arXiv:2508.07347v1 [math.OA] 10 Aug 2025

Remarks on Derivations on Maximal Triangular Operator Algebras

Mark Spivack

August 12, 2025

Abstract

This note concerns bounded derivations on maximal triangular operator algebras on a Hilbert space. Given any bounded derivation δ on a maximal triangular algebra whose invariant lattice is continuous at 1, an operator which is shown to implement δ is constructed explicitly. For a general reducible maximal triangular algebra the same construction yields an operator which is shown to implement any δ, if and only if δ obeys an additional triple product rule.

This work is based on unpublished parts of the author's dissertation [19] and describes a variant of the proof of a more general result in [21] and is thus in effect a footnote to that work. To the best of the author's knowledge the constructive proof and triple product rule have not appeared elsewhere. The work here is not set in the context of the large body of subsequent research, and no claims are made regarding its relationship to later developments.

1 Introduction

Maximal triangular algebras were studied by Kadison and Singer [13] and their properties have been the subject of numerous subsequent studies. In this note we consider derivations on maximal triangular algebras and the questions of when, and by what operators, they are automatically implemented. The results obtained are proofs by construction, i.e. an operator is exhibited explicitly which implements the given derivation.

Derivations, on both bounded and unbounded operator algebras, have been studied extensively (e.g. [2, 5, 6, 12, 16]), in particular on C^* and von Neumann algebras [3, 11, 17], and certain classes of non-self-adjoint algebras [4, 10]. Nest algebras, consisting of all operators in $B(H)$ leaving invariant every element of a given subspace nest and which thus share many properties with maximal triangular algebras, were introduced by Ringrose in [15] (see also [1, 7–9, 14, 18]). Christensen [4] showed that all derivations on nest algebras are implemented. A constructive proof of this result was given in [19, 20], in which the cases of invariant lattices respectively continuous and discrete at 1 were treated separately, and which motivates the approach here.

A maximal triangular algebra is *reducible* if it has any non-trivial invariant subspaces; otherwise it is irreducible. For reducible algebras whose invariant lattices are strongly continuous at 1 we show by construction that any derivation is implemented. Subsequently, for any reducible algebra S with an invariant projection p we construct an operator which implements the given derivation on the subsets pSp and $p^{\perp}Sp^{\perp}$. Since $p^{\perp}Sp = \{0\}$ this leaves only $pSp^{\perp}$. We show that implementation of δ on this set is equivalent to a natural triple product rule on δ. We note that an identical construction for a non-trivial nest algebra yields a proof which no longer depends on whether the invariant lattice is continuous or discrete at 1, and therefore provides an alternative proof.

2 Definitions and preliminaries

Let $B(H)$ be the set of bounded linear operators on a complex Hilbert space H. An algebra $S \subset B(H)$ is *triangular* if the algebra $S \cap S^*$ is maximal Abelian in $B(H)$. Then $S \cap S^*$ is the *diagonal* of S. For any triangular algebra S_1 containing S we have $S \cap S^* = S_1 \cap S_1^*$ since $S \cap S^*$ is maximal Abelian.

Note that $\mathrm{lat}(S) \subset S \cap S^*$: For $p \in \mathrm{lat}(S)$ and $a \in S \cap S^*$, we have $ap = pap, a^*p = pa^*p$, so p commutes with $S \cap S^*$. However $S \cap S^*$ is maximal Abelian, so $p \in S \cap S^*$. For any $p \in \mathrm{lat}(S), a \in S, \quad p^{\perp}ap = p^{\perp}pap = 0$, so that $p^{\perp}Sp = \{0\}$.

A *derivation on* an operator algebra A is a linear map δ from A into $B(H)$ obeying the product rule $\delta(ab) = \delta(a)b + a\delta(b)$. For a projection p in the domain of δ, $\delta(p)$ is an operator which maps the range of p into its kernel, and vice versa, so that $(\delta(p))^2$ commutes with p. (To verify this, consider the identity $\delta(p) = \delta(p^2)$ and apply the product rule.) We identify p with its range where this is unambiguous, and so for example may write $\eta \in p$ when $p\eta = \eta$.

For any $b \in B(H)$ the map d_b defined on a subalgebra A of $B(H)$ by $d_b(a) = ba - ab$ for all $a \in A$ is a derivation. If δ is a derivation on A and $\delta = d_b$ for some $b \in B(H)$ then we say that δ is *implemented*. If also $\delta : A \to A$ then δ is *inner*.

