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DERIVATIONS AND NEST ALGEBRAS ON BANACH SPACE

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ABSTRACT
In this paper the notions of nest and nest algebra are generalised to sets of operators on Banach space. Their fundamental properties are established and continuous derivations on such algebras are shown to be implemented by an operator which can be constructed explicitly.

Introduction

Nest algebras of operators on Hilbert space were introduced in 1965 by Ringrose [4] and generalise in a certain way the set of all $n \times n$ matrices to infinite dimensions. Christensen [1] proved every derivation on a nest algebra to be continuous and implemented and Lance [3] gave another proof of implementation as part of a wider cohomology result.

This paper accomplishes two tasks. The first is to generalise the notion of nest algebras to sets of operators on Banach space, showing the definition to be satisfactory by establishing fundamental properties and giving examples. The second is to prove that every continuous derivation on such a nest algebra is implemented by a bounded operator. The proof constructs this operator explicitly and hence yields a new and slightly more informative proof for nest algebras on Hilbert space. (Note that both [1] and [3] rely on strict Hilbert space techniques and hence cannot be generalised directly.)

Preliminary notation and definitions

Throughout, H denotes a complex Hilbert space, E a complex Banach space and $B(H)$, $B(E)$ the sets of bounded operators on H, E, respectively. By "projection" we mean simply an idempotent. In the context of a Hilbert space

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self-adjoint projections will be referred to as such. $P(E)$ denotes the set of projections in $B(E)$. If p is a projection, then $r(p)$, $n(p)$ refer to its range and null-space respectively, and $p^{\perp}$ to the projection $1-p$. For convenience a projection is sometimes identified with its range, and so we may write $\xi \in p$ to mean $\xi \in r(p)$. If the vectors ξ, η are in H, then $\xi \otimes \eta$ denotes the rank 1 operator defined by $(\xi \otimes \eta)\zeta = \langle \zeta, \xi \rangle \eta$ for all $\zeta \in H$.

Recall that a *nest* on H is a totally-ordered strongly closed set L of self-adjoint projections on H containing 0 and 1. A *nest algebra* N on H is then a set $\mathrm{alg}(L)$ of all operators in $B(H)$ leaving invariant every element of a nest L. We may thus think of N as a flow on L in the direction in which L decreases. A *derivation* on an algebra A in $B(E)$ is a linear map δ from A into $B(E)$ such that $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in A$. δ is *implemented* if it is of the form $\delta(a) = ba - ab$ for all a in A, for some $b \in B(E)$. We denote by d_b the derivation thus defined by b.

Many properties of self-adjoint Hilbert space projections are inapplicable to projections in general. In particular, although the self-adjoint projections in $B(H)$ are exactly those of norm 1, there is no duality in general between closed subspaces of a Banach space and projections of norm 1. (However, see Remark 1 below.) Furthermore, for p, q in $P(E)$, it is not necessarily the case that $r(p) \subset r(q) \Rightarrow n(q) \subset n(p)$. However, $P(E)$ can be partially ordered (see Dunford and Schwartz [2]) as follows: Let p, q be in $P(E)$. Write $p \leqq q$ if $pq = qp = p$. Hence, if $p \leqq q$, then p and q commute, $q - p$ is a projection, and we have both that $r(q)$ contains $r(p)$ and $n(p)$ contains $n(q)$. We can now make the main definitions.

