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PATH INTEGRALS FOR WAVE INTENSITY FLUCTUATIONS IN RANDOM MEDIA

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Approximate expressions for the fourth order moment of a wave propagating in a random medium are derived by using the path integral formulation. These solutions allow the spectrum of intensity fluctuations of a multiply scattered wave to be found, and they are valid at all distances in the medium. The results obtained by path integral methods turn out to be the same as those obtained previously by solving the parabolic partial differential equation for the fourth moment. The spatial frequency spectra of intensity fluctuations are evaluated for a medium in which the irregularities have a single scale and also for one in which there is a range of scale sizes.

1. INTRODUCTION

In recent years there has been an increasing interest in the use of functional integration and Feynman path integral methods to treat problems in various branches of physics. The method was first developed by Wiener in 1922 for the study of Brownian motion and above all by Feynman and Hibbs [1] in quantum mechanics and electrodynamics. Functional integration and path integrals are now among the methods used to treat the problem of propagation in irregular media. The application to electromagnetic wave propagation seems to have been first suggested by Eichmann [2] and was used by Furutsu [3] to find the irradiance distribution function in an extended random medium. Hannay [4] gave a treatment of the path integral method as applied to random media, while Eve [5] used path integral techniques to study light propagation in irregular optical fibres. Zavorotnyi *et al.* [6], Zavorotnyi [7] and Dashen [8] have used path integral techniques to discuss the propagation of waves in an irregular medium.

An advantage of path integral methods in treating problems of propagation in random media is that they often give valuable physical insight and act as a supplement to the more conventional methods. Path integrals, however, can prove difficult to evaluate, and in some cases considerable insight is required to decide which constraints may be applied to the random paths to produce a meaningful answer.

When treating the propagation of a wave field in a random medium the quantities most frequently studied are ensemble average moments of the random field. The first moment gives the mean field, sometimes called the "coherent" or unscattered part of the wave field. The second moment, or product of the field with its complex conjugate at another point in the medium gives the spatial correlation of complex amplitude, and this is frequently used in coherence experiments. The fourth moment yields the square of the wave intensity and is thus used to describe intensity fluctuations due to scattering by

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giving the time-space spectrum and the variance of the fluctuations. Because many measuring instruments are based on square-law detection the fourth moment is of great interest in the theory of random wave propagation and has important implications in practical engineering. The wave field moment have been much studied by using the partial differential equations describing them. Solutions for the fourth moment have been found by this traditional method. However, the increasing interest in functional integration and path integral methods prompts one to ask whether or not this approach can yield new information about the wave moments.

The appropriate path integrals for the first and second moments can be evaluated without too much difficulty. However, the path integral expression for the fourth moment has proved difficult to treat except in the limiting cases where the intensity fluctuations are small, or where the wave has propagated to such large distances that the statistics of the scattered field are approaching the Gaussian limit. Zavorotnyi *et al.* [6], Zavorotnyi [7] and Dashen [8] have used path integral methods to derive an expression for the fourth moment of the wave field as this large distance limit is approached.

In the present paper it is shown how the path integral expression for the fourth moment of the field propagating in a random medium can be evaluated for any distance of propagation in the medium. The results are found explicitly in some particular cases and are shown to agree with similar results obtained by quite different methods. The discussion is conducted with reference to an acoustic wave, since the propagation of sound in a randomly varying ocean poses important problems in the theory of intensity fluctuations. However, the results are applicable to electromagnetic propagaton also.

2. PATH INTEGRAL FOR THE FOURTH MOMENT OF THE WAVE FIELD

Let x, y, z be a Cartesian set of axes in a space filled with medium whose acoustic refractive index

$$n = 1 + n_b = 1 + n_0(x, y, z) + \mu n_1(x, y, z, t) \tag{1}$$

contains weak random irregularities with a spatial scale L. Here $1 + n_0$ is the ensemble average value of n, and $n_0(x, y, z)$ is assumed to vary smoothly on a scale that is large compared with L. Thus $n_0(x, y, z)$ constitutes the mean refractive index profile of the medium. The small scale irregularities are characterized by n_1, a randomly varying quantity with a mean value zero and standard deviation unity, while μ, the rms value of the random fluctuations, is assumed to be very small. Let a monochromatic wave of radian frequency ω be emitted from a source at x_0, y_0 in the plane $z = 0$. The wavelength corresponding to this frequency is $\lambda (k = 2\pi/\lambda)$ when $n = 1$. If

$$p(x, y, z, t) = \mathrm{Re}\,\{A(x, y, z, t) \exp [i(kz - \omega t)]\} \tag{2}$$

is the pressure field at (x, y, z, t) in the medium then A is the complex amplitude of the field associated with p but without the rapidly varying harmonic dependence on z and t. The residual time dependence of A is due to the time variations of refractive index, which are assumed to be very slow compared with ω^{-1}.

The instantaneous intensity of the acoustic wave in the medium varies rapidly like $\exp \{i(kz - \omega t)\}$ with a slowly varying envelope I. From equation (2) this envelope intensity is

$$I_i = A(x_i, y_i, z_i, t_i)A^*(x_i, y_i, z_i, t_i). \tag{3}$$

In investigations of intensity fluctuations the fourth moment of A at some distance z in the medium is used:

$$m = \langle A(x_1, y_1, z, t_1)A^*(x_2, y_2, z, t_2)A(x_3, y_3, z, t_3)A^*(x_4, y_4, z, t_4)\rangle. \tag{4}$$

When

$$(x_4, y_4, z, t_4) = (x_1, y_1, z, t_1), \qquad (x_3, y_3, z, t_3) = (x_2, y_2, z, t_2), \tag{5}$$

the fourth moment (4) reduces to $\langle I_1 I_2\rangle$, the space-time correlation of the intensity. Further, when points 1 and 2 coincide the fourth moment reduces to $\langle I^2\rangle$, the variance of the intensity fluctuations.

The importance of calculating the fourth moment (4) is thus clear, since it allows one to evaluate not only the intensity correlation

$$\rho_I = \langle I_1 I_2\rangle \tag{6}$$

but also the other quantity most frequency used in this area, the normalized variance

$$S_I^2 = (\langle I^2\rangle - \langle I\rangle^2)/\langle I\rangle^2, \tag{7}$$

sometimes referred to as the scintillation index.

2.1. PATH INTEGRALS

The fourth moment (4) can be evaluated as follows by using path integral methods. It is assumed here that the scale size of the irregularities L is large compared with the wavelength of the wave λ: i.e., $kL \gg 1$ so that the angles of scatter are very small. In this case the propagation of A is well described by the parabolic wave equation

$$\partial A(x, y, z, t)/\partial z = (\mathrm{i}/2k)(\partial^2 A/\partial x^2 + \partial^2 A/\partial y^2) + \mathrm{i}kn_b A, \tag{8}$$

with the boundary condition

$$A_0 = (4\pi z)^{-1} \exp\{\mathrm{i}k[(x - x_0)^2 + (y - y_0)^2]/2z\}, \tag{9}$$

corresponding to a point source at $(x_0, y_0, 0)$. Feynman and Hibbs [1] showed that the solution of equation (8) with the boundary condition (9) is given by the limit of the multi-dimensional integral

$$A(\mathbf{r}, z, t) = \lim_{l\to\infty} \frac{\mathrm{i}}{2k} \int_{-\infty}^{\infty} \cdots \int \left(\frac{\mathrm{i}kl}{2\pi z}\right)^l \exp\left\{\frac{\mathrm{i}kz}{l} \sum_{j=1}^{l} \left[\frac{l^2}{2}\left(\frac{\mathbf{r}_j - \mathbf{r}_{j-1}}{z}\right)^2 + n_b(\mathbf{r}_j, z_j, t)\right]\right\} \mathrm{d}\mathbf{r}_1 \cdots \mathrm{d}\mathbf{r}_j \cdots \mathrm{d}\mathbf{r}_{l-1}. \tag{10}$$

Here $\mathbf{r}_j$ is the vector from the z axis to the point (x_j, y_j) in the z_j plane. The points $(\mathbf{r}_j, z_j)$ lie on a path from the source to the observer and expression (10) can be represented as an integral over paths. In the limit as l approaches infinity expression (10) becomes

$$A(\mathbf{r}, z_f, t) = \frac{\mathrm{i}}{2k} \int \cdots \int \exp\left\{\mathrm{i}k \int_0^{z_f} \left[\frac{1}{2}\left(\frac{\mathrm{d}\mathbf{r}(z)}{\mathrm{d}z}\right)^2 + n(\mathbf{r}(z), z, t)\right] \mathrm{d}z\right\} \mathrm{d}(\text{paths}), \tag{11}$$

where the integration is now over all paths connecting $(\mathbf{r}_0, 0)$ and $(\mathbf{r}, z)$. In what follows the time dependence will be suppressed. The physical meaning underlying expressions (10) and (11) becomes clear when one considers Figure 1 in which one of the paths is shown. The position vector of the path $\mathbf{r}(z)$ follows an irregular trajectory from $(\mathbf{r}(0), 0)$ to the observation point $(\mathbf{r}(z_f), z_f)$. On traversing this path a point on a wave front experiences a phase shift, relative to the phase reference e^{ikz}, that is due to two causes.

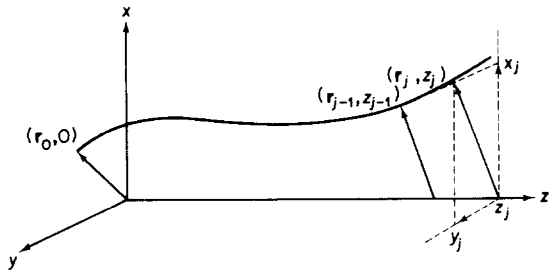

Figure 1. Illustrating a typical path and the co-ordinates used in writing out the path integral (10) and (11).