If ξ, η are non-zero vectors in H, then $< \xi, \eta >$ denotes the inner-product of ξ with η, and $\xi \otimes \eta$ denotes the rank one operator given by $(\xi \otimes \eta)\zeta = < \zeta, \xi > \eta$ for all ζ in H.

We require a few known properties and results (for most of which we omit proofs):

Lemma 1. *[13]. If $p \in \mathrm{lat}(S)$ then $pB(H)p^{\perp} \subset S$.*

Corollary 2. *[13] If $p \in \mathrm{lat}(s)$ then $S\xi \supset p \ \ \forall \xi \in p^{\perp}$.*

Lemma 3. *[13] Lat(S) is totally-ordered.*

Lemma 4. *The commutant S' of S is trivial i.e. $S' = \mathbb{C}$.*

Hence, from a trivial calculation, for any derivation δ on S, any two linear maps which implement δ must differ by some $\alpha \in \mathbb{C}$.

We will also need the following result for derivations acting on projections:

Lemma 5. *[19] Let p be any projection and let δ be any derivation whose domain includes p. Then*
(i) $\delta(p)$ is an operator mapping $r(p)$ into $n(p)$ and vice versa,
(ii) δ is implemented on p by the operator $b = (1 - 2p)\delta(p)$. and
(iii) If $\delta_1(p)$ denotes any operator which maps $r(p)$ into $n(p)$ and vice versa then the linear extension of δ_1 to the algebra (p) generated by p and 1 is a derivation.

Proof. (i) $p^2 = p$, so that

$$\delta(p) = p\delta(p) + \delta(p)p.$$

Pre-multiplying by p we get $p\delta(p)p = 0$, and since δ is defined also on $p^{\perp} = 1 - p$ we get a similar result for $p^{\perp}$, so that $p\delta(p)p = p^{\perp}\delta(p)p^{\perp} = 0$.

(ii) Compute, using (i): $(1 - 2p)\delta(p)p - p(1 - 2p)\delta(p) = \delta(p)p + p\delta(p) = \delta(p)$.

(iii) (p) is just the set $\{\alpha + \beta p : \alpha, \beta \in \mathbb{C}\}$, and since $\delta_1(\mathbb{C})$ must be zero, δ_1 is defined on $\alpha + \beta p \in (p)$ by $\delta_1(\alpha + \beta p) = \beta\delta_1(p)$. Also δ_1 is automatically linear.

Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. Then $\delta_1 \left[(\alpha_1 + \beta_1 p)\left(\alpha_1 + \beta_2 p\right)\right] = \delta_1 \left[\alpha_1\alpha_2 + \left(\alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\beta_2\right) p\right]$ $= \left(\alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\beta_2\right) \delta_1(p)$. and $\delta_1 \left(\alpha_1 + \beta_1 p\right) \left(\alpha_2 + \beta_2 p\right) + \left(\alpha_1 + \beta_1 p\right) \delta_1 \left(\alpha_2 + \beta_2 p\right)$

$= \alpha_2\beta_1\delta_1(p) + \beta_1\beta_2\delta_1(p)p + \alpha_1\beta_2\delta_1(p) + \beta_1\beta_2 p\delta_1(p)$

$= \left(\alpha_1\beta_2 + \alpha_2\beta_1 + \beta_1\beta_2\right) \delta_1(p)$. Hence δ_1 is a derivation. □

3 Derivations on maximal triangular algebras

Throughout the remainder S will denote a reducible maximal triangular algebra. Let $\delta : S \to B(H)$ be a continuous derivation. For the case in which the invariant lattice is strongly continuous at 1 an operator is constructed which implements δ. We also formulate a condition on δ under which, for any reducible maximal triangular algebra, the constructed operator is well-defined and implements δ.

We now proceed with the construction. There is $p \in \operatorname{lat}(S)$ such that $p \neq 0$ or 1 . Let $\xi \in p^{\perp}, \eta \in p$ be unit vectors. Then by Lemma 1 the rank one operator $\xi \otimes \eta$ is in S and $S\xi \supset p$. This immediately gives the first result:

Lemma 6. *For $p \in \operatorname{lat}(S)$ such that $p \neq 0$ or 1, choose unit vector $\xi_0 \in p^{\perp}$ and write $p_0 = \xi_0 \otimes \xi_0$. Define a map b_1 by*

$$b_1 a\xi_0 = \delta\left(ap_0\right)\xi_0 \tag{1}$$

where $a \in S$, $a\xi_0 \in p$, and $b_1 p^{\perp} = 0$. Then b_1 is a well-defined and bounded linear operator, with $\|b_1\| \leqslant \|\delta\|$, and $\delta(a)p = d_{b_1}(a)p \quad \forall a \in S$.

