Wave propagation in finite periodically ribbed structures with fluid loading

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[Two plates]

The paper studies the problem of wave transmission along a fluid-loaded plane elastic membrane supported by a finite array of equally spaced ribs. One of the ribs is driven by a time-harmonic line force and the rest have infinite impedance, so that fluid loading provides the only mechanism for the transmission of energy. Existing solutions for the infinite analogue exhibit a stop/pass band frequency structure, in which the energy is, alternately, exponentially localized around the driving force and constant along the array. However, at pass band frequencies this is inconsistent with numerical studies of finite arrays, which reveal marked amplitude fluctuations. In this paper an exact solution is given for a general finite configuration. This is used to explain and further explore the response. In particular it is shown that as the array length increases the pass band response becomes increasingly sensitive to frequency, and the solution cannot approach an asymptotic limit.

The results give the forces along the array as an interference pattern, which may be thought of as propagating inwards from each end. This solution is obtained by forming a 2×2 matrix which relates the forces at any pair of adjacent ribs to those at the next pair. From the action of this matrix the response can be found everywhere, and the detailed properties of the solution are determined by those of the matrix. Special treatment is needed to deal with the band edges, which conform neither to stop nor pass band behaviour.

1. Introduction

This paper considers the problem of energy transmission in a configuration consisting of a plane elastic membrane supported by a finite array of evenly spaced ribs, one driven by a time-harmonic line force and the rest having infinite mechanical impedance. The only coupling between adjacent bays is provided by fluid loading, and is determined by the properties of a Green function, i.e. the velocity response of an infinite unribbed fluid-loaded membrane to concentrated forcing. The Green function, studied extensively by Crighton (1983, 1984), consists of an acoustic component G_a and a subsonic surface wave component G_s; attention is restricted here to regimes in which G_a can be neglected. Inclusion of the acoustic component or any degree of irregularity in the rib spacing leads to qualitatively different behaviour, at any rate over sufficiently large distances. However, if the acoustic component or the degree of irregularity are appropriately small, results based on the inclusion of only the surface wave component will hold over extensive distances of interest in many applications in aeronautical and marine engineering. Solutions for the infinite analogues have been obtained by Crighton (1984) in the case of a doubly infinite

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array driven at the centre, and by Sobnack (1991) for the semi-infinite case in which the first rib is driven. (This and related problems have been widely studied (see, for example, Woolley 1980; Mace 1980; Crighton & Maidanik 1981; Maidanik & Dickey 1988; Crighton 1989; Mead 1990).) These solutions are expressed in terms of the forces produced at each rib. The results reveal that the frequency range falls into stop bands, in which the response is exponentially localized around the driven rib, and pass bands, in which energy propagates to infinity unattenuated with constant amplitude.

The response of finite structures falls similarly into pass and stop bands. However, while the behaviour in the stop bands corresponds to the infinite analogue, most pass band frequencies exhibit large fluctuations along the array, and at certain frequencies the solutions are stable but distinct according to whether the number of ribs is odd or even (Sobnack 1991). Such features persist as the number of ribs is increased, and thus in general the finite-array solution does not approach a well-defined limit at infinity. The object of this paper is to give the full solution for a general finite configuration and to explain the observed features; in particular we consider the overall behaviour as the array length is increased and the frequency changes, and examine carefully the solution for the band-edges.

The solution may be simply formulated. Suppose the system consists of N ribs, driven at the Mth. The problem is equivalent to solving a system of N linear equations to find the vector $\boldsymbol{F}$ representing the forces at each rib. This system can be arranged (except in the vicinity of the Mth equation) so that any adjacent pair of values F_m, F_{m+1} can be expressed in terms of the next pair F_{m+2}, F_{m+3}. We thus obtain a 2×2 'propagation matrix' P which expresses this relationship and is independent of m. (Such an approach is not new to problems of this type (Hood 1985).) In the stop bands the eigenvalues of this matrix are real and determine the decay rate. In the pass bands they lie on the complex unit circle, and result in unattenuated wave propagation. To explain the amplitude variations, however, one must examine the action of P more closely.

It is useful to think in terms of the waves to either side of the driven rib normalized by the respective end-values F_1 and F_N. In effect these normalized waves propagate inwards, until they encounter the driven rib, at which point the normalizing factors to the left and the right are determined. Thus for a given frequency the pattern of forces evolving from an end-rib is the same (to within a complex multiple) for any length of array and wherever the driven rib is located. The detailed variations along the array in the pass bands can then be characterized as interference between two waves, each due to the propagation of one of the eigenvectors of P. Each of these waves turns out to have constant amplitude, so that the solution at each pass frequency is a simple interference pattern.

From this model a broader picture emerges. The response of the arrays in the pass bands becomes increasingly sensitive to the driving frequency with distance from the ends of the array. This is because the waves at neighbouring frequencies propagate at different speeds, governed by the eigenvalues. It is also found that the amplitude is constant along the array only at the frequencies at the upper edges and middle of the pass bands. At the lower pass band edges it is found that the amplitude decays linearly away from the driven rib, and it will be shown that none of the band edges strictly conforms either to stop band or pass band behaviour. It is further shown that the forces at frequencies close to the lower band edges reach maximum amplitudes which increase unboundedly with array length.

It is feasible to extend this approach to treat more general Green functions. A more important aspect is that it opens the way to an analytical treatment of the localization which is observed in randomly perturbed systems (Sobnack 1991). It is hoped to treat this in another paper. An incidental consequence of the model which should also be mentioned is that it provides the solution for arbitrarily large systems, avoiding the highly inaccurate inversion of large matrices.

The paper is organized as follows: in §2 the governing equations are set out. The general solution is formulated in §3, after derivation of the propagation matrix and its properties. Most of the results of this section apply to any Green function of exponential form. Simple explicit solutions are found in §4 for the special cases which arise at the boundaries between bands and in the middle of the pass bands. The solution in the stop bands reduces to a simple form because for large arrays one of the eigenvalues dominates, and this solution is given in §5. The section also considers further the behaviour of the solution as the frequency changes and as the array length increases. Upper and lower bounds are found for the fluctuating amplitudes at each pass band frequency, and the most relevant quantities which arise in the solution are illustrated. The structural response of the membrane beyond the region supported by ribs is also given.

2. Mathematical formulation

The physical model and the Green function and its properties have been fully described elsewhere (Crighton 1983, 1984; Sobnack 1991) and details will not be repeated at length here. It is assumed that an elastic membrane lies in the plane $y = 0$, with a static compressible fluid in the half-space $y > 0$ and a vacuum in $y < 0$. The membrane is supported by N thin ribs along the lines $x_m = mh$ where $m = 1, \ldots, N$. The rib at x_M for some $M < N$ is driven in the y-direction by a time-harmonic line-force, and all other ribs have infinite mechanical impedance. Fluid loading is therefore the only mechanism by which energy can be transmitted away from the vicinity of the driven rib. A harmonic time dependence $\exp(-\mathrm{i}\omega t)$ is to be understood throughout. The response of the membrane is characterized by the velocity $V(x)$, and the velocity at the mth rib is denoted V_m, so that $V_m = 0$ for $m \neq M$. The force on the membrane due to the mth rib is denoted F_m and the Green function of the fluid-loaded membrane is G. Then $G = G_\mathrm{a} + G_\mathrm{s}$ where G_a and G_s denote the acoustic and subsonic surface wave components respectively. If the ribs are widely spaced in an appropriate sense, then $G(x_n - x_m)$ can be approximated by G_s for $n \neq m$, and this assumption is made here. The main problem with which this paper is concerned is the solution for the forces F_m.

It will be assumed that the velocity V_M at the driven rib is known, and the force F_M required to produce this will be found as part of the solution. (The solution of the related problem in which the force is prescribed is obtained simply by renormalizing throughout by F_M.) The structural response of the membrane is given by

$$V(x) = \sum_{n=1}^{N} F_n G(x - x_n), \tag{1}$$

and the forces are related to the velocities by

$$V_m = \sum_{n=1}^{N} F_n G(x_m - x_n). \tag{2}$$

Under the assumption that $G_a(x)$ can be neglected except for small arguments the Green function has the form $G(x) = A_\infty \exp(i\kappa|x|)$ for $x \neq 0$, and $G(0) = A_0$. The problem may be scaled in such a way that the velocity at x_M is unity, the factors A_∞, A_0 simplify, and the rib-spacing h is subsumed into the scaled driving frequency ϕ. This gives rise to the matrix equation

$$\boldsymbol{W} = A\boldsymbol{F}, \tag{3}$$

where $\boldsymbol{W}$ is the vector of scaled velocities, which takes the value 1 at the M-th place and zero elsewhere, $\boldsymbol{F} = \{F_m\}$ is the vector of forces, with $m = 1, \ldots, N$, and A is the symmetric $N \times N$ matrix given by

$$A_{j,k} = \exp(i\phi|j-k|), \quad \text{for} \quad j \neq k,$$

with diagonal

$$A_{k,k} = 1 - i/\sqrt{3}.$$

For the specific form of this diagonal it is assumed that the fluid and membrane parameters correspond to the low-frequency range of heavy fluid loading. At higher frequencies a more complex form arises; for such frequencies it would be straightforward to derive results similar in structure to those in this paper, although more complicated in detail.

