Left-right splitting for electromagnetic scattering in 3D

M. Spivack, J. Ogilvy and C. Sillence

Abstract: The left–right splitting method and its application to electromagnetic scattering by large 3D scatterers are described. Exact numerical solutions to the governing integral equations can be prohibitively expensive for large scatterers. Under the assumption that energy is predominantly forward-scattered, the solution is expressed as a series of terms, each of which is rapidly and efficiently evaluated. In many cases only one or two terms are needed, and the formulation provides additional physical insight.

1 Introduction

This paper describes the left-right (L–R) splitting algorithm and its application to electromagnetic scattering by 3D scatterers. Surface currents are as usual expressed as the solution of a boundary integral equation, which can become prohibitively expensive for large or complex problems. The L–R operator splitting, used for 2D (scalar) problems in [1, 2] and related methods [3–5], expand the solution of this equation as an operator series in increasing orders of multiple scattering. Each term can be evaluated rapidly, and the approximation is provided by truncating the series.

The approach has proved versatile and has been applied to extended rough surfaces as well as finite scatterers, converging for a wide range of incident angles. The approximation can provide considerable physical insight, and truncation errors can be examined theoretically, in some sense independently of discretisation.

In many cases only a single iteration is needed to provide accurate results. At its simplest the evaluation time scales with $O(f^4)$, where f is frequency. This is sufficient to treat electrically large scatterers, but could be accelerated to $O(f^3)$ by use of fast matrix multiply methods (e.g. fast multipole).

2 Method and applicability

The method arose originally in the treatment of low-grazing-angle scatter by rough surfaces. Suppose cartesian co-ordinates (x,y,z) are specified where z is the vertical, and x is in the plane of incidence and can be thought of as a 'range' direction. If a known radar field is incident on, say, an extended rough surface, the induced surface current can be obtained by solution of the magnetic field integral equation

$$J_{inc}(r_s) = \frac{1}{2}\boldsymbol{J} - \boldsymbol{n} \times \int_{surface} \boldsymbol{J} \times \nabla G dS \qquad (1)$$

or more conveniently in terms of operator notation $\boldsymbol{J}_{inc} = \boldsymbol{A}\boldsymbol{J}$ where $\boldsymbol{r}_s = (x, y, z)$, all terms except $\boldsymbol{J}$ are known, $\boldsymbol{J}_{inc} = \boldsymbol{n} \times \boldsymbol{H}_{inc}$ and $\boldsymbol{n}$ is the surface normal. By splitting the integral into two halves with respect to x to the left and right of the observation point $\boldsymbol{r}_s$, this can be written $\boldsymbol{J}_{inc} = (\boldsymbol{L} + \boldsymbol{R})\boldsymbol{J}$, where the Cauchy principal value is assumed to be in $\boldsymbol{L}$. Formally the solution $\boldsymbol{J} = \boldsymbol{A}^{-1}\boldsymbol{J}_{inc}$ can be expanded and written as a series

$$\boldsymbol{J} = (\boldsymbol{L}^{-1} - \boldsymbol{L}^{-1}\boldsymbol{R}\boldsymbol{L}^{-1} + \ldots)\boldsymbol{J}_{inc} \qquad (2)$$

The key step is that under the observation that the effect of $\boldsymbol{R}$ is in some sense small, this series converges quickly and can be truncated. Physically this corresponds roughly to the assumption that surface interactions are primarily 'from the left', as expected in this scattering regime. More precisely, in many cases R is effectively small by comparison with L for two reasons: first, R excludes the 'diagonal' or principal value, and indeed vanishes in the limit of scattering by an infinite plane; secondly, when applied to a predominantly right-going wave a rapid phase-variation occurs in the integrand. This has been borne out numerically, but in general it cannot be made precise as it depends on the exact scatterer shape and incident field.

Once the surface currents are obtained the scattered electromagnetic fields at all angles are found by integration of these currents over the surface.

Considerable success has been obtained in applying this approach to perfectly conducting rough surfaces and waveguides, and for radar interaction with closed scatterers. For a wide range of incident angles accurate results are obtained using only one or two iterations.

3 Numerical evaluation and cost

Discretisation of the system is straightforward and gives rise to a matrix equation, with $\boldsymbol{A}$, $\boldsymbol{L}$, and $\boldsymbol{R}$ replaced by matrices. The number of surface points N required in the discretisation scales with the square of the frequency f^2. The main computational advantage arises from the cost of inversion of $\boldsymbol{L}$ compared with that of $\boldsymbol{A}$. In effect, $\boldsymbol{A}$ is a full matrix so that the time needed for 'exact' inversion scales with f^6,

IEE Proceedings online no. 20040945

doi:10.1049/ip-smt:20040945

Paper first received 10th March 2004 and in revised form 28th June 2004. Originally published online: 2nd November 2004

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whereas $\boldsymbol{L}$ (which in the 2D case is lower triangular) can be inverted exactly by a marching method of $O(f^4)$. Further terms in the series involve multiplication by $\boldsymbol{R}$ but inversion only of $\boldsymbol{L}$, so computation time is proportional to the number of terms used.

