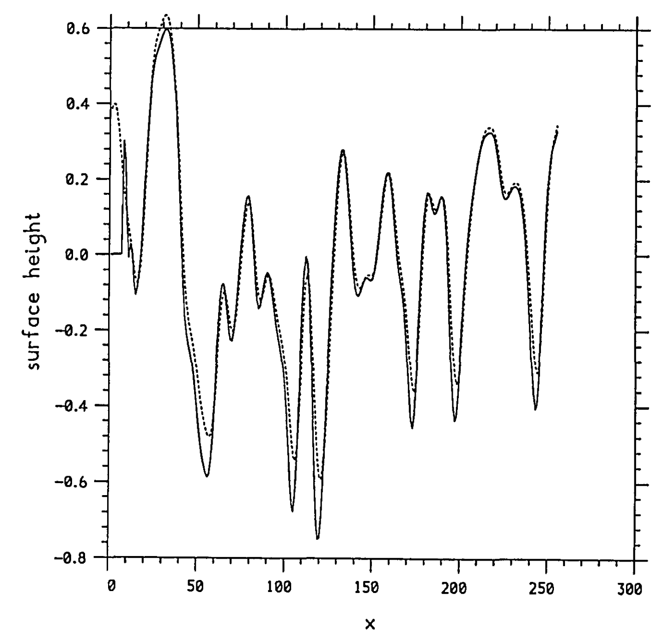


Fig. 3. Comparison as in Fig. 1 for a smoother surface between the $h$ (dotted curve) and its reconstruction $\bar{h}$ (solid curve).

smoother surface with a Gaussian autocorrelation function, $\langle h(x)h(x+\xi)\rangle = \exp(-\xi^2/l^2)$. The two functions again agree closely.

## 4. CONCLUSIONS

The inverse problem for scattering at grazing incidence by one-dimensional surfaces $h$ has been formulated as an integral equation, relating the unknown field derivative to the known complex amplitude of the scattered field along a line in the medium. The equation was solved directly by being coupled to another equation relating the derivative to the surface, and results in which complicated rough surfaces had been accurately recaptured were presented. These results are based on a comparison with simulated data and were found to be highly insensitive to errors in the phase of the data. Nevertheless, the collection of such data in practice would not be trivial. This formulation is easily extended to the case of Neumann boundary conditions, in which $G$ in the governing equations is replaced by its derivative and $\Psi'$ is replaced by the field itself, which again depends almost linearly on the surface.[11] However, it exploits the parabolic equation method valid for grazing incidence and cannot be extended to more general regimes.

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