The scaled frequency ϕ may be assumed to lie in $(0, 2\pi]$. By convention x is taken to increase from left to right. It will be assumed that the driven rib is not at the right-hand end. Define $z = \exp(i\phi)$, and $\beta = A_{k,k} = 1 - i/\sqrt{3}$. This matrix equation represents a system of N linear equations which will be labelled [1] to [N], so that equation [m], say, where $m \neq M$, is

$$0 = z^{m-1}F_1 + \ldots + zF_{m-1} + \beta F_m + zF_{m+1} + \ldots + z^{N-m}F_N \tag{$[m]$}$$

and equation [M] is similar but with 0 on the left replaced by 1.

Since the solution is first to be obtained for the forces normalized by the values at the end-ribs, we define $E_m = F_m/F_N$ for $m \geqslant M$, and similarly $E_m = F_m/F_1$ for $m < M$. (Note that we have chosen to normalize the force at the driven rib by the factor to the right.) It will be shown that, provided $m < M < N-m+1$, $E_m = E_{N-m+1}$.

3. General solution

In this section the expressions which describe the transmission along the structure are derived, and these will later be used to explore in detail the resulting pattern of waves. The first step is to rearrange the equations in §2 so that the force at any rib is expressed completely in terms of those at neighbouring ribs. This is possible because of the form of the underlying Green function, and is the key property. This leads to the definition of the constant 2×2 matrix P, which gives the forces at any pair of adjacent ribs in terms of the next two and whose properties thus govern the nature of the solution. The next step is therefore to identify these properties, by finding the eigenvectors and eigenvalues of P as functions of frequency. When this has been done the forces E_m normalized by the end-values are obtained in terms of powers of the eigenvalues, and this finally allows the normalizing factors to be derived in closed form. Other features such as the maximum and minimum amplitudes along the array will be discussed later.

The derivations are straightforward and details will be kept brief. (There are necessarily a large number of equations: for each of the several cases the forces at the end-ribs and driven rib must be obtained, and these further depend on the location of the driven rib.) Further discussion and illustration of most of the quantities will be given in §5. Unless stated otherwise the expressions obtained apply to the right of the driven rib, that is for F_m where $m > M$, and extend in the obvious way to the left. The word 'amplitude' is used throughout to describe the modulus of the force.

(*a*) *Derivation of propagation matrix*

Consider any equations $[m]$, $[m+1]$ and $[m+2]$ above, where $M < m$. It is convenient to write these in vector form, so that, for example, equation $[m]$ is

$$0 = (z^{m-1}, z^{m-2} \dots z, \beta, z \dots z^{N-m}) \cdot \boldsymbol{F},$$

where β appears in the mth place.

Multiply equation $[m]$ by z and z^2 in turn, where $z = \exp(\mathrm{i}\phi)$, to get

$$0 = (z^m, z^{m-1} \dots z^2, \beta, z, z^2, z^3, z^4 \dots z^{N-m+1}) \cdot \boldsymbol{F}, \tag{4}$$

$$0 = (z^{m+1}, z^m \dots z^3, \beta, z^2, z^3, z^4, z^5 \dots z^{N-m+2}) \cdot \boldsymbol{F}. \tag{5}$$

Subtract (4) from equation $[m+1]$, and (5) from equation $[m+2]$ to give, respectively,

$$0 = (0, 0 \dots 0, z(1-\beta), \beta - z^2, z(1-z^2), z^2(1-z^2) \dots z^{N-m-1}(1-z^2)) \cdot \boldsymbol{F} \tag{6}$$

and

$$0 = (0, 0 \dots 0, z^2 - \beta z^2, z - z^3, \beta - z^4, z - z^5 \dots z^{N-m-2} - z^{N-m+2}) \cdot \boldsymbol{F}. \tag{7}$$

Multiplying (7) by z we can write

$$0 = (0, 0 \dots 0, z^3(1-\beta), z^2(1-z^2), z(\beta - z^4), z^2(1-z^4) \dots z^{N-m-1}(1-z^4)) \cdot \boldsymbol{F} \tag{8}$$

If we put $\gamma = (1-z^4)/(1-z^2) \equiv 1 + z^2$ and multiply (6) by γ we get

$$0 = (0, 0 \dots 0, z(1-\beta)\gamma, (\beta - z^2)\gamma, z(1-z^2)\gamma, z^2(1-z^4) \dots z^{N-m-1}(1-z^4)) \cdot \boldsymbol{F}. \tag{9}$$

Finally we can subtract (9) from (8) and revert to standard notation to get

$$[z^3(1-\beta) - z(1-\beta)\gamma] F_m + [z^2(1-z^2) - (\beta - z^2)\gamma] F_{m+1} + [z(\beta - z^4) - z(1-z^2)\gamma] F_{m+2} = 0. \tag{10}$$

This can be rearranged to give

$$\alpha_2 F_m + \alpha F_{m+1} + \alpha_2 F_{m+2} = 0, \tag{11}$$

where $\alpha = (2-\beta) z^2 - \beta$ and $\alpha_2 = z(\beta - 1)$.

Although this already gives F_m in terms of the values at two adjacent ribs, it is more convenient to express the solution in the form of a propagation matrix. From (11) we have

$$F_m = -F_{m+2} - (\alpha/\alpha_2) F_{m+1}$$

and, provided $m - 1 \neq M$,

$$F_{m-1} = (\alpha/\alpha_2) F_{m+2} + [(\alpha/\alpha_2)^2 - 1] F_{m+1}.$$

If we define $\alpha/\alpha_2 = \omega$ and write the real and imaginary components of ω explicitly, we find that ω is real for all ϕ, and

$$\omega = -2[\cos\phi + \surd 3 \sin\phi] = -4\cos(\phi - \tfrac{1}{3}\pi), \tag{12}$$

so that ω takes values in the range $[-4, 4]$.

The propagation matrix equation is then given by

$$(F_{m-1}, F_m) = P(F_{m+1}, F_{m+2}), \tag{13}$$

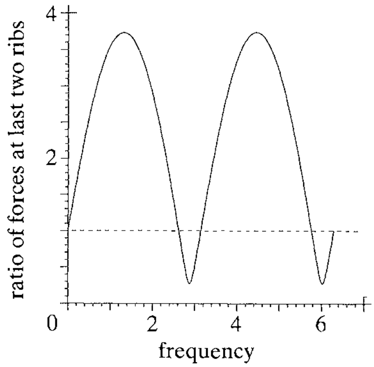

Figure 1. The modulus of the ratio ν (equation (17)) between the forces on the last two ribs as a function of frequency. The frequencies at which ν becomes unity are those for which the amplitudes of the forces are constant along the array.

where P is the matrix

$$P = \begin{pmatrix} \omega^2 - 1 & \omega \\ -\omega & -1 \end{pmatrix}. \tag{14}$$

This holds provided the driven rib is not located between $m-1$ and $m+2$.

The same propagation matrix holds for the incoming wave to the left of the driven rib, so that for $M > m+2$

$$(F_{m+1}, F_{m+2}) = P(F_{m-1}, F_m). \tag{15}$$

The choice of the inward-propagating matrix P rather than the outward-going P^{-1} will shortly become clear; it enables us to specify an 'initial condition' for the wave in the form of the two end values (F_{N-1}, F_N), to within the scalar multiple F_N. (Note that the results which follow can also be derived via the 'square root' of P whose action is $(F_m, F_{m+1}) = P^{\frac{1}{2}}(F_{m+1}, F_{m+2})$ for $M < m-1$.)

(*b*) *Initial condition for the propagation matrix*

The force F_{N-1} can be found in terms of F_N immediately. Multiply equation $[N-1]$ by z and subtract from equation $[N]$ to obtain

$$F_{N-1} = \nu F_N, \tag{16}$$

where

$$\nu = (z^2 - \beta)/z(1-\beta).$$

Then $\nu = -(\omega + z)$, that is

$$\nu = \cos\phi + (2\sqrt{3} - \mathrm{i})\sin\phi. \tag{17}$$

Recalling that $E_m = F_m/F_N$ for all $m > M$, the initial vector for the propagation matrix is

$$(F_{N-1}, F_N) = F_N(E_{N-1}, E_N) = F_N(\nu, 1).$$

Similar equations hold to the left of the driven rib, and in particular $F_2 = \nu F_1$. Note that the forces can only have constant amplitude when $|F_{N-1}| = |F_N|$, i.e. $|\nu| = 1$, and this only happens at the four frequencies $\frac{5}{6}\pi$, π, $\frac{11}{6}\pi$, and 2π (see figure 1).