Definition 1. A *nest* on E is a totally-ordered uniformly-bounded set L of projections in $P(E)$ containing 0 and 1, such that L is complete as an abstract lattice, and for any subset S of L

$$\left(\bigvee_{p \in S} \{p\} \right) E = \bigvee_{p \in S} \{pE\} \quad \text{and} \quad \left(\bigwedge_{p \in S} \{p\} \right) E = \bigwedge_{p \in S} \{pE\}.$$

This is analogous to the concept of completeness applied by Dunford and Schwartz [2; Ch. XVII] to Boolean algebras of projections. By [2] we could have replaced the boundedness condition by the stronger requirement that L belongs to a Boolean algebra which is complete as an abstract lattice. The following can be stated without proof:

Lemma. *L is strongly closed and, in particular, for any directed family $\{p_\alpha\}$ in L the projection $\bigvee \{p_\alpha\}$ is the strong limit of $\{p_\alpha\}$ and is consequently in L.*

DEFINITION 2. A set N of operators in $B(E)$ is a *nest algebra* if $N = \text{alg}(L)$ for some nest L on E.

Henceforth, unless otherwise stated, the context of the work will be E and the terms nest and nest algebra used in the generalised sense.

If p is in a nest L, define p^- by $p^- = \vee \{q \in L : q < p\}$. Then p^- is in L. We can define p^+ analogously.

L can be considered discrete or continuous (from below) at p according to whether or not p^- is distinct from p.

REMARKS.

(1) It might be thought that when E is a Hilbert space, our notion of a nest algebra N is more general than the usual one since L can contain non-self-adjoint projections. However, N is defined by its invariant subspaces, and these form a nest in the usual sense. In fact, every projection p, whose range coincides with that of a projection in L, is in N and is invariant under N.

(2) The following example illustrates the need for our concept of completeness: Consider the Banach space l^∞ of bounded sequences of complex numbers, and let L be the set $\{p_n\} \cup 1$, where p_n is the projection onto the first n coordinates. Then $\vee \{p_n\} = 1$. However, if ξ is the sequence with 1 in each place, then $\|(1 - p_n)\xi\| = 1$ for all n and so p_n does not tend strongly to 1. This situation arises because the subspace $\vee \{r(p_n)\}$ is the set of sequences tending to zero. This has no complement and so is the range of no projection.

Basic properties and examples

All the fundamental properties which might be expected are present. For the Hilbert space versions of several of the following results, see Ringrose [4] or for Corollary 7, Christensen [1]. (The proofs are not essentially very different, but we are concerned here to establish the existence of certain operators, in addition to their membership of a given nest algebra and we cannot, of course, use adjoints.) Whenever a rank 1 operator a is mentioned below with vectors η in its range and ξ such that $a\xi = \eta$, we have in mind the operator $\xi \otimes \eta$ for the case in which E is a Hilbert space. For example, the linear functional f in Lemma 4 below would then be given by $f(\cdot) = \langle \cdot, \eta \rangle$.

Let L be nest, bounded by the real number k, and let $N = \text{alg}(L)$.

LEMMA 3. *If p is a non-zero projection in L, then*

$$pB(E)p^{-\perp} \subset N.$$

For all p in L, $p^{\perp}Np = \{0\}$.

PROOF. (Exactly as in [4] for Hilbert space.)
Let q be in L, and a in $B(E)$.
If $q \leqq p^{-}$, then $pap^{-\perp}q = 0 = qpap^{-\perp}q$.
If $p \leqq q$, then $p = qp$ and so

$$pap^{-\perp}q = qpap^{-\perp}q.$$

Hence $pap^{-\perp}$ is in N.

LEMMA 4. *Let p_α, p_β be non-zero projections in L such that $p_\alpha^- < p_\beta$. Let $\xi \in p_\alpha$, $\eta \in p_\beta - p_\alpha^-$ be unit vectors. Then there is a rank 1 operator a in N such that $a\eta = \xi$, $ap_\alpha^- = ap_\beta^\perp = 0$, and $\| a \| \leqq 2k$. In particular, if $p_\alpha^- < p_\alpha$, then we may take $p_\alpha = p_\beta$ and choose ξ, $\eta \in p_\alpha - p_\alpha^-$.*

