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Direct solution of the inverse problem for rough surface scattering at grazing incidence

Mark Spivack

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 9EW, UK

Abstract. The paper considers the inverse scattering problem for a scalar wavefield incident at grazing angles on a one-dimensional rough surface. The problem is formulated first as a pair of coupled integral equations in two unknown functions, knowledge of which immediately yields the surface. A method is described for the direct approximate solution of this system. Preliminary results are presented in groups of complicated rough surfaces which are closely recaptured in all details except for scale.

1. Introduction

A major aim of the study of surface scattering is the solution of inverse problems, and specifically that of recapturing the surface explicitly from scattered data. There are numerous potential applications, from radar to underwater acoustics. Despite this motivation, in many problems little progress seems to have been achieved for complicated surfaces, and in practice treatment is largely limited to iterative methods (see, for example, [1–3]). For scalar wavefields at arbitrary angles of incidence the scattered field is described by the Helmholtz integral formula [4], which consists of an integral equation and a scattering integral. When the field is incident at small angles to the surface, forward-scattering predominates, and the Helmholtz equations may be replaced by the parabolic equation method [5, 6]. Despite the simplifications offered by the parabolic regime multiple scattering occurs even for very slight roughness, and the dependence of the scattered field upon the surface remains highly nonlinear.

In this paper the inverse problem for a grazing incidence, on a one-dimensional pressure release surface, is formulated as a pair of coupled integral equations, and an approximate method for the direct solution of this system is given. This solution is required to yield the two functions $\partial\Psi/\partial z$ and $\exp(ikh)$ where $h(x)$ is the surface, $\partial\Psi/\partial z$ is the transverse derivative of the field along h, and k is the wavenumber. The form of the surface follows immediately from knowledge of $\exp(ikh)$, or for moderate surface variation by a simple transformation of $\partial\Psi/\partial z$.

This system is obtained by approximating the Green function, in which $\exp(ikh)$ appears, and treating the scattering integrals as integral equations. It is assumed that the incident wavefield is known, together with scattered data along two lines parallel to the mean surface level. The system is nonlinear and ill conditioned, and its treatment presents difficulties which have yet to be fully overcome. However the proposed solution is implemented numerically, and preliminary results are presented in that a complicated, rapidly varying surface is reconstructed from the derivative $\partial\Psi/\partial z$, with errors of scale, but correct in all other details. This is done for moderately rough surfaces, when $\partial\Psi/\partial z$ is almost linear in $h(x)$.

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In section 2 the integral equation description of surface scattering is given. The formulation of the inverse problem in terms of integral equations is given in section 3, and the numerical solution is outlined, together with the computational results.

2. Mathematical formulation

We consider the problem of a scalar (acoustic) time-harmonic wavefield p scattered from a one-dimensional rough surface. The field is at grazing incidence and the Dirichlet boundary condition is assumed. The coordinate axes are x and z, where x is the horizontal $x \geqslant 0$, and z the vertical. For convenience it will be assumed that the mean surface level is at $z = 0$. The source is centred at about $\boldsymbol{r} = (0, z_0)$, with wavenumber k. The rough surface itself is denoted $h(x)$, so that h has mean zero. In the numerical examples, h is drawn from an ensemble of normally distributed and statistically stationary processes, with RMS surface height denoted by ϕ. (This statistical description is used for convenience, but the stochastic nature of the surface is not relevant to the treatment here.)

Since the wavefield propagates predominantly in one direction, it has a slowly-varying part Ψ defined by

$$\Psi(x, z) = p(x, z)\mathrm{e}^{-ikx}.$$

Incident and scattered components Ψ_{inc} and Ψ_{s} are defined similarly, so that $\Psi = \Psi_{\mathrm{inc}} + \Psi_{\mathrm{s}}$. It will be assumed that $\Psi_{\mathrm{inc}}(x, h(x)) = 0$ for $x \leqslant 0$, so that the area of surface illumination is restricted, as for example when the field is a Gaussian beam (see below). Provided angles of incidence and scattering are fairly small, the forward-scattering assumption holds and propagation is well described by the parabolic wave equation (e.g. [7])

$$\frac{\partial \Psi}{\partial x} + 2\mathrm{i}k\frac{\partial^2 \Psi}{\partial z^2} = 0.$$