The behaviour of P must now be characterized in terms of the driving frequency, by identifying its eigenvalues and eigenvectors. Since $(F_{N-2m-1}, F_{N-2m}) = P^m(F_{N-1}, F_N)$ for all m up to the driven rib, it is clear that the eigenvalues largely determine the long-range evolution of the wave. The eigenvectors are needed explicitly because they, and their components which comprise the initial vector (F_{N-1}, F_N), give the detailed solution.

To find these quantities, consider the equation $P(x,y) = \lambda(x,y)$. It is easy to show that the component x can never vanish except when $\omega = 0$; then $P = -1$ and no inconsistency arises by choosing $x \neq 0$. Thus in general we can choose $x = 1$, and simply solve $P(1,y) = \lambda(1,y)$ to give eigenvalues λ_i and representative vectors $(1, y_i)$ in the corresponding eigenspaces. Special cases occur when $\omega = \frac{2}{3}\pi$, π, $\frac{5}{3}\pi$ and 2π, and these will be treated separately.

(c) *Eigenvalues of P*

The determinant of P is 1 so its eigenvalues have product unity. These eigenvalues are given by

$$\lambda_i = \tfrac{1}{2}[-2+\omega^2 \pm \omega\sqrt{(\omega^2-4)}] \tag{18}$$

for $i = 1,2$, where $i = 1$ denotes the one with the positive sign before the square root.

Since ω is real, the eigenvalues are real when $\omega^2 \geqslant 4$. Otherwise the term $\sqrt{(\omega^2-4)}$ is imaginary, and the eigenvalues are complex conjugates with modulus 1. The condition $\omega^2 > 4$ is equivalent to $\phi \in (0, \frac{2}{3}\pi)$ or $\phi \in (\pi, \frac{5}{3}\pi)$. The broad pass band/stop band structure is already apparent: when the eigenvalues are real the larger one λ_i dominates, and P^m acts like λ_i^m on the corresponding eigenvector, which is equivalent to exponential decay from the driven rib. When the values are complex they have modulus one and both eigenvectors propagate without attenuation over any distance. This will be quantified more precisely later, but we can henceforth refer to the frequency ranges $(0, \frac{2}{3}\pi)$, $(\pi, \frac{5}{3}\pi)$ as the stop bands, and $(\frac{2}{3}\pi, \pi)$, $(\frac{5}{3}\pi, 2\pi)$ as the pass bands.

There are six degenerate cases. At the four edge frequencies $\frac{2}{3}\pi$, π, $\frac{5}{3}\pi$, and 2π, $\omega^2 = 4$ and the eigenvalues coincide at the value 1. At the two middle pass band frequencies, $\phi = \frac{5}{6}\pi$ and $\frac{11}{6}\pi$, ω becomes zero and the eigenvalues both take the value -1. The edge frequencies must be treated separately, and except where stated otherwise the expressions derived below do not apply to them. The two middle pass band frequencies will also be dealt with explicitly, but are not excluded from what follows.

(d) *Eigenvectors of P*

Now consider the corresponding eigenspaces. As discussed earlier these can be represented by eigenvectors of the form $\boldsymbol{x}_i = (1, y_i)$, for $i = 1$, 2. The eigenvectors corresponding to the eigenvalues are then $(1, y_i)$ where

$$y_i = -\tfrac{1}{2}\omega \pm \tfrac{1}{2}\sqrt{(\omega^2-4)} \tag{19}$$

and again $i = 1$ corresponds to the positive sign. Then $y_1 y_2 = 1$, and in the pass bands $|y_i| = 1$, so that in that case the two components of each eigenvector $\boldsymbol{x}_i$ have equal amplitude.

These two vectors are not in general orthogonal. They coincide only at the four edge frequencies, when the eigenvalues themselves coincide.

The relationship between λ_i and y_i is particularly convenient. Taking the square of y_i immediately gives

$$y_1^2 = \lambda_2, \quad y_2^2 = \lambda_1 \tag{20}$$

so that $\lambda_i y_i^2 = 1$. (In fact the components y_i are the eigenvalues of the square root operator $P^{\frac{1}{2}}$ mentioned above.) This greatly simplifies the eventual solution.

(e) Projection of initial vector onto eigenvectors

Finally we want to decompose the end vector (F_{N-1}, F_N) in terms of the eigenvectors, since the action of P on these has been found explicitly.

Write $(E_{N-1}, E_N) = \mu_1 \boldsymbol{x}_1 + \mu_2 \boldsymbol{x}_2$. This gives a simple pair of linear equations

$$\mu_1 + \mu_2 = \nu, \quad \mu_1 y_1 + \mu_2 y_2 = 1. \tag{21}$$

Then provided the term $\surd(\omega^2 - 4)$ does not vanish (which again it does only at the band edges)

$$\mu_1 = \frac{1 - \nu y_2}{\surd(\omega^2-4)} = \frac{y_2(z-y_2)}{\surd(\omega^2-4)}, \quad \text{and} \quad \mu_2 = \frac{\nu y_1 - 1}{\surd(\omega^2-4)} = \frac{y_1(y_1-z)}{\surd(\omega^2-4)}. \tag{22}$$

Thus $(F_{N-1}, F_N) = F_N(\mu_1 \boldsymbol{x}_1 + \mu_2 \boldsymbol{x}_2)$.

(f) Solutions for the normalized waves

The first part of the solution can now be formulated, which describes how the vector of forces propagates inwards from the end ribs under the action of P. Now $(E_{N-2m-1}, E_{N-2m}) = P^m(E_{N-1}, E_N)$, so that from (22)

$$(E_{N-2m-1}, E_{N-2m}) = \lambda_1^m \mu_1 \boldsymbol{x}_1 + \lambda_2^m \mu_2 \boldsymbol{x}_2.$$

It is convenient temporarily to consider separately the waves on the odd- and even-numbered ribs, and these are immediately apparent:

$$E_{N-2m} = \lambda_1^m \mu_1 y_1 + \lambda_2^m \mu_2 y_2$$

for $N \geqslant N-2m > M$; and

$$E_{N-2m-1} = \lambda_1^m \mu_1 + \lambda_2^m \mu_2$$

for $N > N-2m-1 > M$. Although these represent two distinct waves their relationship is simplified by (20). Then, except at the four band-edges, the solution to the right of the driven rib normalized by the end force is

$$E_{N-m} = y_2^{m-1} \mu_1 + y_1^{m-1} \mu_2, \tag{23a}$$

where $M < N-m$. Within the pass bands this is a simple interference pattern, which will be explored further in §5.

The solution to the left of M (directly or by symmetry) is similar:

$$E_m = y_2^{m-2} \mu_1 + y_1^{m-2} \mu_2, \tag{23b}$$

for $1 \leqslant m < M$. This equation is enough to characterize most features of the solution which interest us, including phase and amplitude fluctuations in the pass bands and rates of exponential decay in the stop bands. This illustrated in §5 (see for example figures 3 and 4). The equation is independent both of the location of the driving force and of the array length. Thus the pattern of forces which evolves from an end-rib for any two configurations is the same except for a scalar factor.

(g) The force at the driven rib and the factors F_1, F_N

To complete the solution the force F_M at the driven rib and the multiplying factors F_1, F_N to each side of it must be determined. The following equations apply at all frequencies, except those expressions written in closed form, which exclude the band edges.

System driven at an end rib

Suppose first that the driving force is at the beginning of the array, so that $M = 1$. Then E_1 can be obtained from any of the equations [2] to [N]. From equation [2],

$$E_1 = -z^{-1} \sum_{i=0}^{N-2} z^i E_{i+2} + \tfrac{1}{\sqrt{3}} z^{-1} \mathrm{i} E_2. \tag{24}$$

Substituting the expression (23),

$$\begin{aligned} E_1 &= -z^{-1} \sum_{i=0}^{N-2} z^i (\mu_1 y_2^{N-i-3} + \mu_2 y_1^{N-i-3}) + \tfrac{1}{\sqrt{3}} z^{-1} \mathrm{i} E_2 \\ &= -z^{-1} y_2^{N-3} \mu_1 \sum_{i=0}^{N-2} \left(\frac{z}{y_2}\right)^i - z^{-1} y_1^{N-3} \mu_2 \sum_{i=0}^{N-2} \left(\frac{z}{y_1}\right)^i + \tfrac{1}{\sqrt{3}} z^{-1} \mathrm{i} E_2, \end{aligned}$$

in which the geometric series can be summed to obtain

$$E_1 = -z^{-1} \left[\frac{\mu_1 (y_2^{N-1} - z^{N-1})}{y_2 (y_2 - z)} + \frac{\mu_2 (y_1^{N-1} - z^{N-1})}{y_1 (y_1 - z)} \right] + \tfrac{1}{\sqrt{3}} z^{-1} \mathrm{i} E_2.$$

Using (22) this can be written

$$E_1 = -\frac{\beta}{z\sqrt{(\omega^2 - 4)}} (y_1^{N-1} - y_2^{N-1}) - \frac{\mathrm{i}}{\sqrt{3}\sqrt{(\omega^2 - 4)}} (y_1^{N-2} - y_2^{N-2}). \tag{25}$$

Although the denominator $\sqrt{(\omega^2 - 4)}$ vanishes at the band edges, it can be written $y_1 - y_2$ which is a factor of the numerator, and (25) remains bounded. The limiting values at the band edges are given in §4.