Proof. Although p_0 is not necessarily contained in S, if a $\in S$ such that $a\xi_0 \in p$ then $ap_0 = p\left(ap_0\right)p^{\perp} \in S$ and $\|ap_0\| = \|a\xi_0\|_0$. Also $a\xi_0 = 0 \Rightarrow ap_0 = 0$. Thus it follows immediately that b_1 is well-defined and linear. Furthermore

$$\|b_1 a\xi_0\| = \|\delta\left(ap_0\right)\xi_0\| \leqslant \|\delta\|\,\|ap_0\| = \|\delta\|\,\|a\xi_0\|$$

whenever $a\xi_0 \in p$, and so $b_1 \in B(H)$ and $\|b_1\| \leqslant \|\delta\|$. Let $a, c \in S$ such that $c\xi_0 \in p$. Then $ac\xi_0 \in p$, $cp_0 \in S$ and

$$\left(b_1 a - ab_1\right)c\xi_0 = \left[\delta\left(acp_0\right) - a\delta\left(cp_0\right)\right]\xi_0 = \delta(a)c\xi_0.$$

Hence b_1 implements δ on p. □

We can now state the main derivation result for those algebras S with the property that $\operatorname{lat}(S)$ is strongly continuous at 1.

Theorem 7. *Suppose S is such that 1 is the strong limit of projections in lat(S). Then δ is implemented and we can construct $b \in B(H)$ such that $\delta = d_b\,|_S$ and $\|b\| \leqslant 2\|\delta\|$.*

Proof. For any $p_\alpha \in \operatorname{lat}(S)$, $p_\alpha < 1$, we can construct b_α as in Lemma 6 (replacing p, b_1 by p_α, b_α).

As is to be expected since b_α implements δ on p_α and $S' = \mathbb{C}$ it can be shown that $(b_\alpha - b_\beta)\, p_\alpha \in \mathbb{C}p_\alpha$ whenever $p_\alpha < p_\beta < 1$ and that

$$\|(b_\alpha - b_\beta)\, p_\alpha\| \leqslant \|\delta\|.$$

We can thus assume without loss of generality that $(b_\alpha - b_\beta)\, p_\alpha = 0$ whenever $p \leqslant p_\alpha \leqslant p_\beta < 1$ for some fixed non-zero projection $p \in \mathrm{lat}(S)$, and $\|b_\alpha\| \leqslant 2\|\delta\|$. It is then easy to show that the strong limit b for $\{b_\alpha\}$ exists and that $\delta = d_b \mid S$, and $\|b\| \leqslant 2\|\delta\|$ □

For $p \in \mathrm{lat}(S)$, $p \neq 0$ or 1, Lemma 6 gives an operator $b_1 = b_1 p$. Continuing with the construction we define $c_1 = pc_1 p^\perp$ by

$$c_1 p^\perp = -\delta(p)p^\perp, \qquad c_1 p = 0 \ . \tag{2}$$

Then $pc_1p^\perp = c_1p^\perp$ by Lemma 5. It is clear that $c_1 \in B(H)$ and $\|c_1\| \leqslant \|\delta\|$. Now define b_2 by

$$b_2 = b_1 + c_1. \tag{3}$$

Then we have

Theorem 8. *With c_1, and b_2 defined as in (2),(3), b_2 implements δ on the algebra $Sp = pSp$, and $\|b_2\| \leqslant 2\|\delta\|$.*

Proof. Let a ∈ S. With the notation of the construction of b_1, if $\xi \in p$ then

$$\begin{aligned}(b_2ap - apb_2)\,\xi &= (b_2pap - apb_2)\,\xi \\ &= \delta\,(apcp_0)\,\xi_0 - ap\delta\,(cp_0)\,\xi_0, \qquad \text{where} \;\; c \in S, \;\; c\xi_0 = \xi_1, \\ &= \delta(ap)cp_0\xi_0 = \delta(ap)\xi.\end{aligned}$$