The parabolic form G of the Green function may be found by solving the parabolic equation with a source term [5] (or by direct approximation of the full free-space form), and is given by

$$G(x, z; x', z') = \frac{1}{2}\sqrt{\frac{\mathrm{i}}{2\pi k(x - x')}}\exp\left[\frac{\mathrm{i}k(z - z')^2}{2(x - x')}\right] \tag{2.1}$$

when $x' < x$, and $G = 0$ otherwise. The parabolic equation method [5] is then obtained by straightforward analogy with the derivation of the Helmholtz integral formula (for example, [9, 10]), by combining equations for Ψ and G, taking a volume integration, and imposing the boundary conditions. The governing equations [5, 6, 8] are then

$$\Psi_{\mathrm{inc}}(\boldsymbol{r}) = -\int_0^x G(\boldsymbol{r}; \boldsymbol{r}')\frac{\partial \Psi(\boldsymbol{r}')}{\partial z}\,\mathrm{d}x' \tag{2.2}$$

where both $\boldsymbol{r} = (x, h(x))$, $\boldsymbol{r}' = (x', h(x'))$ lie on the surface; and

$$\Psi_{\mathrm{s}}(\boldsymbol{r}) = \int_0^x G(\boldsymbol{r}; \boldsymbol{r}')\frac{\partial \Psi(\boldsymbol{r}')}{\partial z}\,\mathrm{d}x' \tag{2.3}$$

where r' is again on the surface and r is now an arbitrary point in the medium. (The restricted area of illumination enables us to make the lower limits of integration finite.)

Since $\partial\Psi/\partial z$ is considered only at points along the surface, it may be treated here as a function just of x, and denoted Ψ'. Let $\alpha = \frac{1}{2}\sqrt{i/2\pi k}$. The incident field used in the examples later is a Gaussian beam of initial width w, assumed for simplicity to be travelling horizontally, i.e. at zero grazing angle:

$$\Psi_{\mathrm{inc}}(x,z) = \frac{w}{\sqrt{w^2+2ix/k}} \exp\left[-\frac{(z-z_0)^2}{w^2+2ix/k}\right]. \tag{2.4}$$

This field impinges on the surface as it spreads; the pattern of illumination along a *flat* surface is shown in figure 1, for $k = 1$ and $w = 8$. Similar results are obtained for incidence at angles up to around 15° or 20° [5], above which propagation is no longer described accurately by the parabolic wave equation.

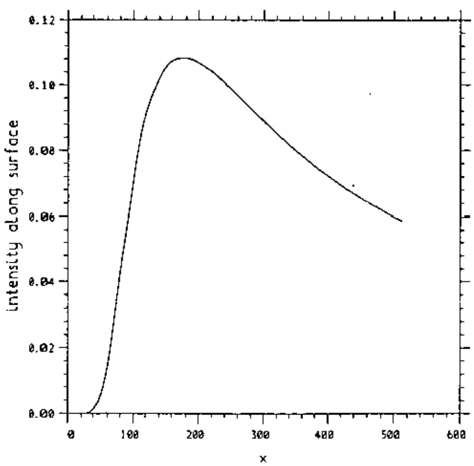

Figure 1. Intensity of incident field along the line $z = 0$, with $k = 1$ and $w = 8$.

3. Inverse problem and solution

3.1. Formulation

Consider the form of the Green function appearing in (2.3), into which h enters once, via r'. Provided z is large compared with the surface variation, the exponent can be approximated (see [11])

$$\frac{ik[z-h(x')]^2}{2(x-x')} \cong ik\,\frac{z^2-2zh(x')}{2(x-x')}. \tag{3.1}$$