The normalizing factor F_N can finally be obtained by dividing equation [1] by F_N and taking the reciprocal of each side:

$$F_N = \left[\beta E_1 + \sum_{i=2}^{N} z^{i-1} E_i \right]^{-1}. \tag{26a}$$

The sum here can again be written in closed form:

$$\begin{aligned} \sum_{i=2}^{N} z^{i-1} E_i &= \sum_{i=0}^{N-2} z^{i+1} E_{i+2} \\ &= z \sum_{i=0}^{N-2} z^i (y_2^{N-i-3} \mu_1 + y_1^{N-i-3} \mu_2) \\ &= \frac{z\mu_1}{y_2} \left(\frac{y_2^{N-1} - z^{N-1}}{y_2 - z} \right) + \frac{z\mu_2}{y_1} \left(\frac{y_1^{N-1} - z^{N-1}}{y_1 - z} \right) \\ &= z(y_1^{N-1} - y_2^{N-1}) / \sqrt{(\omega^2 - 4)}. \end{aligned}$$

Combining this with (25) leads to

$$F_N = \left[\frac{z^2 - \beta^2}{z\sqrt{(\omega^2 - 4)}} (y_1^{N-1} - y_2^{N-1}) - \frac{\mathrm{i}\beta}{\sqrt{3}\sqrt{(\omega^2 - 4)}} (y_1^{N-2} - y_2^{N-2}) \right]^{-1}. \tag{26b}$$

From (25) and (26b) the force F_1 at the driven rib can now be written

$$F_1 = -\frac{\beta\sqrt{3}(y_1^{N-1}-y_2^{N-1})+\mathrm{i}z(y_1^{N-2}-y_2^{N-2})}{\sqrt{3}(z^2-\beta^2)(y_1^{N-1}-y_2^{N-1})-\mathrm{i}\beta z(y_1^{N-2}-y_2^{N-2})}. \tag{27}$$

Similarly the forces $F_m = E_m F_N$ are given for all m by (23a) and (26b).

System driven away from end rib

Now let $M \geqslant 2$. One quantity of interest is the magnitude of the ratio $R = F_1/F_N$. This can be found without first explicitly writing F_M, and the remaining forces can be written in terms of R. If we let $m = M$ we can rearrange equations $[M-1]$–$[M+1]$ as in the derivation of the matrix P (equations (4)–(12)), taking account of the non-zero left hand side of equation $[M]$. Multiplying equation $[M]$ by z and subtracting equation $[M+1]$ we obtain

$$z = z(\beta-1)F_M + (z^2-\beta)F_{M+1} + \sum_{j=2}^{N-M} z^{j-1}(z^2-1)F_{M+j}. \tag{28}$$

Similarly we can express F_M in terms of the forces to the left:

$$z = z(\beta-1)F_M + (z^2-\beta)F_{M-1} + \sum_{k=2}^{M-1} z^{k-1}(z^2-1)F_{M-k}.$$

Subtracting the first from the second and rearranging yields

$$\frac{F_1}{F_N} = \frac{(z^2-\beta)E_{M+1} + \sum_{j=2}^{N-M} z^{j-1}(z^2-1)E_{M+j}}{(z^2-\beta)E_{M-1} + \sum_{k=2}^{M-1} z^{k-1}(z^2-1)E_{M-k}}. \tag{29a}$$

In closed form this becomes

$$R \equiv \frac{F_1}{F_N} = \frac{\sqrt{3}(z^2-\beta)(y_1^{N-M}-y_2^{N-M})-\mathrm{i}z(y_1^{N-M-1}-y_2^{N-M-1})}{\sqrt{3}(z^2-\beta)(y_1^{M-1}-y_2^{M-1})-\mathrm{i}z(y_1^{M-2}-y_2^{M-2})}. \tag{29b}$$

Note that when the middle rib is driven, that is $M = \frac{1}{2}(N+1)$, this reduces trivially to $F_1 = F_N$.

Finally (29) can be used to obtain F_M, and explicit values for F_1 and F_N. Dividing equation $[M+1]$ by F_N and rearranging gives

$$E_M = -z^{-1}\left[\frac{F_1}{F_N}\sum_{j=1}^{M-1} z^{M+1-j}E_j - \tfrac{1}{\sqrt{3}}\mathrm{i}E_{M+1} + \sum_{j=M+1}^{N} z^{j-M-1}E_j\right].$$

From (28) for example

$$\frac{z}{F_N} = z(\beta-1)E_M + (z^2-\beta)E_{M+1} + \sum_{j=2}^{N-M} z^{j-1}(z^2-1)E_{M+j},$$

so that with the above form for E_M

$$F_N = z\left[(1-\beta)\frac{F_1}{F_N}\sum_{j=1}^{M-1} z^{M+1-j}E_j + \tfrac{1}{\sqrt{3}}\mathrm{i}\beta E_{M+1} + \sum_{j=1}^{N-M} z^{j-1}(z^2-\beta)E_{M+j}\right]^{-1}.$$

Evaluating the series eventually yields

$$E_M = -z^{-1}(\omega^2-4)^{-\frac{1}{2}}[z^2R(y_1^{M-1}-y_2^{M-1})+\beta(y_1^{N-M}-y_2^{N-M})+\tfrac{1}{\sqrt{3}}\mathrm{i}z(y_1^{N-M-1}-y_2^{N-M-1})], \tag{30}$$

$$F_N = z\surd(\omega^2-4)\,[z^2(1-\beta)\,R(y_1^{M-1}-y_2^{M-1})+(\tfrac{1}{\sqrt{3}}\mathrm{i}\beta+z^2-\beta) \times(y_1^{N-M}-y_2^{N-M})-\tfrac{1}{\sqrt{3}}\mathrm{i}\beta z(y_1^{N-M-1}-y_2^{N-M-1})]^{-1}, \quad (31)$$

and by symmetry

$$F_1 = z\surd(\omega^2-4)\,[z^2(1-\beta)\,R^{-1}(y_1^{N-M}-y_2^{N-M})+(\tfrac{1}{\sqrt{3}}\mathrm{i}\beta+z^2-\beta) \times(y_1^{M-1}-y_2^{M-1})-\tfrac{1}{\sqrt{3}}\mathrm{i}\beta z(y_1^{M-2}-y_2^{M-2})]^{-1}, \quad (32)$$

where for brevity $R = F_1/F_N$ has been retained and denotes the form (29*b*).

The solutions for the normalized waves and for the forces at the end-ribs and at the driven rib have now been obtained. At the middle and edges of the pass bands, however, the waveforms simplify. In addition those for the stop bands have a much simpler limiting form because one eigenvalue dominates. The solutions will be given in the following sections.

4. Special cases

This section considers the solutions at the frequencies marking the stop/pass band edges and those at the centres of the pass bands. The waveforms in these cases are obtained explicitly, and their limits for large N are found. Discussion of the stop band frequencies will be left until §5. For brevity the derivations will be restricted to the case of a driving force at one end, except for the two upper pass band edges.

At the band-edges the derivation of the eigenvectors breaks down, but (13)–(16) remain valid, and from these we can find the solution directly. When this is done it is found that the solutions at the edge frequencies do not strictly conform either to stop band or pass band behaviour. The upper edges have constant amplitude for each N, but as N increases this amplitude tends to zero; conversely at the lower edges the amplitude decreases from approximately 3 to nearly zero at the end, but does so linearly so that the amplitude at any given rib approaches the value 3 as N increases.

The parameter ω takes the value 2 when $\phi = \pi$ or $\frac{5}{3}\pi$, and $\omega = -2$ when $\phi = \frac{2}{3}\pi$ or 2π. Consider the action of P on any vector (x, y). When $\omega = -2$,

$$P(x,y) = (3x-2y, 2x-y) = (x,y)+2(x-y, x-y) \quad (33a)$$

and when $\omega = 2$

$$P(x,y) = (3x+2y, -2x-y) = (x,y)+2(x+y, -x-y). \quad (33b)$$

Now replace (x, y) by $(E_{N-1}, 1) = (\nu, 1)$ and apply this to the six cases.