PROOF. By the Hahn–Banach Theorem there is $g \in E^*$ such that $g(\eta) = 1$ and $\| g \| = 1$. Define f by

$$f = g \circ (p_\beta - p_\alpha^-)$$

so that $f(\zeta) = g((p_\beta - p_\alpha^-)\zeta)$ for all $\zeta \in E$. Then, clearly, $f \in E^*$ and

$$\| f \| \leqq \| p_\beta - p_\alpha^- \| \leqq 2k.$$

Define a by $a\zeta = f(\zeta)\xi$ for all $\zeta \in E$. Then a is a bounded linear operator of rank 1, $\| a \| = \| f \| \leqq 2k$, and $a\eta = \xi$. Certainly $ap_\alpha^- = ap_\beta^\perp = 0$ since

$$(p_\beta - p_\alpha^-)p_\alpha^- = (p_\beta - p_\alpha^-)p_\beta^\perp = 0.$$

Furthermore, for any $p \in L$,

$$p \leqq p_\alpha^- \Rightarrow ap = ap_\alpha^- p = 0 = pap$$

and

$$p_\alpha \leqq p \Rightarrow p\xi = \xi \Rightarrow pap = ap.$$

Hence $a \in N$.

LEMMA 5. *If p_α, $p_\beta \in L$ such that $0 < p_\alpha < p_\beta$ and ξ, η are unit vectors in p_α, $p_\beta - p_\alpha$ respectively, then there is a rank 1 operator $a \in N$ such that $a\eta = \xi$, $ap_\alpha = ap_\beta^\perp = 0$ and $\| a \| \leqq 2k$.*

PROOF. As for Lemma 4.

COROLLARY 6. *If p is a non-zero projection in L and η is a non-zero vector in $p^{-\perp}$, then $r(p) \subset N\eta$. In particular, if $1^- < 1$, then $N\xi = E$ for all $\xi \in 1^{-\perp}$.*

PROOF. Put $p_\alpha = p$, $p_\beta = 1$ in Lemma 4.

COROLLARY 7. *The commutant N' of N is trivial.*

PROOF. Suppose first that $1^- < 1$. Let $\xi \in E$, $c \in N'$. Choose $\eta \in 1^{-\perp}$ and a rank 1 operator $a \in N$ such that $a\eta = \xi$. Then

$$c\xi = ca\eta = ac\eta$$

the last term being a scalar multiple of ξ. Since this holds for all $\xi \in E$, c must lie in $\mathbf{C}$. Suppose now that $1^- = 1$. Let p be in L, $0 < p < 1$. Choose $\xi \in p$, $\eta \in p^{-\perp}$ and a rank 1 operator $a \in N$ such that $a\eta = \xi$. Again, if $c \in N'$, then $c\xi$ is a scalar multiple of ξ. So cp is a multiple of p. However, the set $\{p \in L : p < 1\}$ tends strongly to 1; so, since $cp \in \mathbf{C}p$ for all $p < 1$, $c \in \mathbf{C}$.

EXAMPLES. Let L^p denote the set of (equivalence classes of) Lebesgue measurable complex-valued functions f on the interval $(0,1)$ such that $|f|^p$ is integrable, where p is any real number such that $p \geqq 1$. Let L^∞ denote the set of essentially bounded functions in L^p considered as the multiplication operators in $B(L^p)$. Consider the set $P = \{\chi_{(0,\lambda)} : \lambda \in (0,1)\}$ of characteristic functions in L^∞. It is easy to see that P is a nest, and so we may take as a nest algebra $N = \operatorname{alg}(P)$. N then contains L^∞ and the left-shift operators. We could, alternatively, choose as a nest any of the numerous sub-nests of P, giving rise to a nest N_1 containing N. For example, the sets

$$P_n = \{\chi_{(0,r/n)} \in P : n \geqq 1,\ r = 0, \dots, n\}$$

are nests and are in the obvious sense discrete, whereas P is everywhere continuous. P is clearly a maximal nest.

Incidentally, the trivial nest $\{0,1\}$ gives rise, of course, to the nest algebra $B(E)$.