The first is the extra geometrical distance covered because of curvature of the path. This is expressed by the first term under the integral sign in the exponent of (11). The second is due to the varying value of the refractive index n over the path and this is taken into account by the other term in the exponent of expression (11). Only one such path is illustrated in Figure 1. The actual value of A observed at $(\mathbf{r}(z_f), z_f)$ is the result of the field traversing all possible paths between source and observer. This is allowed for by adding together the A's resulting from all possible irregular paths between source and observer, each of which would be a different irregular trajectory in Figure 1. This is the meaning of the integral over paths, d(paths).

Note that expression (11) is the limit of the discrete form (10) in which the medium is divided into layers of thickness $z_j - z_{j-1}$. This fact will be used when defining the meaning of a delta function introduced later in the paper. Finally, although the above discussion has been for a point source situated at $(\mathbf{r}(0), 0)$ the case of a plane wave can easily be dealt with as well since this is equivalent to point sources of equal intensity situated uniformly over all elements of the $z = 0$ plane. Thus the plane wave result can be obtained from the point source case by appropriate integration. Similarly, the case of an extended source can be treated since it is a set of point sources with a spatial distribution of intensities in the $z = 0$ plane.

The required expression for the fourth moment (4) is now written down by using expression (11):

$$
\begin{aligned}
\langle \mathrm{E}(\mathbf{r}_1, z_f)\mathrm{E}^*(\mathbf{r}_2, z_f)\mathrm{E}(\mathbf{r}_3, z_f)\mathrm{E}^*(\mathbf{r}_4, z_f)\rangle &= m(\mathbf{r}, z_f) \\
= N_0 \int \cdots \int \exp &\left\{ \frac{ik}{2} \int_0^{z_f} [(\mathbf{r}_1')^2 \, \mathrm{d}z_1 - (\mathbf{r}_2')^2 \, \mathrm{d}z_2 + (\mathbf{r}_3')^2 \, \mathrm{d}z_3 - (\mathbf{r}_4')^2 \, \mathrm{d}z_4] \right\} \\
\times \exp &\left\{ ik \int_0^{z_f} [n_0(\mathbf{r}_1, z_1) \, \mathrm{d}z_1 - n_0(\mathbf{r}_2, z_2) \, \mathrm{d}z_2 + n_0(\mathbf{r}_3, z_3) \, \mathrm{d}z_3 - n_0(\mathbf{r}_4, z_4) \, \mathrm{d}z_4] \right\} \\
\times \exp &\left\{ -\frac{k^2\mu^2}{2} \left\langle \left(\int_0^{z_f} n_1(\mathbf{r}_1, z_1) \, \mathrm{d}z_1 - n_1(\mathbf{r}_2, z_2) \, \mathrm{d}z_2 + n_1(\mathbf{r}_3, z_3) \, \mathrm{d}z_3 - n_1(\mathbf{r}_4, z_4) \, \mathrm{d}z_4 \right)^2 \right\rangle \right\} \\
&\times \mathrm{d}[\mathbf{r}_1] \, \mathrm{d}[\mathbf{r}_2] \, \mathrm{d}[\mathbf{r}_3] \, \mathrm{d}[\mathbf{r}_4]. \qquad (12)
\end{aligned}
$$

The prime is used to denote differentiation with respect to the argument: e.g., $\mathbf{r}_j = \mathbf{r}_j(z)$, and $\mathbf{r}_j' = \mathrm{d}\mathbf{r}_j(z)/\mathrm{d}z$.

Here the four-fold integral over paths has been written formally by using the notation $\mathrm{d}[\mathbf{r}_1] \, \mathrm{d}[\mathbf{r}_2] \, \mathrm{d}[\mathbf{r}_3] \, \mathrm{d}[\mathbf{r}_4]$. The normalizing factor N_0 arises because of the inclusion of $(ikl/2\pi z)$ from expression (10) in the element in path space when passing to the continuous representation (11). The manner in which N_0 is evaluated is best explained in terms of a concrete example later in the paper.

Clearly, the ensemble average affects only the randomly varying components which have been gathered together in the last exponential of expression (12). The fact that the magnitude of these random components is very small compared with unity, or alternatively that they obey Gaussian statistics, allows one to write the ensemble average in the form appearing in expression (12).

2.2. NEW CO-ORDINATE SYSTEM

The following change of variables [9] is now made to facilitate further calculations

$$\mathbf{v}_1 = \tfrac{1}{2}[(\mathbf{r}_1+\mathbf{r}_4)-(\mathbf{r}_2+\mathbf{r}_3)], \qquad \mathbf{v}_2 = \tfrac{1}{2}[(\mathbf{r}_1+\mathbf{r}_2)-(\mathbf{r}_3+\mathbf{r}_4)],$$
$$\mathbf{u}_1 = \tfrac{1}{4}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4)-\mathbf{S}, \qquad \mathbf{u}_2 = (\mathbf{r}_1-\mathbf{r}_4)+(\mathbf{r}_3-\mathbf{r}_2). \tag{13}$$

Here $\mathbf{S}(z)$ is defined as the path of a ray in the absence of random irregularities of refractive index, i.e., when $\mu = 0$, and satisfies the ray equation

$$\mathbf{S}''(z) = \nabla n_0(\mathbf{S}(z); z). \tag{14}$$

The physical meaning of the new variables is clear if one considers Figure 2. $\mathbf{u}_1$ gives the position of the centre of mass of the points $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, on four paths, relative to the deterministic ray path $\mathbf{S}$. $\mathbf{v}_1$ gives the separation between the centre of $\mathbf{r}_1$, $\mathbf{r}_4$, taken as a pair, and $\mathbf{r}_2$, $\mathbf{r}_3$, also taken as a pair. $\mathbf{v}_2$ fulfils the same function for the pairs $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, $\mathbf{r}_4$. Finally $\mathbf{u}_2$ is a measure of the symmetry of arrangement of the above pairs of points.

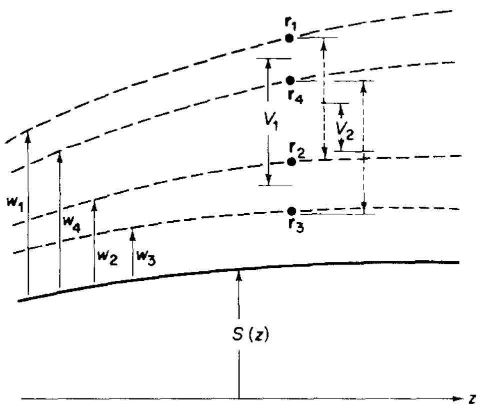

Figure 2. Illustrating the physical meaning of the variables (13) used to simplify the path integral for the fourth moment.

The four variables $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{u}_1$, $\mathbf{u}_2$ are functions of z. The first exponent of expression (12) then becomes, in terms of the new variables, $\exp\{ik\int_0^{z_f}[\mathbf{S}'\cdot\mathbf{u}_2'+\mathbf{u}_1'\cdot\mathbf{u}_2'+\mathbf{v}_1'\cdot\mathbf{v}_2']\,dz\}$. The second exponent of expression (12) can be simplified by expanding n_0 about the ray path $\mathbf{S}(z)$. It can be assumed that the presence of random irregularities leads to small deviations of the paths from the deterministic ray $\mathbf{S}(z)$ and so one introduces

$$\mathbf{w}_j = \mathbf{r}_j - \mathbf{S}, \qquad j = 1, 2, 3, 4. \tag{15}$$

The small deviations $\mathbf{w}_j$ are also shown in Figure 2 and can be expressed in terms of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{u}_1$, $\mathbf{u}_2$ by using equation (13). Let w_{xj}, w_{yj} be the projections of w_j on the x and y axes and introduce the operator

$$\nabla = (w_{xj}\,\partial/\partial x + w_{yj}\,\partial/\partial y). \tag{16}$$

Then the Taylor expansion of n_0 in the variables (15) is

$$n_0(\mathbf{r}_j, z) = n_0|_S + \nabla n_0|_S + \tfrac{1}{2}\nabla^2 n_0|_S. \tag{17}$$

If terms up to and including quadratic are retained in equation (17) the second exponent of expression (12) becomes $\exp\{ik\int_{0'}^{z_f}[\nabla n_0 \cdot \mathbf{u}_2 + \mathbf{u}_1 Q \mathbf{u}_2^T + \mathbf{v}_1 Q \mathbf{v}_2^T]\,dz\}$, where Q is the matrix

$$Q = \begin{bmatrix} \partial^2 n_0/\partial x^2|_S & \partial^2 n_0/\partial x\,\partial y|_S \\ \partial^2 n_0/\partial x\,\partial y|_S & \partial^2 n_0/\partial y^2|_S \end{bmatrix} \tag{18}$$

The new expressions for the first and second exponents of expression (12) can be combined, rearranged, and integrated by parts to give the following expression for the fourth moment:

$$\begin{aligned} m = N_0 \int \cdots \int & \exp\{ik[\mathbf{S}' \cdot \mathbf{u}_2 + \mathbf{v}_2' \cdot \mathbf{v}_1 + \mathbf{u}_2' \cdot \mathbf{u}_1]_{0'}^{z_f}\} \\ & \times \exp\left\{-ik\left[\int_0^{z_f} (\mathbf{v}_2'' - \mathbf{v}_2 Q) \cdot \mathbf{v}_1\,dz + \int_0^{z_f} (\mathbf{u}_2'' - \mathbf{u}_2 Q) \cdot \mathbf{u}_1\,dz\right]\right\} \\ & \times \exp\{-\mathcal{H}(\mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2; z)\}\,d[\mathbf{u}_1]\,d[\mathbf{u}_2]\,d[\mathbf{v}_1]\,d[\mathbf{v}_2]. \end{aligned} \tag{19}$$

Here

$$\begin{aligned} \mathcal{H} = \frac{k^2\mu^2}{2} \int_0^{z_f}\int_0^{z_f} & [D(\tfrac{1}{2}\mathbf{u}_2 + \mathbf{v}_1; z_i, z_j) + D(\tfrac{1}{2}\mathbf{u}_2 - \mathbf{v}_1; z_i, z_j) + D(\mathbf{v}_2 - \tfrac{1}{2}\mathbf{u}_2; z_i, z_j) \\ & + D(\mathbf{v}_2 + \tfrac{1}{2}\mathbf{u}_2; z_i, z_j) - D(\mathbf{v}_1 + \mathbf{v}_2; z_i, z_j) - D(\mathbf{v}_2 - \mathbf{v}_1; z_i, z_j)]\,dz_i\,dz_j, \end{aligned} \tag{20}$$

$$D_{ij} = \langle [n_1(\mathbf{r}_i, z_i) - n_1(\mathbf{r}_j, z_j)]^2 \rangle \tag{21}$$

is the structure function of the random part of the refractive index. The fact that S obeys the ray equation (14) has been used in obtaining equation (19), while the integrals over paths are now in terms of the new variables $\mathbf{u}$, $\mathbf{v}$. It should be noted that the variables $\mathbf{v}_1$ and $\mathbf{v}_2$ appear symmetrically in all the expressions leading up to equation (19) and so could be interchanged without affecting the result.