(*a*) *Upper pass band edges*

$\phi = 2\pi$:

When $\phi = 2\pi$, $\nu = 1$ and from (33*a*) we obtain $P(\nu, 1) = (\nu, 1)$. Then $F_m = F_N$ for all $m > M$, and similarly $F_m = F_1$ for $m < M$. The expression in (29*a*) for the ratio F_1/F_N becomes 1 for all N and M, so that $F_m = F_N$ for all $m \neq M$. Explicit solutions are easily found when these values are substituted in equations $[M]$, $[M+1]$. These equations become

$$1 = \beta F_M+(N-1)\,F_N \quad \text{and} \quad 0 = F_M+(\beta+N-2)\,F_N.$$

The solution is thus

$$F_m = F_N = 3/(1+\mathrm{i}N\surd3) \quad (34a)$$

for all $m \neq M$. This has amplitude $3/\sqrt{(1+3N^2)}$, which tends to zero as N increases. The force F_M at the driven rib is:

$$F_M = F_N(1-N+\tfrac{1}{\sqrt{3}}\mathrm{i}). \tag{34b}$$

Therefore for fixed length N the wave propagates without attenuation, but the amplitude at any rib tends to zero as N approaches infinity. Furthermore the solution is independent of the location of the driving force.

$\phi = \pi$:

When $\phi = \pi$, $\nu = 1$ and (33*b*) again yields $P(\nu, 1) = (\nu, 1)$. Then the forces F_m have amplitude equal to F_N, when $m > M$, with alternating sign,

$$F_m = (-1)^{N-m} F_N, \tag{35a}$$

and similarly $F_m = (-1)^m F_1$ for $m < M$. The amplitudes of the forces at the end-ribs are again equal, but with sign depending on N,

$$F_1 = (-1)^{N-1} F_N. \tag{35b}$$

(*b*) *Lower pass band edges*

$\phi = \tfrac{2}{3}\pi$:

A more surprising situation arises at the lower pass band edges. When $\phi = \tfrac{2}{3}\pi$, $\nu \neq 1$ and (33*a*) gives $P(\nu, 1) = (\nu, 1) + 2(\nu-1, \nu-1)$ and since P acts as the identity on the second term on the right

$$P^m(\nu, 1) = (\nu, 1) + 2m(\nu-1, \nu-1).$$

The wave therefore grows linearly as it propagates away from the end-rib,

$$E_{N-m} = 1 + (N-m)(\nu-1).$$

This corresponds to linear decay from the driven rib and again inspection of the behaviour with increasing N shows that it belongs neither to a stop band nor a pass band.

Now $\nu = 2 - z$, so that

$$E_{N-k} = 1 + k(\nu-1) = 1 + k - kz. \tag{36a}$$

Let $M = 1$. Note that $z^2 = \bar{z}$ and $z^3 = 1$. Equation [2] then gives

$$-zE_1 = \beta[1+(1-z)(N-2)] + \sum_{j=3}^{N} z^{j-2}[1+(1-z)(N-j)].$$

Assuming for the moment that $N-2$ is a multiple of 3, the sum on the right can be written

$$\tfrac{1}{3}(N-2)(z+\bar{z}+1) + \tfrac{1}{6}(1-z)(N-2)[z(N-1)+\bar{z}(N-3)+(N-5)] = \tfrac{1}{3}(N-2)[2z-\bar{z}-1],$$

so that

$$\begin{aligned} E_1 &= -\bar{z}\beta[1+(1-z)(N-2)] - \tfrac{1}{3}(N-2)[2-z-\bar{z}] \\ &= -\bar{z}\beta - \tfrac{1}{3}(N-2)[3\beta(\bar{z}-1)+2-z-\bar{z}]. \end{aligned}$$

Since $\beta = \tfrac{2}{\sqrt{3}}\mathrm{i}\bar{z}$, and $z = \tfrac{1}{2}(\mathrm{i}\sqrt{3}-1)$, this simplifies to

$$E_1 = N - 1 + \tfrac{1}{\sqrt{3}}\mathrm{i}. \tag{36b}$$

From equations [1] and [2]

$$1/F_N = (\beta - z^2)E_1 + z(1-\beta)E_2.$$

Recall that $E_2 = 1+(N-2)(1-z)$ so that, for large N, $E_2 \sim \frac{1}{2}N(3-\mathrm{i}\surd 3)$, which has modulus $\surd 3N$. Then since

$$\beta - z^2 = \tfrac{1}{2}(3+\tfrac{1}{\surd 3}\mathrm{i}) \quad \text{and} \quad z(1-\beta) = -\tfrac{1}{2}(1+\tfrac{1}{\surd 3}\mathrm{i})$$

for large N the above expression yields

$$1/F_N \sim \tfrac{1}{2}N(1+\tfrac{1}{\surd 3}\mathrm{i})$$

so that, finally,

$$F_N \sim \frac{2\surd 3}{N(\surd 3+\mathrm{i})} \tag{36c}$$

and from (36*a*)

$$F_2 = F_N E_2 \sim 3(\surd 3-\mathrm{i})/(\surd 3+\mathrm{i}). \tag{36d}$$

Similarly from (36*b*) the driving force is

$$F_1 = F_N E_1 \sim 2/(1+\tfrac{1}{\surd 3}\mathrm{i}). \tag{36e}$$

This shows that the solution F_m decays linearly in amplitude from approximately 3 at the second rib to nearly zero at the end. Thus in the limit of large N, F_m tends to the value 3 for all m, and in this sense the frequency belongs neither to a pass nor a stop band.

The solution is similar when $N-2$ is no longer a multiple of 3, and identical in the limit of large N. The forces in the general case $M \neq 1$ may be found similarly.

$\phi = \frac{5}{3}\pi$:

At the lower edge of the second pass band, $\phi = \frac{5}{3}\pi$, the solution is similar. Again $\nu \neq 1$ and (33*b*) gives

$$P^m(\nu, 1) = (\nu, 1) + 2m(\nu+1, -\nu-1).$$

Now in this case $\nu = -(z+2)$. The wave again grows linearly but with alternating sign as it propagates away from the end-rib

$$E_{N-k} = (-1)^k(1-k(\nu+1)) = (-1)^k(1+k+kz). \tag{36f}$$

Since $\exp(\frac{5}{3}\mathrm{i}\pi) = -\exp(\frac{2}{3}\mathrm{i}\pi)$, (36*f*) gives the same values for the forces as (36*a*) apart from the alternating sign. The algebra carries through as before to give

$$F_N \sim (-1)^{N-1} 2\surd 3/N(\surd 3+\mathrm{i}), \tag{36g}$$

$$F_2 \sim -3(\surd 3-\mathrm{i})/(\surd 3+\mathrm{i}), \tag{36h}$$

and F_1 is given again by (36*e*).

(*c*) *Middle pass band frequencies*

As indicated earlier solutions for the frequencies in the middle of each pass band, that is $\frac{5}{6}\pi$ and $\frac{11}{6}\pi$, can also be written more simply than the equations in §3. In these cases, as noted by Sobnack (1991), the wave has constant amplitude along the array, but its value depends on whether the length is even or odd. (See Sobnack 1991 for a different approach.)

In each of these cases, ω vanishes, and $\nu = -z$, so that $(E_{N-1}, E_N) = (-z, 1)$, and $P = -1$. The wave therefore has constant amplitude, and takes the form

$$E_{N-2m} = (-1)^m, \quad E_{N-2m-1} = (-1)^{m+1}z \tag{37a}$$

provided M is less than $N-2m$ and $N-2m-1$ respectively, and similarly for the ribs to the left of M.

Now $y_1 = \mathrm{i}$, and $z = \frac{1}{2}(-\sqrt{3}+\mathrm{i})$ when $\phi = \frac{5}{6}\pi$, and $z = \frac{1}{2}(\sqrt{3}-\mathrm{i})$ when $\phi = \frac{11}{6}\pi$. Substituting these values in the respective equations gives the end-values F_1, F_N and the force F_M at the driven rib. Let $M = 1$. If N is even the second term in (26*b*) vanishes, and the first term becomes $\pm\frac{1}{6}(\sqrt{3}-\mathrm{i})$,

$$F_N = \pm 6/(\sqrt{3}-\mathrm{i}), \tag{37b}$$

which has modulus 3, where the plus sign applies in the lower pass band. When N is odd the first term vanishes and

$$F_N = -3/(1+\sqrt{3}\mathrm{i}), \tag{37c}$$

which has modulus $\frac{3}{2}$. Similar simple forms can be found for the force F_1 at the driven rib, and for the corresponding quantities when $M \neq 1$.

These two frequencies thus illustrate vividly the absence of an asymptotic solution as N tends to infinity.

5. Further results and illustrations

The above solutions allow us to examine the waveforms in some detail, and in particular to explore their dependence upon frequency, array length, and position of the driving force. Although the band structure is clear, several important quantities have yet to be evaluated explicitly. These include, in the stop bands, the decay-rate and the amplitudes around the driven rib, and in the pass bands details of the underlying interference pattern. This is the main purpose of this section, which includes several illustrative examples. It is shown in particular how the pass band response fails to approach an asymptotic limit. In addition the questions of maximum and minimum amplitudes along the array, and of periodic solutions, are briefly examined. The final results give the response $V(x)$ of the membrane beyond the ribbed region in terms of the force at the nearest end-rib.