The derivation result

We can now prove the main theorem. Let L, again, be a nest on E bounded by k and let $N = \operatorname{alg}(L)$. Let $\delta : N \to B(E)$ be a continuous derivation.

THEOREM 8. *There exists an operator b in $B(E)$ such that $\delta = d_b \mid N$, and $\| b \| \leqq 2(1+4k)k^2 \| \delta \|$.*

PROOF. Take the two cases, in which L is discrete at 1 and continuous at 1, separately.

Case 1. Suppose $1^- < 1$. Choose any unit vector $\xi \in 1^{-\perp}$. Then by Corollary

6, $N\xi = E$ and in particular Lemma 4 gives us a rank 1 operator p_1 such that $p_1\xi = \xi$. Define a map b on E by

$$bab\xi = \delta(ap_1)\xi \qquad \text{for all } a \in N.$$

b is well defined since $a\xi = 0 \Rightarrow ap_1 = 0$, and b is clearly linear and defined on the whole of E. It is easy to check, referring to Lemma 4, that $\| ap_1 \| \leqq 2k \| a\xi \|$. So b is bounded and $\| b \| \leqq 2k \| \delta \|$ because $\| ba\xi \| = \| \delta(ap_1)\xi \| \leqq \| \delta \| \| ap_1 \|$. Furthermore, $\delta = d_b \mid N$ since, for a, c in N,

$$(ba - ab)c\xi = \delta(acp_1)\xi - a\delta(cp_1)\xi = \delta(a)c\xi.$$

Case 2. Suppose $1^- = 1$. Then, if we write L as the directed set $\{p_\alpha\}$, $\{p_\alpha : p_\alpha < 1\}$ tends strongly to 1. The construction will proceed as follows: for each p_α such that $p_\alpha < 1$, we will define an operator b_α such that $\delta(a)p_\alpha = d_{b_\alpha}(a)p_\alpha$ for all $a \in N$ and in such a way that $\{b_\alpha\}$ has a strong limit.

For each $p_\alpha < 1$, choose p_{α_1}, p_{α_2} in L such that $p_\alpha < p_{\alpha_1} < p_{\alpha_2} < 1$, and unit vectors $\xi_{\alpha_1} \in p_{\alpha_1} - p_\alpha$, $\xi_{\alpha_2} \in p_{\alpha_2} - p_{\alpha_1}$. Now choose a rank 1 operator q_α in N as in Lemma 4 so that $q_\alpha \xi_{\alpha_2} = \xi_{\alpha_1}$ and $\| q_\alpha \| \leqq 2k$.

By Corollary 6, $N\xi_{\alpha_1} \supset r(p_\alpha)$, so we can define b_α on E by

$$b_\alpha a \xi_{\alpha_1} = b_\alpha a q_\alpha \xi_{\alpha_2} = \delta(aq_\alpha)\xi_{\alpha_2} \qquad \text{for all } a \in N$$

such that $a\xi_{\alpha_1} \in p_\alpha$, and

$$b_\alpha p_\alpha^\perp = 0.$$

b_α is well defined since $aq_\alpha = 0$ whenever $a\xi_{\alpha_1} = 0$. b_α is bounded:

$$\| b_\alpha a \xi_{\alpha_1} \| \leqq \| \delta \| \| aq_\alpha \| \leqq 2k \| \delta \| \| a\xi_{\alpha_1} \|$$

so $\| b_\alpha \mid p_\alpha \| \leqq 2k \| \delta \|$, and therefore

$$\| b_\alpha \| = \| (b_\alpha \mid p_\alpha) p_\alpha \| \leqq 2k \| p_\alpha \| \| \delta \| \leqq 2k^2 \| \delta \|.$$

Furthermore, for a, $c \in N$, where $c\xi_{\alpha_1} \in p_\alpha$,

$$(b_\alpha a - ab_\alpha)c\xi_{\alpha_1} = \delta(acq_\alpha)\xi_{\alpha_2} - a\delta(cq_\alpha)\xi_{\alpha_2} = \delta(a)c\xi_{\alpha_1}$$

and so $\delta(a)p_\alpha = d_{b_\alpha}(a)p_\alpha$ for all $a \in N$.