In those cases when the autocorrelation function $R(\mathbf{r}_i, \mathbf{r}_j; z_i, z_j) = \langle n_1(\mathbf{r}_i, z_i) n_1(\mathbf{r}_j, z_j) \rangle$, associated with the structure function D_{ij}, exists H can be simplified. If the spatial scale of the irregularities L is much smaller than the scale on which either n_0 or μ vary in the z direction then we can introduce the variables $z = \frac{1}{2}(z_i + z_j)$, $\zeta = z_i - z_j$, and integrate with respect to ζ. This gives rise to a new correlation function which can be thought of as the projection of R on a plane transverse to the direction of propagation,

$$\rho_{ij} = \int_{-\infty}^{\infty} R(\mathbf{r}_i, \mathbf{r}_j; z, \zeta)\,d\zeta, \tag{22}$$

and $\mathcal{H}$ becomes

$$\mathcal{H} = \int_0^{z_f} H(\mathbf{v}_1(z), \mathbf{v}_2(z); z)\,dz,$$

$$H = k^2\mu^2[2\rho_0 + \rho(\mathbf{v}_2 + \mathbf{v}_1) + \rho(\mathbf{v}_2 - \mathbf{v}_1) - \rho(\tfrac{1}{2}\mathbf{u}_2 - \mathbf{v}_1) - \rho(\tfrac{1}{2}\mathbf{u}_2 + \mathbf{v}_1) - \rho(\mathbf{v}_2 - \tfrac{1}{2}\mathbf{u}_2) - \rho(\mathbf{v}_2 + \tfrac{1}{2}\mathbf{u}_2)]. \tag{23}$$

3. EVALUATION OF THE PATH INTEGRAL FOR THE FOURTH MOMENT

3.1. POINT SOURCE

Using the boundary condition (9) corresponding to a point source situated at $S(0) = (x_0, y_0, 0)$ one finds that

$$\mathbf{u}_1(0) = \mathbf{u}_2(0) = \mathbf{v}_1(0) = \mathbf{v}_2(0) = 0. \tag{24}$$

What follows is concerned, not with the general fourth moment at z_f but rather with quantities involving intensities at two points I_1, I_2. One sees, from equations (5) and (13) that this implies that $\mathbf{u}_2(z_f)$ can be set equal to zero in the solution. If one turns to equation (19) one sees that $\mathbf{u}_1$ in the first exponential is evaluated only at the fixed values it assumes at the initial and final points 0, z_f and so is not a variable in the path integral $d[\mathbf{u}_1]$. This path integral involves only the $\mathbf{u}_1[z]$ in the second exponent of equation (19) which can now be carried out to yield the delta functional $\delta[\mathbf{u}_2'' - \mathbf{u}_2 Q]$. The path integral with respect to $\mathbf{u}_2$ is then carried out and $\mathbf{u}_2$, in view of the delta functional, is given by the solution of the equation

$$\mathbf{u}_2'' = \mathbf{u}_2 Q \tag{25}$$

Since $\mathbf{u}_2(z_f)$ is zero, then $\mathbf{u}_2(z)$, the solution of equation (25), is identically zero for all z so that equation (19) becomes

$$m = N_0 \int \cdots \int \exp\left\{ ik[\mathbf{v}_2'(z) \cdot \mathbf{v}_1(z)]_0^{z_f} - ik \int_0^{z_f} (\mathbf{v}_2'' - \mathbf{v}_2 Q) \cdot \mathbf{v}_1 \, dz] \right\} \exp\left\{ - \int_0^{z_f} H(\mathbf{v}_1, \mathbf{v}_2; z) \, dz \right\} d[\mathbf{v}_1] \, d[\mathbf{v}_2]. \tag{26}$$

where

$$H = 2k^2\mu^2[\rho_0 - \rho(\mathbf{v}_1) - \rho(\mathbf{v}_2) + \tfrac{1}{2}\rho(\mathbf{v}_2 + \mathbf{v}_1) + \tfrac{1}{2}\rho(\mathbf{v}_2 - \mathbf{v}_1)] \tag{27}$$

since ρ is an even function of its argument. It is further assumed that μ^2 can be a slowly varying function of z and also that ρ can depend on z otherwise than through its argument $\mathbf{v}(z)$.

3.2. THE APPROXIMATION OF SMALL DEVIATIONS

Some approximations must be made in order to evaluate expression (26). The path integrals with respect to $\mathbf{v}_1, \mathbf{v}_2$ cannot easily be carried out since the presence of $\mathbf{v}_1(z)$ in $H(\mathbf{v}_1, \mathbf{v}_2, z')$ does not permit one to obtain a delta-functional by integrating the second term of the exponent of expression (26) with respect to $\mathbf{v}_1$. However, progress can be made if one assumes that the presence of the random scattering component distorts the deterministic rays by very small amounts. This enables one to derive an approximate form for $\mathbf{v}_1(z)$ which can later be used to assist in the evaluation of the required path integral expressions. The approximate form of $\mathbf{v}_1(z)$ is denoted by $\mathbf{v}_{10}(z)$ and defined to be the variable $\mathbf{v}_{10}(z)$ in the absence of random scattering but in the presence of the deterministic profile of the refractive index. This corresponds to setting H equal to zero in (26).

In order to determine the form of $\mathbf{v}_{10}(z)$ one can note that since $\mathbf{v}_1(z)$, $\mathbf{v}_2(z)$ appear symmetrically in all expressions leading to expression (26) one can interchange them to obtain, for $H = 0$,

$$m(\mathbf{v}_1(z_f); \mathbf{v}_2(z_f)) = N_0 \int\int \exp\left\{ik[\mathbf{v}_1'(z) \cdot \mathbf{v}_2(z)]_0^{z_f} - ik \int_0^{z_f} (\mathbf{v}_1'' - \mathbf{v}_1 Q) \cdot \mathbf{v}_2 \, \mathrm{d}z\right\} \mathrm{d}[\mathbf{v}_1] \, \mathrm{d}[\mathbf{v}_2]. \tag{28}$$

The path integral with respect to $\mathbf{v}_2$ can now be carried out to yield a delta-functional, as before, and subsequent integration with respect to $\mathbf{v}_1$ means that it is constrained to obey the equation

$$\mathbf{v}_1''(z) = \mathbf{v}_1(z) Q \tag{29}$$

with the condition that

$$\mathbf{v}_1(0) = 0, \tag{30}$$

and also $\mathbf{v}_1(z_f)$ is the value of $\mathbf{v}_1$ in the plane $z = z_f$; i.e., the separation of the two points

$$(\mathbf{r}_1, z_f) = (\mathbf{r}_4, z_f), \qquad (\mathbf{r}_2, z_f) = (\mathbf{r}_3, z_f), \tag{31}$$

at which the intensities I_1, I_2 are observed. The solution of equation (29) subject to condition (30) gives $\mathbf{v}_1(z)$ in the absence of random scattering: i.e., the required $\mathbf{v}_{10}(z)$. The physical significance of $\mathbf{v}_{10}(z)$ is illustrated in Figure 3. The fourth moment m involves

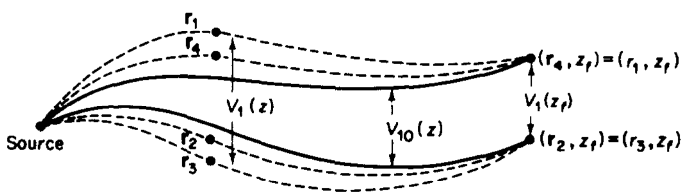

Figure 3. Illustrating the small deviation approximation for $\mathbf{v}_1(z)$ given by the geometrical rays.

four paths from the source point to the two observation points (31) above. These four paths are indicated by the broken lines. From its definition (13) and Figure 2 one sees that $\mathbf{v}_1(z)$ is the separation of the mid-points of the two pairs of paths traversed by $\mathbf{r}_1(z)$, $\mathbf{r}_4(z)$ and $\mathbf{r}_2(z)$, $\mathbf{r}_3(z)$. In the absence of the random component of refractive index the two ray paths from source to the two observation points are the geometrical rays indicated by the heavy lines. They are given by the solutions of the ray equation (14) passing through the source and the two observation points. The solution of equation (29) for the same conditions gives an approximate expression for the separation of these deterministic rays: i.e., $\mathbf{v}_{10}(z)$. One can now proceed on the assumption that the presence of refractive index irregularities causes the rays to deviate by only small amounts from the deterministic paths and so $\mathbf{v}_1(z)$ can be replaced by $\mathbf{v}_{10}(z)$ in the slowly varying term of expression (26): i.e., in H. The other exponent of expression (26) is a rapidly varying term, and here $\mathbf{v}_1(z)$ is not replaced by its approximate form.