Wavelengths and decay rates can be discussed in terms either of the eigenvalues λ_i or their square roots y_i, as in (23). Recall that as the frequency ϕ changes from 0 to 2π the solution sweeps alternately through stop and pass bands. The eigenvalues are at first real, in the stop band, the larger one increasing from 1 to its maximum in the middle and back to 1 at the pass band edge, and then wind once around the complex unit circle and back to unity at the next band edge.

The ratio F_{N-1}/F_N between the forces at the two end ribs is given by ν, and as mentioned above the forces are only constant along the array when $|\nu| = 1$, which happens only at the middle and upper edges of the pass bands. Figure 1 is a graph of this ratio against frequency.

(*a*) *Asymptotic solution in stop bands*

When ϕ is within a stop band the component $\mu_i \boldsymbol{x}_i$ of (E_{N-1}, E_N) associated with the smaller eigenvalue λ_i will decay exponentially as it propagates away from the Nth rib. The other component $\mu_j \boldsymbol{x}_j$ will increase exponentially at the same rate, and so for any frequency the solution at any rib simplifies and approaches a limiting form as the array-length increases. In the lower stop band the eigenvalues are positive and the larger one is λ_1, and in the upper stop band they are negative and λ_2 becomes the larger one.

Assume first that the driven rib is at the left, $M = 1$. The solution for the normalized values E_m from (23) is thus

$$E_m \sim y_j^{N-m-1}\mu_i \tag{38}$$

so that

$$|E_m| \sim |\mu_i y_j^{N-1}|\,\mathrm{e}^{-m\psi},$$

where $\psi = \ln(|y_j|) = \frac{1}{2}\ln(|\lambda_j|)$ defines the decay-rate. Since the eigenvalues are negative in the upper stop band the forces E_m have alternating sign there from one rib to the next. From (38) and (24) the force at the driven rib is

$$E_1 \sim \mu_i(y_j^{N-2} - z^{2-N})/(1-zy_j) - z^{1-N}\beta,$$

which simplifies to

$$E_1 \sim \mu_i y_j^{N-2}/(1-zy_j).$$

This in turn gives rise to a simple form for the end-value F_N,

$$F_N \sim \frac{y_j^{3-N}(y_i - z)(y_j - z)}{\mu_i[\beta(y_j - z) + zy_j(y_i - z)]}.$$

When this is combined with the previous expressions to find $F_m = F_N E_m$ the N-dependence vanishes, and the solution assumes the asymptotic form

$$F_m \sim y_j^{-m+2}\frac{(y_i - z)(y_j - z)}{\beta(y_j - z) + zy_j(y_i - z)} \tag{39a}$$

for $m \geqslant 2$ and

$$F_1 \sim \frac{y_j - z}{\beta(y_j - z) + zy_j(y_i - z)}. \tag{39b}$$

The decay-length, $1/\psi$, is shown in figure 2*a* as a function of frequency. The value reaches a minimum of $1/\ln(2+\sqrt{3})$, at the middle frequency $\phi = \frac{1}{3}\pi$, and tends to infinity at the band edges. The picture for the second stop band is identical, with the minimum decay-length again in the middle, at $\phi = \frac{4}{3}\pi$. The amplitudes of the driving force F_1 and of F_2 from (39*a*) are shown in figure 2*b*. Note that F_2 tends to zero at the lower band edge, which is consistent with the limit from the pass band side, (34*b*).

Now let $M \geqslant 2$. Then it is easily shown from (29) that the ratio F_1/F_N takes the form

$$F_1/F_N \sim y_j^{N-2M+1}. \tag{39c}$$

A more useful measure, however, is the ratio of the forces immediately adjacent to the driving force. Since $F_{M-1}/F_{M+1} \equiv E_{M-1}F_1/E_{M+1}F_N$ it is easy to show that

$$F_{M-1}/F_{M+1} \sim 1 \tag{40}$$

provided the driven rib is reasonably far from either end. Thus, as one would expect in this case, the forces to either side of the driven rib are equal. This is in sharp contrast with the pass bands, as will be shown.

(*b*) *Periodicity, amplitude bounds, and other results in pass bands*

The behaviour which occurs in the pass bands is more complex. Associated with each frequency is a wave, (23), resulting from the interference between the eigencomponents. It is useful to consider the continuous analogue

$$f(x) = y_2^x\mu_1 + y_1^x\mu_2$$

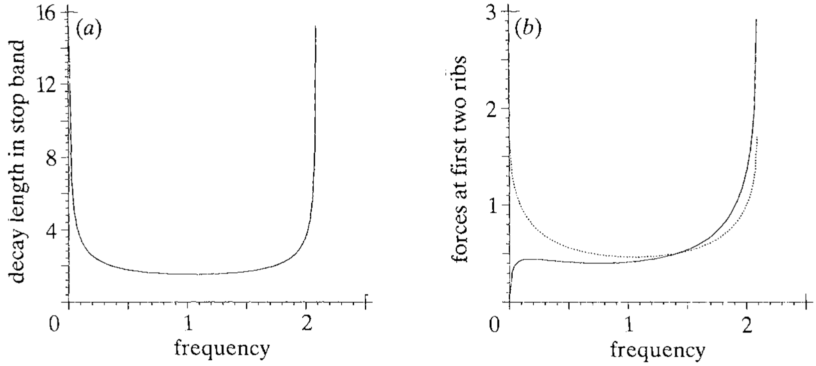

Figure 2. Graphs as functions of frequency in the lower stop band of (*a*) the decay length $1/\psi$, and (*b*) the forces at the driven rib (dotted line) and the adjacent rib (full line) in an array of length 20 with the first rib driven.

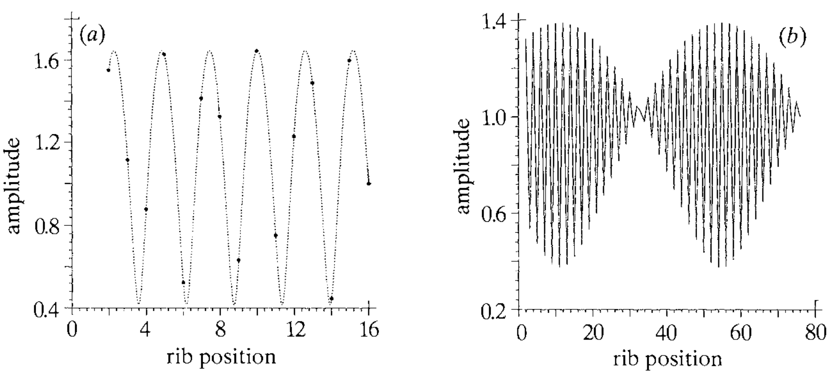

Figure 3. Normalized amplitudes along the array at two pass band frequencies, as a function of rib position (*a*) with $\phi = \frac{7}{9}\pi$ (dots) against the continuous wave $f(x)$, over 15 ribs and (*b*) with $\phi = 2.6$ over 76 ribs. Neither of these includes the driving forces.

of the expression in (23). As the frequency changes across each pass band, the eigenvalues λ_i make one complete circuit of the unit circle, coinciding only at the edges and middle of the band, where they take the values 1 and -1 respectively. Their square roots y_i travel halfway round the circle, from 1 to -1 as frequency increases in the lower pass band and from -1 to 1 in the upper band. The wavelength of the continuous wave approaches infinity at the edges and is at a minimum of 2 in the middle of the bands, so that the rib spacing is never greater than half a wavelength. Figure 3 shows the amplitude of the forces E_m for two typical frequencies, where the driving force at x_M is anywhere to the left. In the first of these, figure 3*a*, $f(x)$ has been superimposed on the forces along the last 15 ribs to the right. Since the normalized wave is independent of the length N, this plot represents the last 15 ribs of any system with $N-M \geqslant 16$. Figure 3*b* shows the response, which in this case is more regular, of a longer array driven at a different frequency.