If $\xi \in p^\perp$ then

$$\begin{aligned}(b_2ap - apb_2)\,\xi \;&=\; -\,apb_2\xi \;=\; ap\delta(p)p^\perp\xi \\ &=\; ap\delta(p)\xi \;=\; \delta(ap)\xi - \delta(ap)p\xi \;=\; \delta(ap)\xi.\end{aligned}$$

Hence $\delta = d_{b_2} \mid Sp$. □

Remark. b_2 clearly implements δ on p since

$$(b_2a - ab_2)\,p = b_2pap - ab_2p = (b_1a - ab_1)\,p \quad \forall a \in S,$$

so we could have stated Lemma 6 with b_2 instead of b_1.

We come now to the final part of the construction. Recall that if q is the rank one operator $\xi \otimes \eta$ then q^* is the rank one operator $\eta \otimes \xi$.

If p_1 is the rank one projection $\xi \otimes \xi$ then $p_1 = q^* q$ and, in general,

$$(\eta \otimes \xi_2)(\xi_1 \otimes \eta) = \xi_1 \otimes \xi_2.$$

This brings us to the following:

Lemma 9. *Choose a fixed unit vector $\eta_1 \in p$ and put $q_1 = \xi_0 \otimes \eta_1$ so that $q_1 \in S$. Define a map c_2 as follows:*

$$c_2 p = 0, \text{ and } pc_2 p^\perp = 0 \tag{4}$$

and for any $\xi \in p^\perp$

$$p_\xi c_2 p^\perp = -q^* \delta(q) p^\perp + q^* \delta(q_1) q_1 q \tag{5}$$

where $q = \xi \otimes \eta_1 = qp^\perp$, and $p_\xi = \xi \otimes \xi$. Then $c_2 \in B(H)$.

(Notice that this definition depends on the fixed vectors $\xi_0 \in p^\perp$, $\eta_1 \in$ p. It is not obvious that the definition is invariant to within a constant additive factor under these choices.)

Proof. c_2 is clearly well-defined. It is easy to see that c_2 is bounded and that for any unit vector η, $\|c_2\eta\| \leqslant 2\|\delta\|$ since $pc_2 = 0$ and, for any unit vector $\xi \in p^\perp$, $\|p_\xi c_2 \eta\| \leqslant 2\|\delta\|$.

It remains to show that c_2 is linear. For this it must be shown that for unit vectors $\xi, \{\xi_i\}_{i\in I}$ in $p^\perp$ such that $\xi = \sum_i \alpha_i \xi_i$, say, the definitions $p_{\xi_i} c_2 p^\perp$ as above lead to the same value for $p_\xi c_2 p^\perp$ as by defining this directly. It will suffice to show this for, say, $\xi = r_\alpha \xi_\alpha + r_\beta \xi_\beta$, where $\|\xi\| = \|\xi_\alpha\| = \|\xi_\beta\| = 1$ and r_α, $r_\beta \in \mathbb{C}$.

Write $p_\alpha = \xi_\alpha \otimes \xi_\alpha$, $p_\beta = \xi_\beta \otimes \xi_\beta$, $p_\xi = \xi \otimes \xi$, $q_\alpha = \xi_\alpha \otimes \eta_1$, $q_\beta = \xi_\beta \otimes \eta_1$, and $q = \xi \otimes \eta_1$.

We have first that, $\forall \zeta \in H$,

$$q\zeta =< \zeta, r_\alpha \xi_\alpha + r_\beta \xi_\beta > \eta = (\bar{r}_\alpha q_\alpha + \bar{r}_\beta q_\beta)\, \zeta.$$

This gives $q^* = r_\alpha q_\alpha^* + r_\beta q_\beta^*$ and

$$p_\xi = q^* q = |r_\alpha|^2 p_\alpha + |r_\beta|^2 p_\beta + r_\alpha \bar{r}_\beta q_\alpha^* q_\beta + \bar{r}_\alpha r_\beta q_\beta^* q_\alpha.$$