3.3. THE FOURIER TRANSFORM OF $m(\mathbf{v}_1, \mathbf{v}_2)$

The path integral expressions discussed above stem from the discrete representation expression (10). The path integrals with respect to $\mathbf{v}_1(z)$ and $\mathbf{v}_2(z)$ are carried out in slabs, transverse to the z axis. A typical slab lying between z_{j-1} and z_j can be seen in Figure 1. In the continuous limit these slabs become infinitesimally thin. Now the path integral with respect to $\mathbf{v}_1(z)$ in expression (26) cannot be carried out in the last of these slabs bordering on z_f since here $\mathbf{v}_1(z)$ has been fixed by the boundary conditions expression (31) and is equal to $\mathbf{v}_1(z_f)$. This difficulty is overcome by taking the Fourier transform of m with respect to $\mathbf{v}_1(z)$ at the entrance to this last slab. This integration allows $\mathbf{v}_1(z)$ to take all values in the last slab and frees the end points of the paths in Figure 3 so that the $\mathrm{d}[\mathbf{v}_1(z)]$ path integral can be carried out. If z_{f-1} is the z coordinate at the entrance to the last slab then the required Fourier transform is

$$
\begin{aligned}
M(\boldsymbol{\nu}, \mathbf{v}_2) &= \frac{1}{(2\pi)^2} \int m(\mathbf{v}_1(z_f); \mathbf{v}_2(z_f))\, \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\mathbf{v}_1(z_{f-1})}\, \mathrm{d}\mathbf{v}_1(z_{f-1}) \\
&= \frac{N_0}{(2\pi)^2} \int\int \exp\{\mathrm{i}k[\mathbf{v}_2'(z)\cdot\mathbf{v}_1(z)]_0^{z_f}\} \exp\left\{-\int_0^{z_f} H(\mathbf{v}_1(z), \mathbf{v}_2(z); z)\, \mathrm{d}z\right\} \\
&\quad \times \exp\left\{-\mathrm{i}k\int_0^{z_f} [\mathbf{v}_2''(z) - \mathbf{v}_2(z)Q - \boldsymbol{\nu}k^{-1}\delta(z - z_{f-1})]\cdot\mathbf{v}_1(z)\, \mathrm{d}z\right\} \\
&\quad \times \mathrm{d}[\mathbf{v}_1(z)]\, \mathrm{d}[\mathbf{v}_2(z)]\, \mathrm{d}\mathbf{v}_1(z_{f-1}).
\end{aligned}
\tag{32}
$$

One replaces $\mathbf{v}_1(z)$ in H of expression (32) by $\mathbf{v}_{10}(z)$ for the reasons outlined above. One also notes that the presence of the Dirac delta function $\delta(z - z_{f-1})$ in the last exponent of expression (32) implies a discontinuity in $\mathbf{v}_2'(z)$ at z_{f-1}. This can be seen more clearly from the equation for $\mathbf{v}_2(z)$ given below. For this reason the first exponential on the right-hand side of expression (32) must be evaluated as

$$
\exp\{\mathrm{i}k([\mathbf{v}_2'(z)\cdot\mathbf{v}_1(z)]_0^{z_{f-1}^{(-)}} + [\mathbf{v}_2'(z)\cdot\mathbf{v}_1(z)]_{z_{f-1}^{(+)}}^{z_f})\}
\tag{33}
$$

where $z_{f-1}^{(-)}$, $z_{f-1}^{(+)}$ are respectively the values of z immediately before and after z_{f-1} where the discontinuity in $\mathbf{v}_2'(z)$ occurs. The quantities $\mathbf{v}_1(z)$, $\mathbf{v}_2(z)$, however, remain continuous at this point. Keeping this in mind, and using equation (30), one obtains the approximate form

$$
\begin{aligned}
M_0(\boldsymbol{\nu}, \mathbf{v}_2) &= \frac{N_0}{(2\pi)^2} \int\int \exp\{\mathrm{i}k[\mathbf{v}_2'(z_{f-1}^{(-)}) - \mathbf{v}_2'(z_{f-1}^{(+)})]\cdot\mathbf{v}_1(z_{f-1}) + \mathrm{i}k\mathbf{v}_2'(z_f)\cdot\mathbf{v}_1(z_f)]\} \\
&\quad \times \exp\left\{-\int_0^{z_f} H(\mathbf{v}_{10}(z), \mathbf{v}_2(z); z)\, \mathrm{d}z\right\} \\
&\quad \times \exp\left\{-\mathrm{i}k\int_0^{z_f} [\mathbf{v}_2''(z) - \mathbf{v}_2(z)Q - \boldsymbol{\nu}k^{-1}\delta(z - z_{f-1})]\cdot\mathbf{v}_1(z)\, \mathrm{d}z\right\} \\
&\quad \times \mathrm{d}\,[\mathbf{v}_1(z)]\, \mathrm{d}[\mathbf{v}_2(z)]\, \mathrm{d}\mathbf{v}_1(z_{f-1})
\end{aligned}
\tag{34}
$$

The path integral with respect to $\mathrm{d}[\mathbf{v}_1(x)]$ can now be done in equation (34) since the $\mathbf{v}_{10}(z)$ in H is tied to the deterministic rays and does not vary with the path integration. This leads to the following delta functional

$$
\delta(\mathbf{v}_2''(z) - \mathbf{v}_2(z)Q - \boldsymbol{\nu}k^{-1}\delta(z - z_{f-1}))
\tag{35}
$$

which, when the path integral with respect to $\mathrm{d}[\mathbf{v}_2(z)]$ is carried out, gives

$$
\begin{aligned}
M_0(\boldsymbol{\nu}, \mathbf{v}_2) &= \{N_0/(2\pi)^2\} \exp\{\mathrm{i}k\mathbf{v}_2'(z_f)\cdot\mathbf{v}_1(z_f)\} \\
&\quad \times \int \exp\left\{-\int_0^{z_f} H(\mathbf{v}_{10}(z), \mathbf{v}_2(z); z)\, \mathrm{d}z\right\} \\
&\quad \times \exp\{\mathrm{i}k[\mathbf{v}_2'(z_{f-1}^{(-)}) - \mathbf{v}_2'(z_{f-1}^{(+)})]\cdot\mathbf{v}_1(z_{f-1})\}\, \mathrm{d}\mathbf{v}_1(z_{f-1})
\end{aligned}
\tag{36}
$$

with the following equation for $\mathbf{v}_2(z)$,

$$\mathbf{v}_2''(z) = \mathbf{v}_2(z)Q + \boldsymbol{\nu} k^{-1}\delta(z - z_{f-1}). \tag{37}$$

The expression (36) is a first approximation to the Fourier transform of the fourth moment (26) with $\mathbf{v}_2(z)$ specified by equation (37) and $\mathbf{v}_{10}(z)$ defined by equation (29). These equations, together with equations (18) and (14), complete the first step in evaluating the intensity fluctuation spectrum by path integral methods.

4. EXPLICIT FORMS OF $M_0(\boldsymbol{\nu}, \mathbf{v}_2)$ FOR SPECIFIC CASES

To illustrate how the formal solution (36) can be applied explicit forms will now be found in two specific cases.

4.1. UNIFORM REFRACTIVE INDEX PROFILE

The first case is that of a randomly inhomogeneous medium in which n_0, the mean refractive index is constant in both space and time. The source is constant in both space and time. The source is situated at the co-ordinate origin while the observing points are at $(\frac{1}{2}x_f, 0, z_f)$, $(-\frac{1}{2}x_f, 0, z_f)$. The deterministic ray path to the first point is, from equation (14)

$$\mathbf{S}(z) = x_f z / 2z_f. \tag{38}$$

Now Q, in equation (18), is zero since n_0 is a constant, so equation (29) becomes $\mathbf{v}_1''(z) = 0$, and $\mathbf{v}_{10}(z)$, which is its solution with initial condition from equation (30) is

$$\mathbf{v}_{10}(z) = \mathbf{v}_1(z_f) z / z_f. \tag{39}$$

Similarly

$$\mathbf{v}_2''(z) = \boldsymbol{\nu} k^{-1}\delta(z - z_{f-1}), \tag{40}$$

which has the solution

$$\mathbf{v}_2(z) = \mathbf{A}z + \mathbf{B} \qquad (0 < z \leq z_{f-1}^{(-)}), \tag{41a}$$

$$\mathbf{v}_2(z) = \mathbf{C}z + \mathbf{D} \qquad (z_{f-1}^{(+)} \leq z \leq z_f). \tag{41b}$$

One sees from expression (40) that there is a discontinuity of magnitude $\boldsymbol{\nu} k^{-1}$ in $\mathbf{v}_2'(z)$ at $z = z_{f-1}$, as mentioned above. Matching $\mathbf{v}_2(z)$ in the plane $z = z_{f-1}$ and using the boundary condtiion at $z = z_f$ we obtain

$$\mathbf{v}_2(z) = (\mathbf{C} - \boldsymbol{\nu} k^{-1})(z - z_{f-1}) - \mathbf{C}(z_f - z_{f-1}) + \mathbf{v}_2(z_f) \qquad (0 < z \leq z_{f-1}^{(-)}), \tag{42a}$$

$$\mathbf{v}_2(z) = \mathbf{C}(z - z_f) + \mathbf{v}_2(z_f) \qquad (z_{f-1}^{(+)} \leq z \leq z_f). \tag{42b}$$