(The error is the factor $\exp[ikh^2/2(x-x')]$; although this exponent becomes large as x' approaches x, the phase variation in G is nevertheless dominated by the approximate exponent (3.1). This may be verified by comparing simulations of Ψ_s for a given rough surface using the full form of G in (2.3) its approximation.) Define the function

$$E(x) = \exp[-\mathrm{i}kh(x)]. \tag{3.2}$$

Suppose now that the scattered field is known at two depths z_1, z_2 in some interval $(0, X)$. The approximation allows us to write the integral (2.3) as

$$\Psi_s(x, z_j) \cong \alpha \int_0^x \left(\exp\left[\frac{\mathrm{i}kz_j^2}{2(x-x')}\right] \Big/ \sqrt{x-x'} \right) E(x')^{z_j/(x-x')} \frac{\partial \Psi(x')}{\partial z} \, \mathrm{d}x' \tag{3.3}$$

for $j = 1, 2$, where the surface-dependent functions E and $\partial\Psi/\partial z$ appearing here are independent of the distance z_j. Thus equation (3.3) for $j = 1, 2$ represents a pair of coupled integral equations, in which the left-hand side is known, and the two unknown functions (including E which forms part of the kernel) appear under the integral sign. To avoid numerical instability (see section 3.2) it will be assumed that $z_1 - z_2$ is not too small. (In solving this system we treat the functions E and $\partial\Psi/\partial z$ as independent, at this stage ignoring their mutual dependence upon the surface. The system would otherwise be overspecified. Similarly, knowledge of Ψ_s at any distance z_1 is formally sufficient, via the wave equation, to define Ψ_s at z_2. In practice however it is a non-trivial task to extend Ψ_s in this way from scattered data on a finite interval.)

From (3.3) a direct solution of the inverse problem will be obtained. If we write these equations as

$$\{\Psi_1, \Psi_2\} \cong A\{E, \partial\Psi/\partial z\}$$

where Ψ_j represents the vector $\Psi_s(x, z_j)$, then we must specify the nonlinear operator A^{-1}. An approximate solution for A^{-1} is described later. Once E is known with sufficient accuracy the surface may be found immediately (to within a constant); alternatively, when the surface is moderately rough, $\partial\Psi/\partial z$ varies approximately linearly with h, and we can write

$$h = L\frac{\partial \Psi}{\partial z}$$

where L is a simple linear transformation which depends upon the source. Explicitly, for small h the incident field (2.4) along the surface can be written as

$$\Psi_{\mathrm{inc}}(x, h(x)) \cong \Psi_{\mathrm{inc}}(x, 0)\left[1 + \frac{2z_0 h(x)}{w^2 + 2\mathrm{i}x/k}\right] \tag{3.4}$$

and in equation (2.2) the variation in G (i.e. the exponent) can be neglected [11] to give

$$\Psi_{\mathrm{inc}}(x, h(x)) \cong -\alpha \int_0^x \frac{1}{\sqrt{x-x'}} \Psi'(x') \, \mathrm{d}x' \tag{3.5}$$

where $\Psi' = \partial\Psi/\partial z$. Thus from (3.4) and (3.5) we can approximate

$$h(x) \cong \left(\left(\int_0^x \frac{1}{\sqrt{x-x'}} \Psi'(x') \, \mathrm{d}x' \Big/ \Psi_{\mathrm{inc}}(x, 0) \right) - 1 \right) \left(\frac{w^2 + 2\mathrm{i}x/k}{2z_0} \right). \tag{3.6}$$

This is an accurate approximation even for fairly rough surfaces. For numerical evaluation the integral in (3.6) is easily discretized, as described later.

4. Numerical solution

In order to illustrate the treatment of a coupled system such as (3.3), and thus carry out a direct solution of the inverse problem, a simple numerical method is now described. Further refinement is needed, but excellent results have nevertheless been obtained.

The principal difficulty presented by (3.3) is that, although $\partial\Psi/\partial z$ and E themselves vary boundedly, E has an exponent which becomes singular. The interval $(0,x)$ is first discretized by evenly spaced points $\{x_r\}$, say, where $r = 0,\ldots,N$. For $j = 1,2$ the integrals (3.3) are written as sums of subintegrals over the corresponding small intervals (x_{r-1}, x_r). Making the naive assumption that both $\partial\Psi/\partial z$ and $E^{z_j/(x-x')}$ vary slowly over each subinterval compared with the deterministic variation, these functions can be taken outside the integrals and for $j = 1,2$ the equations may be written as