Defining $p_\xi c_2 p^\perp$ directly we thus have

$$p_\xi c_2 p^\perp = q^*\delta\left(q_1\right) q_1^* q - q^*\delta(q) p^\perp \tag{6}$$

$$= \left|r_\alpha\right|^2 q_\alpha^*\delta\left(q_1\right) q_1^* q_\alpha + \left|r_\beta\right|^2 q_\beta^*\delta\left(q_1\right) q_1^* q_\beta + r_\alpha \bar{r}_\beta q_\alpha^*\delta(q_1) q_1^* q_\beta \tag{7}$$

$$+ \bar{r}_\alpha r_\beta q_\beta^*\delta\left(q_1\right) q_1^* q_\alpha \tag{8}$$

$$- \left[\left|r_\alpha\right|^2 q_\alpha^{*\delta}\delta\left(q_\alpha\right) + \left|r_\beta\right|^2 q_\beta^*\delta\left(q_\beta\right) + r_\alpha \bar{r}_\beta q_\alpha^*\delta\left(q_\beta\right) + \bar{r}_\alpha r_\beta q_\beta^*\delta\left(q_\alpha\right)\right] \tag{9}$$

On the other hand

$$p_\xi c_2 p^\perp = \left[\left|r_\alpha\right|^2 p_\alpha + \left|r_\beta\right|^2 p_\beta + r_\alpha \bar{r}_\beta q_\alpha^* q_\beta + \bar{r}_\alpha r_\beta q_\beta^* q_\alpha\right] b p^\perp$$

$$= \left|r_\alpha\right|^2 \left[q_\alpha^*\delta\left(q_1\right) q_1^* q_\alpha - q_\alpha^*\delta\left(q_\alpha\right) p^\perp\right] + \left|r_\beta\right|^2 \left[q_\beta^*\delta\left(q_1\right) q_1^* q_\beta - q_\beta^*\delta\left(q_\beta\right) p^\perp\right]$$

$$+ r_\alpha \bar{r}_\beta q_\alpha^* q_\beta \left[q_\beta^*\delta\left(q_1\right) q_1^* q_\beta - q_\beta^*\delta\left(q_\beta\right) p^\perp\right] \tag{10}$$

$$+ \bar{r}_\alpha r_\beta q_\beta^* q_\alpha \left[q_\alpha^*\delta\left(q_1\right) q_1^* q_\alpha - q_\alpha^*\delta\left(q_\alpha\right) p^\perp\right] \tag{11}$$

Notice that $q_\alpha^* q_\beta q_\beta^* = q_\alpha^*$ and $q_\beta^* q_\alpha q_\alpha^* = q_\beta^*$, and these occur in the last two terms of the expression above. Comparing (9) and (11), then, term by term we see that they are equal.

This extends by induction to any finite sum of vectors in $p^\perp$ and hence, since we have shown c_2 to be bounded, to any sum in $p^\perp$. It follows that c_2 is linear and the result follows. □

We now have

Theorem 10. *Define $b = b_2 + c_2$, with b_2 as in equation (3) and c_2 defined as in Lemma 9. Then b implements δ on the algebras Sp and $p^\perp S p^\perp$, and $\|b\| \leqslant 4\|\delta\|$.*

Proof. We already have that $\|b_2\|, \|c_2\| \leqslant 2\|\delta\|$, so $\|b\| \leqslant 4\|\delta\|$.

Let $a \in S$. Then $(bpap - papb) = b_2\, pap - papb_2 = \delta(pap)$ by Theorem 8. To show that $\delta = d_b \mid p^\perp S p^\perp$ we consider various cases.

1) Let $\xi \in p$. Then $\left(bp^\perp ap^\perp - p^\perp ap^\perp b\right)\xi = -p^\perp ap^\perp b\xi = -p^\perp ap^\perp b_1\xi$

$$= -p^\perp ap^\perp \delta(q)\xi_0, \qquad \text{where } q = \xi_0 \otimes \xi_1$$

$$= \delta\left(p^\perp ap^\perp\right) q\xi_0 - \delta\left(p^\perp ap^\perp q\right)\xi_0 = \delta\left(p^\perp ap^\perp\right)\xi.$$

2) Let $\xi \in p^{\perp}$ and consider $\mathrm{pd}_b\left(p^{\perp}ap^{\perp}\right)$:

$$p\left(bp^{\perp}ap^{\perp}\xi - p^{\perp}ap^{\perp}b\xi\right) = pbp^{\perp}ap^{\perp}\xi = b_2p^{\perp}ap^{\perp}\xi$$
$$= -\delta(p)p^{\perp}ap^{\perp}\xi = \delta\left(p^{\perp}\right)p^{\perp}ap^{\perp}\xi$$
$$= \delta\left(p^{\perp}ap^{\perp}\right)\xi - p^{\perp}\delta\left(p^{\perp}ap^{\perp}\right)\xi = p\delta\left(p^{\perp}ap^{\perp}\right)\xi.$$