We determine $\mathbf{C}$ by setting $H = 0$ in (36) and carrying out the integral, remembering that the discontinuity in $\mathbf{v}_2'$ is equal to $\boldsymbol{\nu} k^{-1}$. When the inverse Fourier transform is applied to the result we have an expression for $m(\mathbf{v}_1(z_f); \mathbf{v}_2(z_f))$ in the absence of random scatter. This can then be compared with the corresponding expression that follows directly from expressions (4), (9) and (13),

$$m(\mathbf{v}_1(z); \mathbf{v}_2(z)) = (4\pi z)^{-4} \exp\{ik\mathbf{v}_1(z) \cdot \mathbf{v}_2(z)/z\}$$

to give

$$N_0 = (4\pi z_f)^{-4}$$

$$\mathbf{C} = \mathbf{v}_2(z_f)/z_f. \tag{43}$$

Using expressions (42a), (43) in (36) and letting z_{f-1} approach z_f one obtains, denoting $\mathbf{v}_1(z_{f-1})$ by ξ in this limit,

$$M_0(\boldsymbol{\nu}, \mathbf{v}_2) = \{1/(4\pi z)^4\}\{1/(2\pi)^2\} \exp\{ik\mathbf{v}_2(z_f) \cdot \mathbf{v}_1(z_f)/z_f\}$$
$$\times \int_{-\infty}^{\infty} \exp\left\{-\int_0^{z_f} H(\boldsymbol{\xi}z/z_f, \mathbf{v}_2(z_f)z/z_f - \boldsymbol{\nu}k^{-1}(z-z_f); z)\, dz\right\}$$
$$\times \exp\{-i\boldsymbol{\nu}\cdot\boldsymbol{\xi}\}\, d\boldsymbol{\xi}. \tag{44}$$

The calculation of the fundamental approximation in the case of a medium with a uniform mean refractive index is now complete. The corresponding expression for $m_0(\mathbf{v}_1(z_f), \mathbf{v}_2(z_f))$ is obtained by taking the Fourier transform of expression (44). To compare the result with expressions for $m_0(\mathbf{v}_1(z_f), \mathbf{v}_2(z_f))$ derived by other methods it is convenient to introduce the new variable

$$\mathbf{v}' = \mathbf{v} - k\mathbf{v}_2(z_f)/z_f.$$

Noting that H is a symmetric function of each of its first two arguments one finally obtains

$$m_0(\mathbf{v}_1(z_f)\mathbf{v}_2(z_f)) = \{1/(4\pi z)^4\}\{1/(2\pi)^2\} \int\int \exp$$
$$\left\{-\int_0^{z_f} H(\boldsymbol{\xi}z/z_f, \mathbf{v}_2(z) + \boldsymbol{\nu}'k^{-1}(z_f - z); z)\, dz\right\}$$
$$\times \exp\{-i[k\mathbf{v}_2(z_f)\cdot\boldsymbol{\xi}/z_f + \boldsymbol{\nu}'(\boldsymbol{\xi} + \mathbf{v}_1(z_f))]\}\, d\boldsymbol{\xi}\, d\boldsymbol{\nu}'. \tag{45}$$

In this form expression (45) agrees with the results of Uscinski *et al.* [10] and Macaskill [11].

4.2. LINEAR REFRACTIVE INDEX PROFILE

The second case is a medium in which the mean component n_0 is a linear function of the co-ordinate transverse to the direction of propagation. Take

$$n_0 = a_0 + ax. \tag{46}$$

The ray path in this case is

$$\mathrm{S}(z') = az^2/2 + (x_f/z_f - az_f/2)z \tag{47}$$

with the observer situated at the point $(x_f, 0, z_f)$ on the ray. Now Q is zero in the case of the linear profile expression (46) and so the equations for $\mathbf{v}_2(z)$ and $\mathbf{v}_{10}(z)$ remain the same as in the case of the uniform profile just considered. The solution for $M_0(\boldsymbol{\nu}, \mathbf{v}_2(z_f))$ is thus the same as for the uniform profile and is given by expression (45).

These results show that within the framework of the present approximation the profile n_0 must be at least a quadratic function to give a non-zero Q and affect the form of $M_0(\boldsymbol{\nu}, \mathbf{v}_2(z_f))$ directly. The fourth moment for a linear profile would differ from that for a uniform profile if it were to be calculated for a non-zero value of $\mathbf{u}_2$.

4.3. INTENSITY FLUCTUATIONS

One is now in a position to calculate the intensity correlation (6). It is, from expressions (4) and (5)

$$\rho_I(\mathbf{r}_f) = \int M_0(\boldsymbol{\nu}, 0)\, e^{-i\boldsymbol{\nu}\cdot\mathbf{r}_f}\, d\boldsymbol{\nu}, \tag{48}$$

where $M_0(\boldsymbol{\nu}, 0)$ is given by expression (45) in the cases being considered. The normalized variance S_I^2 is, from expressions (6), (7) and (45)

$$S_I^2 = \int \Phi_0(\boldsymbol{\nu})\, \mathrm{d}\boldsymbol{\nu} - 1, \tag{49}$$

$$\Phi_0(\boldsymbol{\nu}) = \{1/(2\pi)^2\} \int_{-\infty}^{\infty} \exp\left\{ -\int_0^{z_f} H(\boldsymbol{\xi} z/z_f, \boldsymbol{\nu} k^{-1}(z_f - z); z)\, \mathrm{d}z \right\} e^{-i\boldsymbol{\nu}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}. \tag{50}$$

4.4. AN IMPROVED ESTIMATE

A more accurate evaluation of the fourth moment can be obtained by estimating the difference between $m(\mathbf{v}_1, \mathbf{v}_2)$ and its fundamental approximation $m_0(\mathbf{v}_1, \mathbf{v}_2)$. The Fourier transform of this estimate $M_1(\boldsymbol{\nu}, \mathbf{v}_2)$, from Appendix A, is

$$M_1(\boldsymbol{\nu}, \mathbf{v}_2) = M_a(\boldsymbol{\nu}, \mathbf{v}_2) - M_b(\boldsymbol{\nu}, \mathbf{v}_2), \tag{51}$$

$$\begin{aligned} M_a(\boldsymbol{\nu}, \mathbf{v}_2) = {} & \{C_0 N_0/(2\pi)^2\} \int \cdot \int \exp\{-ik[\boldsymbol{\xi} \cdot \mathbf{v}_2(z_f) z_f^{-1} + \boldsymbol{\nu} k^{-1} \cdot \boldsymbol{\xi} - \mathbf{q} k^{-1} \cdot \mathbf{v}_2(z_f)]\} \\ & \times \int_0^{z_f} \exp\left\{ -\int_0^{s} H[(\boldsymbol{\xi} + \mathbf{q} k^{-1}(z_f - s)) z/z_f, \mathbf{v}_2(z_f) + (z_f - z)\boldsymbol{\nu} k^{-1}; z]\, \mathrm{d}z \right\} \\ & \times \exp\left\{ -\int_s^{z_f} H[(\boldsymbol{\xi} + \mathbf{q} k^{-1}(z_f - s)) z/z_f - (z - s)\mathbf{q} k^{-1}, \mathbf{v}_2(z_f) \right. \\ & \left. + \boldsymbol{\nu} k^{-1}(z_f - z); z]\, \mathrm{d}z \right\} \\ & \times F(\mathbf{q})\{1 - \cos[(\boldsymbol{\xi} + \mathbf{q} k^{-1}(z_f - s)) \cdot \mathbf{q} s/z]\}\, \mathrm{d}s\, \mathrm{d}\mathbf{q}\, \mathrm{d}\boldsymbol{\xi} \end{aligned} \tag{52}$$

and

$$\begin{aligned} M_b(\boldsymbol{\nu}, \mathbf{v}_2) = {} & \frac{C_0 N_0}{(2\pi)^2} \int \cdot \int \exp\{-ik[\boldsymbol{\xi} \cdot \mathbf{v}_2(z_f) z_f^{-1} + \boldsymbol{\nu} k^{-1} \cdot \boldsymbol{\xi}]\} \int_0^{z_f} \exp\{ik(\mathbf{v}_2(z_f) \\ & + \boldsymbol{\nu} k^{-1}[z_f - s]) \cdot \mathbf{q} k^{-1}\} \\ & \times \exp\left\{ -\int_0^{z_f} H[\boldsymbol{\xi} z/z_f, \mathbf{v}_2(z_f) + \boldsymbol{\nu} k^{-1}(z_f - z); z]\, \mathrm{d}z \right\} F(\mathbf{q}) \\ & \{1 - \cos[\boldsymbol{\xi} \cdot \mathbf{q} s/z_f]\}\, \mathrm{d}s\, \mathrm{d}\mathbf{q}\, \mathrm{d}\boldsymbol{\xi}, \end{aligned} \tag{53}$$

$$C_0 = 2k^2\mu^2, \qquad F(\mathbf{q}) = \{1/(2\pi)^2\} \int \rho(\mathbf{v})\, e^{i\mathbf{q}\cdot\mathbf{v}}\, \mathrm{d}\mathbf{v}, \tag{54}$$

where $\rho(\mathbf{v})$ is defined by equation (22). The results equations (52) and (53) apply to the case of a point source and a linear profile so N_0 in this case is $(4\pi z)^{-4}$.