3.1 Formulation of matrix equations

The computational details are most easily understood by first describing their application to 2D scattering and then extending to the general 3D case. Consider a surface $h(x,y)$ having corrugations in the y-direction and incident field vector lying in the (x,z)-plane. The y-dependence can then be ignored and the equations for a TM field (say) reduce to

$$H_{inc}(\boldsymbol{r}_s) = H(\boldsymbol{r}_s) - \int_{-\infty}^{\infty} \frac{\partial G(\boldsymbol{r}_s, \boldsymbol{r}')}{\partial \boldsymbol{n}} H(\boldsymbol{r}') d\boldsymbol{S} \qquad (3)$$

where integration is over the surface, G (in this case) a zero-order Hankel function and $\boldsymbol{r}_s = (x, h(x))$, $\boldsymbol{r}' = (x', h(x'))$ both lie on the surface. Equation (3) can be written $H_{inc}(\boldsymbol{r}_s) = (L + R)H$ with L and R defined by

$$\begin{aligned} Lf(x,z) &= -\int_{-\infty}^{x} G(\boldsymbol{r}, \boldsymbol{r}') f(x') dS, \\ Rf(x,z) &= \int_{x}^{\infty} G(\boldsymbol{r}, \boldsymbol{r}') f(x') dS \end{aligned} \qquad (4)$$

and L includes the principal value. This system is then discretised with respect to x, choosing N equally-spaced points $\{x_n\}$. The integral is thus written as a sum of subintegrals, over each of which the unknown part of the integrand $H(\boldsymbol{r}')$ is treated as constant and the remaining part of the integrand is a known function of $h(x)$. Writing this for each of the N surface points gives rise to a matrix equation, say $H_{inc} = AH$. If we define the slope variable $\sigma_n = \sqrt{1 + h'(x_n)^2}$ where $h' = dh/dx$, the matrix entries can be approximated by

$$\begin{aligned} A_{mm} &= -\left[\frac{1}{2} - \frac{\delta}{\sigma_m^2 \pi} h''(x_m)\right]; \quad \text{and} \\ A_{mn} &= \frac{i\delta}{4} \sigma_n \left.\frac{\partial H_0^{(1)}}{\partial n}\right|_{kr_{mn}} \quad \text{for} \quad m \neq n \end{aligned} \qquad (5)$$

It is straightforward to obtain more accurate formulations if necessary. Applying the left–right splitting series (2) to H in place of $\boldsymbol{J}$, the discretised operators L, R give rise to matrices given, respectively, by the lower and upper triangular parts of the matrix A in (5), with the diagonal included in L. As L is lower triangular, it can be inverted rapidly and efficiently by back-substitution giving the first term in the series. (In effect, this 'marches' the solution for the surface current from left to right.) Further terms in the series can be obtained similarly, combining L^{-1} with multiplication by R.

Returning to the general 3D case, (1) transforms to range and transverse co-ordinates x, t

$$\boldsymbol{J}_{inc}(\boldsymbol{r}_s) = \frac{1}{2}\boldsymbol{J} - \boldsymbol{n} \times \int \left[\int_0^{\infty} \gamma(\boldsymbol{r}') \boldsymbol{J} \times \nabla G \, dx'\right] dt \qquad (6)$$

where γ is the Jacobian of the co-ordinate transformation. The operator $\boldsymbol{L}$ is obtained as for the 2D case, replacing the infinite upper limit by x in the inner integral. Discretisation of this system is again straightforward and results in matrix forms for $\boldsymbol{L}$ and $\boldsymbol{R}$. These matrices have dimensions $2M \times 2M$, say, where M is the total number of surface points at which solution is required. The key computational step is the inversion of $\boldsymbol{L}$ which (analogous to 2D) is obtained by marching the surface current in the x-direction.

The system is well-suited to parallelisation and such calculations have been carried out on the Cranfield–Cambridge High-performance computer facility. The marching technique for evaluation of $\boldsymbol{L}^{-1}$ falls into two main parts at each step: a matrix multiplication and an inversion of a 'small' matrix whose dimension is given by the number of surface points in the plane transverse to the x-axis. For example, without further approximation, evaluation for a scatterer of around 220 by 20 wavelengths can be carried out in around 100 minutes using 32 processors on the SunFire machine. Current amplitudes on a waveguide of these dimensions are shown in Fig. 1 as a shaded contour plot. The electric field on a transect some distance along a similar waveguide is shown in Fig. 2.