3) Finally let $\xi \in p^{\perp}$ and consider $p^{\perp}d_b\left(p^{\perp}ap^{\perp}\right)$:
Choose a basis $\{\xi_\alpha\}$ for $p^{\perp}$ and put $p_\alpha = \xi_\alpha \otimes \xi_\alpha$, $q_\alpha = \xi_\alpha \otimes \eta_1 \quad \forall\alpha$. We consider, for each α, $\quad p_\alpha d_b\left(p^{\perp}ap^{\perp}\right)$. First we define the constants $r_{\alpha\beta} \in \mathbb{C}$ by

$$p_\alpha ap_\beta = r_{\alpha\beta}q_\alpha^* q_\beta.$$

(It is easily seen that $p_\alpha ap_\beta$ and $q_\alpha^* q_\beta$ are multiples of the rank one operator $\xi_\beta \otimes \xi_\alpha$.) Note that $q_{\alpha-1} = q_\alpha p_\alpha ap_\beta = r_{\alpha\beta}q_\beta$, and similarly $p_\alpha aq_\beta^* = p_\alpha ap_\beta q_\beta^* = r_{\alpha\beta}q_\alpha^*$.

Then $p_\alpha\left(bp^{\perp}ap^{\perp}\xi - p^{\perp}ap^{\perp}b\xi\right)$

$$= q_\alpha^*\delta\left(q_1\right)q_1^*q_\alpha ap^{\perp}\xi - q_\alpha^*\delta\left(q_\alpha\right)p^{\perp}ap^{\perp}\xi - \sum_\beta p_\alpha p^{\perp}ap_\beta b\xi$$
$$= -q_\alpha^*\delta\left(q_\alpha\right)p^{\perp}ap^{\perp}\xi + \sum_\beta \left[q_\alpha^*\delta\left(q_1\right)q_1^*q_\alpha ap_\beta\xi\right.$$
$$\left.-p_\alpha aq_\beta^\star\delta\left(q_1\right)q_1^\star q_\beta\xi + p_\alpha aq_\beta^\star\delta\left(q_\beta\right)\xi\right]$$
$$= -q_\alpha^*\delta\left(q_\alpha\right)p^{\perp}ap^{\perp}\xi$$
$$+\sum_\beta\left[r_{\alpha\beta}q_\alpha^*\delta\left(q_1\right)q_1^*q_\beta\xi - r_{\alpha\beta}q_\alpha^*\delta\left(q_1\right)q_1^*q_\beta\xi + r_{\alpha\beta}q_\alpha^*\delta\left(q_\beta\right)\xi\right]$$
$$= -q_\alpha^*\delta\left(q_\alpha\right)p^{\perp}ap^{\perp}\xi + \sum_\beta q_\alpha^*\delta\left(r_{\alpha\beta}q_\beta\right)\xi$$
$$= -q_\alpha^*\delta\left(q_\alpha\right)p^{\perp}ap^{\perp}\xi + \sum_\beta q_\alpha^*\delta\left(q_\alpha ap_\beta\right)\xi$$
$$= -q_\alpha^\star\delta\left(q_\alpha\right)p^{\perp}ap^{\perp}\xi + q_\alpha^\star\delta\left(q_\alpha p^{\perp}ap^{\perp}\right)\xi$$
$$= q_\alpha^*q_\alpha\delta\left(p^{\perp}ap^{\perp}\right)\xi = p_\alpha\delta\left(p^{\perp}ap^{\perp}\right)\xi.$$

Hence $\forall\alpha$, $p_\alpha d_b\left(p^{\perp}ap^{\perp}\right)\xi = p_\alpha\delta\left(p^{\perp}ap^{\perp}\right)\xi$, and the proof is complete. □

Since p is invariant under S, and since $p^{\perp}Sp = \{0\}$ and $Sp = pSp$, we can write $S = Sp + pSp^{\perp} + p^{\perp}Sp^{\perp}$. Theorem 10 shows that δ is implemented by b on Sp and $p^{\perp}Sp^{\perp}$, so extending the result to S itself hinges on whether it holds for the remaining term $pSp^{\perp}$. This we have not proved in general. However, we give as the final result a natural condition on δ in the form of triple product rule under which it does indeed hold.