$$\Psi_s(x_n, z_j) \cong \sum_{r=1}^{n} E_r^{z_j/(x_n - X_r)}\Psi'(x_r)\beta_{n,r}(z_j) \tag{3.7}$$

where X_r is the mid-point $(x_r + x_{r+1})/2$, $E_r = E(x_r)$, and

$$\beta_{n,r}(z_j) = \alpha \int_{x_r}^{x_{r+1}} \left(\exp\left[\frac{\mathrm{i}kz_j^2}{2(x-x')}\right] \Big/ \sqrt{x-x'}\right) \mathrm{d}x'$$

for $j = 1,2$. The coefficients β, which depend only on $n - r$, may be found exactly in terms of Fresnel integrals by the change of variable $y = z_j\sqrt{k/\pi}(x - x')^{-1/2}$, followed by an integration by parts. (This type of discretization and 'product integration' rule are widely used in surface scattering, e.g. [5, 6].) We then have a pair of matrix equations, with lower-triangular matrices $\beta(z_j)$, which may be solved simultaneously from the left as follows: Suppose E_i, $\Psi'(x_i)$ have been obtained for $i = 1,\ldots,n-1$. Then

$$\Psi_s(x_n, z_j) - \sum_{r=1}^{n-1} \beta_{n,r}(z_j)E_r^{z_j/(x_n - X_r)}\Psi'(x_r) \cong \beta_{n,n}(z_j)E_n^{z_j/(x_n - X_n)}\Psi'(x_n). \tag{3.8}$$

Dividing through by $\beta_{n,n}(z_j)$, and then dividing each side of the equation for $j = 1$ by that for $j = 2$, the term $\Psi'(x_n)$ drops out and we obtain an expression for

$$E_n^{(z_1 - z_2)/(x_n - X_n)}. \tag{3.9}$$

The approximation for E_n may be obtained from this, and this is then substituted back into (3.8) (either for $j = 1$ or $j = 2$) to find $\Psi'(x_n)$. This yields approximations for the required functions throughout the interval $(0, x)$; the surface itself is then obtained immediately from E, or as the function $L\Psi'$ where L is the transformation (3.6). Note that as z_1 approaches z_2, the exponent in (3.9) tends to zero and the inversion of this expression becomes numerically unstable.

A similar, but more complicated, treatment may be envisaged using more satisfactory approximations, obtained by treating the subintegrals analytically without first assuming E to be slowly varying.

4.1. Preliminary results

This procedure was followed to obtain approximate solutions for $E = \exp(ikh)$ and Ψ', where the surface and scattered wavefield had been produced by simulations using the full form of the parabolic equation method. (Details of this and the generation of random surfaces are given in [6].) In these initial results it was found that E was less well represented than Ψ', and it therefore proved more accurate to reconstruct the surface from Ψ', using (3.6), than from E.

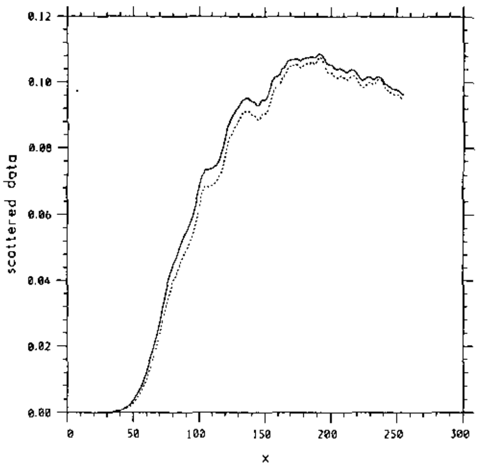

Figure 2. Intensity of scattered components along lines at distances 0.6 (full curve) and 1.2 (broken curve) from the mean surface level.