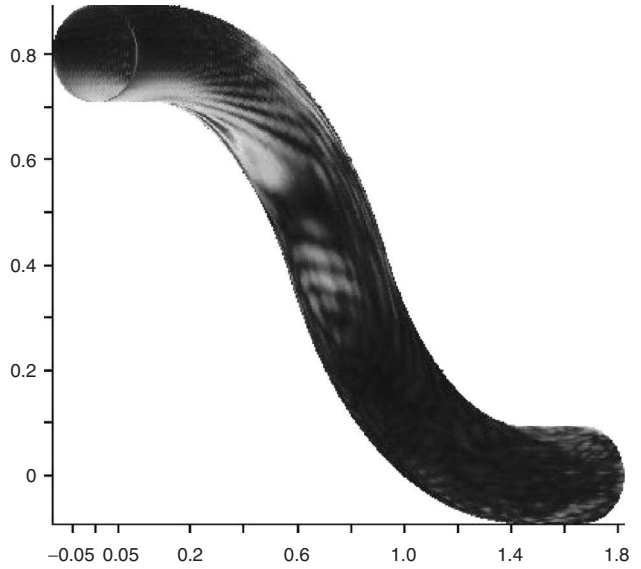

Fig. 1 *Surface current amplitudes on waveguide of length 220 and diameter 20 wavelengths*

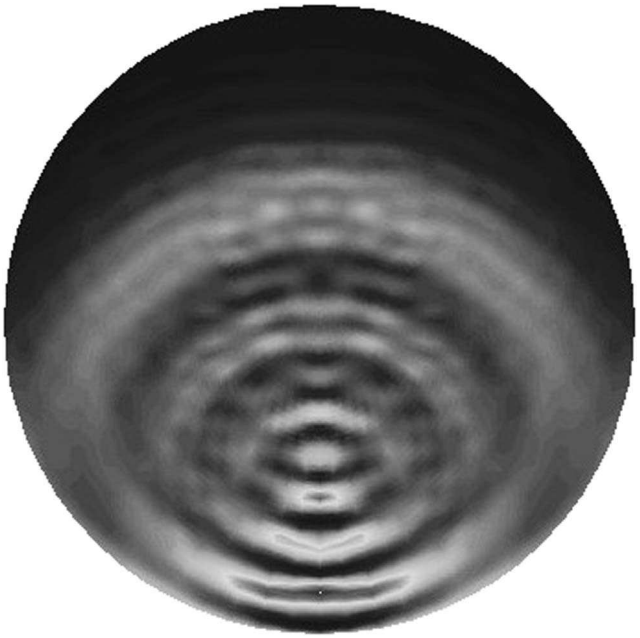

Fig. 2 *Electric field amplitude on plane transect along waveguide of Fig. 1*

For a typical scatterer the matrix inversions form a small part of the overall computation time, compared with the matrix multiplications. Evaluation time for these scales with

$O(f^4)$ because the Green's function must be evaluated for each pair of surface points. The L–R series, before further approximation, allows solution for electrically large scatterers in reasonable times. If necessary accelerated methods could be applied including those based on fast multipole or wavelets to reduce this to $O(f^3)$.

4 Further remarks

The method has been described for the magnetic field integral equation (MFIE) rather than the electric field equation (EFIE). This is in part because of the better convergence properties of MFIE sometimes found with iterative schemes such as standard method of moments. However, it is emphasised that the L–R splitting approach is not restricted to MFIE. Indeed, in treating 2-D problems it was applied with equal success to TE and TM waves represented, respectively, by EIFE and MFIE. Similarly it is applicable to acoustic scattering for which it was originally proposed [1].

Some remarks should be made by comparison of L–R splitting with other techniques such as the multilevel fast multipole method (MLFMM). First, L–R splitting has a fairly clear physical basis in that where energy propagates predominantly in one direction the first term in the series dominates; although convergence may be rapid even when this condition is not strongly satisfied. Secondly, our experience on waveguide problems has been that L–R splitting has been much more efficient in both speed and memory requirements than, for example, MLFMM. (Such comparisons have not yet been made for fully optimised versions.) Thirdly, methods such as FMM are normally applied *in combination* with approximate methods for inversion of the large matrix, whereas L–R splitting is aimed directly at this inversion; therefore as mentioned above acceleration methods such as FMM can be combined with L–R splitting if necessary.

The approach here lends itself naturally to planar or near-planar structures for which it was originally developed, and can in principle treat dielectric or multilayered materials. One way to do this exactly is to consider the full coupled equations for the electric and magnetic currents on each interface. However, the number of unknowns then increases proportionately, leading to a corresponding increase in memory and computation time.

5 Acknowledgments

Development of the left-right splitting method for electromagnetic scattering in 3D was carried out under a continuing research programme funded by BAE Systems Advanced Technology Centre, Filton. The provision of additional computing facilities by the Cranfield–Cambridge HPCF, and work on visualisation of calculated fields, supported by the Cambridge eScience Centre under UK eScience funding, is also acknowledged. The authors are grateful to colleagues in Cambridge and Filton for much help and numerous contributions to this work.

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