To motivate this consider operators $s_1, s_2, s_3 \in S$ and $a \notin S$ such that $s_2 a s_3 \in S$. For example if $s_3 = s_3 p^{\perp}$ and $s_2 = p s_2$ then by Lemma 1 a can be any operator in $B(H)$. Then although $\delta(a)$ is undefined one may ask whether the equation

$$\delta\left(s_1 s_2 a s_3\right) = s_1 \delta\left(s_2 a s_3\right) + \delta\left(s_1\right) s_2 a s_3 \tag{12}$$

holds. In general this would seem to be a very strong restriction, although of course whenever δ is implemented it must hold since an implemented derivation extends automatically to $B(H)$. The result concerns a less restrictive form of condition (12).

Theorem 11 (Triple product rule)**.** *Suppose the vectors $\xi_0 \in p^{\perp}$ and $\eta_1 \in p$, the basis $\{\xi_\alpha\}$ for $p^{\perp}$, and the operator b are all as previously defined. Let η be any vector in p.*

Define the rank one operators

$$q_\alpha = \xi_\alpha \otimes \eta_1, \quad q_1 = \xi_0 \otimes \eta_1, \quad q = \xi_\alpha \otimes \eta, \text{ and } \quad q_2 = \xi_0 \otimes \eta \cdot \tag{13}$$

Then δ is implemented if and only if

$$\delta(q) = \delta\left(q q_\alpha^* q_1\right) q_1^* q_\alpha + q q_\alpha^* \delta\left(q_\alpha\right) - q q_\alpha^* \delta\left(q_1\right) q_1^* q_\alpha. \tag{14}$$

If so then $\delta = d_b \mid S$.

Proof. We have $q = q q_\alpha^* q_1 q_1^* q_\alpha$ and $q_2 = q q_\alpha^* q_1$. All the operators q, q_α, q_1, q_2 and none of their adjoints, are in S.

In addition to the expressions for q, q_2 in terms of the other operators we have $q_\alpha = q_{\alpha_0} q_1^* q_1$ and other such relations which arise wherever the "end-points" of the various operators coincide. So (14) can be written

$$\delta\left(q q_\alpha^* q_1 q_1^* q_\alpha\right) = \delta\left(q q_\alpha^* q_1\right) q_1^* q_\alpha + q q_\alpha^* \delta\left(q_1 q_1^* q_\alpha\right) - q q_\alpha^* \delta\left(q_1\right) q_1^* q_\alpha \cdot$$

Assuming that δ is implemented, say by c, $\delta = d_c$ can be defined on the whole of $B(H)$, and we can compute using δ as if this is so:

$$\begin{aligned}\delta\left(q q_\alpha^* q_1 q_1^* q_\alpha\right) &= \delta\left(q q_\alpha^* q_1\right) q_1^* q_\alpha + q q_\alpha^* q_1 \delta\left(q_1^* q_\alpha\right) \\ &= \delta\left(q q_\alpha^* q_1\right) q_1^* q_\alpha + q q_\alpha^* \delta\left(q_1 q_1^* q_\alpha\right) - q q_\alpha^* \delta\left(q_1\right) q_1^* q_\alpha.\end{aligned}$$

Suppose now that (14) holds.

If $a = \mathrm{pap}^{\perp} \in S$, then $a = \sum_{\alpha} pap_{\alpha}$ where $p_{\alpha} = \xi_{\alpha} \otimes \xi_{\alpha} \forall \alpha$ and each $pap_{\alpha} \in S$. So when considering operators in $pSp^{\perp}$ we may restrict our attention to the rank one operators $\xi_{\alpha} \otimes \eta$, where $\eta \in p$.

Consider, then, $q = \xi_{\alpha} \otimes \eta$, $\eta \in p$. Then q is as in the statement of the theorem.

For any $\xi \in p^{\perp}, q\xi =< \xi,\xi_{\alpha} > \eta$ so

$$\begin{aligned} bq\xi &= b_1 q\xi \;=\; < \xi, \xi_{\alpha} > b_1 \eta \\ &= \; < \xi\xi_{\alpha} > \delta\left(q_2\right)\xi_0 \\ &= \delta\left(q_2\right) q_1^{*} q_{\alpha} \xi . \end{aligned}$$

Then

$$(bq - qb)\xi = b_1 q\xi - qp_{\alpha} b\xi = \left[\delta\left(qq_{\alpha}^{*}q_1\right) q_1^{*} q_{\alpha} + qq_{\alpha}^{\star}\delta\left(q_{\alpha}\right) - qq_{\alpha}^{*}\delta\left(q_1\right) q_1^{*} q_{\alpha}\right]\xi$$

However condition (14) says that this is just $\delta(q)\xi$ which is what we require.