It is clear when we consider the normalized wave as the driving frequency ϕ changes across the pass band that the pattern becomes increasingly sensitive to ϕ as distance from the end-ribs increases. The reason is simply that a small change in ϕ leads to a large change in y_i^n when n is large; in other words the interference patterns (23) at slightly different frequencies propagate at different speeds and eventually

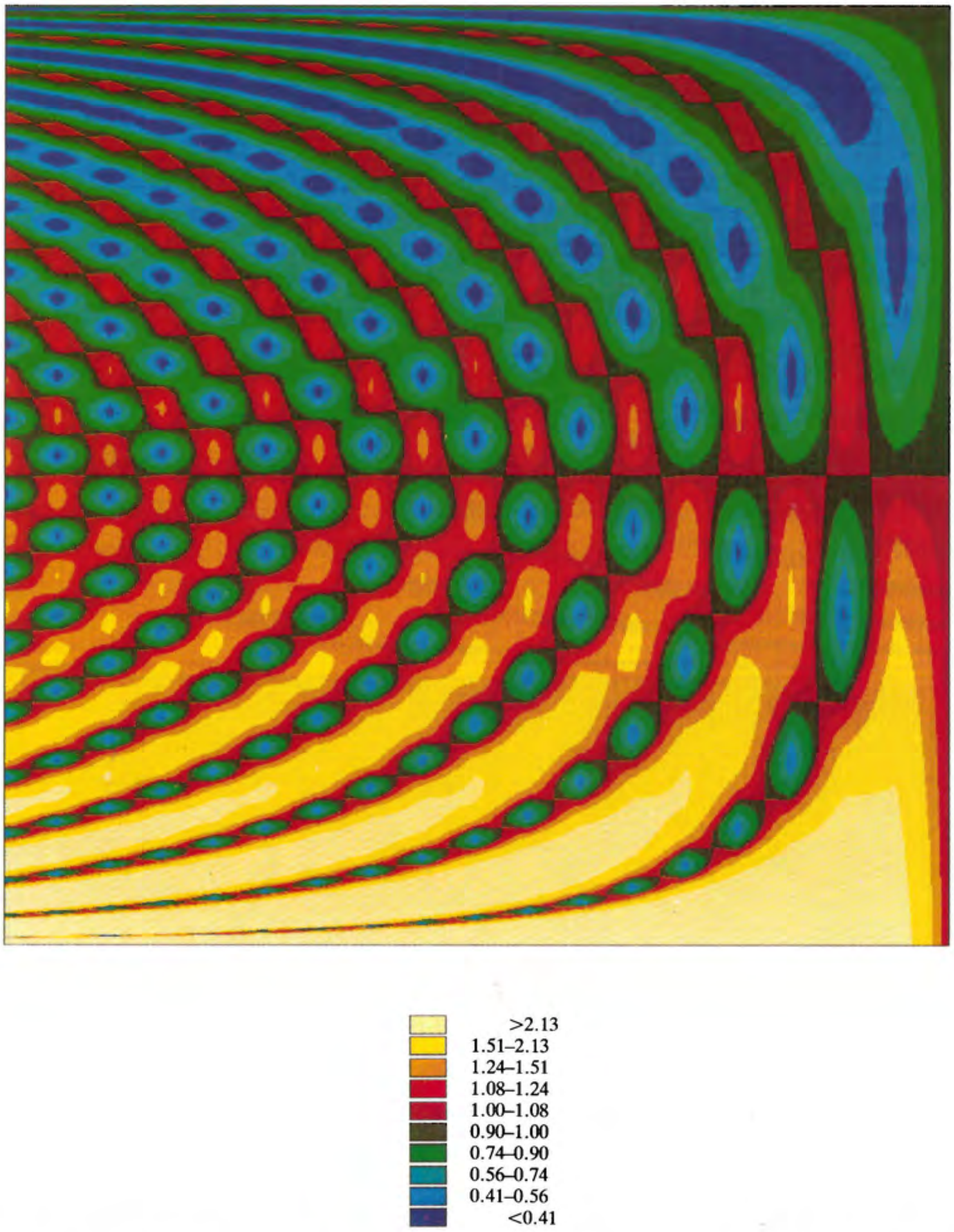

Figure 4. Contour plot of the normalized forces E_m (equation (23*a*)) along an array of 20 ribs of which the first is driven, shown as a function of frequency and of m. The vertical axis represents frequency, rising through the lower pass band from $2\pi/3$ to π, and the horizontal corresponds to rib location m, increasing from left to right and excluding the driven rib.

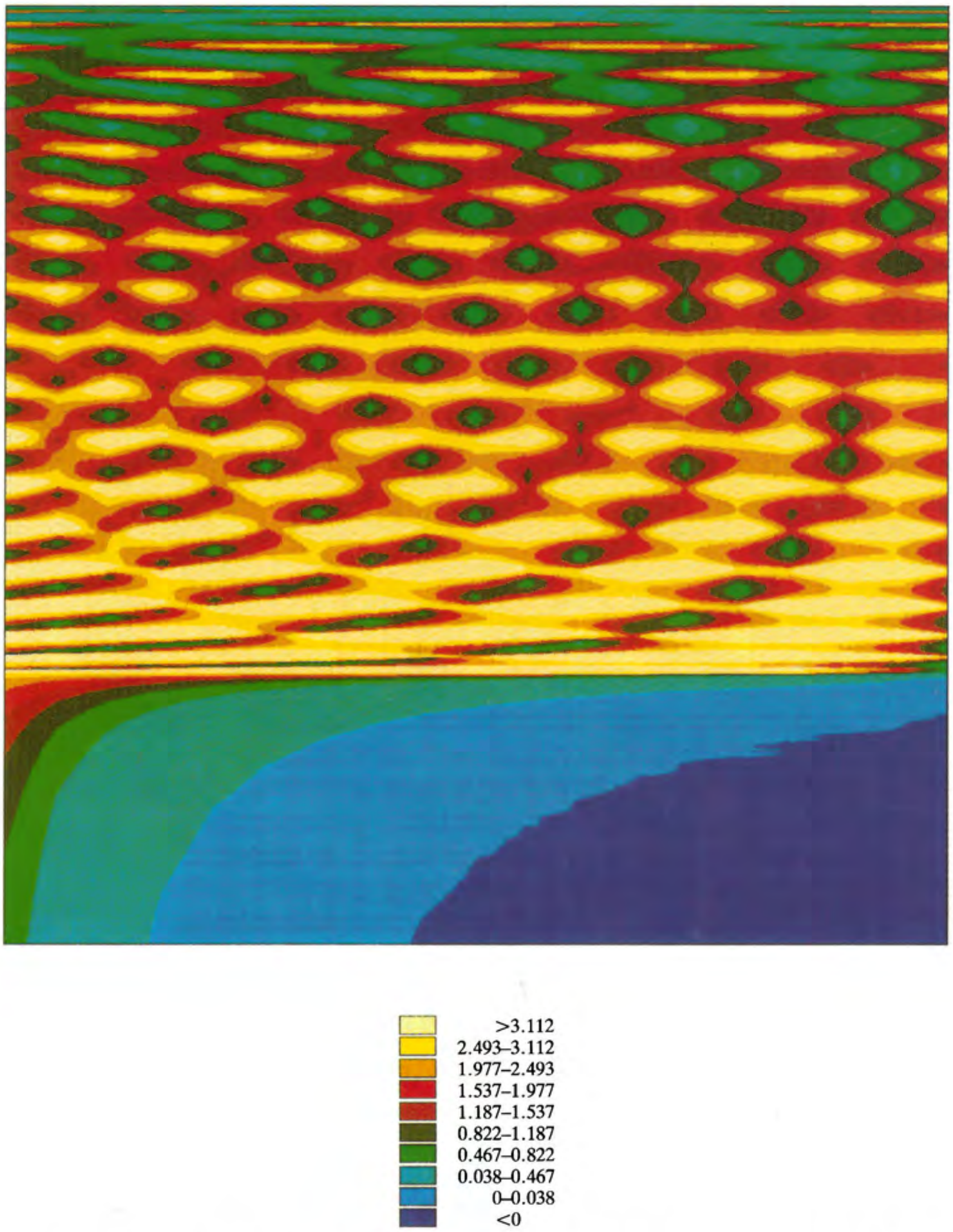

Figure 5. Contour plot of the forces F_m corresponding to the system shown in figure 4, with the frequency range extended downwards to include part of the lower stop band. Again frequency and rib location are represented by the vertical and horizontal axes respectively.

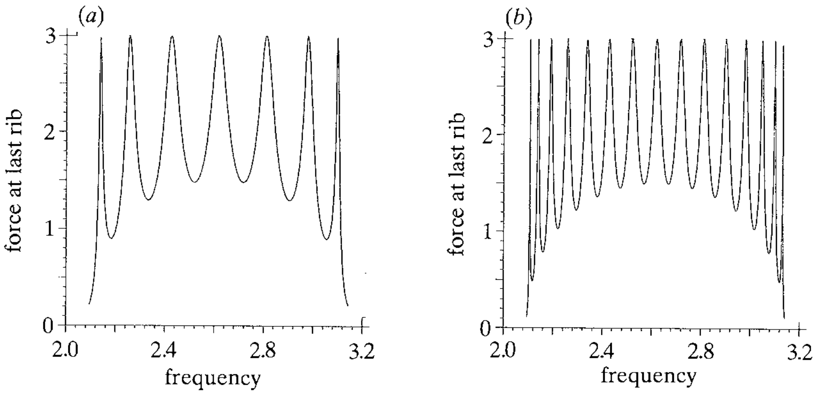

Figure 6. Amplitude of the forces at the end rib in systems driven at the first rib (26) and functions of frequency, for (a) $N = 8$ and (b) $N = 16$.

diverge. This is illustrated in figure 4, plate 1, a contour plot showing the normalized amplitudes $|E_m|$ along the last 19 ribs as a function of m and of frequency across the lower pass band. Again, this plot represents the last 19 ribs to the right of any system with $N-M \geqslant 20$. Figure 5, plate 2, shows the corresponding full (non-normalized) solution $|F_m|$. Here $N = 20$ and the system is driven at the first rib (not shown). The frequency range in this graph has been extended to include part of the lower stop band to illustrate the rapid transition from smooth to fluctuating pass band behaviour. Again, the sensitivity to frequency which is apparent here is greater for longer arrays. (Some values appear negative due to rounding errors on the colour scale.)