We would expect, for $p_\alpha < p_\beta$, that $(b_\alpha - b_\beta)p_\alpha \in \mathbf{C}p_\alpha$, and we can show this directly: Let p_α, p_β be in L, with $p_{\alpha_2} < p_\beta < 1$. Choose a rank 1 operator q in N according to Lemma 5 so that $q\xi_{\beta_1} = \xi_{\alpha_2}$, and $\| q \| \leqq 2k$. Comparing b_α, b_β on p_α, we have, for $a \in N$ such that $a\xi_{\alpha_2} \in p_\alpha$:

$$b_\alpha a\xi_{\alpha_1} = \delta(aq_\alpha)\xi_{\alpha_2} = \delta(aq_\alpha)qq_\beta\xi_{\beta_2}$$

and

$$b_\beta a\xi_{\alpha_1} = b_\beta aq_\alpha qq_\beta\xi_{\beta_2} = \delta(aq_\alpha qq_\beta)\xi_{\beta_2}.$$

Therefore,

$$(b_\beta - b_\alpha)a\xi_{\alpha_1} = aq_\alpha\delta(qq_\beta)\xi_{\beta_2} = \mu a\xi_{\alpha_1},$$

say, where μ is the complex number such that

$$\mu\xi_{\alpha_1} = q_\alpha\delta(qq_\beta)\xi_{\beta_2}.$$

Note that

$$|\mu| = |f(\delta(qq_\beta)\xi_{\beta_2})| \leqq \|f\| \, \|\delta\| \, \|q\| \, \|q_\beta\| \leqq (2k)^3 \|\delta\|,$$

where $f \in E^*$ is defined for q_α as in Lemma 5. This shows what we want for p_α and p_β when $p_{\alpha_2} < p_\beta$. The same thing thus applies even if $p_{\alpha_2} \geqq p_\beta$, since there exists p_γ, say, such that $p_\gamma \geqq \vee\{p_{\alpha_2}, p_{\beta_2}\}$ and both $(b_\alpha - b_\gamma)p_\alpha$ and $(b_\beta - b_\gamma)p_\alpha$ are multiples of p_α.

Now fix p_1, say, in L, $0 < p_1 < 1$, and construct b_1 as above. By the argument above we can suppose, for each $p_\alpha \geqq p_1$, that $(b_\alpha - b_1)p_1 = 0$, by replacing each b_α with $b_\alpha - \mu_\alpha$, where $\mu_\alpha p_1 = (b_\alpha - b_1)p_1$. The bound for the norm of each b_α is then given by

$$\|b_\alpha\| \leqq 2(1+4k)k^2\|\delta\|.$$

So we now have an "increasing" bounded set $\{b_\alpha\}$ of which each b_α implements δ on p_α. There is clearly a strong limit b in $B(E)$ with $\|b\| \leqq 2(1+4k)k^2\|\delta\|$, and $bp_\alpha = b_\alpha p_\alpha$ for all $p_\alpha < 1$. Hence b implements δ on N.

REMARKS.

(1) Let E be a Hilbert space. Then the rank 1 operators referred to in the above proof all have norm 1. The bound for $\|b\|$ then reduces to $2\|\delta\|$. In fact, instead of adjusting each b_α, we could have appealed, in this case, to weak compactness, and this would yield a bound of $\|\delta\|$ for the norm of b. This is essentially Lance's proof [3].

(2) The existing proofs, [1] and [3], of the Hilbert space version of this result also, effectively, consider the continuous and discrete cases separately. As long as L is a non-trivial Hilbert space nest a construction can be made which avoids this [5], but the process is then far more complicated.

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