4.5. PLANE WAVE INCIDENT ON A HALF SPACE

This section is concluded with the results for M_0 and M_1 in the case when a plane wave is incident on a half-space containing the irregular medium. A uniform refractive index profile is assumed. The method of derivation closely follows that of the point source and so details are not given here. For the plane wave case

$$M_0(\boldsymbol{\nu}, \mathbf{v}_2(z_f)) = \frac{1}{(2\pi)^2} \int\int \exp\left\{ -\int_0^{z_f} H[\boldsymbol{\xi}; \mathbf{v}_2(z_f) + \boldsymbol{\nu} k^{-1}(z_f - z); z]\, \mathrm{d}z \right\} \exp\{-i\boldsymbol{\nu} \cdot \boldsymbol{\xi}\}\, \mathrm{d}\boldsymbol{\xi}, \tag{55}$$

$$M_1(\boldsymbol{\nu}, \mathbf{v}_2(z)) = \frac{C_0}{(2\pi)^2}\left[\int\int \exp\left\{-\mathrm{i}[\boldsymbol{\nu}\cdot\boldsymbol{\xi} - \mathbf{q}\cdot\mathbf{v}_2(z_f)]\right\}\right.$$
$$\times \int_0^{z_f} \exp\left\{-\int_0^s H[\boldsymbol{\xi}+\mathbf{q}k^{-1}(z_f-s), \mathbf{v}_2(z_f)+\boldsymbol{\nu}k^{-1}(z_f-z); z]\,\mathrm{d}z\right\}$$
$$\times \exp\left\{-\int_s^{z_f} H[\boldsymbol{\xi}+\mathbf{q}k^{-1}(z_f-z), \mathbf{v}_2(z_f)+\boldsymbol{\nu}k^{-1}(z_f-z); z]\,\mathrm{d}z\right\}$$
$$\times F(\mathbf{q})\{1-\cos[(\boldsymbol{\xi}+\mathbf{q}k^{-1}(z_f-s))\cdot\mathbf{q}]\}\,\mathrm{d}s\,\mathrm{d}\mathbf{q}\,\mathrm{d}\boldsymbol{\xi}$$
$$-\int\cdots\int\int_0^{z_f} \exp\{-\mathrm{i}[\boldsymbol{\nu}\cdot\boldsymbol{\xi} - \mathbf{q}\cdot\mathbf{v}_2(z_f) - \boldsymbol{\nu}\cdot k^{-1}\mathbf{q}(z_f-s)]\}$$
$$\times \exp\left\{-\int_0^{z_f} H[\boldsymbol{\xi}, \mathbf{v}_2(z_f)+\boldsymbol{\nu}k^{-1}(z_f-z); z]\,\mathrm{d}z\right\}$$
$$\left.\times F(\mathbf{q})[1-\cos(\mathbf{q}\cdot\boldsymbol{\xi})]\,\mathrm{d}s\,\mathrm{d}\mathbf{q}\,\mathrm{d}\boldsymbol{\xi}\right]. \tag{56}$$

5. EVALUATION OF FLUCTUATION SPECTRA AND DISCUSSION OF RESULTS

It is important that the theoretical expressions should be capable of being evaluated numerically. Evaluations are presented in this section for two irregular media of practical interest. For the sake of simplicity the case of a half space on which a plane wave is incident is considered, with only one transverse dimension in the x direction. However, the basic methods of evaluation are the same in other cases and the results illustrate well the basic properties of the intensity fluctuation spectra.

5.1. MEDIUM WITH A SINGLE SCALE SIZE

The medium irregularities have the transverse spatial autocorrelation function

$$\rho(x) = \exp\{-x^2/L^2\} \tag{57}$$

where L is the outer scale of the irregularities. From equation (54) the spectrum corresponding to equation (57) is

$$F(q) = (L^2/2\sqrt{\pi})\exp\{-q^2L^2/4\}. \tag{58}$$

5.2. MEDIUM WITH A POWER LAW SPECTRUM

Here the irregularities have the transverse autocorrelation function

$$\rho(x) = (1+|x|/L)\exp\{-|x|/L\}. \tag{59}$$

The spectrum corresponding to equation (59) is

$$F(q) = 2/\pi(1+q^2L^2)^2. \tag{60}$$

5.3. THE FUNDAMENTAL SPECTRUM

The following scaled variables are now introduced:

$$\xi = \xi_x L^{-1}, \qquad v_1 = v_{1x}L^{-1}, \qquad \nu = \nu_x L, \qquad q = q_x L, \qquad Z = z/kL^2. \tag{61}$$

the fundamental spectrum $M_0(\nu)$ equation (55) then becomes

$$M_0(\nu, 0) = \frac{1}{2\pi}\int \exp\left\{-2\Gamma\int_0^{Z_f}[1-g(\xi, \nu(Z_f-Z)]\,\mathrm{d}Z\right\}\exp\{-\mathrm{i}\nu\xi\}\,\mathrm{d}\xi. \tag{62}$$

Here the parameter Γ arises naturally out of the scaling,

$$\Gamma = k^3 \mu^2 \rho_0 L^2, \tag{63}$$

and can be used to distinguish different scattering regimes, and

$$g(x, y) = f(x) + f(y) - \tfrac{1}{2}f(x+y) - \tfrac{1}{2}f(x-y), \qquad f = \rho/\rho_0. \tag{64, 65}$$

The improved estimate $M_1(\nu, 0)$ equation (56) can be rearranged in a straightforward manner and when written in terms of the new variables, equation (61), takes the form

$$\begin{aligned} M_1(\nu, 0) = \frac{4\Gamma}{2\pi} \int_0^{Z_f} \int_{-\infty}^{\infty} \int F(q) \sin^2\left(\frac{q\xi}{2}\right) \exp\left\{-2\Gamma \int_0^{Z_f} [1 - g(\xi, \nu t)]\,\mathrm{d}t\right\} \\ \times [\exp\left\{2\Gamma \int_0^{s} [g(\xi + q[t-s], \nu t) - g(\xi, \nu t)]\,\mathrm{d}t\right\} - 1] \\ \times \exp\{-\mathrm{i}\nu(\xi - qs)\}\,\mathrm{d}q\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned} \tag{66}$$

Numerical evaluation of the fundamental expression (62) for $M_0(\nu, 0)$ presents no difficulty. The improved estimate $M_1(\nu, 0)$, however, is a triple integral and it is advisable that at least one of these be evaluated analytically. In Appendix B it is shown that in the case when $f(\xi)$ can be represented as a Taylor series it is possible to expand the g functions in the exponents of $M_1(\nu)$ equation (66) in small powers of ξ and q and carry out the ξ integral to give

$$M_1 = \frac{\Gamma}{\sqrt{\pi}} \int_0^{Z_f} (K_1 - K_2)\,\mathrm{d}s, \tag{67}$$

where

$$\begin{aligned} K_1 = [\exp(-\nu^2/4\alpha)/\sqrt{\alpha}] \int_{-\infty}^{\infty} F(q) \exp\{-q^2(\gamma - \beta^2/\alpha) + \mathrm{i}\nu q(s + \beta/\alpha)\} \\ \times [1 - \tfrac{1}{2}\{\exp(-[q^2(1+4\mathrm{i}\beta) - 2\nu q]/4\alpha) + \exp(-[q^2(1-4\mathrm{i}\beta) - 2\nu q]/4\alpha)\}]\,\mathrm{d}q \end{aligned} \tag{68}$$

$$\begin{aligned} K_2 = [\exp(-\nu^2/4\alpha)/\sqrt{\alpha}] \int_{-\infty}^{\infty} F(q) \exp\{\mathrm{i}\nu qs\} \\ [1 - \tfrac{1}{2}\{\exp(-[q^2 - 2\nu q]/4\alpha) + \exp(-[q^2 + 2q\nu]/4\alpha)\}]\,\mathrm{d}q. \end{aligned} \tag{69}$$

In expressions (68) and (69)

$$\begin{aligned} \alpha &= \Gamma Z_f[-f''(0) + f''(\nu Z_f)/\nu Z_f], \\ \beta &= \Gamma \nu^{-2}[1 + \tfrac{1}{2}f''(0)(\nu s)^2 - f(\nu s)], \\ \gamma &= \Gamma \nu^{-3}[-\tfrac{1}{3}f''(0)(\nu s)^3 - 2\nu s + 2\mathscr{F}(\nu s)], \qquad \mathscr{F}(\nu s) = \int_0^{\nu s} f(p)\,\mathrm{d}p. \end{aligned} \tag{70}$$

In general the q integrals in expressions (68) and (69) cannot be evaluated analytically but this can be done numerically without trouble. The remaining integral with respect to

s in expression (67) presents no difficulty since the limits are finite and the integrand is slowly varying.

5.3. SPECTRA OF INTENSITY FLUCTUATIONS

The spatial frequency spectra of the intensity fluctuations are shown in Figures 4 and 5 for the two media described above at different distances in the scattering medium. These distances are given as fractions of the distance at which S_I^2, the normalized variance of the fluctuations, is a maximum, z_{f0}. The spectra are shown as the sum of the fundamental and the refined estimate. The fundamental is shown by the broken line and the sum by the full line. The spectra for the medium with the single scale size are shown in Figure 4 and those for the medium with the power law spectrum appear in Figure 5.

It is clear from the figures that the fundamental gives quite an accurate representation of the spectrum. The maximum discrepancy with the refined estimate occurs when the