In the first example a surface was generated with $\phi = 0.23$, and scattering calculated for an incident field due to a source with wavenumber $k = 1$, initial width $w = 8$, located at a distance $z_0 = 22.4$. The random surface was fairly jagged, with an autocorrelation function of the form

$$\langle h(x)h(x+\xi)\rangle = (1 + |\xi|/l)\exp(-|\xi|/l)$$

where l is roughly a wavelength. For this surface figure 2 shows the intensity of the scattered fields at $z_1 = 0.6$ and $z_2 = 1.2$, which represent the given data. (These values of z_j were chosen close to the surface for computational convenience.) In figure 3(*a*) the surface h itself is compared with its solution $\bar{h}$, say, from the approximation of $\partial\Psi/\partial z$. The solution recaptures all the features of $h(x)$ in detail, although it differs in scale. This scaling error becomes clear when h plotted against $\bar{h}$ is rescaled by dividing by a factor 1.8 (figure 3(*b*)). The source of these systematic errors of scale is not yet apparent; more numerical work is needed to explain and correct them. Another (unscaled) example is given in figure 4, this time for smoother surface, with a Gaussian autocorrelation function, $\langle h(x)h(x+\xi)\rangle = \exp(-\xi^2/l^2)$. The agreement is again excellent.

Although our aim is the direct solution of the inverse problem, it should be noted that once the surface approximation has been obtained it may be used with the data in several obvious iterative methods.

5. Conclusions

The inverse problem for scattering at grazing incidence by one-dimensional surfaces h

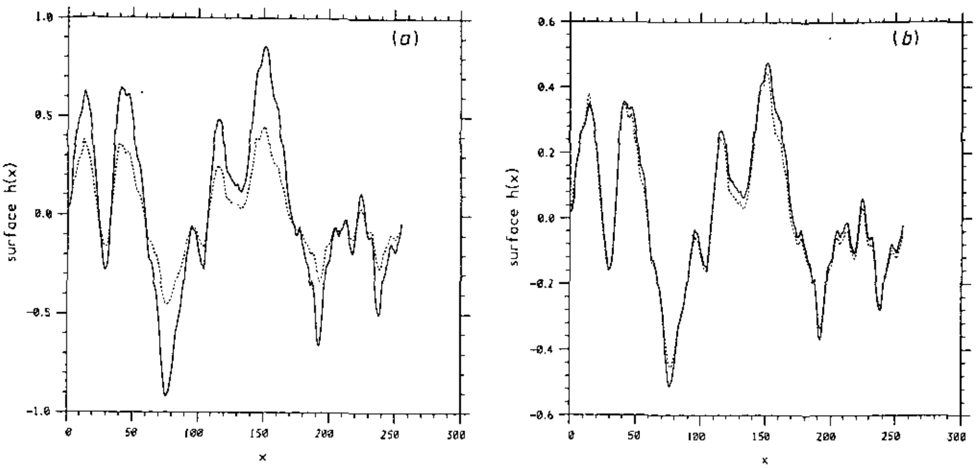

Figure 3. (*a*) Comparison between the surface h (broken curve) and its reconstruction $\bar{h}$ (full curve); (*b*) comparison after rescaling $\bar{h}$ by division by a factor of 1.8.

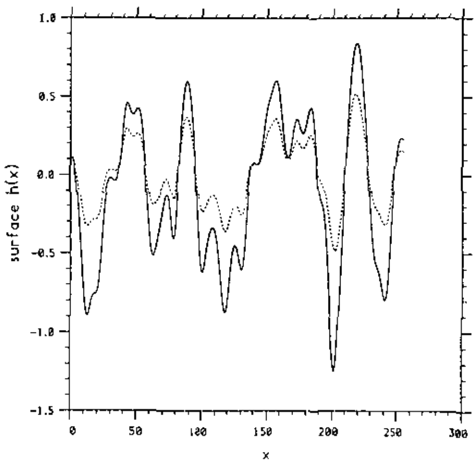

Figure 4. Comparison as in figure 3(*a*) for a smoother surface.

has been reformulated as a pair of coupled integral equations, relating the unknown surface derivative and the function $\exp(ikh)$ to the known value of the scattered field at two lines in the medium. This is based on a simple approximation of the parabolic form of the Green's function in the governing integrals. A direct solution has been given based on additional approximations, and a preliminary numerical scheme. Despite the naive nature of these approximations, a complicated highly varying rough surface is recaptured in all its detail except for scale. The results are based on comparison with simulated data. It is clear that refinement of the numerical scheme is needed, and may be expected to lead to correct scaling. This formulation has natural extensions to scattering at arbitrary angles of incidence, and from two-dimensional surfaces.

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