We have thus shown that if condition (14) holds then $\delta = d_b \mid pSp^{\perp}$ and Theorem 12 shows that b implements δ on the rest of S, and so the proof is complete. □

Remarks.

1) The condition (14) can be shown to be a special case of the product rule (Property D) introduced in [21] and is therefore ostensibly a weaker requirement, although Theorem 11 implies that these conditions are equivalent.

2) The operator b can be constructed in an identical way for any non-trivial nest algebra N since we have used operators in $pB(H)p^{\perp}$ and if $p \in \mathrm{lat}(N)$ such operators are also to be found in N. Furthermore Theorem 11 can be applied to N, and we know that condition (14) of Theorem 11 holds. Thus the construction yields a proof for nest algebras which deals simultaneously with cases discrete and continuous at 1.

3) The definitions and results here lend themselves naturally to Banach space generalization if we should care to imitate the process used in [19, 20] for nest algebras.

Acknowledgments

I remain deeply grateful to Professor Chris Lance for his kind generosity as well as to Professor Roger Plymen for their guidance during the writing of [19] from which this note

is adapted.

References

[1] William Arveson. Operator algebras and invariant subspaces. *Annals of Mathematics*, 100(3):433–532, 1974.

[2] O Bratteli and D Robinson. Operator algebras and quantum statistical mechanics. *Bull. Amer. Math. Soc*, 7:425, 1982.

[3] Ola Bratteli and Derek W Robinson. Unbounded derivations of C*-algebras. *Communications in Mathematical Physics*, 42(3):253–268, 1975.

[4] Erik Christensen. Derivations of nest algebras. *Mathematische annalen*, 229(2):155–161, 1977.

[5] Erik Christensen. Extension of derivations. *Journal of Functional Analysis*, 27(2):234–247, 1978.

[6] Erik Christensen. Extensions of derivations II. *Mathematica Scandinavica*, 50(1):111–122, 1982.

[7] Kenneth R Davidson. Commutative subspace lattices. *Indiana University Mathematics Journal*, 27(3):479–490, 1978.

[8] KR Davidson. Nest algebras, pitman research notes in mathematics, 1988.

[9] JA Erdos. Non-selfadjoint operator algebras. In *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, pages 127–145. JSTOR, 1981.

[10] Frank Gilfeather and David R Larson. Nest-subalgebras of von Neumann algebras: commutants modulo compacts and distance estimates. *Journal of Operator Theory*, pages 279–302, 1982.

[11] B Johnson. Characterization and norms of derivations on von Neumann algebras. In *Algèbres d'Opérateurs: Séminaire sur les Algèbres d'Opérateurs, Les Plans-sur-Bex, Suisse, 13–18 mars 1978*, pages 228–236. Springer, 2006.

[12] Richard V Kadison. Derivations of operator algebras. *Annals of Mathematics*, pages 280–293, 1966.

[13] Richard V Kadison and IM Singer. Triangular operator algebras: fundamentals and hyperreducible theory. *American Journal of Mathematics*, 82(2):227–259, 1960.

[14] E Christopher Lance. Cohomology and perturbations of nest algebras. *Proceedings of the London Mathematical Society*, 3(2):334–356, 1981.

[15] John R Ringrose. On some algebras of operators. *Proceedings of the London Mathematical Society*, 3(1):61–83, 1965.

[16] JR Ringrose. Automatic continuity of derivations of operator algebras. *Journal of the London Mathematical Society*, 2(3):432–438, 1972.

[17] Shôichirô Sakai. Derivations of W*-algebras. *Annals of Mathematics*, pages 273–279, 1966.

[18] Baruch Solel. *Irreducible triangular algebras.* University of Pennsylvania, 1981.

[19] Mark Spivack. *Derivations on operator algebras.* The University of Manchester (United Kingdom), 1982.

[20] Mark Spivack. Derivations and nest algebras on Banach space. *Israel Journal of Mathematics*, 50:193–200, 1985.

[21] Mark Spivack. Certain properties of derivations. *Proceedings of the American Mathematical Society*, 99(4):712–718, 1987.