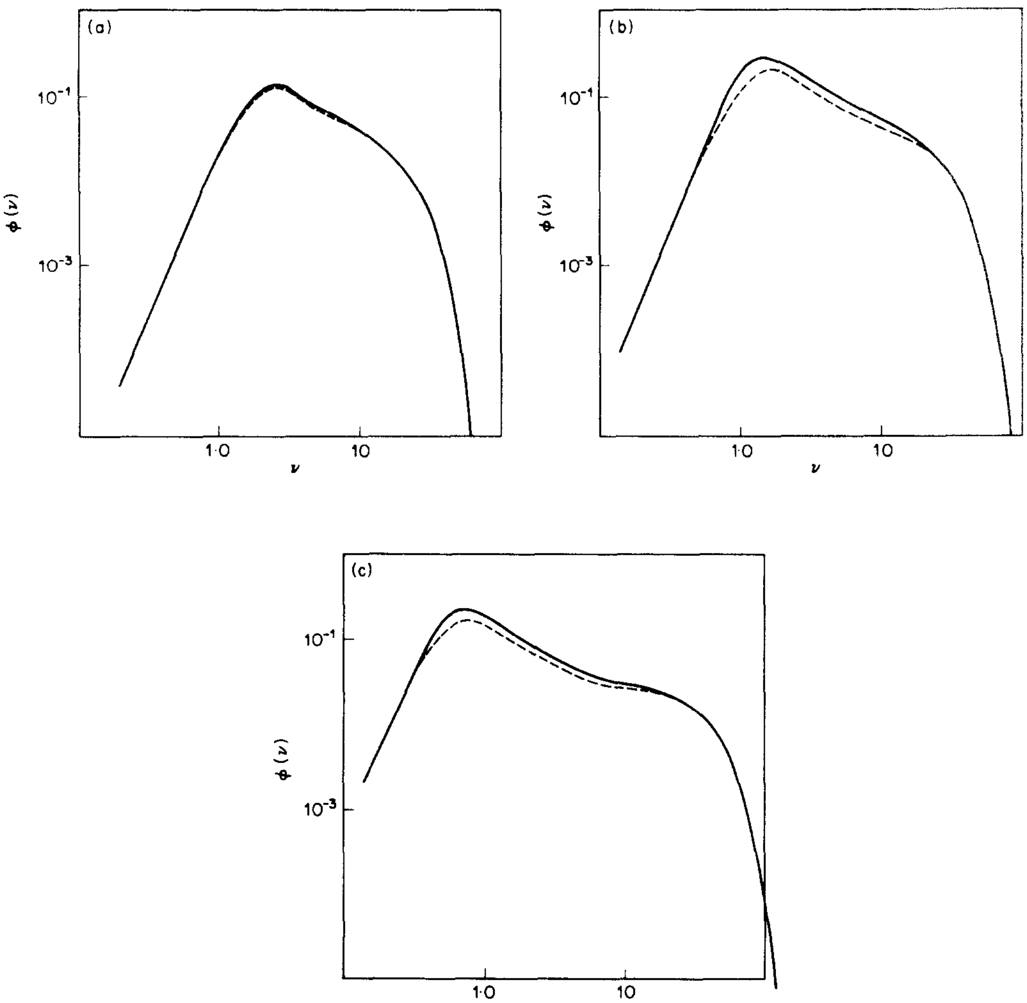

Figure 4. The spatial spectra of intensity fluctuations for the medium with the autocorrelation function (57) and $\Gamma = 1000$ for different distances of propagation z_f/z_{f0} = (a) 0·05, (b) 0·1 and (c) 0·2. The broken line gives the contribution due to the fundamental while the full line gives the refined estimate.

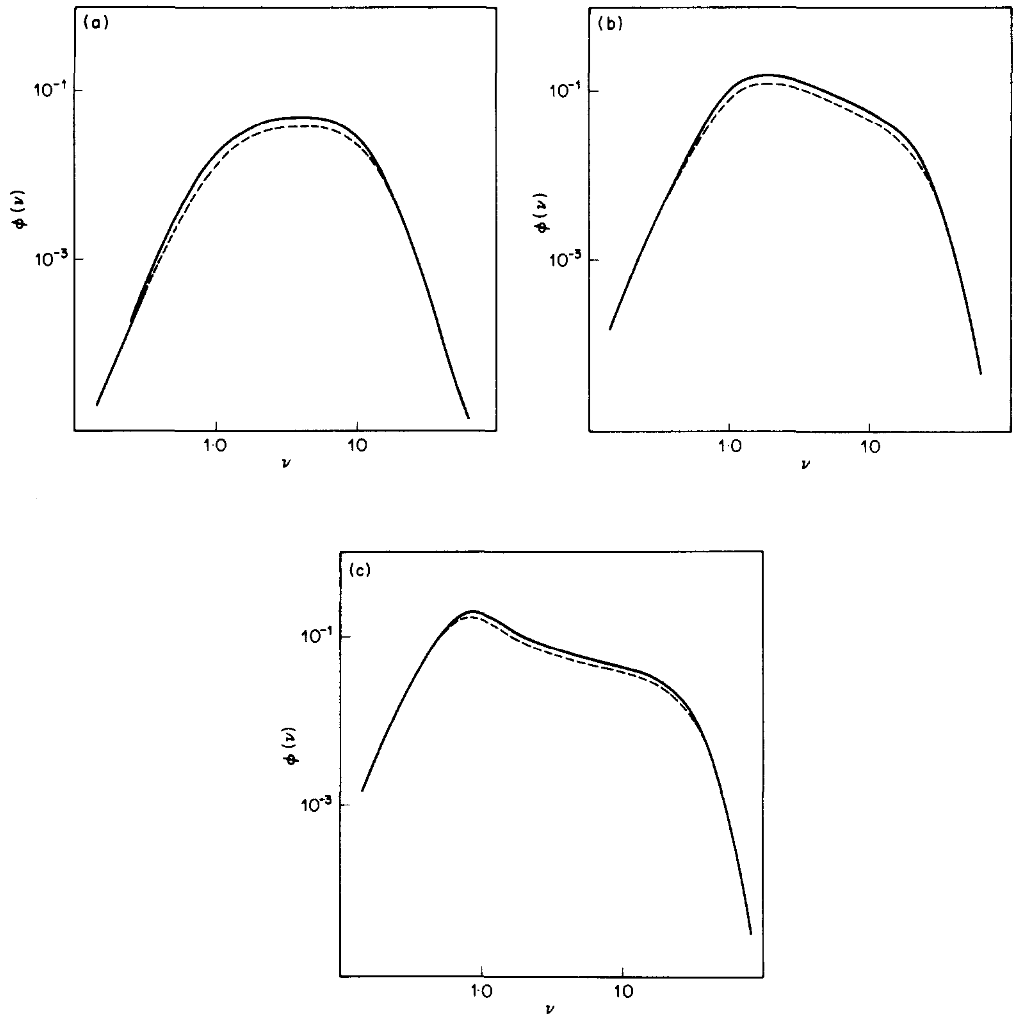

Figure 5. The spatial spectra of intensity fluctuations for the medium with the autocorrelation function (59) and $\Gamma = 1000$ for different distances of propagation z_f/z_{f0} = (a) 0·05, (b) 0·1 and (c) 0·2. The broken line gives the contribution due to the fundamental while the full line gives the refined estimate.

scintillation index is a maximum, i.e., $z_f/z_{f0} = 1$, and is at most of the order of 18% at the peak of the spectrum and much less elsewhere. Thus if high accuracy is not required the fundamental form provides a simple and quick means of investigating how the different autocorrelation functions of medium irregularities affect the form of the intensity fluctuation spectra.

Finally, the variance of the intensity fluctuations S_I^2 obtained by integrating the spectrum over all spatial frequencies is given in Figure 6 as a function of the scaled distance z for the case of the medium with autocorrelation function expression (57) when $\Gamma = 1000$. The maximum discrepancy between the fundamental and the improved estimate is of the order of 15% near the variance peak. The variance obtained by computer simulations of propagation in a randomly irregular medium with the Gaussian autocorrelation function expression (57) by Macaskill and Ewart [12] are also shown in Figure 6.

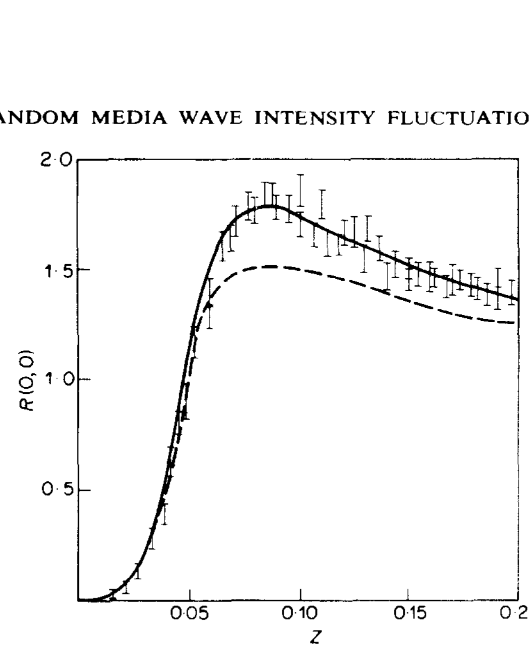

Figure 6. The variance of intensity fluctuations as a function of z_f for $\Gamma = 1000$ in a medium with autocorrelation function (57). The broken line gives the fundamental approximation and the full curve includes the extra contribution leading to the refined estimate. The points and error bars are the results of numerical simulations of the corresponding scattering experiment [12].

5.4. RELATIONSHIP TO THE RESULTS OF OTHER AUTHORS

The fluctuation spectra and variances obtained in the present paper are valid for all values of Γ and at all ranges z in a multiply scattering medium. The spectra obtained by some other authors (Zavorotnyi *et al.* [6], Dashen [8]) are valid in the limit for large distances of propagation when the variance S_I^2 approaches its limiting value of unity. They can be obtained as limiting cases of the expressions derived in the present paper. To illustrate this one can consider $M_1(\nu, 0)$, equation (66), in the case of large z when $g(\xi, \nu z)$, equation (64), becomes simply $f(\xi)$. The result then obtained for the spectrum $\Phi_1(\nu)$ in this limit is,

$$\begin{aligned}\Phi_1(\nu) = (\Gamma/\pi) \int\int\int_0^{Z_f} F(q)\{1-\cos[q(\xi+qs)]\} \\ \times\left[\exp\left\{-2\Gamma\left[Z_f-(Z_f-s)f(\xi+qs)-\int_0^s f(\xi+qZ)\,\mathrm{d}Z\right]\right\}\right. \\ \left.-\exp\{-2\Gamma[1-f(\xi+qs)]\}\right]\mathrm{e}^{-\mathrm{i}\nu\xi}\,\mathrm{d}\xi\,\mathrm{d}q\,\mathrm{d}s. \qquad (71)\end{aligned}$$

If this expression and the large range form of the fundamental term expression (62) are combined,

$$\Phi(\nu) \approx \Phi_0(\nu) + \Phi_1(\nu), \qquad (72)$$

then the resulting expression for $\Phi(\nu)$ corresponds to that obtained by Zavorotnyi *et al.* [6] when $v_2 = 0$. In spectral form it is

$$\begin{aligned}\Phi(\nu) \approx (1/2\pi) \int \exp\{-2\Gamma Z_f[1-\rho(\xi)]\}\,\mathrm{e}^{-\mathrm{i}\nu\xi}\,\mathrm{d}\xi + (\Gamma/\pi)\int\int\int_0^{Z_f} F(q) \\ \times\{1-\cos[q(\xi+qs)]\} \\ \times\exp\left\{-2\Gamma\left[Z_f-(Z_f-s)f(\xi+qs)-\int_0^s f(\xi+qZ)\,\mathrm{d}Z\right]\right\}\mathrm{e}^{-\mathrm{i}\nu\xi}\,\mathrm{d}\xi\,\mathrm{d}q\,\mathrm{d}s. \\ (73)\end{aligned}$$

Integration of expression (73) with respect to ν gives the variance S_I^2, the first term of which is the saturation value of unity, while the second gives the way in which this asymptotic value is approached at large range.