It is less obvious that the same is true of the end-values F_N as N increases. Figure 6 shows the modulus of F_N in the lower pass band, for $N = 8$ and $N = 16$. The force oscillates increasingly rapidly as N increases, reaching a maximum value of 3 for all N. (For comparison, the case $N = 20$ corresponds to the right-hand edge of figure 5.) At the band edges, as has been seen, the amplitude F_N at the end-rib tends to zero as N increases. At the same time, however, *maxima* of F_N occur increasingly close to the edge frequencies.

Consider briefly those frequencies ϕ for which the normalized wave E_m is periodic in m. These cases correspond to those values of y_i which are roots of unity, and they can be found by solving $y^n = 1$ for a given n. This will not be done here. However, it is seen immediately from (25), (26*b*) that the forces at the driven and end ribs are also periodic, with the same period. Therefore these periodic frequencies give rise to fully periodic solutions.

A problem of greater interest is to estimate the maximum and minimum amplitudes attained by E_m or F_m along the array for each frequency as the array length is increased. If y_i is *not* a root of unity then clearly the forces E_m will eventually explore almost all possible values of the continuous wave $f(x)$ (see figure 3). More precisely, for any x we can find values k such that E_{N-k} is arbitrarily close to $f(x)$. Therefore at non-periodic frequencies the supremum and infimum of the amplitudes of E_m are just those of the wave $f(x)$ itself. Furthermore since there are countably many periodic frequencies this holds for almost all ϕ in the pass bands.

Now, since $y_2 = y_1^{-1}$ and $|y_1| = |y_2| = 1$,

$$|f(x)| = |\mu_1 + y_2^{-2x}\mu_2| = |\mu_1| \times |1 + y_2^{-2x}\mu_2/\mu_1|.$$

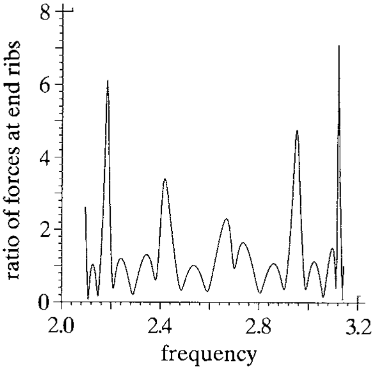

Figure 7. Modulus of the ratio F_1/F_N between the two end forces (29) for an array of length 20 with the driving force located at the sixth rib.

This reaches a maximum when

$$y_2^{-2x}\mu_2/\mu_1 = |\mu_2/\mu_1|,$$

and a minimum when

$$y_2^{-2x}\mu_2/\mu_1 = -|\mu_2/\mu_1|.$$

The supremum E_+ and infimum E_- of $|E_m|$ are thus almost everywhere

$$E_+ = |\mu_1|+|\mu_2|, \quad E_- = ||\mu_1|-|\mu_2||. \tag{41}$$

Thus, to the right of the driving force, for any N the amplitudes $|F_m|$ are bounded above by $F_+ = F_N E_+$, and below by $F_- = F_N E_-$. The ratio E_+/E_- between these quantities describes the maximum possible relative amplitude change along the array; this value is approached asymptotically for large N at almost all frequencies. Now, $E_- \leqslant |\nu|$ and is therefore uniformly bounded. However, $|\mu_i|$ tends to infinity as the frequency approaches either of the lower pass band edges (with the inverse square root of the frequency difference). Thus although the solutions at the lower edges have a finite limit as $N \to \infty$, the quantities E_+ and the ratio E_+/E_- grow unboundedly as these edges are approached. Furthermore, when $M = 1$, say, since the end amplitude $|F_N|$ has peaks of 3 arbitrarily close to the edges as N increases, the *actual* maxima F_+ also tend to infinity. Similarly E_+/E_- tends to infinity (again as an inverse square root) as the upper edge is approached since E_- tends to zero. This represents an extreme departure from the solutions obtained for infinite arrays. Since E_+, E_- are independent of N and the amplitudes $|F_N|$ are uniformly bounded by 3, the function $3E_+$ is a convenient upper-bound on the amplitudes for all N.

Now consider the forces at the end ribs when the source of excitation is located away from the end, $M \neq 1$. The amplitude of the ratio F_1/F_N (29) in the pass band fluctuates with frequency and with position of M. Figure 7 shows this ratio as a function of frequency for $N = 20$ and $M = 6$. This quantity also fluctuates increasingly rapidly as N increases.

(c) *Structural response beyond the array*

Although the main concern of this paper is to characterize the forces along the array, it is also of some interest to find the response of the membrane beyond the region supported by ribs. This is easily found from the equation

$$V(x) = \sum_{m=1}^{N} F_m G(x-x_m)$$

given in §2. Let x' first be the scaled distance of x beyond x_N, so that $x' = x/h - N$. Then provided x' is reasonably large the Green function $G(x-x_N)$ can be approximated by G_s, and the above equation can be written

$$V(x) = \sum_{m=1}^{N} F_m \exp(\mathrm{i}\phi[x'+N-m]) = \exp(\mathrm{i}\phi x') \sum_{m=1}^{N} F_m z^{N-m}.$$

Now equation [N] can be written

$$0 = \sum_{m=1}^{N} F_m z^{N-m} - \tfrac{1}{\sqrt{3}}\mathrm{i} F_N$$

and substituting this in the previous equation gives

$$V(x) = \tfrac{1}{\sqrt{3}}\mathrm{i} \exp(\mathrm{i}\phi x') F_N, \tag{42a}$$

where F_N is given by (26*b*) when $M = 1$ and (31) otherwise. The response $V(x)$ is therefore a wave with constant amplitude $\frac{1}{\sqrt{3}}|F_N|$ propagating with wavenumber ϕ. As we have seen this amplitude is approximately zero in the stop bands and in the pass bands is extremely sensitive to frequency, oscillating when $M = 1$ with a maximum of $\sqrt{3}$. Similarly, the solution for $x < x_1$, provided $M \neq 1$, is

$$V(x) = \tfrac{1}{\sqrt{3}}\mathrm{i} \exp(\mathrm{i}\phi x'') F_1, \tag{42b}$$

where x'' is the scaled distance $x'' = 1 - x/h$, and F_1 is given by (32). Finally, when $M = 1$, (42*b*) must be modified slightly because of the non-zero velocity which appears on the left-hand-side of equation [1], to obtain

$$V(x) = (1 + \tfrac{1}{\sqrt{3}}\mathrm{i}) \exp(\mathrm{i}\phi x'') F_1, \tag{42c}$$

where F_1 is now given by (27).

6. Summary and concluding remarks

The full solution has been obtained for a finite array of equally spaced ribs driven at any given location, and the resulting stop/pass band structure has been investigated. The general case, excluding the boundaries between bands, is described by equations (23)–(32) in §3. Equations for these boundaries and other special cases are given in §4, and limiting forms for the stop bands in §5. In the stop bands the energy is exponentially and, for large arrays, symmetrically localized around the driven rib, and in the pass bands it propagates unattenuated, but with large amplitude fluctuations at almost all frequencies. The solution is conveniently interpreted in terms of a wave propagating inwards from each free end, which interacts with the driven rib and in doing so acquires a complex multiplying factor. These waves propagate at speeds which change with the driving frequency, and as a consequence the response becomes highly sensitive to changes in frequency as the array length increases. This remark extends to the structural response of the membrane beyond the ribbed region. Thus it is clear both from the normalized forces (23) and from the multiplying end values (e.g. (26)) that in the pass bands the response cannot approach a meaningful limit as the array length is increased. The absence of an asymptotic limit is also apparent from the solutions at the middle of the bands (37) and from the response near the edges (the functions E_+, E_+/E_- from (41)).

The method of solution is possible because the underlying Green function G has the form $G_s(x) \sim a \exp(bx)$ for $x \neq 0$. Although the solutions in this paper have been formulated for one such Green function, most of the results in §3 hold for any function of the above form. Detailed properties of the solution, however, such as the band structure and the behaviour within the bands depend through ω upon the relationship between $\exp(i\phi)$ and β.

These results do not elucidate the response in régimes in which the acoustic component G_a plays a significant role, or those in which the ribs are irregularly spaced. Such structures, studied for example by Crighton (1983, 1984) and Sobnack (1991), exhibit essentially different behaviour for sufficiently large arrays. One might extend the approach to include G_a; this gives rise, approximately, to coupling over a finite number n of ribs, analogous to (11), which yields a $(n-1)\times(n-1)$ propagation matrix. The difficulty then arises of specifying the initial $(n-1)$-dimensional vector for the matrix. On the other hand when the ribs are irregularly spaced this procedure will lead to a set of propagation matrices which vary along the array, and which are not statistically independent. It is hoped to address these problems in future work.

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