Exactly the same procedure can be applied in the case of a point source starting from expressions (50), (52) and (53). The asymptotic large range form resulting is identical with that obtained by Dashen [8].

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APPENDIX A

An estimate of m_1, the difference between m and its fundamental approximation m_0, is now made. Let

$$m_1(\mathbf{v}_1, \mathbf{v}_2) = m(\mathbf{v}_1, \mathbf{v}_2) - m_0(\mathbf{v}_1, \mathbf{v}_2) \tag{A1}$$

and so

$$M_1(\boldsymbol{\nu}, \mathbf{v}_2) = M(\boldsymbol{\nu}, \mathbf{v}_2) - M_0(\boldsymbol{\nu}, \mathbf{v}_2). \tag{A2}$$

To make an estimate of M_1 one can first note that since equation (26) is symmetric in $\mathbf{v}_1$, $\mathbf{v}_2$ they can be reversed without changing the value of $m(\mathbf{v}_1, \mathbf{v}_2)$. When this is done and the Fourier transform of the resulting expression taken one has

$$M(\boldsymbol{\nu}, \mathbf{v}_2) = N_0 \int \cdots \int \exp\left\{ik[\mathbf{v}_1'(z_f)\cdot\mathbf{v}_2(z_f) - \int_0^{z_f} (\mathbf{v}_1'' - \mathbf{v}_1 Q)\cdot \mathbf{v}_2 \, dz + \boldsymbol{\nu} k^{-1}\mathbf{v}_1(z_f)]\right\} \exp\left\{-\int_0^{z_f} H(\mathbf{v}_1, \mathbf{v}_2; z)\, dz\right\} d[\mathbf{v}_1]\, d[\mathbf{v}_2]\, d\mathbf{v}_1(z_f). \tag{A3}$$

Now M_0 can be given the formal representation

$$M_0(\boldsymbol{\nu}, \mathbf{v}_2) = N_0 \int \cdots \int \exp\left\{ ik[\mathbf{v}_1'(z_f) \cdot \mathbf{v}_2(z_f) - \int_0^{z_f} (\mathbf{v}_1'' - \mathbf{v}_1 Q) \cdot \mathbf{v}_2 \, dz + \boldsymbol{\nu} k^{-1} \cdot \mathbf{v}_1(z_f)] \right\} \times \exp\left\{ -\int_0^{z_f} H(\mathbf{v}_1, \mathbf{v}_{20}; z) \, dz \right\} d[\mathbf{v}_1] \, d[\mathbf{v}_2] \, d\mathbf{v}_1(z_f), \tag{A4}$$

where $\mathbf{v}_{20}(z)$ is the value of $\mathbf{v}_2(z)$ obtained when evaluating the zeroth approximation. For example in the case of the uniform and linear profiles it is, from equations (47) and (50),

$$\mathbf{v}_{20}(z) = \boldsymbol{\nu} k^{-1}(z_f - z) + \mathbf{v}_2(z_f). \tag{A5}$$

The validity of expression (A4) can be verified by carrying out the path integral with respect to $\mathbf{v}_2$ and then solving the resulting equation for $\mathbf{v}_1(z)$. Using equations (A3) and (A4) in equation (A2) and expanding the exponential terms involving h one obtains, on retaining only the first term in the expansion,

$$M_1(\boldsymbol{\nu}, \mathbf{v}_2) \approx N_0 \int \cdots \int \exp\left\{ ik[\mathbf{v}_1'(z_f) \cdot \mathbf{v}_2(z_f) - \int_0^{z_f} [\mathbf{v}_1'' - \mathbf{v}_1 Q) \cdot \mathbf{v}_2 \, dz + \boldsymbol{\nu} k^{-1} \cdot \mathbf{v}_1(z_f)] \right\} \exp\left[-\int_0^{z_f} H(\mathbf{v}_1, \mathbf{v}_{20}; z) \, dz \right] \int_0^{z_f} \{ H[\mathbf{v}_1(s), \mathbf{v}_{20}(s); s] - H[\mathbf{v}_1(s), \mathbf{v}_2(s); s] \} \, ds \, d[\mathbf{v}_1] \, d[\mathbf{v}_2] \, d\mathbf{v}_1(z_f). \tag{A6}$$

Now the two final quantities H can be written as sums of the correlation function of the medium ρ, and when the spectral representation

$$F(\mathbf{q}) = [1/(2\pi)^2] \int \rho(\mathbf{v}) \exp(iq \cdot \mathbf{v}) \, d\mathbf{v}. \tag{A7}$$

is introduced one has

$$M_1 = M_a - M_b, \tag{A8}$$

$$M_a(\boldsymbol{\nu}, \mathbf{v}_2) = 2k^2\mu^2 N_0 \int \cdots \int \exp\{ ik[\mathbf{v}_1'(z_f) \cdot \mathbf{v}_2(z_f) + \boldsymbol{\nu} k^{-1} \cdot \mathbf{v}_1(z_f)] \} \times \int_0^{z_f} \exp\left\{ -ik \int_0^{z_f} [\mathbf{v}_1'' - \mathbf{v}_1 Q + \mathbf{q} k^{-1} \delta(z-s)] \cdot \mathbf{v}_2(z) \, dz \right\} \times \exp\left\{ -\int_0^{z_f} H[\mathbf{v}_1(z, s), \mathbf{v}_{20}(z); z] \, dz \right\} \times F(\mathbf{q})[1 - \cos(\mathbf{v}_1(s) \cdot \mathbf{q})] \, ds \, d\mathbf{q} \, d[\mathbf{v}_1] \, d[\mathbf{v}_2] \, d\mathbf{v}_1(z_f) \tag{A9}$$

$$M_b(\boldsymbol{\nu}, \mathbf{v}_2) = 2k^2\mu^2 N_0 \int \cdots \int \exp\{ ik[\mathbf{v}_1'(z_f) \cdot \mathbf{v}_2(z_f) + \boldsymbol{\nu} k^{-1} \cdot \mathbf{v}_1(z_f)] \} \times \int_0^{z_f} \exp\left\{ -ik\left[\mathbf{q} k^{-1} \cdot \mathbf{v}_{20}(s) + \int_0^{z_f} (\mathbf{v}_1'' - \mathbf{v}_1 Q) \cdot \mathbf{v}_2(z) \, dz \right] \right\} \times \exp\left\{ -\int_0^{z_f} H[\mathbf{v}_1(z), \mathbf{v}_{20}(z); z] \, dz \right\} \times F(\mathbf{q})[1 - \cos(\mathbf{v}_1(s) \cdot \mathbf{q})] \, ds \, d\mathbf{q} \, d[\mathbf{v}_1] \, d[\mathbf{v}_2] \, d\mathbf{v}_1(z_f). \tag{A10}$$

The path integrals with respect to $\mathbf{v}_2$ and $\mathbf{v}_1$ can now be carried out in both M_a and M_b to give $\mathbf{v}_1$. The procedure is similar to that employed in section 4 of the paper. In M_a, $\mathbf{v}_1(z')$ is evaluated for two regions of z' because of the discontinuity in the equation for $\mathbf{v}_1(z)$ introduced by the delta-function. The boundary conditions used are that $\mathbf{v}_1(z)$ is continuous at $z=s$, is equal to zero at $z=0$, and must reduce to the form in the zeroth approximation when $s=z_f$. Also the discontinuity in the first derivative of $\mathbf{v}_1(z)$ at $z=s$ is equal to $-\mathbf{q}/\mathbf{k}$. This gives finally for $\mathbf{v}_1(z)$ in M_a

$$\mathbf{v}_1(z)=[\mathbf{v}_1(z_f)+\mathbf{q}k^{-1}(z_f-s)]z/z_f \qquad (0\leqslant z\leqslant s)$$

$$\mathbf{v}_1(z)=[\mathbf{v}_1(z_f)+\mathbf{q}k^{-1}(z_f-s)]z/z_f-(z-s)\mathbf{q}k^{-1} \qquad (s\leqslant z\leqslant z_f) \tag{A11}$$

while in M_b

$$\mathbf{v}_1(z)=\mathbf{v}_1(z_f)z/z_f. \tag{A12}$$

Use of these results in equations (A9) and (A10) leads to equations (52) and (53).

APPENDIX B

The exponential terms in expression (66) are combined and it is noted that, since ΓZ is assumed to be large, the first of the resulting exponential terms,

$$I_1=\exp\left\{-2\Gamma\left(\int_0^s [1-g(\xi+q[y-s],\,\nu y)+\int_s^{z_f}[1-g(\xi,\,\nu y)]\right)\mathrm{d}y,\right. \tag{B1}$$

is negligible except for values of g close to unity. Inspection of the form of g equation (64) shows that this occurs for small values of the first of the arguments of g.

The main contribution to I_1 is thus obtained by expanding the functions g in equation (B1) in Taylor series in the first of their arguments and then carrying out the integrals with respect to y. This leads eventually to

$$I_1=\exp\{-(\alpha\xi^2+2\beta\xi q+\gamma q^2)\}, \tag{B2}$$

where the coefficients α, β, γ are given by equations (70). The same procedure can be followed with the second of the exponential terms to give

$$I_2=\exp\{-\alpha\xi^2\}, \tag{B3}$$

so that equation (66) becomes

$$M_{(1)}^{iv}=\frac{\Gamma}{\pi}\int_0^{z_f}\int_{-\infty}^{\infty}\int F(q)\sin^2(q\xi/2)[I_1-I_2]\,\mathrm{e}^{-i\nu(\xi-qs)}\,\mathrm{d}\xi\,\mathrm{d}q\,\mathrm{d}s. \tag{B4}$$

The integrals with respect to ξ are of standard form and can easily be carried out to yield the results in expressions (68) and (69).